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SIMPLE QUOTIENT ALGEBRAS AND SUBALGEBRAS OF JACOBIAN ALGEBRAS

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Starting from an associative commutative algebra A and its commuting derivations, in [1] V. T. Filippov constructed a certain n-Lie algebra A^* whose n-ary operation bases on the notion of Jacobian. In [2] this algebra was called the Jacobian algebra. In the same article, the class of the Jacobian algebras $A_G(h_1, \dots, h_n, t)$ was distinguished and the question was raised of describing simple factors of these algebras.

In the present article we consider the Jacobian algebra $A_R(h_1, \dots, h_n, t)$, where R is the field of real numbers and $h_i(x) = x_i$ is the ith projection of a vector $x = (x_1, \dots, x_n) \in R^n$. This n-Lie algebra is denoted by $A(n,t)$. In Theorem 1 we prove simplicity of the quotient algebra of $A(n,t)$ by a one-dimensional ideal. Next, we distinguish some class of subalgebras $E(n,t,J)$ of $A(n,t)$ (see the definition below) and establish isomorphism between some algebras of this class. In particular, we prove that over an algebraically closed field they all are isomorphic (for a fixed $n \in N$, $n > 2$). In Theorem 2 we prove simplicity of the n-Lie algebra $E(n,t,J)$ over an arbitrary field Φ of characteristic 0. In the case of a field Φ of characteristic $p > 0$, we construct examples of simple finite-dimensional n-Lie algebras of dimensions $p^n - 1$, $p^n - 2$, p^{n-1}, and $p^{n-1} - 1$.

We now recall some definitions. Let Ψ be an associative commutative ring with unity. As usual, by an *Ω-algebra* we mean a unitary Ψ-module furnished with a system Ω of polylinear n-ary algebraic operations. An *n-Lie algebra* is an Ω-algebra L with one anticommutative n-ary operation $[x_1, \dots, x_n]$ satisfying the identity

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n].$$

Let Φ be a field and let L be an arbitrary n-Lie algebra over Φ. Henceforth we assume that $n > 2$.

A subalgebra I of L is called an *ideal* if $[I, L, \dots, L] \subseteq I$. The subalgebra $L^1 = [L, \dots, L]$ of L is called the *derived algebra* of L. The algebra L is called *simple* if $L^1 \neq 0$ and L lacks ideals other than 0 or L.

Henceforth we denote by $\langle w_v; v \in \Upsilon \rangle$ the vector space that is spanned by the family of vectors $\{w_v; v \in \Upsilon\}$.

Unless otherwise stated, from now on we assume that Φ is a field of characteristic 0 which includes R, R is as usual the field of real numbers, and R^n is the abelian group of n-rows with entries in R.

Let $X = \{x_1, \dots, x_n\}$ be a set of variables and let $A(n)$ be the associative commutative Φ-algebra generated by all powers $x_i^{a_i}$, where $a_i \in R$ and $x_i \in X$. If we denote an arbitrary basis element $x_1^{a_1} \dots x_n^{a_n}$ of $A(n)$ by $x^{(a)}$, where $a = (a_1, \dots, a_n) \in R^n$, then $A(n) = \langle x^{(a)} : a \in R^n \rangle$ with the following multiplication table for the basis elements: $x^{(a)} x^{(b)} = x^{(a+b)}$.

Observe that if $\varepsilon_i = (0, \dots, \overset{i}{1}, \dots, 0) \in R^n$ then $x_i = x^{(\varepsilon_i)}$. As usual, the partial derivatives $\frac{\partial}{\partial x_i} : x^{(a)} \mapsto a_i x^{(a - \varepsilon_i)}$ are written on the left and denoted by ∂_i; i.e., $\partial_i x^{(a)} = a_i x^{(a - \varepsilon_i)}$.

