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SOBOLEV SPACES AND (P, Q)-QUASICONFORMAL MAPPINGS OF CARNOT GROUPS[†)‡)]

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In 1968 at the first Donetsk colloquium on mapping theory Yu. G. Reshetnyak stated the problem of describing all isomorphisms φ^* between the homogeneous Sobolev spaces L_n^1 which are generated by quasiconformal mappings φ of the Euclidean space $\mathbb{R}^n$ by the rule $\varphi^*(u) = u \circ \varphi$. It was shown in [1] that these isomorphisms are exactly the latticial isomorphisms of the spaces L_n^1. The approach in [1] to Reshetnyak's problem is natural to consider in the context of the preceding results (see, for instance, [2, pp. 419–420]). The theorems by Banach, Stone, Eilenberg, Arens and Kelley, Hewitt, and Gel'fand and Kolmogorov provide conditions on various structures of the space $C(S)$ of continuous functions whose isomorphisms determine the topological space S up to homeomorphism. We recall Stone's result according to which $C(S)$, regarded as a lattice ordered group, determines S. On the other hand, M. Nakai [3] and L. Lewis [4] established that the isomorphism between two Royden algebras is equivalent to the quasiconformal equivalence of the domains of definition. Distinguishing in the homogeneous Sobolev space L_n^1 two structures, the structure of a vector lattice and the structure of a seminormed space, we now obtain a situation close to Stone's article in an algebraic sense and to Nakai's article in a metric sense. This view of the problem is most natural as allowing us to reconstruct a mapping despite keeping at a minimum "information" for finding the mapping, as well as to prove its continuity, and to discover its metric properties.

The following problem arises in the framework of the approach of [1] to Reshetnyak's problem: what are the metric and analytical properties of a measurable mapping φ inducing the isomorphism φ^* by the rule $\varphi^*(f) = f \circ \varphi$, $f \in L_n^1$. Taking various function spaces L_n^1, we arrive at different problems: the Sobolev spaces W_p^1, $p > n$, were considered in [5]; the homogeneous Besov spaces $b_p^l(\mathbb{R}^n)$, $n > 1$, $lp = n$, for $p = n + 1$ in [6] and for $p > n + 1$ in [7]; the Sobolev spaces W_p^1, $n - 1 < p < n$, in [8]; the Sobolev spaces W_p^1, $1 \le p < n$, (and the spaces of potentials) in [9, 10]; and the three-index scales of Nikol'skiĭ–Besov spaces and Lizorkin–Triebel spaces (and their anisotropic analogs) in [11]. In [12], the theory of multipliers was applied to the change-of-variable problem in Sobolev spaces. The results of [5–11] factually assert that, depending on the relation between the order of smoothness, the summability exponent, and the dimension, the fact that the operator φ^* in an isomorphism implies quasiconformality or quasi-isometry of the mapping in a metric on the domain which is adequate to the geometry of the function space in question.

Qualitatively new effects appear in this problem when we study the analytical and metric properties of homeomorphisms inducing *bounded operators* between Sobolev spaces. We recall the main result of [13, 14]:

Theorem 1. *Suppose that $\varphi : \Omega \to \Omega'$ is a homeomorphism between spatial domains $\Omega, \Omega' \subset \mathbb{R}^n$, $n \ge 2$. Then the following assertions are equivalent:*

(1) *the mapping φ induces the bounded operator $\varphi^* : L_p^1(\Omega') \to L_p^1(\Omega)$, $p \in [1, \infty)$, by the rule $\varphi^*(f) = f \circ \varphi$;*

(2) *the mapping φ belongs to $L_{1,\mathrm{loc}}^1$ and $|\nabla\varphi(x)|^p \le K_p |\det \nabla\varphi(x)|$ almost everywhere in Ω, $p \in [1, \infty)$.*

[†)] To the unfaded memory of Sergeĭ L'vovich Sobolev.

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For $p \in (1, \infty)$, claims (1) and (2) are equivalent to the assertion:
(3) *the inequality*

$$\operatorname{cap}_p(\varphi^{-1}(F), \varphi^{-1}(G)) \le C_p \operatorname{cap}_p(F, G)$$

holds for every ring (F, G) with F a continuum and $G \subset \Omega'$.

(Here $\nabla\varphi(x)$ is the formal Jacobian matrix (defined almost everywhere) and $|\nabla\varphi(x)|$ is the norm of the linear operator corresponding to this matrix. See § 4 for the definition of a ring and the p-capacity of a ring.)

Observe that for $p = n$ the claims of Theorem 1 transform into well-known definitions of a quasiconformal mapping (see, for instance, [1, 15]). As is well known, a quasiconformal mapping can also be defined in purely metric terms as a mapping with bounded distortion (see [15–17]). An analog of the metric definition for $p \ne n$ is known only for homeomorphisms inducing the bounded operator $\varphi^* : L^1_p(\Omega') \to L^1_p(\Omega)$ for $n - 1 < p < \infty$ [18]. Theorem 1 was generalized in [19] to the class of homeomorphisms inducing the bounded operator $\varphi^* : L^1_p(\Omega') \to L^1_q(\Omega)$ for $1 \le q < p < \infty$. For $n-1 < q < p = n$, this class of homeomorphisms coincides with the class of mean quasiconformal mappings which were studied by many authors under certain analytical constraints (see, for instance, [20]). Some applications of mean quasiconformal mappings to embedding theorems for Sobolev classes are exposed in [21].

Quasiconformal mappings in non-Riemannian metrics were first considered by G. D. Mostow [22] in 1972. Quasiconformal mappings on Carnot groups appear naturally in connection with the rigidity problem in rank 1 symmetric spaces [22, 23] and the change-of-variable problem in Sobolev spaces for a nonholonomic metric [11, 24]. We refer to [25] ([11, 23, 24, 26–29]) for various aspects of the theory of quasiconformal mappings on the Heisenberg (Carnot) groups and relevant questions of analysis.

A ***stratified homogeneous group*** [30], alternatively a *Carnot group* [23], is a connected simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra $\mathcal{G}$ splits into the direct sum $V_1 \oplus \dots \oplus V_m$ of vector spaces such that $[V_1, V_k] = V_{k+1}$ for $1 \le k \le m - 1$, $[V_1, V_m] = \{0\}$, and $\dim V_1 \ge 2$. Such an algebra is endowed with the natural family of the dilations $\delta_t = \exp(A \log t)$, where A is the linear operator defined as $Ax = kx$ for $x \in V_k$. Let $X_{11}, \dots, X_{1n_1}$ be vector fields constituting a basis for the space V_1. Since these fields generate V_1; for each i, $1 < i \le m$, we can choose a basis X_{ij}, $1 \le j \le n_i = \dim V_i$, for V_i which is formed by commutators of the fields $X_{1k} \subset V_1$ of order j. Since the algebra $\mathcal{G}$ is nilpotent, the exponential mapping $\exp : \mathcal{G} \to \mathbb{G}$ is a diffeomorphism and the mappings $\exp \circ \delta_t \circ \exp^{-1}$, denoted henceforth by the same symbol δ_t, are group automorphisms of $\mathbb{G}$. This implies in particular that every element $x \in \mathbb{G}$ can be written as $\exp(\sum x_{ij} X_{ij})$, $1 \le i \le m$, $1 \le j \le n_j$. The numbers $\{x_{ij}\}$ are called the *coordinates* of $x \in \mathbb{G}$.

Fix a bi-invariant Haar measure on $\mathbb{G}$ (it is generated by the Lebesgue measure on $\mathcal{G}$ by means of the exponential mapping). We normalize the Lebesgue measure so that the ball $B(0, 1)$ has measure 1. Then $|B(0, r)| = r^\nu$. The number $\nu = \operatorname{trace} A$ is called the ***homogeneous dimension*** of $\mathbb{G}$. Clearly, $|\delta_t E| = t^\nu |E|$.

The Euclidean space $\mathbb{R}^n$ with its standard structure is an example of an abelian group: the vector fields $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, have no nontrivial commutation relations and constitute a basis for the corresponding Lie algebra. The Heisenberg group $\mathbb{H}^n$ is an example of a nonabelian Carnot group. Its Lie algebra has dimension $2n + 1$ and its center is one-dimensional. If $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ is a basis for the Heisenberg algebra then the only nontrivial commutation relations are $[X_i, Y_i] = T$, $i = 1, \dots, n$, whereas all other brackets are zero.

A homogeneous norm on a group $\mathbb{G}$ is a continuous function $\rho : \mathbb{G} \to [0, \infty)$ of class C^∞ on $\mathbb{G} \setminus \{0\}$, where 0 is the identity of $\mathbb{G}$, possessing the following properties:

(a) $\rho(x) = \rho(x^{-1})$ and $\rho(\delta_t(x)) = t\rho(x)$;

(b) $\rho(x) = 0$ if and only if $x = 0$;

(c) there exists a constant $c > 0$ such that $\rho(x_1 x_2) \le c(\rho(x_1) + \rho(x_2))$ for all $x_1, x_2 \in \mathbb{G}$.

A homogeneous norm is defined in a nonunique fashion; however, arbitrary two homogeneous norm are equivalent. A homogeneous norm determines some homogeneous metric ρ (denoted by the same letter) as follows: given two points $x, y \in \mathbb{G}$, put $\rho(x, y) = \rho(x^{-1}y)$. Given the metric, we routinely define the spheres $S(x,t)$, the balls $B(x,t)$, and the topology which turns out to be equivalent to the Euclidean topology.

The *Carnot–Carathéodory distance* $d(x,y)$ between two points $x,y\in\mathbb{G}$ is defined to be the greatest lower bound of the lengths of all horizontal curves with endpoints x and y, where the length is measured in the Riemannian metric with respect to which the vector fields $X_{11},\dots,X_{1n_1}$ are orthonormal and a horizontal curve is a piecewise smooth path whose tangent vector belongs to V_1. The distance $d(x,y)$ is always a finite left-invariant metric with respect to which the group of the automorphisms δ_t is a group of dilations with coefficient t: $d(\delta_t x,\delta_t y)=td(x,y)$. We put $d(x)=d(0,x)$ by definition. It is easy to demonstrate that the distances $d(x,y)$ and $\rho(x,y)$ are equivalent.

