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ON A MULTIDIMENSIONAL SYSTEM OF HYPERGEOMETRIC DIFFERENTIAL EQUATIONS

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Introduction

Interest has recently been aroused in studying hypergeometric functions of many variables. In the one-dimensional case the generalized hypergeometric differential equation has the following form [1, p. 77]:

$$xP\left(x\frac{d}{dx}\right)y(x) - Q\left(x\frac{d}{dx}\right)y(x) = 0,$$

where P and Q are polynomials:

$$P(z) = a\prod_{k=1}^{p}(z-\alpha_k), \quad Q(z) = \prod_{k=1}^{q}(z-\beta_k).$$

In the multidimensional case there are several approaches to the notion of hypergeometric function: such functions can be defined to be the sums of power series of a certain form (the so-called Γ-series) [2–4], solutions to systems of differential equations [1, 5, 6], the Euler-type integrals [7, 8], and the Mellin–Barnes integrals [6].

Multidimensional systems of differential equations of hypergeometric type appear in some problems of mathematical physics. In particular, such equations arise in superstring theory while studying the Ukawa connection constants [7].

In the present article, as a multidimensional analog of the generalized hypergeometric differential equation we consider the Horn hypergeometric system [4]:

$$Q_i\left(x\frac{\partial}{\partial x}\right)x_i^{-1}y(x_1,\dots,x_n) = P_i\left(x\frac{\partial}{\partial x}\right)y(x_1,\dots,x_n), \quad i=1,\dots,n. \tag{1}$$

Here $x\frac{\partial}{\partial x} = (x_1\frac{\partial}{\partial x_1},\dots,x_n\frac{\partial}{\partial x_n})$, and P_i and Q_i are polynomials. Henceforth we assume that P_i and Q_i are representable as the products of linear factors and that P_i has no common divisors with Q_i, $i=1,\dots,n$.

In § 1, we exhibit an integral representation for solutions to (1) and write down some system of difference equations whose fulfillment is a sufficient condition for the integral in question to satisfy the Horn system of equations. In § 2, we expose the main result of the article (Theorem 1) which contains a criterion for solvability of the corresponding system (6) of difference equations. *The necessary and sufficient conditions of this criterion are the agreement conditions* (8) *on the polynomials* P_i *and* Q_i. *Moreover, if a solution to* (6) *exists then it is determined uniquely up to a factor satisfying the periodicity condition* $\phi(s+e_i)\equiv\phi(s)$ *for all* $i\in 1,\dots,n$. In § 3, we state conditions under which the involved integral transformation exists. In § 4, we exhibit a method for representing a solution to (1) in the form of a multiple series in the case of simple singularities. In the theory of Gel′fand and his coauthors [3–5], simple singularities correspond to the so-called nonresonance case. The solutions to (1) that can be found by means of the integral representation are expressed by Horn series in the case of simple singularities (Theorem 3). Here by a Horn series we mean a power series whose coefficients are the ratios of products of Γ-functions whose arguments depend linearly on the summation variables $m_1,\dots,m_n$.

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§ 1. Integral Representation for Solutions to the System of Differential Equations

We seek a solution to (1) in the form

$$y(x) = \int_C \varphi(s) x^s \, ds, \tag{2}$$

where $s = (s_1, \dots, s_n)$, $x^s = x_1^{s_1} \dots x_n^{s_n}$, $ds = ds_1 \dots ds_n$, and C is some n-dimensional contour that is specified in the process of solving (1) and satisfies the following conditions:

A. For every $i = 1, \dots, n$ the contour C_i resulting from translating C by the basis vector $-e_i$ (-1 at the ith position) is equivalent (homologous) to C.

B. The integrand in (2) decreases rapidly enough on C: the product of the integrand and an arbitrary monomial $s_1^{k_1} \dots s_n^{k_n}$ is bounded.

It follows from A and B that

$$\left(x_i \frac{\partial}{\partial x_i}\right)^k y(x) = \int_C s_i^k \varphi(s) x_1^{s_1} \dots x_n^{s_n} \, ds_1 \dots ds_n. \tag{3}$$

The condition A also guarantees the equality

$$x_i^{-1} y(x) = \int_C \varphi(s + e_i) x^s \, ds. \tag{4}$$

From (3) and (4) we obtain

$$\left(Q_i\left(x\frac{\partial}{\partial x}\right)x_i^{-1} - P_i\left(x\frac{\partial}{\partial x}\right)\right)y(x) = \int_C (\varphi(s+e_i)Q_i(s) - \varphi(s)P_i(s))x^s \, ds. \tag{5}$$

It follows from (5) that the function $y(x)$ in (2) meets (1) if $\varphi(s)$ satisfies the system of the difference equations

$$\frac{\varphi(s + e_i)}{\varphi(s)} = \frac{P_i(s)}{Q_i(s)}, \quad i = 1, \dots, n. \tag{6}$$

Thus, solutions to (1) in the class of functions admitting the integral representation (2) can be obtained by solving (6).

§ 2. Solution of the System of Difference Equations

Let us clarify the properties of (6). In the one-dimensional case, (6) transforms into the single equation

$$\frac{\varphi(s+1)}{\varphi(s)} = \frac{P(s)}{Q(s)} \tag{7}$$

in one variable s. Given expansions of the polynomials P and Q,

$$P(s) = a \prod_{k=1}^{p} (s - \alpha_k), \quad Q(s) = \prod_{k=1}^{q} (s - \beta_k),$$

the general solution to (7) has the form

$$\varphi(s) = a^s \frac{\prod_{i=1}^{p} \Gamma(s-\alpha_i)}{\prod_{j=1}^{q} \Gamma(s-\beta_j)} \phi(s),$$

where $\phi(s)$ is an arbitrary function with period 1 [1, p. 77]. The condition that the polynomials P_i and Q_i are representable as the products of linear factors enables us to use the information on the general form of a solution to (7) effectively for solving the multidimensional system (6).