Fix $t' = (t_1, \dots, t_n) \in R^n$ and define the following n-ary operation on the underlying space of the

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algebra $A(n)$:

$$[x^{(\mathbf{a}_1)}, \dots, x^{(\mathbf{a}_n)}] = \begin{vmatrix} x_1^{t_1}\partial_1 x^{(\mathbf{a}_1)} & \dots & x_n^{t_n}\partial_n x^{(\mathbf{a}_1)} \\ \vdots & \ddots & \vdots \\ x_1^{t_1}\partial_1 x^{(\mathbf{a}_n)} & \dots & x_n^{t_n}\partial_n x^{(\mathbf{a}_n)} \end{vmatrix}. \tag{1}$$

Operation (1) defines on the underlying space of $A(n)$ the Ω-algebra which we denote by $A(n,t)$, where $t = t' - \mathbf{e}$, $\mathbf{e} = (1, \dots, 1) \in R^n$. It is easy to verify that the mappings $D_i : x^{(a)} \mapsto x_i^{t_i}\partial_i x^{(a)}$ are commuting derivations of $A(n)$. Therefore, by Theorem 1 of [2] the Ω-algebra $A(n,t)$ is an n-Lie algebra. Notice that the n-Lie algebra $A(n,t)$ is isomorphic to the algebra $A_R(h_1, \dots, h_n, t)$ of [2], where $h_i(x)$ is the ith projection of a vector $x = (x_1, \dots, x_n) \in R^n$. By Theorem 2 of [2], the algebra $A(n,t)$ is a Jacobian algebra.

Henceforth, given a matrix $(a_{ij}) \in \Phi_n$ with entries in Φ, we denote by $|a_{ij}|$ the determinant of (a_{ij}).

Lemma 1. *In the algebra $A(n,t)$*

$$[x^{(a_1)}, \dots, x^{(a_n)}] = |a_{ij}| x^{(a_1 + \dots + a_n + t)}. \tag{2}$$

PROOF. The claim follows from [2, Lemma 1].

Let U be some fixed basis for the algebra L. If $u \in L$ is an arbitrary nonzero element and $u = \sum_{i=1}^k \alpha_i u_i$, where u_i are distinct elements of U and $\alpha_i \neq 0$, then we call k the *length* of u and denote it by $h(u)$.

Put

$$\widetilde{A}(n,t) = \begin{cases} A(n,0) & \text{if } t = 0; \\ \langle x^{(a)} : a \in R^n \setminus \{t\} \rangle & \text{if } t \neq 0. \end{cases}$$

Using (2), we can easily verify that $\widetilde{A}(n,t)$ is a subalgebra of $A(n,t)$ for every $t \in R^n$.

Let $\overline{A}(n,t) = \widetilde{A}(n,t)/\Phi x^{(0)}$ be the quotient algebra of the n-Lie algebra $\widetilde{A}(n,t)$ by the one-dimensional ideal $\Phi x^{(0)}$. By definition,

$$\overline{A}(n,t) = \langle \bar{x}^{(a)} = x^{(a)} + \Phi x^{(0)} : a \in R' = R^n \setminus \{0, t\} \rangle.$$

Theorem 1. *For arbitrary $n > 2$ and $t \in R^n$, the n-Lie algebra $\overline{A}(n,t)$ is simple.*

PROOF. Let $\mathbf{J}$ be a nonzero ideal of $\overline{A}(n,t)$ and let k be the least length of elements of $\mathbf{J}$. Demonstrate that $k = 1$.

Assume that $k > 1$ and let u be an arbitrary element of $\mathbf{J}$ of length k: $u = \sum_{i=1}^k \alpha_i \bar{x}^{(u_i)}$, where $u_i \in R'$ and $\alpha_i \in \Phi$. Two cases are possible:

I. $\langle u_1 \rangle \neq \langle u_2 \rangle$. In this case there are $\mathbf{a}_3, \dots, \mathbf{a}_n \in R'$ such that $\dim\langle u_1, u_2, \mathbf{a}_3, \dots, \mathbf{a}_n \rangle = n$. Then