Further, we consider a family Γ of curves constituting a smooth fibration of an open set $A\subset\mathbb{G}$. Usually, the fibers $\gamma\in\Gamma$ are the integral curves of some smooth vector field V whose values at all points belong to V_α. If we denote the flow corresponding to this field by the symbol φ_s, then each fiber has the form $\gamma(s)=\varphi_s(p)$, where p belongs to some surface S transversal to V and the parameter s varies in some interval $I\subset R$. We suppose that the fibration Γ of A is furnished with a measure $d\gamma$ satisfying the inequality

$$c_0|B|^{\frac{\nu-\alpha}{\nu}}\le\int\limits_{\gamma\in\Gamma,\gamma\cap B(x,r)\neq\varnothing}d\gamma\le c_1|B|^{\frac{\nu-\alpha}{\nu}}$$

for sufficiently small balls $B=B(x,r)\subset\mathbb{G}$ with constants c_0 and c_1 independent of $B(x,r)$. For the fibration determined by a vector field $V\in V_\alpha$, the measure $d\gamma$ can be obtained as the interior multiplication $i(V)$ of V with the bi-invariant volume form dx.

Let D be a domain in $\mathbb{G}$. A locally-summable function $f:D\to\mathbb{R}$ belongs to $L^1_p(D)$ if the weak derivatives $X_{1j}f$, $j=1,\dots,n_1$, along the vector fields X_{1j} belong to $L_p(D)$. On using the averaging method well-known in Euclidean space, it can be demonstrated that every function f in $L^1_p(D)$ can be approximated by functions $f_k\in C^\infty(D)$ so that $f_k\to f$ in $L_1(U)$ for every domain $U\Subset D$ (the notation $U\Subset D$ means that U is compactly embedded in D) and $X_{1j}f_k\to X_{1j}f$ in $L_p(D)$, $j=1,\dots,n_1$. The space L^1_p is furnished with the seminorm

$$\|f\mid L^1_p(D)\|=\Big(\int\limits_D|\nabla_{\mathcal{L}}f|^p(x)\,dx\Big)^{1/p},$$

where $\nabla_{\mathcal{L}}f=(X_{11}f,\dots,X_{1n_1}f)$ is called the *subgradient* of f.

A mapping $\varphi:D\to\mathbb{G}$ is called *absolutely continuous* on lines ($\varphi\in ACL(D)$) if for every domain U, $\overline{U}\subset D$, and the fibration Γ determined by a left-invariant vector field X_{1j}, $j=1,\dots,n_1$, the mapping φ is absolutely continuous on $\gamma\cap U$ with respect to the one-dimensional Hausdorff measure H^1 for $d\gamma$-almost all curves $\gamma\in\Gamma$. Such a mapping φ has the derivatives $X_{1j}\varphi\in V_1$ along the vector field X_{1j}, $j=1,\dots,n_1$, almost everywhere in D [23]. A mapping $\varphi:D\to\mathbb{G}$ belongs to the Sobolev class $W^1_{p,\mathrm{loc}}(D)$ if $\rho(\varphi(x))\in L_{p,\mathrm{loc}}(D)$, $\varphi\in ACL(D)$, and $X_{1j}\varphi\in L_{p,\mathrm{loc}}(D)$, $j=1,\dots,n_1$. Given a domain $U\subset D$, $\overline{U}\subset D$, we consider the norm

$$\|\varphi\mid W^1_p(U)\|=\|\rho(\varphi(\cdot))\mid L_p(U)\|+\Big(\int\limits_U|\nabla_{\mathcal{L}}\varphi|^p(x)\,dx\Big)^{1/p},$$

where the matrix $\nabla_{\mathcal{L}}\varphi(x)=(X_{1i}\varphi_{1j}(x))$, $i,j=1,\dots,n_1$, called the *(formal) horizontal differential* of φ at x, determines the linear operator $\nabla_{\mathcal{L}}\varphi:V_1\to V_1$ of the horizontal space V_1 [23] for almost all x and $|\nabla_{\mathcal{L}}\varphi|$ is the norm of this operator.

In [24, Proposition 5], it was demonstrated that mappings of the Sobolev class on a Carnot group can be characterized in terms of the properties of the coordinate functions: a continuous mapping $\varphi:\Omega\to\mathbb{G}$ belongs to $W^1_{p,\mathrm{loc}}(\Omega)$ if and only if $\varphi\in HW^1_{p,\mathrm{loc}}(\Omega)$. Here a continuous mapping $\varphi:\Omega\to\mathbb{G}$ *belongs to the class* $HW^1_{p,\mathrm{loc}}(\Omega)$ if

(1) the coordinate functions φ_{ij} belong to $ACL(\Omega)$ for all i and j;

(2) $\varphi_{1j} \in W^1_{p,\mathrm{loc}}(\Omega)$, $j = 1, \dots, n_1$;

(3) the vector $X_{1k}\varphi = (X_{1k}\varphi_{ij})$ belongs to V_1 for almost all $x \in \Omega$, $k = 1, \dots, n_1$ (the weak contactness condition).

In the present article, generalizing the results of [13, 14, 19, 24, 26] in particular, we present equivalent geometric and analytical properties of homeomorphisms inducing bounded operators between Sobolev spaces on Carnot groups. The main results of the article for $p = q$ were announced in [26].

§ 1. Sobolev Spaces and Relevant Classes of Mappings

We say that a mapping $\varphi : D \to D'$ *generates the bounded embedding operator* $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 \le q \le p \le \infty$, by the rule $\varphi^* f = f \circ \varphi$ if there is a constant $K < \infty$ such that $\|\varphi^* f \mid L^1_q(D)\| \le K \|f \mid L^1_p(D')\|$ for every function $f \in L^1_p(D')$.

Proposition 1. *Suppose that* $\varphi : D \to D'$ *generates the bounded embedding operator* $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 \le q \le p < \infty$. *Then* $\varphi \in ACL(D)$.

To prove this proposition, we need the following two lemmas:

A nonnegative function Φ defined on open sets of D and taking finite values is called *quasiadditive* (*additive*) if the inequality $\sum_{i=1}^{\infty} \Phi(A_i) \le \Phi(A)$ (the equality $\sum_{i=1}^{\infty} \Phi(A_i) = \Phi(\bigcup_{i=1}^{\infty} A_i)$) holds for every collection of pairwise disjoint open sets $\{A_i\}$, $A_i \subset A \subset D$, $i \in \mathbb{N}$.

Lemma 1. *Suppose that* $\varphi : D \to D'$ *generates the bounded embedding operator* $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 \le q < p \le \infty$. *Then*

$$\Phi(A') = \sup_{f \in \overset{\circ}{L}{}^1_p(A')} \left(\frac{\|\varphi^* f \mid \overset{\circ}{L}{}^1_q(\varphi^{-1}(A'))\|}{\|f \mid \overset{\circ}{L}{}^1_p(A')\|} \right)^{\kappa}, \quad \text{where } \kappa = \begin{cases} \frac{pq}{p-q} & \text{for } p < \infty, \\ q & \text{for } p = \infty, \end{cases}$$

is a bounded additive function defined on open sets of D'.

PROOF. It is obvious that $\Phi(A'_1) \le \Phi(A'_2)$ if $A'_1 \subset A'_2$.

Suppose that A'_i, $i \in N$, are pairwise disjoint open sets in D' and let $A'_0 = \bigcup_{i=1}^{\infty} A'_i$ and $A_i = \varphi^{-1}(A'_i)$, $i = 0, 1, \dots$. Consider a function $f_i \in \overset{\circ}{L}{}^1_p(A'_i)$ such that the conditions $\|\varphi^* f_i \mid \overset{\circ}{L}{}^1_q(A_i)\| \ge (\Phi(A'_i)(1 - \frac{\varepsilon}{2^i}))^{\frac{1}{\kappa}} \|f_i \mid \overset{\circ}{L}{}^1_p(A'_i)\|$ and $\|f_i \mid \overset{\circ}{L}{}^1_p(A'_i)\|^p = \Phi(A'_i)(1 - \frac{\varepsilon}{2^i})$ for $p < \infty$ ($\|f_i \mid \overset{\circ}{L}{}^1_p(A'_i)\| = 1$ for $p = \infty$), $\varepsilon \in (0, 1)$, are satisfied simultaneously. Putting $f_N = \sum_{i=1}^{N} f_i$ and applying Hölder's inequality for $1 \le q < p \le \infty$ (the case of equality), we obtain

$$\begin{aligned}
\left\| \varphi^* f_N \mid \overset{\circ}{L}{}^1_q \left(\bigcup_{i=1}^{N} A_i \right) \right\| &\ge \left(\sum_{i=1}^{N} \left(\Phi(A'_i) \left(1 - \frac{\varepsilon}{2^i}\right) \right)^{\frac{q}{\kappa}} \|f_N \mid \overset{\circ}{L}{}^1_p(A'_i)\|^q \right)^{1/q} \\
&= \left(\sum_{i=1}^{N} \Phi(A'_i) \left(1 - \frac{\varepsilon}{2^i}\right) \right)^{\frac{1}{\kappa}} \left\| f_N \mid \overset{\circ}{L}{}^1_p \left(\bigcup_{i=1}^{N} A'_i \right) \right\| \\
&\ge \left(\sum_{i=1}^{N} \Phi(A'_i) - \varepsilon \Phi(A'_0) \right)^{\frac{1}{\kappa}} \left\| f_N \mid \overset{\circ}{L}{}^1_p \left(\bigcup_{i=1}^{N} A'_i \right) \right\|.
\end{aligned}$$

Hence,

$$\Phi(A'_0)^{\frac{1}{\kappa}} \ge \sup \frac{\left\| \varphi^* f_N \mid \overset{\circ}{L}{}^1_q \left(\bigcup_{i=1}^{N} A_i \right) \right\|}{\left\| f_N \mid \overset{\circ}{L}{}^1_p \left(\bigcup_{i=1}^{N} A'_i \right) \right\|} \ge \left(\sum_{i=1}^{N} \Phi(A'_i) - \varepsilon \Phi(A'_0) \right)^{\frac{1}{\kappa}},$$

where the least upper bound is calculated over all functions $f_N \in \overset{\circ}{L}{}^1_p(\bigcup_{i=1}^N A'_i)$ of the above form. Since N and ε are arbitrary, we have proven that Φ is quasiadditive. The reverse inequality is immediate.

