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TOPOLOGICAL AND GEOMETRICAL PROPERTIES OF MAPPINGS WITH SUMMABLE JACOBIAN IN SOBOLEV CLASSES. I

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Let $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \geq 2$, be a continuous mapping. Then f is *open* if the image of an open set is open; and f is *discrete* if the inverse image $f^{-1}(y)$ of every point $y \in \mathbb{R}^n$ consists of isolated points. The aim of the present article is to indicate analytical conditions on a mapping which guarantee certain topological properties.

It is convenient to state analytical requirements on $f : G \to \mathbb{R}^n$ in terms of Sobolev spaces. We suppose that all coordinate functions f_i of $f = (f_1, \dots, f_n)$ belong to the Sobolev space $W^1_{p,\mathrm{loc}}(G)$. Thereby the formal Jacobian matrix $Df(x) = (\frac{\partial f_i}{\partial x_j})$, $i, j = 1, \dots, n$, and the Jacobian determinant $J(x, f) = \det Df(x)$ are defined almost everywhere in G. The norm $|Df(x)|$ of $Df(x)$ is the norm of the linear operator determined by this matrix in $\mathbb{R}^n$.

A modern way of studying the topological characteristics of mappings by means of their analytical properties was paved by Yu. G. Reshetnyak while working on the problems of the theory of spatial mappings with bounded distortion [1]. Recall that a mapping $f : G \to \mathbb{R}^n$ is a *mapping with bounded distortion* if the following conditions are satisfied:

(1) $f \in W^1_{n,\mathrm{loc}}(G)$;

(2) there is a constant $K \in [1, \infty)$ such that $|Df(x)|^n \leq KJ(x, f)$ almost everywhere in G.

The least constant in this inequality is called the *quasiconformality coefficient.* Yu. G. Reshetnyak proved that a mapping with bounded distortion is continuous, open, and discrete [1]. The key point of the proof is a close connection between mappings of this class with quasilinear elliptic equations and nonlinear potential theory, and the method is widely used in the topic under consideration. Observe that continuity of a mapping with bounded distortion ensues from a more general result of [2] (a simpler proof of the corresponding theorem of [2] was given in [3]).

It is convenient to write down analytical constraints on f as the requirement of finiteness of various norms of the local distortion

$$K(x) = \frac{|Df(x)|^n}{J(x, f)} < \infty$$

almost everywhere in G. Thus, the inequality $1 \leq K(x) < \infty$ for almost every $x \in G$ means that $J(x, f) > 0$ almost everywhere on the set $\{x : Df(x) \neq 0\}$. We put $K(x) = 1$ at the points where the numerator and the denominator vanish simultaneously. A mapping $f \in W^1_{n,\mathrm{loc}}(G)$ has bounded distortion if and only if $K(x) \in L_\infty(G)$.

The necessity of studying the topological properties of mappings arises also in the problems of nonlinear elasticity [4–10]. It was shown in [4, 5] that boundedness of $K(x)$ is too burdensome in problems of nonlinear elasticity: the situation is typical in which the function $K(x)^p$ is integrable for some $p < \infty$. It was established in [7] that a continuous nonconstant plane mapping f satisfying the conditions $f \in W^1_{2,\mathrm{loc}}(G)$, $G \subset \mathbb{R}^2$, and $K(x) \in L_{1,\mathrm{loc}}(G)$ is open and discrete. The proof of this result grounds on the two-dimensional theory of Beltrami equations and relies on the fact that such a mapping can be represented as a composite of some analytic function and homeomorphism (thus, an analog of the Stoilov factorization theorem is valid for mappings in this class). In [10], Reshetnyak's

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theorem was generalized to nonconstant mappings $f \in W^1_{n,\mathrm{loc}}(G)$, $G \subset \mathbb{R}^n$, with $K(x) \in L_{p,\mathrm{loc}}(G)$, $p > n-1$.

In connection with the problems of nonlinear elasticity, J. Ball [4, 5] defined the mapping classes

$$\mathscr{A}_{p,q}(\Omega) = \{f \in W^1_p(\Omega) : \operatorname{adj} Df \in L_q\},$$

where $p \geq n-1$, $q \geq p/(p-1)$, and the *adjugate* matrix $\operatorname{adj} Df$ of Df is defined from the condition $Df(x) \operatorname{adj} Df(x) = J(x,f)\,\mathrm{Id}$ for almost all x. Thus, $J(x,f) \in L_1(\Omega)$ if $f \in \mathscr{A}_{p,q}$.

A mapping $f : G \to \mathbb{R}^n$ is *quasilight* if the connected components of the inverse image $f^{-1}(y)$ are compact for each point $y \in f(G)$.

It was proven in [9] that each continuous quasilight mapping $f \in \mathscr{A}_{p,q}$, with $K(x) \in L^{n-1+\varepsilon}(\Omega)$, $\varepsilon > 0$, is open and discrete. The methods of [9, 10] are a further development of Yu. G. Reshetnyak's arguments of [1].

In the present article, we obtain topological results for mappings $f \in W^1_{q,\mathrm{loc}}(G)$ under the following constraints:

$(M1)$ $q \geq n-1$ for $n=2$ and $q > n-1$ for $n \geq 3$;

$(M2)$ $J(x,f) \geq 0$;

$(M3)$ $J(x,f) \in L_{1,\mathrm{loc}}(G)$;

$(M4)$ $J(x,f) = 0$ almost everywhere on a set $A \subset G$, $|A| > 0$, implies $Df(x) = 0$ almost everywhere on A;

$(M5)$ $f : G \to \mathbb{R}^n$ is continuous;

$(M6)$ $f : G \to \mathbb{R}^n$ possesses at least one of the following properties:

(a) the mapping is almost absolutely continuous (see the definition below);

(b) $\operatorname{adj} Df \in L_{q,\mathrm{loc}}$, $q = \frac{n}{n-1}$.

It is well known that, for each $f \in W^1_{q,\mathrm{loc}}(G)$, there is an increasing sequence $\{A_k\}$ of closed sets such that the restriction $f|_{A_k}$ is Lipschitz continuous for every k and the set $S = G \setminus \bigcup_k A_k$ has measure zero. We call a mapping $f \in W^1_{q,\mathrm{loc}}(G)$ *almost absolutely continuous* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every collection $\{B(x_i, r_i)\}$ of pairwise disjoint balls with $x_i \in S$ for all i, the condition $\sum_i |B(x_i, r_i)| < \delta$ implies $\sum_i (\operatorname{osc}_{B(x_i,r_i)} f)^n < \varepsilon$. Applying Besikovich's theorem, we can easily verify that $|f(S)|$ has measure zero; consequently, each almost absolutely continuous mapping $f \in W^1_{q,\mathrm{loc}}(G)$ satisfies Luzin's condition $\mathscr{N}$.

The first condition guarantees existence of a $\mathscr{K}^*$-differential [1] and that each set of zero capacity is totally disconnected (for $q = n-1$ this properties hold only for $n = 2$!). The second condition is used in the proof of monotonicity and preservation of orientation (see § 1 below). The third condition is natural and is due to the fact that local summability of the Jacobian is guaranteed only for $q \geq n$. If a mapping satisfies the fourth condition then we say that it has *finite distortion.* In [2], it was proven in particular that every mapping of the class $W^1_{n,\mathrm{loc}}(G)$ only satisfying $(M2)$ and $(M4)$ is monotone and consequently has a continuous representative. It turns out (Theorem 3) that monotonicity is enjoyed by the mappings of the class $W^1_{q,\mathrm{loc}}(G)$ satisfying some of the conditions $(M1)$–$(M6)$ (see § 1). However, in this case a quasicontinuous representative may have discontinuities on a set of q-capacity zero, $n-1 < q < n$ [3]. Since continuity of f is essential for some results, we impose the condition $(M5)$ on f. The condition $(M6)$ plays the role of a regularity condition in the results obtained. Probably, it is not optimal. The question of whether $(M6)$ can be relaxed is of interest in its own right and remains still open. Observe that a mapping $f \in W^1_{n,\mathrm{loc}}(G)$ satisfying $(M2)$ and $(M4)$ also satisfies $(M3)$, $(M5)$, and $(M6)$. Therefore, Theorem 1 stated below covers both Reshetnyak's theorem and the results of [7, 10]. Moreover, it turns out that the above-cited results of [9] are valid under weaker assumptions.

The main result of the present article for mappings satisfying the above-listed conditions is as follows:

Theorem 1. *Suppose that $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \geq 2$, is a nonconstant mapping of the class $W^1_{1,\mathrm{loc}}(G)$ which satisfies (M2)–(M6) and that $K(x) \in L_{p,\mathrm{loc}}$ for some $n-1 \leq p \leq \infty$ if $n = 2$ and $n - 1 < p \leq \infty$ if $n \geq 3$. Then f*

(1) belongs to $W^1_{q,\mathrm{loc}}(G)$ with $q = \frac{np}{p+1}$;

(2) is open and discrete;

(3) is differentiable almost everywhere in Ω in the classical sense.

EXAMPLE. An important class of mappings satisfying the hypothesis of Theorem 1 consists of the mappings with bounded q-dilatation (in another terminology, q-quasiregular mappings). These are mappings f of the Sobolev class that, alongside $(M2)$–$(M6)$, satisfy the pointwise inequality $|\nabla f|^q \leq KJ(x, f)$ almost everywhere, where K is a constant and $n - 1 \leq q \leq n$ for $n = 2$ and $n - 1 < q \leq n$ for $n \geq 3$. For $q = n$ this class coincides with the class of mappings with bounded distortion [1]. It is immediately verified that $K(x) \in L_{p,\mathrm{loc}}$, where $p = \frac{q}{n-q}$.

REMARK 1. In the case of $p = n$ we obtain a new proof of openness and discreteness for mappings with bounded distortion (quasiregular mappings in the terminology of [11, 12]) which does not use the approximation of a mapping by smooth mappings.

The method of the present article bases on the change-of-variable formula with the multiplicity function and degree of a mapping (Theorem 2). Using this formula, we can prove that a nonconstant mapping of the class $W^1_{q,\mathrm{loc}}(G)$ satisfying $(M1)$–$(M4)$ and $(M6\mathrm{b})$ is monotone in G (Theorem 3). Hence, we infer in particular that the coordinate functions of such mapping are monotone and consequently continuous everywhere except for a set of p-capacity zero for $n - 1 < p < n$ (are continuous everywhere for $p = n$). Monotonicity of mappings of the class $W^1_{q,\mathrm{loc}}(G)$, $n - 1 < q < n$, with nonnegative Jacobian and finite distortion was proven in [3] by another method under the assumption $\operatorname{adj} Df \in L_{r,\mathrm{loc}}(G)$, $r > q/(q-1)$.

Furthermore, in Theorem 4 we establish that a mapping only satisfying $(M1)$–$(M6)$ preserves orientation. In [9], this property was proven for mappings of the class $W^1_q(G)$ under the following additional assumptions: $\operatorname{adj} Df(x) \in L_q(G)$, $q \geq p/(p-1)$, and the Hausdorff 1-measure of $f^{-1}(y)$ equals zero for each $y \in \mathbb{R}^n$.

In § 2, we introduce condition $(M7)$ which describes geometrical and topological properties (including quasilightness) of a mapping. Assuming this condition, we can prove the claim of Theorem 1 without involving the ideas and methods of the theory of quasilinear elliptic equations (the proof bases only on the change-of-variable formula).