$$\begin{aligned} v = [u, \bar{x}^{(u_2)}, \bar{x}^{(\mathbf{a}_3)}, \dots, \bar{x}^{(\mathbf{a}_n)}] &= \sum_{i=1}^k \alpha_i [\bar{x}^{(u_i)}, \bar{x}^{(u_2)}, \bar{x}^{(\mathbf{a}_3)}, \dots, \bar{x}^{(\mathbf{a}_n)}] \\ &= \alpha_1 [\bar{x}^{(u_1)}, \bar{x}^{(u_2)}, \bar{x}^{(\mathbf{a}_3)}, \dots, \bar{x}^{(\mathbf{a}_n)}] + \sum_{i=3}^k \alpha_i [\bar{x}^{(u_i)}, \bar{x}^{(u_2)}, \bar{x}^{(\mathbf{a}_3)}, \dots, \bar{x}^{(\mathbf{a}_n)}] \in \mathbf{J} \end{aligned}$$

and $1 \leq h(v) < h(u)$, which contradicts the choice of u.

II. $\langle u_1 \rangle = \langle u_2 \rangle$. Put $\alpha\varepsilon_r = (0, \dots, \overset{r}{\alpha}, \dots, 0) \in R'$. Take $\alpha \in R$ and $r \in \{1, \dots, n\}$ such that $\langle \alpha\varepsilon_r \rangle \neq \langle u_1 \rangle$ and $\langle \alpha\varepsilon_r - t \rangle \neq \langle u_1 \rangle$. Then there are $\mathbf{a}_2, \dots, \mathbf{a}_{n-1} \in R'$ such that $\dim\langle \alpha\varepsilon_r - t, u_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1} \rangle = n$ and $\mathbf{a}_n = \alpha\varepsilon_r - t - \sum_{i=2}^{n-1} \mathbf{a}_i \in R'$. We have

$$[\bar{x}^{(u_i)}, \bar{x}^{(\mathbf{a}_2)}, \dots, \bar{x}^{(\mathbf{a}_n)}] = \gamma_i \bar{x}^{(u_i + \mathbf{a}_2 + \dots + \mathbf{a}_n + t)} = \gamma_i \bar{x}^{(u_i + \alpha\varepsilon_r)}. \tag{3}$$

where $\gamma_i \in \Phi$, $i = 1, \dots, k$, $\gamma_1\gamma_2 \neq 0$, and $\langle u_1 + \alpha\varepsilon_r \rangle \neq \langle u_2 + \alpha\varepsilon_r \rangle$. Using (3), we obtain

$$v = [u, \bar{x}^{(\mathbf{a}_2)}, \dots, \bar{x}^{(\mathbf{a}_n)}] = \sum_{i=1}^{k} \alpha_i [\bar{x}^{(u_i)}, \bar{x}^{(\mathbf{a}_2)}, \dots, \bar{x}^{(\mathbf{a}_n)}] = \sum_{i=1}^{k} \alpha_i \gamma_i \bar{x}^{(v_i)} \in \mathbf{J},$$

where $\alpha_i\gamma_i \neq 0$ for $i = 1, 2$ and $v_i = u_i + \alpha\varepsilon_r$ for $i = 1, \dots, k$. Thus, $v \in \mathbf{J}$, $2 \leq h(v) \leq h(u)$, $\langle v_1 \rangle \neq \langle v_2 \rangle$, and we arrive at the first case.

Hence, $k = 1$ and $\bar{x}^{(\mathbf{a}_1)} \in \mathbf{J}$ for some $\mathbf{a}_1 \in R'$. Demonstrate that in this event $\bar{x}^{(c)} \in \mathbf{J}$ for every $c \in R'$.

Two cases are possible:

I. $\langle c - t \rangle \neq \langle \mathbf{a}_1 \rangle$. Let $\mathbf{a}_2, \dots, \mathbf{a}_{n-1} \in R'$ be such that $\dim\langle \mathbf{a}_1, \dots, \mathbf{a}_{n-1}, c - t \rangle = n$ and $\mathbf{a}_n = c - t - \sum_{i=1}^{n-1} \mathbf{a}_i \in R'$. Then $[\bar{x}^{(\mathbf{a}_1)}, \dots, \bar{x}^{(\mathbf{a}_n)}] = |a_{ij}| \bar{x}^{(\mathbf{a}_1 + \dots + \mathbf{a}_n + t)} = |a_{ij}| \bar{x}^{(c)} \in \mathbf{J}$. Since $|a_{ij}| \neq 0$, we have $\bar{x}^{(c)} \in \mathbf{J}$.