A *regularized distance* from a point $z \in \mathbb{G}$ is a function $\tilde{d}_z : \mathbb{G} \to [0, \infty)$ possessing the following properties: $\tilde{d}_z \in C^\infty(\mathbb{G})$, $c_1\rho(x,z) \leq \tilde{d}_z(x) \leq c_2\rho(x,z)$, and $|\nabla_{\mathcal{L}}\tilde{d}_z|(x) \leq c_3$, where $x \in \mathbb{G}$ and the constants c_1, c_2, and c_3 are independent of z. A regularized distance is defined as in Euclidean space [31] by means of the Whitney partition [30] of the domain $\mathbb{G} \setminus \{z\}$.

Lemma 2. *Suppose that $\varphi : D \to D'$ is a homeomorphism possessing the following property: there is a function $g \in L_{q,\mathrm{loc}}(D)$, $q \geq 1$, such that, for some countable everywhere dense set of points $z \in \mathbb{G}$, the function $[\varphi]_z(x) = \tilde{d}_z(\varphi(x))$ belongs to $L^1_q(D)$ and*

$$|\nabla_{\mathcal{L}}[\varphi]_z|(x) \leq Kg(x) \quad \textit{almost everywhere in} \quad D,$$

with some constant K independent of z. Then $\varphi \in ACL(D)$ and $|\nabla_{\mathcal{L}}\varphi|(x) \leq K'g(x)$, with some constant K' independent of φ and g.

PROOF. Consider the fibration Γ_j of D generated by some vector field X_{1j}. The function $[\varphi]_z|_\gamma$ is absolutely continuous on $d\gamma$-almost all lines γ of Γ_j whose choice is independent of z, $g|_\gamma \in L_q$, and $|X_{1j}[\varphi]_z|_\gamma|(x) \leq Kg|_\gamma(x)$ almost everywhere. Hence, for every segment $[x,y]$ of γ we have

$$\big|[\varphi]_z|_\gamma(y) - [\varphi]_z|_\gamma(x)\big| \leq K \int\limits_{[x,y]} g\, dt.$$

Thus, the increment of the function $[\varphi]_z|_\gamma$ along γ is controlled by the integral of g independently of the choice of z. Consequently, we can pass to the limit in z and, since the set of the points z is dense in $\mathbb{G}$, the last inequality is valid for every point $z \in \mathbb{G}$. Putting $z = \varphi(x)$, we obtain

$$\rho(\varphi(x), \varphi(y)) \leq c_1^{-1}K \int\limits_{[x,y]} g\, dt.$$

This implies absolute continuity of φ on almost all lines of the horizontal fibration as well as the estimate $|X_{1j}\varphi| \leq K'g$ in D. Lemma 2 is proven.

We say that a mapping $\varphi : D \to D'$ *satisfies Luzin's condition* $(\mathcal{N})$ if the image of each set of measure zero is a set of measure zero. Given $\varphi : D \to D'$, denote by $\mathcal{J}_\varphi(x)$ the volume derivative

$$\mathcal{J}_\varphi(x) = \lim_{r\to 0} \frac{|\varphi(B(x,r))|}{r^\nu}.$$

It is well known [32] that there is a Borel set S of measure zero outside which $\varphi : D \to D'$ satisfies Luzin's condition $(\mathcal{N})$ and the change-of-variable formula is valid:

$$\int\limits_D f \circ \varphi(x)\mathcal{J}_\varphi(x)\, dx = \int\limits_{D'} f(y)\chi(y)\, dy,$$

where $\chi(y)$ is the characteristic function of the set $D' \setminus \varphi(S)$.

PROOF OF PROPOSITION 1. By Lemma 2, it suffices to verify existence of a function $g \in L_{q,\mathrm{loc}}(D)$ such that $|\nabla_{\mathcal{L}}(u \circ \varphi)|(x) \leq Kg(x)$ almost everywhere in D for all functions $u \in C^\infty(D)$ such that $\|u \mid L^1_\infty(D')\| \leq 1$. Fix a point $y_0 \in D'$ and take the function $\eta(y) \doteq \xi(\delta_r(y_0^{-1}y))$, with $B(y_0, 2r) \subset \mathbb{G}$, where $\xi \in C_0^\infty(\mathbb{G})$ is a truncator such that $\xi|_{B(0,1)} \equiv 1$ and $\xi|_{\mathbb{G}\setminus B(0,2)} \equiv 0$.

Using Lemma 1, for $q \le p < \infty$ we then obtain ($B = B(y_0, r)$)

$$\left(\int\limits_{\varphi^{-1}(B)} |\nabla_{\mathcal{L}}(u \circ \varphi)|^q(x)\, dx \right)^{\frac{1}{q}} \le \Phi(2B)^{\frac{p-q}{pq}} \left(\int\limits_{2B} |\nabla_{\mathcal{L}}(u - u(y_0))\eta(y)|^p\, dy \right)^{\frac{1}{p}}$$

$$\le \Phi(2B)^{\frac{1}{\kappa}} \left(\int\limits_{2B} |\nabla_{\mathcal{L}} u(y)|^p \eta^p(y)\, dy \right)^{\frac{1}{p}} + \Phi(2B)^{\frac{1}{\kappa}} \left(\int\limits_{2B} |\nabla_{\mathcal{L}} \eta|^p (u - u(y_0)^p\, dy \right)^{\frac{1}{p}}$$

$$\le \Phi(2B)^{\frac{1}{\kappa}} (|2B|^{\frac{1}{p}} + (c_1 r^{-1} c_2 r |2B|^{\frac{1}{p}})) = C\Phi(2B)^{\frac{1}{\kappa}} |B|^{\frac{1}{p}}.$$

(If $p=q$ then we put $\Phi(2B)^{\frac{1}{\kappa}} = \|\varphi^*\|$ for all balls B.) This implies in particular that $|\nabla_{\mathcal{L}}(u \circ \varphi)|^q(x) = 0$ for almost all $x \in Z \cup S$, with $Z = \{x \in D \,:\, \mathcal{J}_\varphi(x) = 0\}$, since $|\varphi(Z \setminus S)| = 0$ by the change-of-variable formula. To verify this, fix $\varepsilon > 0$. There is a collection $\{B_i = B(x_i, r_i)\}$ of balls covering the set $\varphi(Z \setminus S)$ and possessing the following property: the balls $2B_i = B(x_i, 2r_i)$ constitute a covering of finite multiplicity and the multiplicity of the covering depends only on the algebraic and geometric properties of the group $\mathbb{G}$ and $\sum r_i^\nu < \varepsilon$. From the last inequalities for $q < p$ we obtain

$$\int\limits_{Z} |\nabla_{\mathcal{L}}(u \circ \varphi)|^q(x)\, dx \le \sum_{i=1}^{\infty} \int\limits_{\varphi^{-1}(B_i)} |\nabla_{\mathcal{L}}(u \circ \varphi)|^q(x)\, dx$$

$$\le C \sum_{i=1}^{\infty} \Phi(2B_i)^{\frac{q}{\kappa}} |B_i|^{\frac{q}{p}} \le C \left(\sum_{i=1}^{\infty} \Phi(2B_i) \right)^{\frac{q}{\kappa}} \left(\sum_{i=1}^{\infty} |B_i| \right)^{\frac{q}{p}}.$$

Since $\sum_{i=1}^{\infty} \Phi(2B_i) \le C'\Phi(D')$ and $\varepsilon > 0$ is an arbitrary number, the sought equality is proven. (For $q = p$, the estimate is even simpler.)

Thus, we arrive at the relations

$$\int\limits_{\varphi^{-1}(B)} |\nabla_{\mathcal{L}}(u \circ \varphi)|^q(x)\, dx = \int\limits_{\varphi^{-1}(B) \setminus (Z \cup S)} \frac{|\nabla_{\mathcal{L}}(u \circ \varphi)|^q(x) \mathcal{J}_\varphi(x)}{\mathcal{J}_\varphi(x)}\, dx$$

$$= \int\limits_{B \setminus \varphi(Z \cup S)} \frac{|\nabla_{\mathcal{L}}(u \circ \varphi)|^q}{\mathcal{J}_\varphi} (\varphi^{-1}(y))\, dy \le C\Phi(2B)^{\frac{q}{\kappa}} |B|^{\frac{q}{p}}, \quad p < \infty.$$

From the Lebesgue theorem on differentiation of integrals we obtain

$$\frac{|\nabla_{\mathcal{L}}(u \circ \varphi)|^q}{\mathcal{J}_\varphi} (\varphi^{-1}(y)) \le C\Phi'(y)^{\frac{q}{\kappa}} \quad \text{for almost all} \quad y \in D' \setminus \varphi(Z \cup S)$$

or $|\nabla_{\mathcal{L}}(u \circ \varphi)|(x) \le C^{\frac{1}{q}} \Phi'(\varphi(x))^{\frac{1}{\kappa}} \mathcal{J}_\varphi(x)^{\frac{1}{q}}$ for $q < p$ ($|\nabla_{\mathcal{L}}(u \circ \varphi)|(x) \le C^{\frac{1}{p}} \times \mathcal{J}_\varphi(x)^{\frac{1}{p}}$ for $q = p$) for almost all $x \in D$, since $|S| = 0$. Putting

$$g(x) = \begin{cases} C^{\frac{1}{q}} \Phi'(\varphi(x))^{\frac{1}{\kappa}} \mathcal{J}_\varphi(x)^{\frac{1}{q}} & \text{for } q < p, \\ C^{\frac{1}{p}} \mathcal{J}_\varphi(x)^{\frac{1}{p}} & \text{for } q = p, \end{cases}$$

we find that $g \in L_{q,\mathrm{loc}}(D)$. Consequently, the conditions of Lemma 2 are satisfied and therefore $\varphi \in ACL(D)$.