As is well known, the connection between mappings with bounded distortion and nonlinear elliptic equations bases on the property that the columns of the matrix $\operatorname{adj} Df(x) = \{A_{ij}(x)\}$ are divergence-free fields; i.e.,

$$\int_G \sum_{i=1}^{n} A_{ij} \frac{\partial \varphi}{\partial x_i}\, dx = 0$$

for every function $\varphi \in C_0^\infty(G)$ and every $j = 1, \dots, n$. This property is a particular instance of the more general relation $\operatorname{div}((\operatorname{adj} Df(x))V \circ f) = [(\operatorname{div} V) \circ f]J(x, f)$ in the distributional sense, where $f : G \to \mathbb{R}^n$ is a mapping of the class $W^1_{n,\mathrm{loc}}(G)$ and V is a C^1-smooth vector field. For smooth mappings this property can be proved by straightforward calculations utilizing equality between mixed second-order derivatives. In § 3, we give a new proof of this result which bases on topological invariants and so makes it possible to extend the method to objects of noncommutative geometry (for instance, Carnot groups).

The proof of Theorem 1 is given in § 4.

In the second part of the article, we introduce the class of continuous, open, and discrete mappings with bounded (q, s)-distortion $1 \leq q \leq p < \infty$ (for $q = p = n$ this is exactly the classical class of mappings with bounded distortion [1]) and study some properties of these mappings. In particular, we indicate conditions under which these mappings satisfy Luzin's condition $\mathscr{N}$ and the condition $\mathscr{N}^{-1}$. Moreover, we establish capacitary estimates, local distortion estimates, and Liouville-type theorems.

§ 1. Properties of Mappings of the Class $W^1_{q,\mathrm{loc}}(G)$, $n-1 < q \le n$

Suppose that $f : G \to \mathbb{R}^n$ is a mapping of the class $W^1_{q,\mathrm{loc}}(G)$ whose summability exponent satisfies $(M1)$. Henceforth we consider a quasicontinuous representative of this class. Thus, for each $\varepsilon > 0$, there is an open set U, $\mathrm{cap}(U; W^1_q(G)) < \varepsilon$, such that f is continuous outside U (for the definition of capacity, see § 3). If $x \in G$ then the restriction $f|_{S(x,r)}$, $B(x,r) \subset G$, is continuous for almost every r.

A function $f \in W^1_{q,\mathrm{loc}}(G)$ is *spherically monotone* (or simply *monotone*) if, for every point $x \in G$, there is a number $r_x > 0$ such that the ball $B(x,r)$ lies in G and for almost every $r \in (0, r_x)$ the following inequalities are valid on the ball:

$$\operatorname*{ess\,sup}_{z\in B(x,r)} f(z) \le \sup_{z\in S(x,r)} f(z) \quad \text{and} \quad \operatorname*{ess\,inf}_{z\in B(x,r)} f(z) \ge \inf_{z\in S(x,r)} f(z).$$

(It was shown in [13, Proposition 2] that each function of the class $f \in W^1_{q,\mathrm{loc}}(G)$ which is weakly monotone in the sense of [3] is also spherically monotone.) For $n-1 < q < n$ each spherically monotone function is continuous everywhere except for a set of Hausdorff dimension at most $n-q$ [3].

It is well known that a monotone function of the class $f \in W^1_{q,\mathrm{loc}}(G)$, $n-1 < q \le n$, is differentiable almost everywhere [1, 12]. Here we present another proof of this property (which seems shorter).

Proposition 1. *Every spherically monotone function $f \in W^1_{q,\mathrm{loc}}(G)$, $n-1 \le q \le \infty$ for $n = 2$ and $n-1 < q \le \infty$ for $n \ge 3$, is differentiable almost everywhere.*

Proof. The Sobolev inequality holds for each point $x \in G$ and almost every radius $r \in (1, r_x)$:

$$(\operatorname*{osc}_{S(x,r)} f)^q \le C r^{q-(n-1)} \int\limits_{S(x,r)} |\nabla f|^q \, dS.$$

From this inequality we can obtain the following estimate [13, Proposition 3]:

$$(\operatorname*{osc}_{S(x,r)} f)^q \le C r^{q-n} \int\limits_{\{z: r<|x-z|<2r\}} |\nabla f|^q \, dz$$

for every $r < r_x/2$. In [13, Proof of Proposition 3] it was also noted that the refined function f satisfies the relations $\inf_{z\in S(x,r)} f(z) \le f(y) \le \sup_{z\in S(x,r)} f(z)$ for all $y \in B(x,r)$ and almost every $r \in r_x$. Hence,

$$\begin{aligned}
\overline{\lim_{z\to x}} \left(\frac{|f(z)-f(x)|}{|z-x|} \right)^q &\le \overline{\lim_{r\to 0}} \left(\frac{\sup\{|f(z)-f(y)| : z, y \in B(x,r)\}}{r} \right)^q \\
&\le \overline{\lim_{r\to 0}} \left(\frac{\sup\{|f(z)-f(y)| : z, y \in S(x,r)\}}{r} \right)^q \\
&\le C \overline{\lim_{r\to 0}} \frac{1}{r^n} \int\limits_{B(x,2r)} |\nabla f|^q \, dy \le \tilde{C} M(|\nabla f|^q)(x),
\end{aligned}$$

where $M(g)(x)$ is the maximal function. Since the maximal function is finite almost everywhere, the last inequalities imply validity of the hypothesis of Stepanov's theorem for f.

Now, we give a change-of-variable formula in the Lebesgue integral in the form we need below. Recall that if $g : A \to \mathbb{R}$ is a measurable function on a measurable set A then the function $N(y, g, A) = \mathrm{card}\{g^{-1}(y) \cap A\}$ is the *multiplicity function* or *Banach indicatrix*.

Proposition 2 [14]. *Suppose that a mapping $f : G \to \mathbb{R}^n$ has partial derivatives almost everywhere in G and the Jacobian $J(x, f)$ is locally summable in G. Then*

(1) *there is a Borel set $E_f \subset G$ of measure zero such that the mapping f_E, equal to f outside E_f and zero on E, satisfies Luzin's condition $\mathscr{N}$;*

(2) *for each measurable set $A \subset G$ and every measurable real function $u : \mathbb{R}^n \to \mathbb{R}$, the functions $(u \circ f)(x)|J(x, f)|$ and $u(y)N(y, f_E, A)$ are measurable; moreover, if one of them is integrable (integrability of $(u \circ f)(x)|J(x, f)|$ is considered on A) then so is the other and the following equality holds:*

$$\int_A (u \circ f)(x)|J(x, f)|\, dx = \int_{\mathbb{R}^n} u(y)N(y, f_E, A)\, dy = \int_{\mathbb{R}^n} u(y)N(y, f, A \setminus E)\, dy. \tag{1}$$

Recall that a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is called the *$\mathscr{K}$-differential* of a mapping $f : G \to \mathbb{R}^n$ at a point $x \in G$ if

$$\varlimsup_{t \to 0} \sup_{X \in S(0,1)} \left| \frac{f(x + tX) - f(x)}{t} - L(X) \right| = 0.$$

Recall that a mapping in $W^1_{q,\mathrm{loc}}(G)$, $n - 1 \le q \le n$ for $n = 2$ and $n - 1 < q \le n$ for $n \ge 3$, has a $\mathscr{K}$-differential at almost every point of G and the matrix of the $\mathscr{K}$-differential coincides with the formal Jacobian matrix [15, Theorem 1] (for $q > n$ the mapping is differentiable almost everywhere in the ordinary sense; see, for instance, [1]).

A mapping $f : G \to \mathbb{R}^n$ is *$\mathscr{K}^*$-differentiable* at a point $x \in G$, if it possesses the following properties:

(1) f is continuous on the spheres $S(x, r) \subset G$ for $r \in (0, r_x) \setminus E_x$, where r_x is a positive number and $E_x \subset (0, r_x)$ is a set of measure zero;

(2) f has all partial derivatives at x;

(3) the linear mapping $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix of partial derivatives satisfies

$$\varlimsup_{\substack{t \to 0 \\ t \in (0, r_x) \setminus E_x}} \sup_{X \in S(0,1)} \left| \frac{f(x + tX) - f(x)}{t} - Df(x)(X) \right| = 0.$$

The definition implies that a mapping f, *$\mathscr{K}^*$-differentiable* at a point $x \in G$, has the *$\mathscr{K}$-differential* at x equal to $Df(x)$.

REMARK 2. We can guarantee existence of a $\mathscr{K}^*$-differential almost everywhere on G for the mappings $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \ge 2$, of the class $W^1_{q,\mathrm{loc}}(G)$ for some $n - 1 \le q \le \infty$ if $n = 2$ and $n - 1 < q \le \infty$ if $n \ge 3$, because, as mentioned, the mappings of this class enjoy all of the above properties. Moreover, the product of a monotone function $f \in W^1_q(G)$, $q > n - 1$, and a continuous mapping $g \in W^1_q(G)$, $q > n - 1$, is $\mathscr{K}^*$-differentiable almost everywhere in G as well, while fg is certainly summable only to the power $q/2$. The definition of *$\mathscr{K}^*$-differentiability* is justified by the fact that many arguments below (see, for instance, Theorem 2) rely only on the above-listed properties (1)–(3) rather than on the membership of a mapping in the corresponding Sobolev class.

The following proposition is a restatement of one result of [16]:

Proposition 3. *Suppose that a continuous mapping $f : G \to \mathbb{R}^n$ is $\mathscr{K}^*$-differentiable almost everywhere in G and has locally summable Jacobian $J(x, f)$ in G. Then*

(1) *there is a Borel set $E_f \subset G$ of measure zero such that f satisfies Luzin's condition $\mathscr{N}$ outside E_f;*

(2) *for each compact domain $D \subset G$ such that $\overline{D} \subset G$ and $|\partial D| = 0$ and every continuous real function u such that $u|_{f(\partial D)} = 0$ and the function $y \longmapsto u(y)\mu(y, f, D)$ is integrable in $\mathbb{R}^n$, the function $(u \circ f)(x)J(x, f)$ is integrable on $D \setminus f^{-1}(f(E_f))$ and the following equality is valid:*

$$\int_{D \setminus f^{-1}(f(E_f))} (u \circ f)(x)J(x, f)\, dx = \int_{\mathbb{R}^n} u(y)\mu(y, f, D)\chi(y)\, dy, \tag{2}$$

where χ is the characteristic function of the set $f(G) \setminus f(E_f)$;

(3) *the following formula holds for almost every $y \in \mathbb{R}^n \setminus f(\partial D \cup E_f)$:*

$$\mu(y, f, D) = \sum_{x \in f^{-1}(y) \cap D} \operatorname{sgn} J(x, f).$$

In particular, if f has nonnegative Jacobian then

$$\mu(y, f, D) = N(y, f, D)$$

for almost every $y \in \mathbb{R}^n \setminus f(\partial D \cup E_f)$.

Here $\mu(\cdot, f, D)$ is the degree of f. For the definition and properties of degree see, for instance, [1, 11, 12].

Observe that the restriction of the integration domain on the left-hand side of (2) essentially reduces the scope of its applicability. For instance, if f satisfies the condition $\mathscr{N}$ then $J(x, f) = 0$ on $f^{-1}(f(E_f))$ and therefore the integration domain on the left-hand side of (2) coincides with D (in this case formula (2) is well known). We now distinguish a class of mappings for which the integration domain on the left-hand side of (2) coincides with D.

A mapping $f : G \to \mathbb{R}^n$ which is $\mathscr{K}^*$-differentiable almost everywhere in G and has locally summable Jacobian $J(x, f)$ in G is *stable* if $J(x, f) = 0$ almost everywhere on the set $f^{-1}(f(E_f))$, where E_f is the set of Proposition 2. Thus, a mapping satisfying Luzin's condition $\mathscr{N}$ is always stable. Consequently, if f in the change-of-variable formula (2) is stable then the integration domain on the left-hand side of (2) coincides with D.