II. $\langle c - t \rangle = \langle \mathbf{a}_1 \rangle$. There is $d \in R'$ such that $\langle d \rangle \neq \langle \mathbf{a}_1 \rangle$ and $\langle d - t \rangle \neq \langle \mathbf{a}_1 \rangle$. Then, by case I, $\bar{x}^{(d)} \in \mathbf{J}$. Since $\langle c - t \rangle \neq \langle d \rangle$, we have $\bar{x}^{(c)} \in \mathbf{J}$ by case I.

Thus, $\mathbf{J} = \overline{A}(n, t)$, and since $\mathbf{J}$ is an arbitrary nonzero ideal of $\overline{A}(n, t)$, the algebra $\overline{A}(n, t)$ is simple. The proof of the theorem is over.

Take $t \in Z^n$. Observe that $\overline{A}_Z(n, t) = \langle \bar{x}^{(a)} \in \overline{A}(n, t) : a \in Z^n \rangle$ is a subalgebra of $\overline{A}(n, t)$. So the following assertion holds:

Theorem 1′. *For arbitrary $n > 2$ and $t \in Z^n$, the n-Lie algebra $\overline{A}_Z(n, t)$ is simple.*

PROOF repeats that of Theorem 1 verbatim.

In the case of a field Φ of characteristic $p > 0$, instead of $A(n)$ we consider the algebra $A_p(n)$ of truncated polynomials in variables $X = \{x_1, \dots, x_n\}$ which is generated by all powers $x_i^{a_i}$, where $a_i \in Z_p$. Thus,

$$A_p(n) = \langle x^{(a)} : a \in Z_p^n \rangle, \quad x^{(a)} x^{(b)} = x^{(a+b)}.$$

Fix $t \in Z_p^n$ and define an n-ary operation on the underlying space of the algebra $A_p(n)$ by the formula (1). Denote the so-obtained Ω-algebra by $A_p(n, t)$. As before, $A_p(n, t)$ is an n-Lie Jacobian algebra.

Put

$$\widetilde{A}_p(n, t) = \begin{cases} A_p(n, 0) & \text{if } t = 0; \\ \langle x^{(a)} : a \in Z_p^n \setminus \{t\} \rangle & \text{if } t \neq 0. \end{cases}$$

The following assertion is valid:

Theorem 1″. *For arbitrary $n > 2$ and $t \in Z_p^n$, the quotient algebra of the n-Lie algebra $\widetilde{A}_p(n, t)$ by the one-dimensional ideal $\Phi x^{(0)}$ is simple.*

PROOF is analogous to that of Theorem 1.

As a corollary, we obtain examples of simple finite-dimensional n-Lie algebras of dimensions $p^n - 1$ and $p^n - 2$.

Take $t = (t_1, \dots, t_n) \in R^n$ and $J \subseteq \{1, \dots, n\}$, $1 \leq \operatorname{card} J \leq n$. Henceforth by t_J we mean the real number defined by the formula

$$t_J = (1 - n)^{-1} \sum_{j \in J} t_j. \tag{4}$$

Consider the following class of subalgebras of $A(n, t)$:

$$E(n, t, J) = \Big\langle x^{(a)} \in A(n, t) : \sum_{j \in J} a_j = t_J \Big\rangle. \tag{5}$$

It is easy to see that $E(n,t,J)$ is a subalgebra of $A(n,t)$.

Let $E(n) = \langle x^{(a)} : a \in R^{n-1}\rangle$ be a vector space over a field Φ. Furnish $E(n)$ with the n-ary operation

$$[x^{(\mathbf{a}_1)},\dots,x^{(\mathbf{a}_n)}] = (-1)^{n-j}t_J \begin{vmatrix} a_{11} & \cdots & a_{1\,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n\,n-1} & 1 \end{vmatrix} x^{(\mathbf{a}_1+\dots+\mathbf{a}_n+\tilde{t})}, \tag{6}$$

where $j \in J$ and $\tilde{t} = (t_1,\dots,t_{j-1},t_{j+1},\dots,t_n) \in R^{n-1}$. Operation (6) defines on the space $E(n)$ the Ω-algebra which we denote by $E_J(n,t,j)$. In the case when $\tilde{t}=0$, we denote the algebra $E_J(n,t,j)$ by $E(n,r)$, where $r = (-1)^{n-j}t_J \in R$.