If $\varphi \in ACL(\Omega)$ then the formal horizontal differential $\nabla_{\mathcal{L}}\varphi(x)$ generates the homomorphism $D\varphi$, called the *formal differential*, of the Lie algebra V [28, Theorem 4]. The quantity $\mathcal{J}(x,\varphi) = \det D\varphi$ is called the *Jacobian* of φ at x.

Introduce the characteristic

$$K_p(x) = \inf\{K(x) : |\nabla_{\mathcal{L}}\varphi|^p(x) \le K(x)|\mathcal{J}(x,\varphi)|\}.$$

Theorem 2. *A homeomorphism $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$ if and only if $\varphi \in ACL(D)$ and the quantities*

$$K_{p,p} = \|K_p(\cdot) \mid L_\infty(D)\|^{\frac{1}{p}} \quad \text{for } 1 \le q = p < \infty,$$

$$K_{p,q} = \left\|K_p(\cdot) \mid L_{\frac{q}{p-q}}\right\|^{\frac{1}{p}} \quad \text{for } 1 \le q < p < \infty$$

are finite. The norm of the operator $\varphi^* : L^1_p(D') \to L^1_q(D)$ is equivalent to $K_{p,q}$.

PROOF. The membership $\varphi \in ACL(D)$ is proven in Proposition 1. We turn to proving the other assertions of the theorem.

Necessity: THE CASE OF $1 \le q \le p < \infty$. By Lemma 1, the inequality $\|\varphi^* f \mid \overset{\circ}{L}{}^1_q(\varphi^{-1}(A))\| \le \Phi(A)^{1/\kappa}\|f \mid \overset{\circ}{L}{}^1_p(A)\|$ holds for every function $f \in \overset{\circ}{L}{}^1_p(A)$ with $q \le p$, where $A \subset D'$ is an open subset (for $q = p$ we put $\Phi(A)^{\frac{1}{\kappa}} = \|\varphi^*\|$). Fix a truncator $\eta \in C_0^\infty(\mathbb{G})$ which equals 1 on $B(0,1)$ and 0 outside $B(0,2)$. Insert the functions $h_{1j}(y) = (y_0^{-1}y)_{1j}\eta(\delta_r^{-1}(y_0^{-1}y))$, $j = 1,\dots,n_1$, in the above inequality. Here the symbol $(y_0^{-1}y)_{1j}$ stands for the $1j$th coordinate function of the mapping $y_0^{-1}y$. Then we arrive at the inequality

$$\left(\int\limits_{\varphi^{-1}(B(y_0,r))} |\nabla_{\mathcal{L}}\varphi|^q\,dx\right)^{1/q} \le C\Phi(B(y_0,2r))^{1/\kappa}(r^\nu)^{1/p}, \tag{1}$$

where C is some constant depending only on ν and p.

If φ does not satisfy condition $(\mathcal{N})$ then, by the change-of-variable theorem of [28], there is a Borel set E of measure zero such that the formula

$$\int\limits_D (g \circ \varphi)|\mathcal{J}(x,\varphi)|\,dx = \int\limits_{D'} g(y)\chi(y)\,dy \tag{2}$$

is valid with $\chi(\cdot)$ the characteristic function of the set $D' \setminus \varphi(E)$.

Put $Z = \{x \in D \setminus E \mid \mathcal{J}(x,\varphi) = 0\}$. Show that

$$\int\limits_Z |\nabla_{\mathcal{L}}\varphi|^p\,dx = 0. \tag{3}$$

By (2), we have $|\varphi(Z \setminus E)| = 0$. Fix $\varepsilon > 0$ and fix an open set $U \supset \varphi(Z \setminus E)$, $|U| < \varepsilon$. There is a covering $\{B(x_i,r_i)\}$ of finite multiplicity of U by the balls such that $B(x_i,2r_i)$ as well constitute a covering of finite multiplicity of U and $\sum r_i^\nu < N\varepsilon$ (the multiplicity N of the covering is independent of U). Then from (1) we obtain

$$\int\limits_Z |\nabla_{\mathcal{L}}\varphi|^q(x)\,dx = \int\limits_{Z\setminus E} |\nabla_{\mathcal{L}}\varphi|^q(x)\,dx \le \sum_{i=1}^{\infty} \int\limits_{\varphi^{-1}(B(y_i,r_i))} |\nabla_{\mathcal{L}}\varphi|^q(x)\,dx$$

$$\le \begin{cases} C^p\|\varphi^*\|^p \sum\limits_{i=1}^{\infty} r_i^\nu & \text{for } q = p, \\ C^q \sum\limits_{i=1}^{\infty} \Phi(B(y_i,2r_i))^{\frac{q}{\kappa}}(r_i^\nu)^{\frac{q}{p}} \le \widetilde{C}^q\Phi(D'))^{\frac{p-q}{p}}\left(\sum\limits_{i=1}^{\infty} r_i^\nu\right)^{\frac{q}{p}} & \text{for } q < p. \end{cases}$$

Since $\Phi(D') < \infty$ and $\varepsilon > 0$ is arbitrary, (3) is proven and consequently $|\nabla_{\mathcal{L}}\varphi| = 0$ almost everywhere on Z.

For $q = p$ we apply (2) to the left-hand side of (1). Then the Lebesgue theorem on differentiation of integrals implies that

$$\frac{|\nabla_{\mathcal{L}}\varphi|^p(\varphi^{-1}(y))}{|\mathcal{J}(\varphi^{-1}(y),\varphi)|} \le C^p\|\varphi^*\|^p \quad \text{almost everywhere in } D' \setminus \varphi(E \cup Z).$$

Let $S \subset D' \setminus \varphi(E \cup Z)$ be the set of measure zero on which the last inequality is not valid and let χ_S be the characteristic function of S. Then $|\mathcal{J}(x,\varphi)| = 0$ almost everywhere on $\varphi^{-1}(S)$ by (2). Therefore, $\varphi^{-1}(S) \subset Z$ and $|\nabla_{\mathcal{L}}\varphi|^p(x) \le C^p\|\varphi^*\|^p|\mathcal{J}(x,\varphi)|$ almost everywhere in D.

In the case of $q < p$ we rewrite (1) as

$$\int\limits_{\varphi^{-1}(B(y_0,r))} |\nabla_{\mathcal{L}}\varphi|^q\,dx \le \widetilde{C}^q\left(\frac{\Phi(B(y_0,2r))}{|B(y_0,2r)|}\right)^{\frac{q}{\kappa}} r^\nu$$

and apply (2) to the left-hand side of this relation:

$$\int\limits_{\varphi^{-1}(B(y_0,r))} |\nabla_{\mathcal{L}}\varphi|^q(x)\,dx = \int\limits_{\varphi^{-1}(B(y_0,r))\setminus Z} |\nabla_{\mathcal{L}}\varphi|^q(x)\,dx$$
$$= \int\limits_{B(y_0,r)} \frac{|\nabla_{\mathcal{L}}\varphi|^q(\varphi^{-1}(y))}{|\mathcal{J}(\varphi^{-1}(y),\varphi)|}\chi(y) \le \widetilde{C}^q\left(\frac{\Phi(B(y_0,2r))}{|B(y_0,2r)|}\right)^{\frac{q}{\kappa}} r^\nu.$$

The Lebesgue theorem on differentiation of integrals and the properties of the derivative of a countably-additive set function [32] imply that

$$\left(\frac{|\nabla_{\mathcal{L}}\varphi|^q(\varphi^{-1}(y))}{|\mathcal{J}(\varphi^{-1}(y),\varphi)|}\right)^{p/(p-q)} \chi(y) \le \widetilde{C}^\kappa\Phi'(y) \quad \text{almost everywhere in } D'.$$

Integrating the inequality over D', we obtain

$$K_{p,q}^{\frac{pq}{p-q}} = \int\limits_{D\setminus Z} \left(\frac{|\nabla_{\mathcal{L}}\varphi|^p}{|\mathcal{J}(x,\varphi)|}\right)^{\frac{q}{p-q}} dx = \int\limits_{D'} \left(\frac{|\nabla_{\mathcal{L}}\varphi|^q(\varphi^{-1}(y))}{|\mathcal{J}(\varphi^{-1}(y),\varphi)|}\right)^{\frac{p}{p-q}} \chi(y)\,dy$$
$$\le \widetilde{C}^\kappa \int\limits_{D'} \Phi'(y)\,dy \le \widetilde{C}^\kappa\Phi(D') \le \widetilde{C}^\kappa\|\varphi^*\|^{\frac{pq}{p-q}}.$$

Sufficiency: Show that the inequality $\|\varphi^* f \mid L^1_q(D)\| \le K_{p,q}\|f \mid L^1_p(D')\|$, $q \le p$, holds for every function $f \in L^1_p(D') \cap C^\infty(D')$. Since $f \circ \varphi$ belongs to the class $ACL(D)$, we have

$$\|\varphi^* f \mid L^1_q(D)\| \le \left(\int\limits_D (|\nabla_{\mathcal{L}} f||\nabla_{\mathcal{L}}\varphi|)^q\,dx\right)^{1/q}$$
$$= \left(\int\limits_{D\setminus Z} |\nabla_{\mathcal{L}} f|^q|\mathcal{J}(x,\varphi)|^{q/p}\frac{|\nabla_{\mathcal{L}}\varphi|)^q}{|\mathcal{J}(x,\varphi)|^{q/p}}\,dx\right)^{1/q}.$$

Using Hölder's inequality for $q \le p$, we derive the estimate

$$\|\varphi^* f \mid L^1_q(D)\| \le \Big(\int\limits_{D\setminus Z} \Big(\frac{|\nabla_{\mathcal{L}}\varphi|^p}{|\mathcal{J}(x,\varphi)|}\Big)^{q/(p-q)} dx\Big)^{(p-q)/pq} \Big(\int\limits_{D\setminus Z} |\nabla_{\mathcal{L}} f|^p |\mathcal{J}(x,\varphi)|\,dx\Big)^{1/p} \qquad (4)$$

(for $q = p$ the left factor equals $K_{p,p}$). Applying (2) to the right factor, we obtain the following estimate for the norm: $\|\varphi^*\| \le K_{p,q}$.