Let $D \Subset G$ be a compact domain in G. If $f : \partial D \to \mathbb{R}^n$ is a continuous mapping then the image $f(\partial D)$ is called a *cycle*. For an arbitrary continuous extension $F : D \to \mathbb{R}^n$ of f we can define the degree $\mu(y, F, D)$ of F at the points $y \in \mathbb{R}^n \setminus f(\partial D)$. The *linking number* $\nu(y, f(\partial D))$ of y with respect to the cycle $f(\partial D)$ equals $\mu(y, F, D)$ for all $y \in \mathbb{R}^n \setminus f(\partial D)$. Obviously, the definition of the linking number is independent of the extension of f. Moreover the following proposition is valid:

Proposition 4. *If $f : D \to \mathbb{R}^n$ has a nondegenerate $\mathscr{K}^*$-differential L at a point $x \in D$ then there is a sequence of positive numbers r_n, $n \in \mathbb{N}$, with $\lim_{n\to\infty} r_n = 0$ such that the linking number $\nu(y, f(S(x, r_n)))$ of $y = f(x)$ with respect to the cycle $f(S(x, r_n))$ equals $\operatorname{sign}\det Df(x)$ for all $n \in \mathbb{N}$.*

We need some definition of [8] for stating Theorem 2. Suppose that D is a compact domain with smooth boundary. A function $u : \partial D \to \mathbb{R}$ belongs to the class $W^1_p(\partial D)$ $(L_p(\partial D))$ if for every local coordinate system $(U_\alpha, \varphi_\alpha)$, $W_\alpha = \varphi_\alpha(U_\alpha) \subset \mathbb{R}^{n-1}$, of the manifold ∂D the composite $u \circ \varphi_\alpha^{-1}$ belongs to $W^1_p(W_\alpha)$ $\left(u \circ \varphi_\alpha^{-1} \in L_p(W_\alpha)\right)$. We say that $u \in \mathscr{A}_{p,q}(\partial D)$ if $u \in W^1_p(\partial D; \mathbb{R}^n)$ (i.e., each coordinate function of u belongs to $W^1_p(\partial D; \mathbb{R})$) and $|\operatorname{adj} Du| \in L_q(\partial D)$.

The following assertion can be viewed as an extended version of the change-of-variable formula of [2] ([6]) for mappings of the class $W^1_{n,\mathrm{loc}}(G)$ $(\mathscr{A}_{p,q})$.

Theorem 2. *Suppose that a mapping $f : G \to \mathbb{R}^n$ satisfies one of the following conditions:*

(1) *$f : G \to \mathbb{R}^n$ is a continuous stable mapping;*

(2) *$f \in \mathscr{A}_{p,q}(D)$, where $D \subset G$ is a compact domain with smooth boundary, $p \geq n-1$, and $q \geq \frac{n}{n-1}$; moreover, the trace of f on ∂D belongs to $\mathscr{A}_{p,q}(\partial D)$ and is continuous.*

Then, for every continuous bounded real function u such that $u|_{f(\partial D)} = 0$ and the function $y \longmapsto u(y)\mu(y, f, D)$ is integrable in $\mathbb{R}^n$, the function $(u \circ f)(x)J(x, f)$ is integrable on D and the following equality is valid:

$$\int_D (u \circ f)(x) J(x, f)\, dx = \int_{\mathbb{R}^n} u(y)\nu(y, f(\partial D))\chi(y)\, dy. \tag{3}$$

Here χ is the characteristic function of the set $f(G) \setminus f(E_f)$ under condition (1) *of the theorem and χ is identically unity in $\mathbb{R}^n$ under condition* (2) *of the theorem.*

PROOF. Under condition (2), Theorem 2 ensues from [8, Theorem 5.1], wherein the formula

$$\int_D (u \circ f)(x) J(x,f)\, dx = \mu(y_0, f, D) \int_{\mathbb{R}^n} u(y)\, dy$$

was proven for every bounded smooth function f whose support lies in the connected component of the complement $\mathbb{R}^n \setminus f(\partial D)$ containing y_0.

Prove the theorem, assuming condition (1). Given a measurable function $v\colon D \to \mathbb{R}$ such that $v(x) \geq 0$ for almost all x, construct the function $y \longmapsto N_f(y, v, D)$ as follows: If $N(y, f, D) < \infty$ then let $N_f(y, v, D)$ equal the sum of all values of the function $v(x)$ at the points of the set $f^{-1}(y) \cap D$. If $N(y, f, D) = \infty$ then the value $N_f(y, v, D)$ equals the limit of the sum of the values of $v(x)$ over all finite subsets of $f^{-1}(y) \cap D$.

If $v(x)$ is the indicator of a set A, $\overline{A} \subset D$, then, recalling that $J(x, f) = 0$ almost everywhere on $f^{-1}(f(E_f))$, we can rewrite the formula of Proposition 2 as

$$\int_D v(x) |J(x,f)|\, dx = \int_{\mathbb{R}^n} N_f(y, v, D) \chi(y)\, dy.$$

Hence, it is clear how to validate this formula for linear combinations of the indicators of finitely many measurable sets (for simple functions). Approximating an arbitrary nonnegative measurable function v by an increasing sequence $\{v_m\}$ of simple functions and using Beppo Levy's theorem, we find that

$$\int_D v(x) |J(x,f)|\, dx = \int_{\mathbb{R}^n} N_f(y, v, D) \chi(y)\, dy.$$

If v is an arbitrary measurable function then we put by definition

$$N_f(y, v, D)(y) = N_f(y, v^+, D)(y) - N_f(y, v^-, D)(y).$$

Take v to be a measurable function $u \circ f \operatorname{sgn} J(x, f)$. Then

$$\int_D u \circ f J(x,f)\, dx = \int_D v(x) |J(x,f)|\, dx = \int_{\mathbb{R}^n} N_f(y, v, D) \chi(y)\, dy.$$

It remains to establish that $N_f(y, v, D)\chi(y) = u(y)\nu(y, f(\partial D))\chi(y)$ at almost all points $y \in \mathbb{R}^n$.

Let E_1 be the set of the points $x \in D$ at which f has no $\mathscr{K}^*$-differential, $E_2 = \{x \in D : J(x,f) = 0\}$, and $E_3 = \{y \in \mathbb{R}^n \setminus f(E_f) : N(y, f, D) = \infty\}$, where E_f is the set from Proposition 2. Since $\overline{D} \subset G$ is compact, the function $N(y, f, D)$ is summable and hence $|E_3| = 0$. Put $S = E_3 \cup f(E_1) \cup f(E_2) \cup f(E_f)$. Then $|S \setminus f(E_f)| = 0$ by (1). Take an arbitrary $y \in \mathbb{R}^n \setminus (S \cup f(\partial D))$. The set $f^{-1}(y) \cap D$ is finite, f has a $\mathscr{K}^*$-differential at each point of $f^{-1}(y) \cap D$, $J(x, f) \neq 0$ for all $x \in f^{-1}(y) \cap D$, and all points of this set are interior points of D. Let $a_1, \dots, a_N$ be all points of the set $f^{-1}(y) \cap D$. By the definition of the $\mathscr{K}^*$-differential, there is a sequence of closed balls $B^i_m = B(a_i, r_{i,m})$, $m \in \mathbb{N}$, such that their radii vanish as $m \to \infty$ and $\nu(y, f(\partial B^i_m)) = \operatorname{sgn} J(a_i, f)$ by Proposition 4. For a sufficiently large m the balls B^i_m and B^j_m, $i \neq j$, are disjoint. Then the following relations hold for almost all $y \in \mathbb{R}^n \setminus f(E_f)$:

$$u(y)\nu(y, f(\partial D)) = u(y) \sum_{i=1}^N \nu(y, f(\partial B^i_m)) = u(y) \sum_{i=1}^N \operatorname{sgn} J(a_i, f) = N_f(y, v, D).$$

The theorem is proven.

A mapping $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \geq 2$, of the class $W^1_{1,\mathrm{loc}}(G)$ which is $\mathscr{K}^*$-differentiable almost everywhere in G is *monotone* in G [2] if, for every point $x \in G$ and almost every $r \in (0, r_x)$, $r_x > 0$, the measure of the inverse image $f^{-1}(V_0)$ of the unbounded component V_0 of the complement of the cycle $f(S(x,r))$ equals zero. This definition of monotonicity for mappings of the class $W^1_n(G)$ was introduced in [2].

Theorem 3. *Suppose that $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \geq 2$, is a nonconstant mapping of the Sobolev class $W^1_{q,\mathrm{loc}}(G)$ which satisfies $(M1)$–$(M5)$ and the stability condition or $(M1)$–$(M4)$ and $(M6b)$. Then f is monotone in G and differentiable in the classical sense almost everywhere in G.*

Using the same method as in Theorem 3, we can prove the following corollary:

Corollary 1. *Suppose that a nonconstant mapping $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \geq 2$, of the class $W^1_{1,\mathrm{loc}}(G)$ is $\mathscr{K}^*$-differentiable almost everywhere in G, satisfies $(M2)$–$(M5)$ and the stability condition. Then f is monotone in G.*

From Theorem 3 and Corollary 1 we derive

Corollary 2. *The coordinate functions of f are monotone.*

Theorem 3 ensues from the following lemmas:

Lemma 1. *Suppose that $D \Subset G$, $|\partial D| = 0$, is a compact domain such that the restriction $f|_{\partial D}$ is continuous. If V is a connected component of the open set $\mathbb{R}^n \setminus f(\partial D)$ such that $\nu(y, f(\partial D)) \neq 0$, $y \in V$, then $|V \setminus f(D)| = 0$.*

PROOF. If $|V \setminus f(E_f)| = 0$, where E_f is the set from Proposition 2, then we have nothing to prove. Otherwise we consider a compact set $A \subset V \setminus f(D)$, $|A| \neq 0$. Fix a point $y \in V$. Consider the characteristic function $\xi_A(z)$ of A and let $\xi_k(z)$ be a sequence of compactly-supported continuous functions such that $\xi_k|_{f(\partial D)} \equiv 0$ for all k and $\lim_{k\to\infty} \xi_k(z) = \xi_A(z)$ pointwise. Inserting the function $\xi_k(z)$ in (3) and passing to the limit over $k \to \infty$, we obtain

$$\int\limits_{f^{-1}(A)\cap D} J(x,f)\,dx = \int\limits_V \nu(y, f(\partial D))\xi_A(z)\chi(z)\,dz = \nu(y, f(\partial D))|A| \tag{4}$$

(under condition $(M6\mathrm{b})$, the function $\chi(z)$ is identically unity). Since $f^{-1}(A) \cap D = \varnothing$, the right-hand side of (4) may vanish only in the case of $|A| = 0$. The lemma is proven.

Lemma 2. *Suppose that $D \Subset G$, $|\partial D| = 0$, is a compact convex domain which satisfies the conditions of Theorem 2, f is nonconstant, and $|f(\partial D)| = 0$. If V is the exterior connected component for the cycle $f(\partial D)$ then $|f^{-1}(V) \cap D| = 0$.*

PROOF. It follows from the hypothesis of the lemma that there is a component of $\mathbb{R}^n \setminus f(\partial D)$ on which $\nu(y, f(\partial D)) \neq 0$. (Otherwise from (3) we could infer $\int_D J(x,f)\,dx = 0$, whence $Df(x) = 0$ almost everywhere in D and therefore f could be constant on D.)