Lemma 2. *For every $j_0 \in J$, the isomorphism holds: $E_J(n,t,j_0) \cong E(n,t,J)$.*

PROOF. Suppose that $x^{(\mathbf{a}_i)} \in E(n,t,J)$ with $\mathbf{a}_i = (a_{i1},\dots,a_{in}) \in R^n$ and $i = 1,\dots,n$. Denote by A_i the ith column of the matrix $A = (a_{ij})$. As follows from (5), $\sum_{j\in J} A_j = (t_J,\dots,t_J)^\top = t_J\mathbf{e}^\top$. From here and elementary properties of determinants we obtain

$$|A| = t_J|A_1 \dots A_{j_0-1}\mathbf{e}^\top A_{j_0+1}\dots A_n| = (-1)^{n-j_0}t_J|A_1\dots A_{j_0-1}A_{j_0+1}\dots A_n\mathbf{e}^\top|. \tag{7}$$

Define the linear mapping $\varphi : E(n,t,J) \mapsto E_J(n,t,j_0)$ that acts at the basis elements by the rule $\varphi(x^{(a)}) = x^{(\bar{a})}$, where $a = (a_1,\dots,a_n) \in R^n$ and $\bar{a} = (a_1,\dots,a_{j_0-1},a_{j_0+1},\dots,a_n) \in R^{n-1}$. Demonstrate that φ is an isomorphism. Using (6) and (7), we derive

$$[\varphi(x^{(\mathbf{a}_1)}),\dots,\varphi(x^{(\mathbf{a}_n)})] = [x^{(\bar{\mathbf{a}}_1)},\dots,x^{(\bar{\mathbf{a}}_n)}]$$
$$= (-1)^{n-j_0}t_J \begin{vmatrix} a_{11} & \cdots & a_{1\,j_0-1} & a_{1\,j_0+1} & \cdots & a_{1n} & 1 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n\,j_0-1} & a_{n\,j_0+1} & \cdots & a_{nn} & 1 \end{vmatrix} \varphi(x^{(\mathbf{a}_1+\dots+\mathbf{a}_n+t)})$$
$$= \varphi(|A|x^{(\mathbf{a}_1+\dots+\mathbf{a}_n+t)}) = \varphi([x^{(\mathbf{a}_1)},\dots,x^{(\mathbf{a}_n)}]).$$

Hence, φ is a homomorphism.

It follows from (5) that $x^{(a)} \in E(n,t,J)$ if and only if $a_{j_0} = t_J - \sum_{j\in J\setminus\{j_0\}} a_j$. Therefore, $a_i \in R$ can be taken arbitrarily if $i = 1,\dots,n,\ i \neq j_0$. Hence φ is an epimorphism.

If $x^{(a)} \neq x^{(c)}$ then by (5) $\varphi(x^{(a)}) \neq \varphi(x^{(c)})$. Thereby φ is a monomorphism. The proof of the lemma is complete.

Call n-Lie algebras A and B over a field Φ *parametrically isomorphic* if there are a one-to-one mapping $\varphi : A \mapsto B$ and a nonzero element $v \in \Phi$ such that $[\varphi(\mathbf{a}_1),\dots,\varphi(\mathbf{a}_n)] = v\varphi([\mathbf{a}_1,\dots,\mathbf{a}_n])$ for all $\mathbf{a}_1,\dots,\mathbf{a}_n \in A$. In this case we write $A \overset{v}{\cong} B$.

The definition readily implies that if $A \overset{v}{\cong} B$ and $B \overset{u}{\cong} C$ then $A \overset{vu}{\cong} C$.

Lemma 3. *Let A and B be n-Lie algebras over a field Φ. Suppose that the equation $x^{n-1} - v^{-1} = 0$ is solvable in Φ for some $v \in \Phi$. If $A \overset{v}{\cong} B$ then $A \cong B$.*

PROOF. Let φ be a one-to-one mapping from the algebra A into B such that $[\varphi(a_1),\dots,\varphi(a_n)] = v\varphi([a_1,\dots,a_n])$ for all $a_1,\dots,a_n \in A$.