To extend the so-obtained estimate to all functions $f \in L^1_p(D')$, $1 < q \le p < \infty$, we approximate f by a sequence of smooth functions $f_n \in L^1_p(D')$ so as to have $\|f - f_n \mid L^1_p(D')\| \to 0$ and $f - f_n \to 0$ quasi-everywhere in D' as $n \to \infty$ [33]. Since the inverse image $\varphi^{-1}(E)$ of a set $E \subset D'$ of zero capacity has zero capacity, we have $\varphi^*(f_n) \to \varphi^*(f)$ quasi-everywhere in D. Hence, we conclude that the extension of the operator φ^* from the subspace $f \in L^1_p(D') \cap C^\infty(D')$ to $f \in L^1_p(D')$ by continuity coincides with the substitution operator φ^*: $\varphi^*(f) = f \circ \varphi$ (since from each sequence converging in $L^1_p(D)$, $1 < p$, we can extract a subsequence which converges quasi-everywhere).

If $1 = q < p$ then, using the method described in the preceding case, we can extend this operator to all functions of the Sobolev class under consideration: the only difference is that from a sequence $\varphi^*(f_n)$, with f_n converging quasi-everywhere in $L^1_p(D')$, we can extract a subsequence which converges almost everywhere in D.

If $q = p = 1$ then we should replace the capacitary characteristic of convergence with a coarser one: if a sequence $f_n \in L^1_1(D')$ converges to $f \in L^1_1(D')$ in $L^1_1(D')$ then some its subsequence converges almost everywhere. To complete the proof, it suffices to use the following property [34]: the inverse image of a set of measure zero under the mapping $\varphi : D \to D'$ inducing the bounded operator $\varphi^* : L^1_1(D') \to L^1_1(D)$ is a set of measure zero.

Corollary 1. *An additive set function $U \mapsto \Phi(U)$, with $U \subset D'$ an open set, considered in Lemma 1 is absolutely continuous. Moreover, the set function $V \mapsto \Phi(\varphi(V))$, with $V \subset D$ an open set, is absolutely continuous too.*

PROOF. It is obvious that the estimate (4) is valid not only for D but also for an arbitrary open set $\varphi^{-1}(U)$, where $U \subset D'$ is an open set. Consequently,

$$\|\varphi^* f \mid \overset{\circ}{L}{}^1_q(\varphi^{-1}(U))\| \le \Big(\int\limits_{\varphi^{-1}(U)\setminus(Z\cup E)} \Big(\frac{|\nabla_{\mathcal{L}}\varphi|^p}{|\mathcal{J}(x,\varphi)|}\Big)^{q/(p-q)} dx\Big)^{(p-q)/pq} \|f \mid \overset{\circ}{L}{}^1_p(U)\|.$$

Hence, $\Phi(U) \le \|K_p(\cdot) \mid L_{\frac{q}{p-q}}(\varphi^{-1}(U \setminus \varphi(Z \cup E)))\|^{\frac{p-q}{q}}$. Since the set function $D' \setminus \varphi(Z \cup E) \supset A' \mapsto |\varphi^{-1}(A')|$ is absolutely continuous, the function defined on open sets $U \subset D'$ by the rule $U \mapsto \|K_p(\cdot) \mid L_{\frac{q}{p-q}}(\varphi^{-1}(U \setminus \varphi(Z \cup E)))\|^{\frac{p-q}{q}}$ is absolutely continuous. Since it dominates $\Phi(U)$, the latter is absolutely continuous too.

Similarly, we verify that the estimate $\Phi(\varphi(V)) \le \|K_p(\cdot) \mid L_{\frac{q}{p-q}}(V \setminus (Z \cup E))\|^{\frac{p-q}{q}}$, is valid for every open set V, whence we conclude that the set function $V \mapsto \Phi(\varphi(V))$ is absolutely continuous.

§ 2. Capacity and Mappings Generating Embeddings of the Sobolev Spaces

Recall the concept of capacity [33]. A *condenser* in a domain $D \subset \mathbb{G}$ is a pair (F_0, F_1) of disjoint connected closed sets $F_0, F_1 \subset D$. A continuous function $u \in L^1_p(D)$ is called *admissible* for a condenser

(F_0, F_1) if the set $F_i \cap D$ is contained in some connectedness component of the set $\mathrm{Int}\{x \mid u(x) = i\}$, $i = 0, 1$. The *p-capacity* of a condenser (F_0, F_1) in the space $L^1_p(D)$ is the number

$$\mathrm{cap}_p(F_0, F_1; D) = \inf \|u \mid L^1_p(D)\|^p,$$

where the infimum is calculated over all functions admissible for the condenser (F_0, F_1). A function $v \in L^1_p(D)$ is called an *extremal function* for a condenser (F_0, F_1) if

$$\int\limits_{D\setminus(F_0\cup F_1)} |\nabla_{\mathcal{L}} v|^p \, dx = \mathrm{cap}_p(F_0, F_1; D)$$

and $v - w \in \overset{\circ}{L}{}^1_p(D \setminus (F_0 \cup F_1))$ for every function w admissible for the pair (F_0, F_1).

Denote by $E_p(D)$ the set of extremal functions for the p-capacity of all pairs of connected compact sets $F_0, F_1 \subset D$ with nonempty interior whose boundary points are regular with respect to the open set $D \setminus (F_0 \cup F_1)$ (see the definition in [35, 36]) (by Wiener's test [35], the regularity condition guarantees continuity of a solution to the corresponding Dirichlet problem at the boundary points of F_0 and F_1 and consequently the possibility of continuous extension by zero (unity) to the set F_0 (F_1)). Thus, the functions of the class $E_p(D)$ are also admissible for the corresponding condensers. As in the Euclidean case [37] the following theorem is valid:

Theorem 3 [33]. *Assume that $1 < p < \infty$. There is a countable collection of functions $v_i \in E_p(D)$, $i \in \mathbb{N}$, such that, for every function $u \in L^1_p(D)$ and every $\varepsilon > 0$, u is representable as $u = c_0 + \sum_{i=1}^\infty c_i v_i$ and $\|u \mid L^1_p(D)\| \le \sum_{i=1}^\infty \|c_i v_i \mid L^1_p(D)\| \le \|u \mid L^1_p(D)\| + \varepsilon$.*

Theorem 4. *A homeomorphism $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 < q \le p < \infty$, if and only if the inequalities $\mathrm{cap}_p(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D) \le K^p \,\mathrm{cap}_p(F_0, F_1; D')$ for $1 < q = p < \infty$ and $\mathrm{cap}_q^{1/q}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D) \le \Phi(D' \setminus (F_0 \cup F_1))^{\frac{p-q}{pq}} \,\mathrm{cap}_p^{1/p}$ $(F_0, F_1; D')$ for $1 < q < p < \infty$ are valid for every condenser $(F_0, F_1) \subset D$, where Φ is a bounded quasiadditive function.*

PROOF. *Necessity*: Suppose that u is an admissible function for a condenser (F_0, F_1). Then the composition $u \circ \varphi$ is admissible for the condenser $(\varphi^{-1}(F_0), \varphi^{-1}(F_1))$. Hence, $\mathrm{cap}_p^{1/p}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D) \le K \|u \mid L^1_p(D')\|$ for $q = p$ and $\mathrm{cap}_q^{1/q}(\varphi^{-1}(F_0), \varphi^{-1}(F_1); D) \le \|u \circ \varphi \mid L^1_q(D)\| \le \Phi(D' \setminus (F_0 \cup F_1))^{(p-q)/pq} \|u \mid L^1_p(D')\|$ for $q < p$. Since the admissible function u is arbitrary, we can take the greatest lower bound on the right-hand sides of these inequalities and thereby prove the necessity part.

Sufficiency: Suppose that u_i is an extremal function for a condenser $(F_0^i, F_1^i) \subset D'$ and w_i is an extremal function for $(\varphi^{-1}(F_0^i), \varphi^{-1}(F_1^i)) \subset D$. Then

$$\|c_i w_i \mid L^1_q(A_i)\| \le \begin{cases} K \|c_i u_i \mid L^1_p(A_i')\| & \text{for } q = p, \\ \Phi(A_i')^{(p-q)/pq} \|c_i u_i \mid L^1_p(A_i')\| & \text{for } q < p, \end{cases}$$

where $A_i' = D' \setminus (F_0^i \cup F_1^i)$, $A_i = \varphi^{-1}(A_i')$, and c_i is some constant. Take the qth power of the inequality and sum the resulting terms:

$$\sum_{i=1}^\infty \|c_i w_i \mid L^1_q(A_i)\|^q \le \begin{cases} K^p \left(\sum\limits_{i=1}^\infty \|c_i u_i \mid L^1_p(A_i')\|^p \right)^{1/p} & \text{for } q = p, \\ \sum\limits_{i=1}^\infty \Phi(A_i')^{(p-q)/pq} \|c_i u_i \mid L^1_p(A_i')\|^q & \text{for } q < p. \end{cases}$$

Applying Hölder's inequality, for $q < p$ we obtain

$$\left(\sum_{i=1}^{\infty}\|c_i u_i \mid L^1_q(A_i)\|^q\right)^{1/q} \le \Phi\left(\bigcup_{i=1}^{\infty} A'_i\right)^{(p-q)/pq} \left(\sum_{i=1}^{\infty}\|c_i u_i \mid L^1_p(A'_i)\|^p\right)^{1/p}.$$

Using Theorem 3, we can now validate the inequality $\|\varphi^* f \mid L^1_q(D)\| \le K\|f \mid L^1_p(D')\|$, $q \le p$, for every function $f \in L^1_p(D')$ as in [37].