Assuming $(M6\mathrm{a})$, we have $|(V \cap f(D)) \setminus f(E_f)| = 0$. Indeed, if $|(V \cap f(D)) \setminus f(E_f)| > 0$ then, applying (4) to the characteristic function $\xi_A(z)$ of a compact set $A \subset (V \cap f(D)) \setminus f(E_f)$, $|A| \neq 0$, we obtain

$$0 = \int\limits_{f^{-1}(A)\cap D} J(x,f)\,dx = \int\limits_V \nu(y, f(\partial D))\xi_A(z)\chi(z)\,dz = \nu(y, f(\partial D))|A|, \quad y \in V, \tag{5}$$

for $\nu(y, f(\partial D)) = 0$. Since $J(x,f) \geq 0$, we have $J(x,f)|_{f^{-1}(A)\cap D} = 0$ almost everywhere and from (1) we deduce $|A| = 0$. Since A is an arbitrary compact set in $(V \cap f(D)) \setminus f(E_f)$, we have $|(V \cap f(D)) \setminus f(E_f)| = 0$. Hence, $|(f^{-1}(V) \cap D) \setminus f^{-1}(f(E_f))| = 0$ and therefore $J(x,f)|_{f^{-1}(V)\cap D} = 0$ due to the stability condition.

Suppose to a contradiction that $|f^{-1}(V)\cap D|>0$. Consider a set $L\subset f^{-1}(V)\cap D$ of positive measure such that all points of L are Lebesgue points for the Jacobian and $\mathscr{H}^*$-differentiability points for f. It is obvious that

$$|(f^{-1}(V)\cap D)\setminus L|=0. \tag{6}$$

Consider $x\in L$ and t_x such that $f|_{S(x,t_x)}$ is continuous and $f(S(x,t_x))\subset V$. Suppose that W is a connected component of the open set $\mathbb{R}^n\setminus f(S(x,t_x))$ for which $\nu(y,f(S(x,t_x)))\neq 0$, $y\in W$. By Lemma 1, $|W\setminus f(B(x,t_x))|=0$.

Assuming (M6b), by what was said in the beginning of the proof from the inclusion $W\subset V$ we derive $J(x,f)|_{f^{-1}(W)\cap B(x,t_x)}=0$.

Assuming (M6a), we apply (3) to D and a continuous function u such that $u(x)>0$ at all points $x\in W$ and $u(x)=0$ at the points $x\notin W$. We obtain

$$\int\limits_{D\cap f^{-1}(W)}(u\circ f)(x)J(x,f)\,dx=0. \tag{7}$$

Since $(u\circ f)(x)>0$ on $B(x,t_x)\cap f^{-1}(W)$, we also arrive at $J(x,f)|_{f^{-1}(W)\cap B(x,t_x)}=0$.

Now, consider a connected component W of an open set $\mathbb{R}^n\setminus f(S(x,t_x))$ for which $\nu(y,(S(x,t_x)))=0$, $y\in W$.

Apply (3) to $D=B(x,t_x)$ and a continuous function u such that $u(x)>0$ at all points $x\in W$ and $u(x)=0$ at the points $x\notin W$. We obtain

$$\int\limits_{f^{-1}(W)\cap B(x,t_x)}(u\circ f)(x)J(x,f)\,dx=0,$$

which again implies $J(x,f)|_{f^{-1}(W)\cap B(x,t_x)}=0$.

Since $|f^{-1}(f(\partial D))|=0$, we have $J(x,f)|_{B(x,t_x)}=0$ by the above. Hence, $Df(x)=0$ almost everywhere on $B(x,t_x)$, since distortion is finite (condition (M4)). Consequently, L lies in the open set $U=\bigcup_{x\in L}B(x,t_x)$ on which $Df(x)=0$. For this reason, the range of the mapping $f|_U$ is at most countable; moreover, $f(U)\subset V$.

Fix a point $x_0\in U$ and a sphere $S(x_0,t)\subset U$. Suppose that $a\in D$ is an arbitrary point such that $f(a)$ belongs to some bounded component of the complement $\mathbb{R}^n\setminus f(D)$. Joining a with the points x of the sphere $S(x_0,t)$ by segments l_x and using absolute continuity of f on almost all segments, we conclude that, on almost all segments l_x (with respect to the surface measure on the sphere $S(x_0,t)$), there exists a set of positive measure whose image is outside $f(U)$ but still lies in V. Applying Fubini's theorem, we obtain $|(f^{-1}(V)\cap D)\setminus L|>0$, which contradicts (6). The lemma is proven.

Recall that a mapping preserves orientation if the degree $\mu(y,f,D)$ of the mapping is positive for each compactly embedded subdomain $D\Subset G$ and every point $y\in f(D)\setminus f(\partial D)$ (for the definition and the properties of degree see, for instance, [1, 11, 12]).

Theorem 4. *Suppose that a nonconstant mapping $f:G\to\mathbb{R}^n$, $G\subset\mathbb{R}^n$, satisfies (M1)–(M5) and either is stable or satisfies (M6b). Then f preserves orientation and is differentiable almost everywhere.*

PROOF OF THEOREM 4. Fix an arbitrary point $y\in f(D)\setminus f(\partial D)$ and the connected component V of the open set $\mathbb{R}^n\setminus f(\partial D)$ containing y. It is impossible that $J(x,f)=0$ almost everywhere on $f^{-1}(V)$, for the partial derivatives of f would otherwise vanish on $f^{-1}(V)$ and therefore the set V would be at most countable, which is false.

We similarly exclude the possibility $|V\setminus f(E_f)|=0$, since by stability $J(x,f)=0$ almost everywhere on $f^{-1}(V)$ which leads to a contradiction as in the preceding case.

Insert in (4) the characteristic function $\xi_A(z)$ of a compact set $A \subset V \setminus f(E_f)$ of positive measure. Then $|f^{-1}(A)| > 0$ and $J(x,f) > 0$ almost everywhere on $f^{-1}(A)$ by Proposition 2. We obtain

$$0 < \int\limits_{f^{-1}(A)\cap D} J(x,f)\,dx = \int\limits_A \mu(y,f,D)\chi(z)\,dz \le \mu(y,f,D)|V \setminus f(E_f)|.$$

Hence, the degree is a positive function of y.

The same arguments prove the following

Corollary 3. *Suppose that a continuous nonconstant mapping $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \ge 2$, of the class $W^1_{1,\mathrm{loc}}(G)$ is $\mathscr{K}^*$-differentiable almost everywhere in G, satisfies $(M2)$–$(M5)$ and is stable. Then f preserves orientation.*

§ 2. Openness and Discreteness of Quasilight Mappings

In addition to $(M1)$–$(M6)$, we introduce one more condition on f.

$(M7)$ The mapping $f : G \to \mathbb{R}^n$ is continuous and satisfies one of the following topological assumptions:

(a) the connected components of the set $f^{-1}(y)$ are compact for each $y \in f(G)$;

(b) for each point y, there is a compact domain $D \Subset G$ such that $y \in f(D)$ and the multiplicity $N(z,f,D)$, $z \in \mathbb{R}^n \setminus f(\partial D)$, of the mapping is bounded almost everywhere in some neighborhood of y.

Recall that a mapping $f : G \to \mathbb{R}^n$ for which the connected components of the inverse image $f^{-1}(y)$ are compact for each $y \in f(G)$ is called *quasilight.*

Introduce the characteristic

$$K_s(x;f) = \inf\{k(x) : |\nabla_{\mathscr{L}} f|^s(x) \le k(x)J(x,f)\}.$$

Clearly, $K_s(x;f) = 0$ for almost all $x \in Z = \{x \in G : J(x,f) = 0\}$. Observe that $K_n(x;f)$ differs from $K(x)$ only by $K_n(x;f) = 0$ on Z (recall that $K(x) = 1$ for almost all $x \in Z$).

Theorem 5. *Let $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \ge 2$, be a nonconstant mapping of the class $W^1_{1,\mathrm{loc}}(G)$ which satisfies $(M2)$–$(M5)$ and $(M7)$ and is stable. Suppose that*

(1) $K_s(x;f) \in L_{\infty,\mathrm{loc}}(G)$ in the case of $n-1 < q = s \le n$;

(2) $K_s(x;f) \in L_{\frac{q}{s-q},\mathrm{loc}}(G)$ in the case of $n-1 < q < s \le n$ (for $n = 2$ the summability exponent q may equal 1).

Then f

(1) belongs to $W^1_{q,\mathrm{loc}}(G)$;

(2) is open and discrete;

(3) is differentiable almost everywhere in Ω.

Differentiability and preservation of orientation ensue from Theorem 4. To prove openness and discreteness, it suffices to demonstrate that the inverse image of each point is totally disconnected. The remaining items are proved in the assertions below. Denote by Z the set $\{z \in G : J(z,f) = 0\}$.

Lemma 3. *Suppose that $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \ge 2$, is a mapping of the class $W^1_{1,\mathrm{loc}}(G)$ which satisfies $(M2)$–$(M5)$ and for which*

(1) $K_s(x;f) \in L_{\infty,\mathrm{loc}}(G)$ if $1 \le q = s < \infty$;

(2) $K_s(x;f) \in L_{\frac{q}{s-q},\mathrm{loc}}(G)$ if $1 \le q < s < \infty$.

Fix a compact domain $D \Subset G$ and an arbitrary domain $D' \subset \mathbb{R}^n$ such that $D' \supset f(\overline{D})$.

Then

(1) for every function $u \in W^1_\infty(D')$, the composite $u \circ f$ belongs to $W^1_q(D)$ and the following inequality holds:

$$\|u \circ f \mid L^1_q(D)\| \le K_{q,s}(f;D)^{\frac{1}{s}} \Bigg(\int\limits_{D'} |\nabla u(y)|^s N(y,f_E,D)\,dy\Bigg)^{\frac{1}{s}}, \tag{8}$$

where

$$K_{q,s}(f;D) = \begin{cases} \|K_s(\cdot;f) \mid L_\infty(D)\| & \text{for } 1 \le q = s < \infty, \\ \|K_s(\cdot;f) \mid L_{\frac{q}{s-q}}(D)\| & \text{for } 1 \le q < s < \infty; \end{cases}$$

(2) *if D is such that $N(y, f_E, D) \in L_\infty(\mathbb{R}^n)$ then, for every function $u \in L^1_s(D')$, the composite $u \circ f$ belongs to $L^1_q(D)$ and the following inequality holds:*

$$\|u \circ f \mid L^1_q(D)\| \le (K_{q,s}(f;D)\|N(y, f_E, D) \mid L_\infty(\mathbb{R}^n)\|)^{\frac{1}{s}}\|u \mid L^1_s(D')\|;$$

(3) *the composite $u \circ f$ can be differentiated by the classical rule: $\nabla(u \circ f)(x) = Df(x)^T \nabla u(f(x))$ almost everywhere in D.*

In particular, f belongs to $W^1_{q,\mathrm{loc}}(G)$. (For the notation $N(y, f_E, D)$ see Proposition 2.)

PROOF. We verify the assertions of the lemma for $u \in W^1_\infty(D')$. Since $u \circ f$ belongs to the class $ACL(D)$ and f has finite distortion, the derivatives of $u \circ f$ are calculated by the classical formulas; moreover,

$$\|u \circ f \mid L^1_q(D)\| \le \left(\int\limits_D (|\nabla u|(f(x))|Df|)^q(x)\,dx\right)^{\frac{1}{q}}$$

$$= \left(\int\limits_{D\setminus Z} |\nabla u|^q(f(x))|J(x,f)|^{\frac{q}{s}} \frac{|Df(x)|)^q}{|J(x,f)|^{\frac{q}{s}}}\,dx\right)^{\frac{1}{q}}.$$

Using Hölder's inequality with exponents s/q and $s/(s-q)$, we derive

$$\|u \circ f \mid L^1_q(D)\| \le \left(\int\limits_{D\setminus Z} \left(\frac{|Df(x)|^s}{|J(x,f)|}\right)^{\frac{q}{s-q}} dx\right)^{\frac{s-q}{sq}} \left(\int\limits_D |\nabla u|^s(f(x)) \cdot |J(x,f)|\,dx\right)^{\frac{1}{s}}$$

(for $q = s$ the left factor equals $K_{s,s}$). Applying (1) to the right factor, we obtain the sought estimate for the norm. Putting $u = x_i$, $i = 1, \dots, n$, we see that $\partial f_i/\partial x_j \in L_{q,\mathrm{loc}}(G)$.