Define the mapping ψ by the rule: $\psi(b) = ub$ for every $b \in B$, where $u \in \Phi$ is a root of the equation $x^{n-1} - v^{-1} = 0$. Then the mapping $\theta = \psi\varphi : A \mapsto B$ is one-to-one and $[\theta(a_1),\dots,\theta(a_n)] = u^n[\varphi(a_1),\dots,\varphi(a_n)] = u^n v\varphi([a_1,\dots,a_n]) = u^{n-1}v\theta([a_1,\dots,a_n]) = \theta([a_1,\dots,a_n])$, since $u^{n-1} = v^{-1}$. Hence θ is an isomorphism. The proof of the lemma is complete.

Proposition 1. *Let $t = (t_1,\dots,t_n) \in R^n$, $n > 2$. For every $J \subseteq \{1,\dots,n\}$, there is $r \in R$ such that $E(n,t,J) \cong E(n,r)$.*

PROOF. Construct the following chain of sets J_k inductively.

Take $j_1 \in J$. Put $J_0 = \{j_1\}$. If J_k has been constructed then put $J_{k+1} = J_k \cup \{j_{k+2}\}$, where $j_{k+2} \in \{1, \dots, n\}$, $j_{k+2} \notin J_k$, and $t_{j_{k+2}} \neq 0$. If there is no such j_{k+2} then put $s = k$ and finish the construction.

Put $t^k = (t_1^k, \dots, t_n^k) \in R^n$, $k = 0, \dots, s$, where

$$t_i^k = \begin{cases} 0 & \text{if } i \in J_k \setminus \{j_{k+1}\}, \\ t_i & \text{in the opposite case.} \end{cases}$$

Observe that $t_{J_k}^k \neq 0$.

Prove by induction that, for every $i = 0, \dots, s$ there exists $v_i \in R$ such that $E(n,t,J) \overset{v_i}{\cong} E(n,t^i,J_i)$.

Lemma 2 implies that $E(n,t,J) \overset{v_0}{\cong} E(n,t^0,J_0)$ for some $v_0 \in R$.

Suppose that $E(n,t,J) \overset{v_{k-1}}{\cong} E(n,t^{k-1},J_{k-1})$, $v_{k-1} \in R$, $k < s$. Then, using Lemma 2, obtain $E(n,t^{k-1},J_{k-1}) \cong E_{J_{k-1}}(n,t^{k-1},j_k) \overset{v_k}{\cong} E_{J_k}(n,t^k,j_k) \cong E_{J_k}(n,t^k,j_{k+1}) \cong E(n,t^k,J_k)$, which was to be proven.

Thus, $E(n,t,J) \overset{v_s}{\cong} E(n,t^s,J_s)$.

It remains to observe that, by Lemma 2, $E(n,t^s,J_s) \overset{v_{s+1}}{\cong} E(n,r')$ for some $r' \in R$ and $E(n,t,J) \cong E(n,r)$ for some $r \in R$. The proof of the proposition is over.

Corollary. *If the field Φ is algebraically closed then $E(n,t,J) \cong E(n,1)$.*

PROOF. The claim is immediate from Lemma 3 and Proposition 1.

Theorem 2. *For arbitrary $n > 2$ and $r \in R$, the n-Lie algebra $E(n,r)$ is simple.*

PROOF. Let $\mathbf{J}$ be a nonzero ideal of the algebra $E(n,r)$. Suppose that $x^{(\mathbf{a}_1)} \in \mathbf{J}$ for some $\mathbf{a}_1 \in R^{n-1}$. Let $c \in R^{n-1}$ be an arbitrary element. Demonstrate that $x^{(c)} \in \mathbf{J}$.