§ 3. Definition of (p, q)-Quasiconformal Homeomorphisms and Their Properties

Given $\varphi : D \to D'$, introduce the characteristic $L_\varphi(x,r) = \max_{\rho(x,y)=r} \rho(\varphi(x), \varphi(y))$, provided that r is sufficiently small. Fix a constant $\lambda \ge 1$. A homeomorphism $\varphi : D \to D'$ is called (p,q)-*quasiconformal*, $1 \le q \le p < \infty$, if there exist a constant $K < \infty$ and (for $q < p$) a bounded additive absolutely continuous function Φ defined on open subsets of D and such that

$$\varlimsup_{r\to 0} \frac{L^p_\varphi(x,r)r^{\nu-p}}{|\varphi(B(x,\lambda r))|} \Big/ \left(\frac{\Phi(B(x,\lambda r))}{|B(x,r)|}\right)^{\frac{p-q}{q}} \le K$$

for all points $x \in D$ (for $p = q$, the function in the denominator is assumed to be equal to unity [26, 27]; see [18] for the case of $\mathbb{G} = \mathbb{R}^n$).

Let γ be some curve in $\mathbb{G}$ joining points x_1 and x_2. According to [27], a mapping φ is called α-*absolutely continuous* ($\alpha \ge 1$) on γ if for every $\varepsilon > 0$ there is $\delta > 0$ such that the inequality $\sum_{i\ge 1} \rho(\varphi(a_i), \varphi(b_i))^\alpha < \varepsilon$ is valid for an arbitrary collection of segments (a_i, b_i) of γ such that $\sum_{i\ge 1} \rho(a_i, b_i)^\alpha < \delta$. The concept of α-absolute continuity is connected with the concept of the α-dimensional Hausdorff measure H^α defined for an arbitrary set $A \subset \mathbb{G}$ as

$$H^\alpha(A) = \lim_{\varepsilon\to 0} H^\alpha_\varepsilon(A) = \lim_{\varepsilon\to 0} \inf\Big\{\sum_j (\operatorname{diam} B_j)^\alpha : A \subset \bigcup_j B_j,\ \operatorname{diam} B_j < \varepsilon\Big\}.$$

Observe the following fact:

Lemma 3 [27]. *Suppose that $v \in V_\alpha$, $\alpha = 1, \dots, m$, is a left-invariant vector field and $\gamma(s)$ is some integral line of this field. Then $\gamma(s)$ has finite α-dimensional Hausdorff measure.*

An important property of (p,q)-quasiconformal mappings is α-absolute continuity along the lines of the fibration Γ generated by a left-invariant vector field $X_{\alpha\tau} \in V_\alpha$.

Theorem 5 (the ACL-property). *Suppose that $\varphi : D \to D'$ is a (p,q)-quasiconformal homeomorphism, $1 < q \le p < \infty$. Then, for an arbitrary natural $1 \le \alpha < q$, the mapping φ is α-absolutely continuous on almost all lines of the fibration Γ generated by the left-invariant vector field $X_{\alpha\tau} \in V_\alpha$ ($1 \le \tau \le n_\alpha$). Moreover, every (p,q)-quasiconformal mapping is $\mathcal{P}$-differentiable almost everywhere in D.*

PROOF. Fix some field $X_{\alpha\tau}$ and let Γ be the fibration generated by this field. Take the cube $Q = Y\gamma_0$, where $\gamma_0 = \exp_s X_{\alpha\tau}$, $|s| \le M$, and Y is the hyperplane transversal to $X_{\alpha\tau}$:

$$Y = (a; 0; b) \mid x_{\alpha\tau} = 0,\ |a| \le M,\ |b| \le M$$

(with these notations, $a = (x_{ij})$, $1 \le i \le \alpha$, $1 \le j \le n_i$, $j \ne \tau$ for $i = \alpha$ and $b = (x_{ij})$, $\alpha < i \le m$, $1 \le j \le n_i$).

Given a point $y \in Y$, denote by γ_y the element $y\gamma_0$ of the fibration which starts at y. Thus, Q is the union of all such segments of integral lines. Consider the tubular surface of the fiber γ_y of radius r: $E(y, r) = \gamma_y B(\epsilon, r) \cap Q$.

The following lemma is valid:

Lemma 4 [27]. *Let Φ be a quasiadditive function on $\mathbb{G}$. Then $\overline{\lim}_{r\to 0}\frac{\Phi(E(y,r))}{r^{\nu-\alpha}}<\infty$ for $d\gamma$-almost all $y\in Y$.*

PROOF OF THEOREM 5. The case $1<p=q<\infty$ was considered in [27] under more general assumptions on the topological properties of mappings. The method of [27] for proving absolute continuity applies also to the case of $1<q<p$.

Assume that $1<q<p<\infty$. Take a point $y\in Y$ so that the assertion of Lemma 4 hold for the element γ_y of the fibration. Fix a compact set $F\subset\gamma_y$ and a number $C>K$. Observe that F is the union of the following increasing sequence of closed sets:

$$F_l=\left\{x\in F\;\middle|\;\left(\frac{L^p_\varphi(x,r)r^{\nu-p}}{|\varphi(B(x,\lambda r))|}\right)^q\le C^q\left(\frac{\Phi(B(x,\lambda r))}{r^\nu}\right)^{p-q}\text{ for all } r<\frac{1}{l},\ l\in\mathbb{N}\right\},$$

(the proof of closure of F_l bases on absolute continuity of the set function Φ). Fix l, $\varepsilon>0$, and $t>0$. There is $\delta>0$ such that for each $0<r<\min(\delta,1/l)$ there exists a numeric sequence $s_j\in F_l$, $j=1,\dots,N$, such that the balls $B_j=B(x_j,r)$, where $x_j=\gamma_y(s_j)$, cover F_l (moreover, the sequence $\{s_j\}$ is chosen so that each point of F_l be contained in at most two balls), $Nr^\alpha\le H^\alpha(F_l)+\varepsilon$, and $\rho(\varphi(x_j),\varphi(y))<t$, $y\in B(x_j,r)$. Then

$$L^q_\varphi(x_j,r)\le C^{\frac{q}{p}}r^{q-\alpha}\left(\frac{\Phi(\lambda B_j)}{r^{\nu-\alpha}}\right)^{\frac{p-q}{p}}\left(\frac{\Psi(\lambda B_j)}{r^{\nu-\alpha}}\right)^{\frac{q}{p}},$$

where the additive function Ψ is defined on Borel sets by the relation $\Psi(A)=|\varphi(A)|$ and $\lambda B_j=B_j(x_j,\lambda r_j)$.

The balls $B(\varphi(x_j),L_\varphi(x_j,r))$ obviously cover the image $\varphi(F_l)$. Therefore,

$$\begin{aligned}H^\alpha_t(\varphi(F_l))^{\frac{q}{\alpha}}&\le\left(\sum_{j=1}^N L^\alpha_\varphi(x_j,r)\right)^{\frac{q}{\alpha}}\le N^{\frac{q-\alpha}{\alpha}}\sum_{j=1}^N L^q_\varphi(x_j,r)\\&\le C^{\frac{q}{p}}N^{\frac{q-\alpha}{\alpha}}r^{q-\alpha}\left(\sum_{j=1}^N\frac{\Phi(\lambda B_j)}{r^{\nu-\alpha}}\right)^{\frac{p-q}{p}}\left(\sum_{j=1}^N\frac{\Psi(\lambda B_j)}{r^{\nu-\alpha}}\right)^{\frac{q}{p}}\\&\le\text{const}(H^\alpha(F_l)+\varepsilon)^{\frac{q-\alpha}{\alpha}}\left(\frac{\Phi(\lambda B_j)}{(\lambda r)^{\nu-\alpha}}\right)^{\frac{p-q}{p}}\left(\frac{\Psi(\lambda B_j)}{(\lambda r)^{\nu-\alpha}}\right)^{\frac{q}{p}}.\end{aligned}$$

Passing to the limit as $r\to 0$ and afterwards letting ε and t tend to zero, we obtain

$$H^\alpha(\varphi(F_l))^{\frac{q}{\alpha}}\le\text{const}(H^\alpha(F_l))^{\frac{q-\alpha}{\alpha}}(\Phi'(y))^{\frac{p-q}{p}}(\Psi'(y))^{\frac{q}{p}}$$

for an arbitrary l. Since $\varphi(F)$ is the limit of the increasing sequence $\varphi(F_l)$, the last inequality holds for F. Consequently, $\varphi\in ACL(D)$.

We are left with demonstrating that φ is $\mathcal{P}$-differentiable almost everywhere in D. From the condition of (p,q)-quasiconformality we obtain the inequality

$$\left(\frac{L_\varphi(x,r)}{r}\right)^{pq}\le C^q\left(\frac{\Phi(\lambda B_j)}{r^\nu}\right)^{p-q}\left(\frac{\Psi(\lambda B_j)}{r^\nu}\right)^q.$$

Passing to the limit as $r\to 0$, we arrive at the relation

$$\overline{\lim_{r\to 0}}\left(\frac{L_\varphi(x,r)}{r}\right)^{pq}\le\text{const}\,\Phi'(x)^{p-q}\Psi'(x)^q$$

for almost all points of D; so φ is $\mathcal{P}$-differentiable almost everywhere in D [23, 28].

The following theorem establishes a connection between the (p,q)-quasiconformal mappings and the mappings generating embedding of the Sobolev spaces.

Theorem 6. *Suppose that $\varphi : D \to D'$ is a (p,q)-quasiconformal homeomorphism, $1 < q \le p$. Then φ generates the bounded embedding operator*

$$\varphi^* : L^1_p(D') \to L^1_q(D), \quad 1 < q \le p < \infty.$$

PROOF. Since φ is $\mathcal{P}$-differentiable, we have

$$\lim_{r\to 0} \frac{L^p_\varphi(x,r)}{r^p} = |\nabla_{\mathcal{L}}\varphi(x)|^p, \quad \lim_{r\to 0} \frac{|\varphi(B(x,\lambda r))|}{r^\nu} = \lambda^\nu |\mathcal{J}(x,\varphi)|.$$

If $1 < q = p < \infty$ then $|\nabla_{\mathcal{L}}\varphi(x)|^p \le K\lambda^\nu |\mathcal{J}(x,\varphi)|$ almost everywhere in D. In the case of $1 < q < p < \infty$ we have $|\nabla_{\mathcal{L}}\varphi(x)|^p \le K\lambda^\nu |\mathcal{J}(x,\varphi)| {\Phi'}^{\frac{p-q}{q}}(x)$ in D. Hence, $K_{p,q} = \|K_p(\cdot) \mid L_{\frac{q}{p-q}}\|^{\frac{1}{p}} < \infty$. Since φ belongs to the class ACL by Theorem 5, the conditions of Theorem 2 are satisfied and consequently φ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 < q \le p < \infty$.