Prove the second assertion of the lemma. If $u \in L^1_s(D')$ then there is a sequence u_k of smooth functions converging to u quasieverywhere (see below) and in the $L^1_s(D')$ norm. To prove item (2), observe that the sequence $u_k \circ f$ is bounded in $L^1_q(D)$ and converges to $u \circ f$ quasieverywhere. Using Poincaré's inequality, we infer that the composite $u \circ f$ is locally integrable in D. Thus, $u \circ f \in L^1_q(D)$ and item (2) is proven.

On the other hand, we can differentiate the composite $u_k \circ f(x)$ by the classical formula for almost all $x \in D$. Let $S \subset D'$ be the set of points at which u has no derivative. It follows from the change-of-variable formula that the Jacobian vanishes almost everywhere on the set $A = \{x : x \in f^{-1}(S)\}$ (which may have positive measure). By finiteness of distortion, all partial derivatives of the coordinate functions of f therefore vanish almost everywhere on A. For this reason, the limit of the sequence $\nabla(u_k \circ f)(x) = Df(x)^T \nabla u_k(f(x))$ in $L_q(D)$ equals $\nabla(u \circ f)(x) = Df(x)^T \nabla u(f(x))$; moreover, $\nabla(u \circ f)(x) = 0$ almost everywhere on A. We are left with observing that $Df(x)^T \nabla u(f(x))$ is the weak derivative of the function $u \circ f$.

By [1, 17], Theorem 1 will be proven if we establish that f preserves orientation and the inverse image $f^{-1}(y)$ is totally disconnected for each $y \in \mathbb{R}^n$.

Preservation of orientation for the mappings in the class in question ensues from Lemma 3 and Theorem 4.

Lemma 4. *The inverse image $f^{-1}(y)$ is totally disconnected for each $y \in \mathbb{R}^n$.*

PROOF. To establish the required property, we show that the inverse image $f^{-1}(y)$ has q-capacity zero; whence, using the well-known properties of capacity [1], we infer that in the case of $q > 1$ the Hausdorff $(n - q)$-measure of $f^{-1}(y)$ equals zero. Since $n - q < 1$, $f^{-1}(y)$ is totally disconnected. If $n = 2$ and $q = 1$ then the linear Hausdorff measure of a set of zero 1-capacity equals zero [18]; consequently, this set has no degenerate continua as connected components.

We recall the basic facts of capacity theory of [19] which are needed in the proof of the lemma. Suppose that $\mathbb{M}$ is a Riemannian space. Denote by $F(\mathbb{M})$ some normed space whose elements are continuous functions on $\mathbb{X}$. The algebraic operations in $F(\mathbb{M})$ are defined in a standard manner.

Suppose that, together with each function $u \in F(\mathbb{M})$, the space $F(\mathbb{M})$ contains the modulus $|u|$. Thus, $F(\mathbb{M})$ is a vector lattice with respect to the pointwise order relation between functions. Moreover, suppose that the norm and the order are connected as follows: there is a continuous monotone increasing function $\alpha : [0, \infty) \to [0, \infty)$ satisfying the conditions $\alpha(0) = 0$, $\alpha(t) \to \infty$ as $t \to \infty$, and

$$\alpha(\|\max(u, v)\|) + \alpha(\|\min(u, v)\|) \leq \alpha(\|u\|) + \alpha(\|v\|),$$

where $u, v \in F(\mathbb{M})$ are arbitrary functions.

EXAMPLE 1. Consider the collection of functions $\varphi : \mathbb{M} \to \mathbb{R}$ belonging to the intersection $F(\mathbb{M}) = C(\mathbb{M}) \cap W^1_\infty(\mathbb{M})$ and having the finite norm $\|\varphi \mid W^1_q(\mathbb{M})\| = (\|\varphi \mid L_q(\mathbb{M})\|^q + \|\nabla\varphi \mid L_q(\mathbb{M})\|^q)^{\frac{1}{q}}$ ($\|\varphi \mid L^1_q(\mathbb{M})\| = \|\nabla\varphi \mid L_q(\mathbb{M})\|$). Take α to be $\alpha(t) = t^q$. The closure of $F(\mathbb{M})$ in the norm under consideration coincides with the Sobolev space $W^1_q(\mathbb{M})$ $\left(L^1_q(\mathbb{M})\right)$, $1 \leq q < \infty$.

EXAMPLE 2. Suppose that $\mu : \mathbb{M} \to \mathbb{R}$ is an arbitrary nonnegative summable function on $\mathbb{M}$. Take $F(\mathbb{M})$ to be the class of compactly-supported functions in $\overset{\circ}{L}{}^1_q(\mathbb{M}; \mu) = C_0(\mathbb{M}) \cap W^1_\infty(\mathbb{M})$ with the finite norm $\left\|\varphi \mid \overset{\circ}{L}{}^1_q(\mathbb{M}; \mu)\right\| = \|\nabla\varphi \mid L_q(\mathbb{M})\|$ and let $\alpha(t) = t^q$. If $\mu \equiv 1$ then put $\overset{\circ}{L}{}^1_q(\mathbb{M}; 1) = \overset{\circ}{L}{}^1_q(\mathbb{M})$.

EXAMPLE 3. Fix a compact set $\omega \subset \mathbb{M}$ with nonempty interior. Consider the subspace $L^1_q(\omega; \mathbb{M})$ of the space $L^1_q(\mathbb{M})$ of Example 1 which is constituted by the functions vanishing on ω and which is endowed with the norm $\left\|\varphi \mid L^1_q(\omega; \mathbb{M})\right\| = \|\nabla\varphi \mid L_q(\mathbb{M})\|$ and the same function α.

Suppose that e is a compact subset of $\mathbb{M}$. The set of *F-admissible* functions for $e \subset \mathbb{M}$ is the collection $A(e; F(\mathbb{M})) = \{u \in F(\mathbb{M}) : u \geq 1 \text{ on } e\}$ and the *capacity* of e with respect to $F(\mathbb{M})$ is

$$\operatorname{cap}(e; F(\mathbb{M})) = \inf\{\alpha(\|u\|) : u \in A(e; F(\mathbb{M}))\}.$$

If $A(e; F(\mathbb{M})) = \varnothing$ then we put $\operatorname{cap}(e; F(\mathbb{M})) = \infty$. The capacity defined on compact sets extends routinely to arbitrary sets $E \subset \mathbb{M}$ (see [19], wherein it is proven in particular that the so-defined capacity is a generalized Choquet capacity).

EXAMPLE 4. The capacity of a set $E \subset \mathbb{M}$ with respect to the space $W^1_q(\mathbb{M})$ of Example 1 is called the *Sobolev capacity* of E and is denoted by $\operatorname{cap}\left(E; W^1_q(\mathbb{M})\right)$. The class of admissible functions for the capacity of a compact set $e \subset \mathbb{M}$ is $A\left(e; W^1_q(\mathbb{M})\right) = \left\{u \in C(\mathbb{M}) \cap W^1_\infty(\mathbb{M}) : u \geq 1 \text{ on } e\right\}$.

EXAMPLE 5. The capacity of a set $E \subset \mathbb{M}$ with respect to the space $\overset{\circ}{L}{}^1_q(\mathbb{M}; \mu)$ of Example 2 is sometimes called the *weighted variational capacity* of the condenser $(E, \mathbb{M})$ and denoted by $\operatorname{cap}\left(E; \overset{\circ}{L}{}^1_q(\mathbb{M}; \mu)\right)$. The class of admissible functions of $(e, \mathbb{M})$, where e is a compact set, is $A\left(e; \overset{\circ}{L}{}^1_q(\mathbb{M}; \mu)\right) = \left\{u \in C_0(\mathbb{M}) \cap W^1_\infty(\mathbb{M}) : u \geq 1 \text{ on } e\right\}$.

EXAMPLE 6. Fix a compact set $\omega \subset \mathbb{M}$ with nonempty interior. The capacity of a set $E \subset \mathbb{M} \setminus \omega$ with respect to $L^1_q(\omega; \mathbb{M})$ (see Example 3) is called the capacity of the condenser $(\omega, E; \mathbb{M})$ and denoted by $\operatorname{cap}\left(\omega, E; L^1_q(\mathbb{M})\right)$. The class of admissible functions of $(\omega, e; \mathbb{M})$ for a compact set e is $A\left(e; L^1_q(\omega; \mathbb{M})\right) = \left\{u \in C(\mathbb{M}) \cap W^1_\infty(\mathbb{M}) : u \geq 1 \text{ on } e,\ u = 0 \text{ on } \omega\right\}$.

We say that a set $E \subset \mathbb{M}$ *has capacity zero* if $\operatorname{cap}(E; F(\mathbb{M})) = 0$. A property is said to hold *quasieverywhere* on $\mathbb{M}$ if it holds everywhere except for a set of capacity zero. It is well known that a countable union of sets of capacity zero has capacity zero.

Since Poincaré's inequality is valid in bounded domains $\Omega \subset \mathbb{R}^n$, it is clear that on bounded domains the collections of sets of zero capacities $\operatorname{cap}(E; W_q^1(\Omega))$ and $\operatorname{cap}(E; \overset{\circ}{L}{}_q^1(\Omega))$ coincide. In the following lemma we indicate a condition for a set to have measure zero. In particular, we prove that the collections of sets of capacity zero in Examples 4–6 on bounded domains in $\mathbb{R}^n$ coincide.

Lemma 5. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $E \subset \Omega$, $\omega \subset \Omega$ is a compact set with nonempty interior, and a number $b \in \mathbb{R}$ and a constant K are such that, for each $a > b$, there is a lower semicontinuous function $u_a \in L_q^1(\Omega)$ with the properties $u|_E \geq a$, $u|_\omega \leq b$, and $\|u_a \mid L_q^1(\Omega)\| \leq K$. Then $\operatorname{cap}(E; W_q^1(\Omega)) = 0$, $\operatorname{cap}(E; \overset{\circ}{L}{}_q^1(\Omega)) = 0$, and $\operatorname{cap}((\omega, E; L_q^1(\Omega)) = 0$.*

Proof. Let $D \Subset \Omega$ be a compact domain with smooth boundary, $\omega \subset D$, and let Q be a minimal cube with sides parallel to the coordinate axes which contains D. There exists a bounded linear extension operator $\operatorname{ext} : L_q^1(D) \to L_q^1(Q)$, $1 \leq q \leq \infty$, such that $\operatorname{ext} u_a \in L_q^1(Q) \cap W_\infty^1(Q)$ if $u_a \in L_q^1(Q) \cap W_\infty^1(Q)$. Consider the function $v_a = \frac{\max(u_a, b) - b}{a-b}$. It is obvious that $\|v_a \mid L_q^1(Q)\| \leq \frac{\|\operatorname{ext}\| K}{a-b}$ for every $a > b$. Moreover, the set $V_a = \{x : v_a > 1 - \delta\}$ is open and includes E, where $\delta \in (0,1)$ is an arbitrary number. On the other hand, the compact domain ω includes some ball $B \subset D$ on which $v_a = 0$. By a version of Poincaré's inequality (see, for instance, [18]), we have the inequality

$$\left(\int_Q |g|^{q^*}\, dx\right)^{\frac{1}{q^*}} \leq C l(Q)^{n/q^*} \left(\int_Q |\nabla g|^q\, dx\right)^{\frac{1}{q}} \tag{9}$$

in which $q^* \in [1, qn/(n-q)]$, $l(Q)$ is the side length of Q, and $2Q$ is the cube with the same center as Q and with sides twice as large as those of Q, where $g \in L_q^1(Q)$ is an arbitrary function vanishing on B. Hence, $\operatorname{cap}(E \cap D; W_q^1(Q)) = 0$, since

$$\operatorname{cap}(E \cap D; W_q^1(Q)) \leq \operatorname{cap}(V_a \cap D; W_q^1(Q)) \leq C \frac{\|\operatorname{ext}\| K}{(a-b)(1-\delta)},$$

where C is some constant and $a \in \mathbb{R}$, $a > b$, is an arbitrary number (the function $\frac{v_a}{1-\delta}$ can be mollified if need be). Multiplying the result of mollification by a suitable truncator, we can prove that $\operatorname{cap}(E \cap D; \overset{\circ}{L}{}_q^1(G)) = 0$ and $\operatorname{cap}(\omega, E \cap D; L_q^1(G)) = 0$. Since D is an arbitrary domain, Lemma 5 is proven.