Two cases are possible:

I. $\mathbf{a}_1 = 0$. Choose $\mathbf{a}_2, \dots, \mathbf{a}_{n-1} \in R^{n-1}$ such that $\dim\langle \mathbf{a}_2, \dots, \mathbf{a}_{n-1}, c + \mathbf{a}_1\rangle = n-1$. Put $\mathbf{a}_n = c - \sum_{i=1}^{n-1} \mathbf{a}_i$. Then $[x^{(\mathbf{a}_1)}, \dots, x^{(\mathbf{a}_n)}] = \alpha x^{(c)} \in \mathbf{J}$ for some nonzero $\alpha \in \Phi$. Hence, $x^{(c)} \in \mathbf{J}$.

II. $\mathbf{a}_1 \neq 0$. Considering $\mathbf{a}_1 \neq 0$ and $c = 0$ in the proof of case I, we infer that $x^{(0)} \in \mathbf{J}$. Hence, $x^{(c)} \in \mathbf{J}$ by case I.

Thus, if $x^{(\mathbf{a}_1)} \in \mathbf{J}$ for some $\mathbf{a}_1 \in R^{n-1}$ then $x^{(c)} \in \mathbf{J}$ for all $c \in R^{n-1}$ and thereby $\mathbf{J} = E(n,r)$.

Suppose that there exists an element $u \in \mathbf{J}$ such that $h(u) = \min\{h(v) : v \in \mathbf{J}\}$ and $h(u) = k > 1$. Assume that $u = \sum_{i=1}^k \alpha_i x^{(u_i)}$, where $u_i \in R^{n-1}$ and $\alpha_i \in \Phi$. In accordance with case II, we may assume that $u_1 = 0$. Let $\mathbf{a}_2 = u_2$ and let $\mathbf{a}_3, \dots, \mathbf{a}_n \in R^{n-1}$ be such that $\dim\langle \mathbf{a}_2, \dots, \mathbf{a}_n\rangle = n-1$. Then

$$v = [u, x^{(\mathbf{a}_2)}, \dots, x^{(\mathbf{a}_n)}] = \sum_{i=1}^k \alpha_i r \begin{vmatrix} u_{i1} & \cdots & u_{i\,n-1} & 1 \\ u_{21} & \cdots & u_{2\,n-1} & 1 \\ a_{31} & \cdots & a_{3\,n-1} & 1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{n\,n-1} & 1 \end{vmatrix} x^{(u_i + u_2 + \mathbf{a}_3 + \ldots + \mathbf{a}_n)}.$$

It is easy to see that $v \in \mathbf{J}$ and $1 \leq h(v) < h(u)$, and we arrive at a contradiction to the choice of the element $u \in \mathbf{J}$. Hence, if $\mathbf{J} \neq 0$ then $x^{(a)} \in \mathbf{J}$ for some $a \in R^{n-1}$. However, in that event $\mathbf{J} = E(n,r)$ as was proven earlier. The proof of the theorem is over.

Let $t \in Z^n$ and $J \subseteq \{1, \dots, n\}$ be such that t_J is a nonzero integer. Then $E_Z(n,t,J) = \langle x^{(a)} \in E(n,t,J);\ a \in Z^n\rangle$ is a nonabelian subalgebra of $E(n,t,J)$.

Theorem 2′. *The n-Lie algebra $E_Z(n,t,J)$ is simple.*

PROOF is analogous to that of Theorem 2.

In the case of a field Φ of characteristic $p > 0$, we as before construct the n-Lie algebra $E_p(n,t,J)$ which is a subalgebra of $A_p(n,t)$ by using reduction modulo p. By analogy to Theorem 2, we prove the following

Theorem 2″. *Let $J \subseteq \{1,\dots,n\}$ and $t \in Z_p^n$ be such that the algebra $E_p(n,t,J)$ is not abelian. If $n \not\equiv 0 \pmod p$ then $E_p(n,t,J)$ is a simple n-Lie algebra. If $n \equiv 0 \pmod p$ then the derived algebra of $E_p(n,t,J)$ is simple.*

As a corollary we obtain examples of simple finite-dimensional n-Lie algebras of dimensions p^{n-1} and $p^{n-1} - 1$.

In conclusion, we note that the idea of considering the n-Lie algebras $A(n,t)$ and $E(n,t,J)$ was proposed to the author by V. T. Filippov to whom the author is grateful for supervision.

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