The converse assertion is valid under some additional constraints on the exponents p and q.

Theorem 7. *Suppose that $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $\nu - 1 < q \le p < \infty$. Then φ is (p,q)-quasiconformal.*

PROOF. Consider a ball $B(x_0,\lambda r)$, $\lambda > 1$, in D. Suppose that a point $y_1 \in \varphi(S(x_0,r))$ is such that $L_\varphi(x_0,r) = \rho(\varphi(x_0), y_1)$. Denote by y_2 a point in $\varphi(S(x_0,\lambda r))$ that is most distant from y_1 and denote by y_3 a point in $\varphi(S(x_0,r))$ that is most distant from y_2. In D', consider the continua

$$F_1 = \{y \in D' \mid \rho(y,y_2) \le \rho(y_2,y_3)\} \cap \varphi(B(x_0,\lambda r)),$$
$$F_0 = \{y \in D' \mid \rho(y,y_2) \ge \rho(y_2,y_1)\} \cap \varphi(B(x_0,\lambda r)).$$

Under the above conditions, the function $\eta(x) = (cL_\varphi(x_0,r))^{-1}(\min(d(x,F_0)), cL_\varphi(x_0,r))$ (c is some constant) is admissible for the capacity of the pair of these continua in the Sobolev space $L^1_p(\varphi(B(x_0,\lambda r)))$. Since the operator $\varphi^* : L^1_p(\varphi(B(x_0,\lambda r))) \to L^1_q(B(x_0,\lambda r))$ is bounded by Theorem 2 and its norm $\|\varphi^*\|$ is bounded by $cK_{p,p}$ for $q = p$ and $c_1\Phi(\varphi(B(x_0,\lambda r)))^{\frac{p-q}{pq}}$ for $q < p$ (c_1 is some constant), from the estimate for the Teichmüller capacity [24] we obtain

$$cr^{\frac{\nu-q}{q}} \le \operatorname{cap}_q^{\frac{1}{q}}(\varphi^{-1}(F_0),\varphi^{-1}(F_1); B(x_0,\lambda r))$$
$$\le \|\varphi^*\| \operatorname{cap}_p^{\frac{1}{p}}(F_0,F_1;\varphi(B(x_0,\lambda r))) \le \|\varphi^*\| \left(\frac{|\varphi(B(x_0,\lambda r))|}{L_\varphi(x_0,r)}\right)^{\frac{1}{p}}.$$

Consequently,

$$\frac{L^p_\varphi(x_0,r)r^{\nu-p}}{|\varphi(B(x_0,\lambda))|} \le \begin{cases} c_1 K_{p,p} & \text{for } q = p, \\ c_1\left(\frac{\Phi(\varphi(B(x_0,\lambda r)))}{r^\nu}\right)^{\frac{p-q}{q}} & \text{for } q < p. \end{cases}$$

Passing to the limit as $r \to 0$, we obtain (p,p)-quasiconformality of φ for $q = p$ and the inequality

$$\overline{\lim_{r\to 0}} \frac{L^p_\varphi(x,r)r^{\nu-p}}{|\varphi(B(x,\lambda r))|} \Big/ \left(\frac{\Psi(B(x,\lambda r))}{|B(x,r)|}\right)^{\frac{p-q}{q}} \le c_2 \quad \text{everywhere in the domain } D$$

for $q < p$, where the set function Ψ, defined for open subsets U of D by the relation $\Psi(U) = \Phi(\varphi(U))$, is bounded, additive, and, by Corollary 1, absolutely continuous. Hence, φ is (p,q)-quasiconformal.

§ 4. Further Properties of (p, q)-Quasiconformal Homeomorphisms

In this section, we need an estimate from below for the capacity of a ring. By a ring in $\mathbb{G}$ we mean a pair $R = (E, G)$ of sets, where the set $G \subset \mathbb{G}$ is open and $E \subset G$ is compact. The quantity

$$\operatorname{cap}_p(E, G) = \inf \int_G |\nabla_{\mathcal{L}} u|^p \, dx, \quad 1 \le p < \infty,$$

where the greatest lower bound is calculated over all continuous functions $u \in \overset{\circ}{L}{}^1_p(G)$, $u|_F \ge 1$, is called the *p-capacity* of the ring $R = (E, G)$.

Lemma 5. *If the set E is connected and $G \subset \{x : \rho(x, E) \le c_0 \operatorname{diam} E\}$, where c_0 is a sufficiently small number depending only on the constant in the generalized triangle inequality, then*

$$\operatorname{cap}_p^{\nu-1}(E, G) \ge c \frac{(\operatorname{diam} E)^p}{|G|^{p-(\nu-1)}}$$

for $\nu - 1 < p < \infty$, where the constant c depends only on ν and p.

PROOF. Since the left- and right-hand sides of the inequality under proof are invariant under left translations and have the same degree of homogeneity under dilations, it suffices to prove the lemma in the case of $\operatorname{diam} E = \rho(0, \sigma) = 1$ for some points $0, \sigma \in E$. Take a point $\sigma^{-1} \in S(0, 1)$. Then $1 = \operatorname{diam} E \le c_1(r_2 - 1)$ and $S(\sigma^{-1}, r) \cap (\mathbb{G} \setminus G) \ne \varnothing$ for $1 \le r \le r_2$, where $r_2 = \rho(\sigma^{-1}, \sigma) = \rho(\sigma^2)$. Fix an arbitrary point $x_r \in E \cap S(\sigma^{-1}, r)$ and denote by $P(r)$ the set

$$\{\xi \in S(\sigma^{-1}, r) : \rho(\xi, x_r) \le \rho(x_r, (\mathbb{G} \setminus G) \cap S(r))\}.$$

Every function $u \in \overset{\circ}{L}{}^1_p(G) \cap C(G))$ such that $u \ge 1$ on E takes the value 0 on the sphere $S(\sigma^{-1}, r)$, $1 \le r \le r_2$, (the choice of c_0 in the conditions of the lemma is determined by this requirement). Therefore, the following inequality is valid for almost all $r \in (1, r_2)$ [24, Theorem 1]:

$$\int_{S(r) \cap G} M_{\gamma r}(|\nabla_{\mathcal{L}} u|)^p(\xi) d\sigma_r(\xi) \ge c_2 \omega_r(P(r))^{\frac{\nu-1-p}{\nu-1}},$$

where ω_r is the measure on $S(\sigma^{-1}, r)$ associated with the "spherical" coordinate system [24]. (Here $\gamma > 1$ is some constant and $M_\delta g$ denotes the maximal function defined for every locally summable function g as

$$M_\delta g(x) = \sup \left\{ |B(x, r)|^{-1} \int_{B(x,r)} |g| \, dx : r \le \delta \right\},$$

where $B(x, r) = \{y \in \mathbb{G} : \rho(x^{-1} y) < r\}$ is the ball of radius r centered at $x \in \mathbb{G}$.) Consequently,

$$\int_G M_{\gamma r}(|\nabla_{\mathcal{L}} u|)^p \, dx \ge c_2 \int_{r_1}^{r_2} \omega_r(P(r))^{\frac{\nu-1-p}{\nu-1}} \, dr.$$

Furthermore,

$$(\operatorname{diam} E)^p \le c_1 \left(\int_{r_1}^{r_2} dr \right)^p \le c_1 \left(\int_{r_1}^{r_2} \omega_r(P(r)) \, dr \right)^{p-(\nu-1)} \left(\int_{r_1}^{r_2} \omega_r(P(r))^{\frac{\nu-1-p}{\nu-1}} dr \right)^{\nu-1}$$
$$\le \frac{c_1}{c_2} |G|^{p-(\nu-1)} \left(\int_G |M_{\gamma r} \nabla_{\mathcal{L}} u|^p \, dx \right)^{\nu-1}.$$

Applying the maximal function theorem, we obtain

$$\left(\int_G |\nabla_{\mathcal{L}} u|^p\,dx\right)^{\nu-1} \geq c\frac{(\operatorname{diam} E)^p}{|G|^{p-(\nu-1)}}$$

for every function $u \in \overset{\circ}{L}{}^1_p(G) \cap C(G)$ such that $u = 1$ on E.

Theorem 8. *Suppose that $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $\nu - 1 < q \leq p < \infty$. Then the inverse mapping $\varphi^{-1} : D' \to D$ is an ACL-mapping.*

PROOF. Fix some field $X_{\alpha\tau}$, and let Γ be the fibration generated by this field. Take the cube $Q = Y\gamma_0$, where $\gamma_0 = \exp_s X_{\alpha\tau}$, $|s| \leq M$, and Y is the transversal hyperplane to $X_{\alpha\tau}$:

$$Y = \{(a; 0; b) : x_{\alpha\tau} = 0,\ |a| \leq M,\ |b| \leq M\}$$

(with the above notations $a = (x_{\alpha\tau})$, $1 \leq i \leq \alpha$, $1 \leq j \leq n_i$, $j \neq \tau$ for $i = \alpha$ and $b = (x_{ij})$, $\alpha < i \leq m$, $1 \leq j \leq n_i$).