We continue the proof of Theorem 5. Let f be a mapping satisfying the hypothesis of Theorem 5. Fix a compact domain $D \Subset G$ with $f(D) \setminus f(\partial D) \neq \varnothing$ and fix an arbitrary bounded domain Ω containing $f(\overline{D})$. Consider the space $\overset{\circ}{L}{}_p^1(\Omega; \mu)$ of Example 2 with the weight function μ defined by

$$\mu(y) = \begin{cases} \mu(y, f, D) & \text{if } y \in f(D) \setminus f(\partial D), \\ 1 & \text{if } y \in (\Omega \setminus f(\overline{D})) \cup f(\partial D). \end{cases} \tag{10}$$

Recall that, by $(M7)$, we have

(1) for $y \in f(G)$, there is a compact domain $D \Subset \mathbb{R}^n$ such that $y \in f(D) \setminus f(\partial D)$;

or

(2) there is a compact domain $D \Subset \mathbb{R}^n$ such that $y \notin f(\partial D)$ and the function $\mu(z, f, D)$ is bounded in some neighborhood W of y.

The first condition holds in the case when each connected component of the inverse image $f^{-1}(y)$ is compact [20]; i.e., the mapping is light.

Recall that a series in a normed space is *norm convergent* if the series of the norms of its terms converges.

Lemma 6. *Assume that $y \in f(D) \setminus f(\partial D)$. There exists a series $\sum_{k=1}^{\infty} \varphi_k$ which is norm convergent in $\overset{\circ}{L}{}^1_p(\Omega; \mu)$ and possesses the following properties:*

(1) *the terms φ_k of the series are nonnegative functions;*

(2) *the sum of the series is a lower semicontinuous function equal to ∞ at y.*

PROOF. In the first of the five cases listed before the statement of the lemma, take W to be the bounded connected component of the open set $\mathbb{R}^n \setminus f(\partial D)$ containing y. Then, for each $k \in \mathbb{N}$, there is a continuous nonnegative function $\varphi_k \in \overset{\circ}{W}{}^1_\infty(W)$ such that $\varphi_k(y) \geq 1$ and

$$\|\varphi_k \mid \overset{\circ}{L}{}^1_p\Omega; \mu)\|^p = \mu(y, f, D) \int\limits_W |\varphi_k|^p \, dx \leq 1/2^{kp}, \quad k \in \mathbb{N}. \tag{11}$$

Existence of such function follows from the fact that the capacity of a singleton with respect to $\overset{\circ}{L}{}^1_p(\Omega)$ equals zero for every domain Ω containing the given point. The series $\sum_{k=1}^{\infty} \varphi_k$ is norm convergent and possesses the required properties.

In the second case we take $W \subset \Omega$ to be a neighborhood of y in which the function $\mu(z, f, D)$ is bounded by some constant M. The further arguments are similar to the above with the only difference that (11) is replaced with the inequality $\|\varphi_k \mid \overset{\circ}{L}{}^1_p(W; \mu)\|^p \leq M \int_W |\varphi_k|^p \, dx \leq 1/2^{kp}$, $k \in \mathbb{N}$.

Lemma 7. *Assume that $y \in f(D) \setminus f(\partial D)$. Then $\operatorname{cap}(f^{-1}(y); W^1_q(G)) = 0$.*

PROOF. Observe that $f^{-1}(y)$ is a relatively closed subset of G. If φ is the sum of the series in Lemma 6 then the function $\varphi \circ f = \sum_{k=1}^{\infty} \varphi_k \circ f$ is lower semicontinuous and equals infinity at the points of $f^{-1}(y)$; moreover, the series is norm convergent by Lemma 3. Thus, $\varphi \circ f \in W^1_{q,\mathrm{loc}}(G)$. In view of Lemma 5, the q-capacity of the set $f^{-1}(y) \cap D$ then equals zero for each compact domain $D \Subset G$ with smooth boundary. Consequently, it equals zero for the whole domain G. Covering $f^{-1}(y)$ by a countable collection of domains $D_n \Subset G$ with smooth boundaries and using countable semiadditivity of capacity, we obtain $\operatorname{cap}(f^{-1}(y); W^1_q(G)) = 0$. The lemma is proven.

The proof of Theorem 5 is complete.

§ 3. Solutions to Quasilinear Elliptic Equations and the Change-of-Variable Formula

As is well known, the connection between mappings with bounded distortion and nonlinear elliptic equations bases on the property that the columns of the matrix $\operatorname{adj} Df(x) = \{A_{ij}(x)\}$ are divergence-free fields; i.e.,

$$\int\limits_G \sum_{i=1}^n A_{ij} \frac{\partial \varphi}{\partial x_i} \, dx = 0$$

for every function $\varphi \in C_0^\infty(G)$ and every $j = 1, \dots, n$. This property can be proved for smooth mappings by straightforward calculation and then extended by continuity (using approximation) to a suitable Sobolev class (see, for instance, [21]). Here we give a new proof of this result by using the change-of-variable formula (3) (see Corollary 4 below).

Lemma 8. *Suppose that $f : G \to \mathbb{R}^n$ is a mapping of the class $W^1_{q,\mathrm{loc}}(G)$, where $q \geq n-1$ for $n = 2$ and $q > n-1$ for $n \geq 3$, and $u : G \to \mathbb{R}$ is a function which is $\mathscr{K}^*$-differentiable almost everywhere in G, vanishes outside $\omega \Subset G$, and is such that the mapping $f_u : G \to \mathbb{R}^n$, $f_u = (f_1, \dots, f_{j-1}, u, f_{j+1}, \dots, f_n)$, is continuous and stable; moreover,*

$$\sum_{i=1}^n A_{ij} \frac{\partial u}{\partial x_i} \in L_1(\omega) \quad \text{for some } j = 1, \dots, n.$$

Then

$$\int_G \sum_{i=1}^n A_{ij}\frac{\partial u}{\partial x_i}\,dx = 0. \tag{12}$$

PROOF. Denote by A_{ij} the entries of the matrix $\operatorname{adj} Df(x)$. Note that the mapping $f_u : G \to \mathbb{R}^n$ is $\mathscr{K}^*$-differentiable almost everywhere in G and the Jacobian $J(x, f_u)$ of this mapping is nothing but the integrand of (12). Consider a compact domain $D \Subset G$ such that $\omega \Subset D$, $|\partial D| = 0$, $u|_{\partial D} \equiv 0$, and the restriction $f_u|_{\partial D}$ is continuous. Then the cycle $f_u(\partial D)$ lies in the $(n-1)$-dimensional plane and therefore $\nu(y, f_u(\partial D)) = 0$ for every point $y \in \mathbb{R}^n \setminus f_u(\partial D)$. Thus, the conditions of Theorem 2 are satisfied and the right-hand side of (3) for the mapping $f_u : D \to \mathbb{R}^n$ equals zero. Thus, (12) is proven.

Corollary 4. *Suppose that $f : G \to \mathbb{R}^n$ is a mapping of the class $W^1_{q,\mathrm{loc}}$, with $q \geq n-1$. Then the columns of the matrix* $\operatorname{adj} Df$ *are divergence-free fields.*

PROOF. Fix an arbitrary function $\varphi \in C_0^\infty(G)$ vanishing outside some compact domain $\omega \Subset G$. Consider a sequence $f_k : \omega \to \mathbb{R}^n$ of smooth mappings which converges in $W^1_{n-1}(\omega)$ to f. Observe that we can choose f_k so that $f_k \in W^1_q(\omega)$ for some $q > n-1$. By Lemma 8,

$$\int_G \sum_{i=1}^n A_{k,ij}\frac{\partial \varphi}{\partial x_i}\,dx = 0,$$

where $A_{k,ij}$ are the entries of the matrix $\operatorname{adj} Df_k(x)$. Passing to the limit as $k \to \infty$, we come to the desired relation.

Theorem 6. *Suppose that $f : G \to \mathbb{R}^n$ is a continuous mapping of the class $W^1_{q,\mathrm{loc}}(G)$, where $q \geq n-1$ for $n = 2$ and $q > n-1$ for $n \geq 3$, and the Jacobian $J(x, f)$ is locally summable on the open set $W = G \cap f^{-1}(\Omega)$. Let $V : \Omega \to \mathbb{R}^n$ be a vector field $V = (v_1, \dots, v_n)$ of class C^1. If f is almost absolutely continuous then*

$$\operatorname{div}((\operatorname{adj} Df(x))V \circ f) = [(\operatorname{div} V) \circ f]J(x, f) \tag{13}$$

in the distributional sense on W.

REMARK 3. Formula (13) is of interest in its own right and its proof under other analytical assumptions bases on approximation of a mapping by smooth mappings (see [6, 8]). For mappings of the class $W^1_{n,\mathrm{loc}}$ we can deduce this formula from Corollary 4 by a standard passage to the limit. The proof below grounds only on Lemma 8. Bearing in mind applications of this formula to Carnot groups, we are interested in conditions under which we can prove it without passing to the limit. Below (in Lemma 9) we present a condition [8, Theorem 3.2] under which (13) is valid without Luzin's condition $\mathscr{N}$ for f.

PROOF. Fix a function $\varphi \in C_0^\infty(W)$. We have to prove that

$$\int_W \sum_{i=1}^n\sum_{j=1}^n A_{ij} v_j \circ f \frac{\partial \varphi}{\partial x_i}\,dx = -\int_W [(\operatorname{div} V) \circ f] J(x, f)\varphi(x)\,dx. \tag{14}$$

We can transform the integrand on the left-hand side of (14) as follows:

$$\begin{aligned}\sum_{i,j=1}^n A_{ij} v_j \circ f\frac{\partial\varphi}{\partial x_i} &= \sum_{i,j=1}^n A_{ij}\frac{\partial}{\partial x_i}((v_j\circ f)\varphi) - \sum_{i,j=1}^n A_{ij}\frac{\partial}{\partial x_i}(v_j \circ f)\varphi \\ &= \sum_{i,j=1}^n A_{ij}\frac{\partial}{\partial x_i}((v_j\circ f)\varphi) - \sum_{i,j,k=1}^n A_{ij}\left(\frac{\partial v_j}{\partial y_k}\right)\circ f\frac{\partial f_k}{\partial x_i}\varphi \\ &= \sum_{i,j=1}^n A_{ij}\frac{\partial}{\partial x_i}((v_j\circ f)\varphi) - \sum_{j=1}^n \left(\frac{\partial v_j}{\partial y_j}\right)\circ f J(x, f)\varphi. \end{aligned} \tag{15}$$

Since $A_{ij}v_j \circ f\frac{\partial\varphi}{\partial x_i} \in L_1(W)$ for all i and j and $\sum_{j=1}^{n}\left(\frac{\partial v_j}{\partial y_j}\right) \circ fJ(x,f)\varphi \in L_1(W)$; to prove (14), we only have to establish that

$$\sum_{j=1}^{n}\int_W \sum_{i=1}^{n} A_{ij}\frac{\partial}{\partial x_i}((v_j \circ f)\varphi)\,dx = 0. \tag{16}$$

Note that the integrand in (16) is a summable function for every fixed j (to verify this, it suffices to consider (16) with V replaced by the new field V_j obtained from V by substituting zero for all but the jth components) and satisfies the conditions of Lemma 8: to this end, it suffices to consider the function $v_j \circ f$ in place of u in Lemma 8. It is immediately checked that, for a fixed j, the integrand in (16) is the Jacobian of the continuous mapping $F_{j,\varphi} = (f_1,\dots,f_{j-1},(v_j\circ f)\varphi, f_{j+1},\dots,f_n)$. Observe that

$$(f_1,\dots,f_{j-1},v_j\circ f, f_{j+1},\dots,f_n) = G_j \circ f,$$

where $G_j(y) = (y_1,\dots,y_{j-1},v_j(y),\dots,y_n)$ is a mapping satisfying the Lipschitz condition. The composite $G_j \circ f$ is continuous and almost absolutely continuous. Hence, the mapping $F_{j,\varphi}$ is continuous and $\mathscr{K}^*$-differentiable, satisfies Luzin's condition $\mathscr{N}$ (see below), and meets the hypothesis of Theorem 2. Thus, the hypothesis of Lemma 8 is satisfied; consequently, (16) ensues from (12).

It remains to demonstrate that the mapping $F_{j,\varphi}$ satisfies Luzin's condition $\mathscr{N}$. Let A_k and S be the sets mentioned in the definition of almost absolute continuity. The restriction $F_{j,\varphi}|_{A_k}$ is Lipschitz continuous and therefore satisfies Luzin's condition $\mathscr{N}$ on A_k. We are left with verifying that $F_{j,\varphi}|_S$ satisfies Luzin's condition $\mathscr{N}$. Given $\varepsilon > 0$, find $\delta > 0$ from the condition of almost absolute continuity of the mapping $G_j \circ f$. Let $\{B(x_i,r_i)\}$, $x_i \in S$ for all i, be an arbitrary collection of pairwise disjoint balls such that $\sum_i |B(x_i,r_i)| < \delta$. Estimate the sum $\sum_i(\operatorname{osc}_{B(x_i,r_i)} F_{j,\varphi})^n$. If $x \in B(x_i,r_i)$ then

$$|F_{j,\varphi}(x) - F_{j,\varphi}(x_i)|^n \le C(\sup_{x\in G}|\varphi|^n(x)(\operatorname*{osc}_{B(x_i,r_i)}(G_j\circ f))^n$$
$$+ \sup_{x\in\operatorname{supp}\varphi}|G_j\circ f(x)|^n(\operatorname*{osc}_{B(x_i,r_i)}\varphi)^n) \le \widetilde{C}\Big(\delta + \sum_i |B(x_i,r_i)|\Big) \le 2\widetilde{C}\delta.$$

Hence, $F_{j,\varphi}$ satisfies the condition $\mathscr{N}$.

REMARK 4. The almost absolute continuity condition is used in the present article exactly once; namely, to verify the following claim in the end of the proof Theorem 6: *if a continuous mapping $f : G \to \mathbb{R}^n$, $f = (f_1,\dots,f_i,\dots,f_n)$, of a suitable class satisfies Luzin's condition $\mathscr{N}$ then the mapping $f = (f_1,\dots,\varphi f_i,\dots,f_n)$ as well satisfies Luzin's condition $\mathscr{N}$ for every function $\varphi \in C_0^\infty(G)$.* Surely, the almost absolute continuity condition may be replaced with another condition guaranteeing this claim.

In the following assertion we show how (13) can be derived from the above results in the situation under study.

Corollary 5 [1,11,12,21]. *Suppose that $f : G \to \mathbb{R}^n$ is a mapping of the class $W^1_{n,\mathrm{loc}}$ satisfying (M2) and (M4). Then* (13) *is valid for every C^1-smooth vector field V.*

PROOF. Observe that, under the conditions of the corollary, f is monotone and continuous by [2], satisfies Luzin's condition $\mathscr{N}$ (see [13, 22]), and enjoys the property of Remark 4, the latter proven by means of the estimate for a monotone function which is exhibited in the proof of Proposition 1. Since $J(x,f) \in L_{1,\mathrm{loc}}(G)$, all prerequisites for implementation of the proof of Theorem 6 in the situation under consideration are satisfied.

Lemma 9 [8, Theorem 3.2]. *Suppose that $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \ge 2$, is a nonconstant mapping of the class $\mathscr{A}_{q,s}(G)$, where $q \ge n-1$ and $s \ge \frac{n}{n-1}$. Then* (13) *is valid for every C^1-smooth vector field V with bounded derivative.*

Corollary 6. *Suppose that $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \ge 2$, is a nonconstant mapping of the class $W^1_{q,\mathrm{loc}}(G)$ satisfying (M1)–(M6a). Then (13) is valid for every C^1-smooth vector field V.*

Corollary 7. *Suppose that $f : G \to \mathbb{R}^n$ is a continuous mapping of the class $W^1_{q,\mathrm{loc}}(G)$ satisfying (M4), where $q \geq n-1$ for $n = 2$ and $q > n-1$ for $n \geq 3$, and the Jacobian $J(x, f)$ is locally summable on the open set $W = G \cap f^{-1}(\Omega)$. Let $V : \Omega \to \mathbb{R}^n$ be a vector field $V = (v_1, \dots, v_n)$ of the class $L_{\infty,\mathrm{loc}}(\Omega)$ such that $\operatorname{div} V = 0$ in the weak sense on Ω. If f is almost absolutely continuous then*

$$\operatorname{div}((\operatorname{adj} Df(x))V \circ f) = 0$$

in the distributional sense on W.

PROOF. Fix a function $\varphi \in C_0^\infty(W)$. Put $V_\varepsilon = (M_\varepsilon v_1, \dots, M_\varepsilon v_n)$, where M_ε is the Sobolev mollification on Ω with parameter $\varepsilon < \operatorname{dist}(f(\operatorname{supp}\varphi), \partial\Omega)$. Then $\operatorname{div} V_\varepsilon = 0$ in the conventional sense and we can apply Theorem 6 to V_ε. Thus,

$$\int\limits_W \sum_{i=1}^n \sum_{j=1}^n A_{ij}(M_\varepsilon v_j) \circ f \frac{\partial \varphi}{\partial x_i}\, dx = 0.$$

Since the mapping satisfies Luzin's condition $\mathscr{N}$, $J(x, f) = 0$ almost everywhere on the inverse image $f^{-1}(S)$ of a set S of measure zero. Therefore, $A_{ij}(x) = 0$ almost everywhere on the same inverse image. Consider an arbitrary sequence V_{ε_n} converging to V everywhere on $f(\operatorname{supp}\varphi)$ except for a set Z of measure zero. Then $M_\varepsilon v_j \circ f$ converges everywhere outside $f^{-1}(S)$ to the function $v_j \circ f$ bounded on $\operatorname{supp}\varphi$. By the Lebesgue dominated convergence theorem, we can pass to the limit and finish the proof of the corollary.

REMARK 5. In terms of exterior differential forms, (13) represents the equality $df^*\omega = f^*d\omega$ in the weak sense for a form ω of degree $n-1$ whose coefficients belong to the corresponding class. In Theorem 6, Lemma 9, and Corollaries 5–7, we thus give conditions on a mapping and the coefficients of a form for exterior derivation and pull-back to commute (cf. [1]).

Define the matrix

$$G(x) = \begin{cases} J(x, f)^{\frac{2}{n}} (Df(x)^T Df(x))^{-1} & \text{if } J(x, f) > 0, \\ \mathrm{Id} & \text{otherwise.} \end{cases} \tag{17}$$

The matrix $G(x)$ is symmetric, has determinant 1, and characterizes the local deviation of f from a conformal mapping. From the definition of distortion we obtain the estimate

$$\frac{1}{C_n (K(x))^{\frac{2}{n}}} |\xi|^2 \leq \langle G(x)\xi, \xi\rangle \leq C_n K^{2-\frac{2}{n}}(x) |\xi|^2, \tag{18}$$

where C_n is a constant depending only on the dimension n.

Suppose that v is a real-valued smooth function on $\mathbb{R}^n$. Consider $u = v \circ f$. By the chain rule (Lemma 3), we have $\nabla u(x) = Df(x)^T (\nabla v)(f(x))$. The connection between mappings with bounded distortion and extremals of the Dirichlet integral established by Yu. G. Reshetnyak [1] is a consequence of the formula

$$\langle G(x)\nabla u(x), \nabla u(x)\rangle^{\frac{n-2}{2}} G(x)\nabla u(x) = \operatorname{adj} Df(x) |\nabla v(f(x))|^{n-2} \nabla v(f(x)). \tag{19}$$

It follows from (13) and (19) that if a function $v \in C^1(\Omega)$ is n-harmonic, i.e., if v is a solution to the equation $\operatorname{div}(|\nabla v(x)|^{n-2}\nabla v(x)) = 0$ in a domain $\Omega \subset \mathbb{R}^n$, then u is a weak solution to the equation

$$\operatorname{div}(A(x, \nabla u)) = 0 \tag{20}$$

on $f^{-1}(\Omega) \cap G$, where the mapping $A(x, \xi) = \langle G(x)\xi, \xi\rangle^{\frac{n-2}{2}} G(x)\xi$ satisfies the conditions

$$\frac{1}{C_n K(x)} |\xi|^n \leq A(x, \xi) \cdot \xi \leq C_n K^{n-1}(x) |\xi|^n$$

which can be verified by means of (17) and (18).

The case of $K(x) \in L_\infty(G)$ corresponds to a mapping with bounded distortion. In this case (20) is a quasilinear elliptic equation and the regularity properties of its solutions are well known (see, for instance, [11]). In particular, solutions to this equation satisfy the weak Harnack inequality which implies the strict maximum principle for the coordinate functions.

§ 4. Proof of Theorem 1

Suppose that $f : G \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$, $n \geq 2$, is a nonconstant mapping of the class $W^1_{1,\mathrm{loc}}(G)$ which satisfies $(M2)$–$(M6)$ and that $K(x) \in L_{p,\mathrm{loc}}$ for some $n-1 \leq p \leq \infty$ if $n = 2$ and $n - 1 < p \leq \infty$ if $n \geq 3$. Then $f \in W^1_{q,\mathrm{loc}}(G)$ with $q = \frac{np}{p+1}$ by Lemma 3; moreover, f is monotone (and consequently locally bounded) by Theorem 3, preserves orientation, and is almost everywhere differentiable by Theorem 4. The theorem will be proven if we establish that the inverse image $f^{-1}(y)$ is totally disconnected for every point $y \in \mathbb{R}^n$. Since all arguments are of a local nature, without loss of generality we may assume that a nonconstant mapping f is defined on a compact domain $D \Subset G$, $y = 0 \in f(D)$, and $f(D) \subset B(0, e^{-e}) = \Omega'$. To prove that the inverse image $f^{-1}(0)$ is totally disconnected, it suffices to validate the estimate

$$\int\limits_{D'} \left|\nabla \log\log \frac{1}{|f(x)|}\right|^s dx < \infty \tag{21}$$

for every compact domain $D' \Subset D$, where $n - 1 < s \leq q \leq n$ for $1 < q$ and $s = 1$ for $q = 1$, $n = 2$. Indeed, the function $u = \log\log \frac{1}{|f(x)|}$ is lower semicontinuous in D and $u|_{f^{-1}(0)} \equiv \infty$; therefore, $\mathrm{cap}_s(f^{-1}(0)) = 0$ by Lemma 5. Hence, $f^{-1}(0)$ is totally disconnected. To prove (21), we use the special approximation of $\log \frac{1}{|y|}$ of [10].