Given a point $y \in Y$, denote by γ_y the element $y\gamma_0$ of the fibration which starts at y. Thus, Q is the union of all such intervals of integral lines. Consider the following tubular surface of the fiber γ_y of radius r:

$$E(y, r) = \gamma_y B(e, r) \cap Q.$$

Take a point $y \in Y$ so that the assertion of Lemma 4 hold for γ_y. On γ_y, take arbitrary pairwise disjoint closed segments $\Delta_1, \dots, \Delta_k$ of lengths $b_1, \dots, b_k$. Denoting by R_i the open set of points at a distance less than a given $r > 0$ from Δ_i, consider the condenser (Δ_i, R_i). Suppose that $r > 0$ is so small that the sets $R_1, \dots, R_k$ are pairwise disjoint, the condenser $(\varphi^{-1}(\Delta_i), \varphi^{-1}(R_i))$ satisfies the conditions of Lemma 5, and $r < cb_i$, $i = 1, \dots, k$, where c is a suitable constant. Then

$$\operatorname{cap}_p(\Delta_i, R_i; D') \leq \frac{|R_i|}{r^p} \leq c_1 b_i r^{\nu-1-p},$$

$$\operatorname{cap}_q(\varphi^{-1}(\Delta_i), \varphi^{-1}(R_i); D) \geq c_2 \frac{(\operatorname{diam}\varphi^{-1}(\Delta_i))^{q/(\nu-1)}}{|\varphi^{-1}(R_i)|^{(1-\nu+q)/(\nu-1)}}.$$

By Theorem 4, from the last two inequalities we derive

$$\operatorname{diam}\varphi^{-1}(\Delta_i) \leq c_3 r^{\frac{(\nu-1-p)(\nu-1)}{p}} \Phi(R_i)^{\frac{(p-q)(\nu-1)}{pq}} |\varphi^{-1}(R_i)|^{\frac{1-\nu+q}{q}} b_i^{\frac{\nu-1}{p}}.$$

Summing over $i = 1, \dots, k$, applying Hölder's inequality, and using the definition of a quasiadditive function, we obtain

$$\sum_{i=1}^k \operatorname{diam}\varphi^{-1}(\Delta_i)$$

$$\leq c_4 \left(\frac{\Phi(E(y, r))}{r^{\nu-1}}\right)^{\frac{(p-q)(\nu-1)}{pq}} \left(\frac{|\varphi^{-1}(E(y, r)|}{r^{\nu-1}}\right)^{\frac{1-\nu+q}{q}} \left(\sum_{i=1}^k b_i\right)^{\frac{\nu-1}{p}}$$

Letting r tend to zero, we find that

$$\sum_{i=1}^k \operatorname{diam}\varphi^{-1}(\Delta_i) \leq c_5 \left(\sum_{i=1}^k b_i\right)^{(\nu-1)/p}$$

whence $\varphi^{-1} \in ACL(D')$.

Theorem 9. *Suppose that $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $\nu - 1 < q < p < \infty$. Then the inverse mapping $\varphi^{-1} : D' \to D$ generates the bounded embedding operator ${\varphi^{-1}}^* : L^1_r(D') \to L^1_s(D)$, where $r = \frac{q}{q-\nu+1}$ and $s = \frac{p}{p-\nu+1}$; moreover, φ^{-1} is an (r,s)-quasiconformal homeomorphism.*

PROOF. Theorems 2, 7, and 5 imply that φ is an ACL-homeomorphism differentiable almost everywhere. By Theorem 8, φ^{-1} belongs to $ACL(D')$. Given φ, put $s = l_\varphi(x,t)$, where $l_\varphi(x,t) = \min_{\rho(x,y)=t} \rho(\varphi(x), \varphi(y))$. Then $t = L_{\varphi^{-1}}(y,s)$. We have

$$\frac{L_{\varphi^{-1}}(y,s)}{s} = \frac{t}{l_\varphi(x,t)} = \frac{tL^{\nu-1}_\varphi(x,t)}{l_\varphi(x,t)L^{\nu-1}_\varphi(x,t)}$$
$$\leq \frac{tL^{\nu-1}_\varphi(x,t)}{|\varphi(B(x,t))|} = \frac{L^{\nu-1}_\varphi(x,t)}{t^{\nu-1}} \frac{t^\nu}{|\varphi(B(x,t))|}.$$

Therefore, the inequality $|\nabla_{\mathcal{L}}\varphi^{-1}(y)| \leq \frac{|\nabla_{\mathcal{L}}\varphi|^{\nu-1}}{|\mathcal{J}(x,\varphi)|}$ is valid for almost all $x \in D \setminus (E \cup Z)$ and $y = \varphi(x) \in D' \setminus \varphi(E \cup Z)$, with E and Z defined in the proof of Theorem 2. This inequality leads to the relation

$$\int\limits_{D'\setminus\varphi(E)} \left(\frac{|\nabla_{\mathcal{L}}\varphi^{-1}(y)|^r}{|\mathcal{J}(y,\varphi^{-1})|}\right)^{s/(r-s)} dy \leq \int\limits_{D\setminus Z} \left(\frac{|\nabla_{\mathcal{L}}\varphi|^p}{|\mathcal{J}(x,\varphi)|}\right)^{q/(p-q)} dx.$$

(To prove it, we have to apply the change-of-variable formula (2) and the equalities $\frac{rs}{r-s} = \frac{pq}{(p-q)(\nu-1)}$ and $|\mathcal{J}(\varphi(x), \varphi^{-1})|^{-1} = |\mathcal{J}(x,\varphi)|$ for almost all $x \in D \setminus (E \cup Z)$ and $y = \varphi(x) \in D' \setminus \varphi(E \cup Z)$.) Using Theorem 2, we conclude that φ^{-1} generates the bounded embedding operator ${\varphi^*}^{-1} : L^1_r(D) \to L^1_s(D')$. By Theorem 7, φ^{-1} is an (r,s)-quasiconformal homeomorphism.

Observe some geometric properties of (p,q)-quasiconformal homeomorphisms. The results of [11, 24, 34] yield the following theorem:

Theorem 10. *Suppose that $\varphi : \mathbb{G} \to \mathbb{G}$ generates the bounded embedding operator $\varphi^* : L^1_p(\mathbb{G}) \to L^1_q(\mathbb{G})$, $\nu < q < p < \infty$ ($\nu - 1 < q < p < \nu$). Then the inequality $\rho(\varphi(a), \varphi(b))^{\frac{p-\nu}{p}} \leq c\rho(a,b)^{\frac{q-\nu}{q}}$ $\left(\rho(\varphi^{-1}(a), \varphi^{-1}(b))^{\frac{r-\nu}{r}} \leq c\rho(a,b)^{\frac{s-\nu}{s}}\right.$ with $r = q/(q-\nu+1)$ and $\left.s = p/(p-\nu+1)\right)$ holds for arbitrary two points $a, b \in \mathbb{G}$, where c is some constant depending on $\|\varphi^*\|$, ν, p, and q.*

Theorem 11. *Suppose that $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L^1_p(D') \to L^1_q(D)$, $1 \leq q \leq p < \infty$. Then, for each measurable set $E \subset D'$, we have the inequalities $|\varphi^{-1}(E)|^{\frac{\nu-q}{\nu q}} \leq cK^{\frac{p-q}{pq}}|E|^{\frac{\nu-p}{\nu p}}$ if $1 \leq q \leq p < \nu$ and $|E|^{\frac{p-\nu}{\nu p}} \leq cK^{\frac{p-q}{pq}}|\varphi^{-1}(E)|^{\frac{q-\nu}{\nu q}}$ if $\nu < q \leq p < \infty$.*

PROOF. The case of $p = q$ was settled in [34]. Fix a truncator $\eta \in C^\infty_0(\mathbb{G})$ equal to 1 on $B(0,1)$ and 0 outside $B(0,2)$. Inserting the function $f(y) = \eta(\delta^{-1}_r(y_0^{-1}y))$ in the inequality $\|\varphi^* f \mid \overset{\circ}{L}{}^1_q(\varphi^{-1}(U))\| \leq \Phi(U)^{\frac{p-q}{pq}} \|f \mid \overset{\circ}{L}{}^1_p(U)\|$, with $U \subset D'$ an open set, we obtain

$$\|\varphi^* f \mid \overset{\circ}{L}{}^1_q(\varphi^{-1}(B(y_0, 2r)))\| \leq c\Phi(B(y_0, 2r))^{\frac{p-q}{pq}} |B(y_0, r)|^{\frac{\nu-p}{\nu p}}.$$

On the other hand, the function $\varphi^* f$ equals 1 on $\varphi^{-1}(B(y_0, r))$. The embedding theorem in L_α, $\alpha = \frac{\nu q}{\nu - q}$ [38], implies that

$$|\varphi^{-1}(B(y_0, r))|^{\frac{\nu-q}{\nu q}} \leq c\Phi(B(y_0, 2r))^{\frac{p-q}{pq}} |B(y_0, r)|^{\frac{\nu-p}{\nu p}}$$

for each ball $B(y_0, r) \subset D'$ such that $B(y_0, 2r) \subset D'$ (here $1 \le q < p < \nu$). Applying Hölder's inequality, we derive the inequality

$$\left(\sum_{i=1}^{\infty} |\varphi^{-1}(B_i(y_i, r_i))|\right)^{\frac{\nu-q}{\nu q}} \le c\Phi\left(\bigcup_{i=1}^{\infty} B_i(y_i, 2r_i)\right)^{\frac{p-q}{pq}} \left(\sum_{i=1}^{\infty} |B_i(y_i, r_i)|\right)^{\frac{\nu-p}{\nu p}}$$

for each collection $\{B_i(y_i, r_i)\}$ of balls covering E and such that the multiplicity of the covering $\{B_i(y_i, 2r_i)\}$ is finite. Hence, $|\varphi^{-1}(E)|^{\frac{\nu-q}{\nu q}} \le cK^{\frac{p-q}{pq}}|E|^{\frac{\nu-p}{\nu p}}$. Now, the case of $\nu < q < p < \infty$ ensues from Theorem 9.

Corollary 2. *Suppose that $\varphi : D \to D'$ generates the bounded embedding operator $\varphi^* : L_p^1(D') \to L_q^1(D)$, $1 \le q < p < \infty$. Then φ satisfies Luzin's condition $(\mathcal{N}^{-1})$ for $1 \le q \le p < \nu$, in particular, $|\mathcal{J}(x, \varphi)| \ne 0$, and satisfies Luzin's condition $(\mathcal{N})$ for $\nu < q \le p < \infty$.*

Some applications of (p, p)-quasiconformal homeomorphisms to the classification of Riemannian manifolds can be found in [39].

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