Lemma 10 [10]. *For each $0 < a < e^{-e}$ the function $\Phi_a : \Omega' \to \mathbb{R}$, defined by the formulas*

$$\Phi_a(y) = \begin{cases} \log \frac{1}{|y|} & \text{if } r = |y| > a, \\ \log \frac{1}{a} - \left(\frac{|y|-a}{a}\right) + \frac{(|y|-a)^2}{2a^2} & \text{if } \frac{a}{2} < |y| < a, \\ \log \frac{1}{a} + \log 2 + \frac{1}{2} + (5 - 12\log 2)\frac{|y|^2}{a^2} & \\ \quad +4(-7 + 12\log 2)\frac{|y|^4}{a^4} + 8(5 - 8\log 2)\frac{|y|^6}{a^6} & \text{if } |y| < \frac{a}{2}, \end{cases}$$

possesses the following properties:

(i) $\Phi_a \in C^2(\Omega')$;
(ii) $\Phi_a(y) \geq e$ *for each* $y \in \Omega'$;
(iii) Φ_a *is radial;*
(iv) $\Phi'_a(r) = \Phi'_a(|y|) \leq 0$;
(v) Φ_a *is n-superharmonic;*
(vi) $\log \frac{1}{a} \leq \Phi_a(y) \leq \log \frac{1}{a} + \frac{1}{2} + \log 2$ *for each* $|y| \leq a$;
(vii) $\Phi_a(y) = \log \frac{1}{|y|}$ *for* $a \leq |y| < e^{-e}$;
(viii) $|\nabla\Phi_a(y)|^{n-2}\nabla\Phi_a(y) \in C^1(\Omega')$.

To prove (21), fix an arbitrary nonnegative function $\eta \in C_0^\infty(D)$, $\eta \geq 0$, and the function Φ_a, $0 < a < e^{-e}$, of Lemma 9. We can derive (21) by passing to the limit as $a \to \infty$ in the estimate

$$\int\limits_D |\nabla(\log(\Phi_a \circ f))(x)|^s \eta^s(x)\, dx$$
$$\leq C_n \left(\int\limits_D |\nabla\eta(x)|^n K^{n-1}(x)\, dx\right)^{\frac{s}{n}} \left(\int\limits_D K^{\frac{s}{n-s}}(x)\, dx\right)^{\frac{n-s}{n}} \tag{22}$$

for some $1 \leq s \leq 2$ if $n = 2$ and $n - 1 < s \leq n$ if $n > 2$ such that $\frac{s}{n-s} \leq p$. Applying Hölder's inequality to the left-hand side of (22), we obtain

$$\int_D |\nabla(\log(\Phi_a\circ f))(x)|^s\eta^s(x)\,dx = \int_D |\nabla(\log(\Phi_a\circ f))(x)|^s\eta^s(x)K(x)^{\frac{s}{n}}\frac{dx}{K(x)^{\frac{s}{n}}}$$

$$\leq \Bigl(\int_D |\nabla(\log(\Phi_a\circ F))(x)|^n\eta^n(x)\frac{dx}{K(x)}\Bigr)^{\frac{s}{n}}\Bigl(\int_D K^{\frac{s}{n-s}}(x)\,dx\Bigr)^{\frac{n-s}{n}}.$$

Inequality (22) will be proven, if we establish the estimate

$$\int_D |\nabla(\log\Phi\circ f)(x)|^n\eta^n(x)\frac{dx}{K(x)} \leq C_n\int_D |\nabla\eta(x)|^nK^{n-1}(x)\,dx \tag{23}$$

in which we should replace Φ with the function Φ_a.

REMARK 6. In the case of mappings of the class $W^1_{n,\mathrm{loc}}$, (22) and (23) were proven in [10] (see below). The authors of [10] treat (13) as a differential equation and take φ in (14) to be a test function of the form $\eta\Phi^m\circ f$. We cannot proceed in this way under the conditions of Theorem 1, since we cannot guarantee convergence of the integrals in (14). Our method bases on the further employment of Lemmas 8 and 9 and calculations (15). Inequality (23) is a weak Harnack type inequality and is of interest in its own right. In the following assertion we present conditions under which we can prove (23).

Lemma 11. *Suppose that the conditions of Theorem 1 are satisfied. Suppose that a function $\Phi\in C^2(\Omega')$ possesses the following properties: $\Phi\geq\delta>0$, Φ is n-superharmonic, and the vector field $|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)\in C^1(\Omega')$ has bounded derivative. Then (23) holds for every function $\eta\in C_0^\infty(D)$, $\eta\geq 0$, where $D=f^{-1}(\Omega')\subset G$.*

PROOF. Fix a function $\eta\in C_0^\infty(D)$, $\eta\geq 0$, and a function $\Phi\in C^2(\Omega')$ which possesses the following properties: $\Phi\geq\delta>0$, Φ is n-superharmonic, and the vector field $|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)\in C^1(\Omega')$ has bounded derivative.

Insert the compactly-supported test function $\varphi(x)=\nu^n\Phi^{1-n}(f(x))$ in (15). Using Lemma 3, we find its gradient

$$\nabla\varphi(x) = n\eta^{n-1}(x)\Phi^{1-n}(f(x))\nabla\eta(x) - (n-1)\eta^n(x)\Phi^{-n}(f(x))(Df(x))^T\nabla\Phi(f(x))$$

and insert it in (15), assuming for a moment that V is an arbitrary C^1-smooth vector field. We obtain

$$\begin{aligned}&-(n-1)\langle\operatorname{adj}Df(x)V(f(x)),(Df(x))^T\nabla\Phi(f(x))\rangle\eta^n(x)\Phi^{-n}(f(x))\\&\qquad+n\langle\operatorname{adj}Df(x)(V(f(x))),\nabla\eta(x)\rangle\eta^{n-1}(x)\Phi^{1-n}(f(x))\\&=\sum_{i,j=1}^n A_{ij}\frac{\partial}{\partial x_i}(v_j(f(x))\nu^n\Phi^{1-n}(f(x))) - \operatorname{div}V(f(x))\eta^n(x)\Phi^{1-n}(f(x))J(x,f)\,dx.\end{aligned}$$

Since $Df(x)\operatorname{adj}Df(x)=J(x,f)\operatorname{Id}$, transforming the first summand on the left-hand side of the above equality, we find that

$$\begin{aligned}&(n-1)\langle V(f(x)),\nabla\Phi(f(x))\rangle\eta^n(x)\Phi^{-n}(f(x))J(x,f)\\&\qquad-n\langle\operatorname{adj}Df(x)(V(f(x))),\nabla\eta(x)\rangle\eta^{n-1}(x)\Phi^{1-n}(f(x))\\&=-\sum_{i,j=1}^n A_{ij}\frac{\partial}{\partial x_i}(v_j(f(x))\nu^n\Phi^{1-n}(f(x))) + \operatorname{div}V(f(x))\eta^n(x)\Phi^{1-n}(f(x))J(x,f).\end{aligned}\tag{24}$$

Consider the field V_j whose all but the jth components are zero and the component v_j is the jth component of the vector field $|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)$. Then we can easily see that both summands on the left-hand side of (24) and the second summand on the right-hand side of (24) are summable functions. Therefore, so is the function $\sum_{i=1}^{n} A_{ij}\frac{\partial}{\partial x_i}(v_j(f(x))\nu^n\Phi^{1-n}(f(x)))$. Note that both functions $v_j \circ f$ and $\Phi^{1-n}\circ f$ belong to $W^1_{q,\mathrm{loc}}$ and moreover are differentiable almost everywhere (as composites of a smooth function and an almost everywhere differentiable mapping). Thereby the product $v_j(f(x)\nu^n\Phi^{1-n}(f(x))$ as well is differentiable almost everywhere in D (see Remark 2). Moreover, the mapping

$$x \mapsto (f_1(x),\dots,f_{j-1}(x),\nu^n(x)((v_j\Phi^{1-n})\circ f)(x), f_{j+1}(x),\dots,f_n(x))$$

is $\mathscr{K}^*$-differentiable and satisfies Luzin's condition $\mathscr{N}$ under the assumption ($M6$) (see the end of the proof of Theorem 8). Thus, the conditions of Corollary 6 are satisfied. Hence,

$$\int\limits_D \sum_{i=1}^{n} A_{ij}\frac{\partial}{\partial x_i}(v_j(f(x))\nu^n\Phi^{1-n}(f(x)))\,dx = 0.$$

If ($M6$b) is satisfied then the vanishing of this integral ensues from Lemma 9. Indeed, take the vector field V in Lemma 9 to be $(0,\dots,0,v_j\Phi^{1-n},0,\dots,0)$ (the nonzero component occupies the jth place). Then (13) holds for this vector field. Substituting the function ν^n for the test function φ in (14), from (15) and (13) we infer that the integral in question vanishes.

Since we can take j to be an arbitrary number from 1 to n, we have

$$\begin{aligned}(n-1)\int\limits_D &\langle V(f(x)),\nabla\Phi(f(x))\rangle\eta^n(x)\Phi^{-n}(f(x))J(x,f)\,dx\\ -n\int\limits_D &\langle \operatorname{adj} Df(x)(V(f(x)));\nabla\eta(x)\rangle\eta^{n-1}(x)\Phi^{1-n}(f(x))\,dx\\ &= \int\limits_D \operatorname{div} V(f(x))\eta^n(x)\Phi^{1-n}(f(x))J(x,f)\,dx. \end{aligned} \tag{25}$$

Since Φ is n-superharmonic, $\operatorname{div}|\nabla\Phi(y)|^{n-2}\nabla\Phi(y)\le 0$. Putting $V = |\nabla\Phi(y)|^{n-2}\nabla\Phi(y)$, from (25) we arrive at the inequality

$$\int\limits_D \frac{|\nabla\Phi(f(x))|^n}{\Phi^n(f(x))}\eta^n(x)J(x,f)\,dx \le \frac{n}{n-1}\int\limits_D |\operatorname{adj} Df(x)|\frac{|\nabla\Phi(f(x))|^{n-1}}{\Phi^{n-1}(f(x))}|\nabla\eta(x)|\eta^{n-1}(x)\,dx.$$

Using the estimate $|\operatorname{adj} Df(x)|^{\frac{n}{n-1}} \le c_n|Df(x)|^n = c_nK(x)J(x,f)$ with some constant c_n, depending only on n, and applying Hölder's inequality to the right-hand side of the last inequality, we obtain

$$\int\limits_D \frac{|\nabla\Phi(f(x))|^n}{\Phi^n(f(x))}\eta^n(x)J(x,f)\,dx \le C_n\int\limits_D |\nabla\eta(x)|^nK^{n-1}(x)\,dx.$$

Finally, to derive (23), it suffices to recall the relations $J(x,f) = \frac{|Df(x)|^n}{K(x)}$ and $|\nabla(\Phi\circ f)(x)| \le |Df(x)||\nabla\Phi(f(x))|$.

Lemma 11 is proven.

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