Observe that not every system of the form (6) has a solution. For instance, the system

$$\frac{\varphi(s+e_1)}{\varphi(s)} = s_2, \quad \frac{\varphi(s+e_2)}{\varphi(s)} = 1$$

is unsolvable, since the second equation implies that $\varphi(s)$ is periodic in s_2 and hence $\frac{\varphi(s+e_1)}{\varphi(s)}$ as well is a periodic function in s_2; however, the right-hand side of the first equation is not periodic in s_2.

A criterion for solvability of (6) is given in the following theorem:

Theorem 1. *For solvability of* (6), *it is necessary and sufficient that the following agreement conditions be satisfied*:

$$\frac{P_i(s+e_j)Q_i(s)}{Q_i(s+e_j)P_i(s)} = \frac{P_j(s+e_i)Q_j(s)}{Q_j(s+e_i)P_j(s)}, \quad i,j = 1,\dots,n,\ i \neq j. \tag{8}$$

Moreover, if a solution to (6) *exists then it is unique up to an arbitrary factor* $\phi(s)$ *which meets the periodicity conditions* $\phi(s+e_i) = \phi(s)$ *for all* $i = 1,\dots,n$.

PROOF. *Necessity*: Suppose that $\varphi(s)$ satisfies (6). Increasing the argument s in the ith equation of (6) by the increment e_j, we obtain

$$\frac{\varphi(s+e_i+e_j)}{\varphi(s+e_j)} = \frac{P_i(s+e_j)}{Q_i(s+e_j)}. \tag{9}$$

Multiplying (9) by the jth equation of (6), we find that

$$\frac{\varphi(s+e_i+e_j)}{\varphi(s)} = \frac{P_i(s+e_j)P_j(s)}{Q_i(s+e_j)Q_j(s)}. \tag{10}$$

Similarly, increasing the argument s in the jth equation of (6) by the increment e_i and multiplying the resultant equality by the ith equation of (6), we arrive at the equality

$$\frac{\varphi(s+e_j+e_i)}{\varphi(s)} = \frac{P_j(s+e_i)P_i(s)}{Q_j(s+e_i)Q_i(s)}. \tag{11}$$

The left-hand sides of (10) and (11) coincide. Equating the right-hand sides, we obtain (8). By the arbitrariness of i and j ($i \neq j$), we have proven the necessity of the agreement conditions for solvability of (6).

Sufficiency: To prove sufficiency, we need the following lemma:

Lemma 1. *Suppose that the system* (6) *satisfies the agreement conditions. Then every linear factor on the right-hand side of the ith equation of* (6) *depends on* s_i, $i = 1,\dots,n$.

PROOF OF LEMMA 1. Since all equations (and variables) in (6) have the same shape, it suffices to demonstrate that each linear factor on the right-hand side of the first equation depends on s_1. Suppose to the contrary that

$$P_1 = K_1 L_1, \quad Q_1 = M_1 N_1, \tag{12}$$

where K_1, L_1, M_1, and N_1 are polynomials such that every linear factor of K_1 and M_1 depends on s_1 and every linear factor of L_1 and N_1 is independent of s_1. For the fixed value $j = 1$, the agreement conditions (8) involve the equations

$$\frac{P_1(s+e_i)Q_1(s)}{Q_1(s+e_i)P_1(s)} = \frac{P_i(s+e_1)Q_i(s)}{Q_i(s+e_1)P_i(s)}, \quad i = 2, \dots, n. \tag{13}$$

Using (12), we can rewrite (13) as

$$\frac{K_1(s+e_i)L_1(s+e_i)M_1(s)N_1(s)}{M_1(s+e_i)N_1(s+e_i)K_1(s)L_1(s)} = \frac{P_i(s+e_1)Q_i(s)}{Q_i(s+e_1)P_i(s)}, \quad i = 2, \dots, n. \tag{14}$$

After all cancellations in (14), every linear factor on the right-hand side of (14) must depend on s_1 or the right-hand side must be constant. By assumption, every linear factor of K_1 and M_1 depends on s_1; therefore, the same is true for the fraction $\frac{K_1(s+e_i)M_1(s)}{M_1(s+e_i)K_1(s)}$. Consequently (by the uniqueness theorem for the expansion of a polynomial into the product of linear factors and in view of independence of L_1 and N_1 of s_1), we have

$$\frac{L_1(s+e_i)N_1(s)}{N_1(s+e_i)L_1(s)} = \text{const}. \tag{15}$$

Since P_1 and Q_1 have no common divisors, so are L_1 and N_1 as well. Therefore, (15) implies that

$$\frac{L_1(s+e_i)}{L_1(s)} = \text{const}, \quad \frac{N_1(s)}{N_1(s+e_i)} = \text{const}.$$

Hence, L_1 and N_1 are independent of s_i. Indeed, from these equalities we infer that L_1 and N_1 are periodic in s_i; however, a polynomial is periodic in one of its arguments if and only if it is independent of this argument. Since i is an arbitrary index in the set $\{2, \dots, n\}$, L_1 and N_1 depend on none of the variables s_i, $i = 1, \dots, n$ (they are independent of s_1 by assumption); i.e., $L_1 = \text{const}$ and $N_1 = \text{const}$. This is exactly the condition that every linear factor of P_1 and Q_1 depends on s_1.

Similar arguments lead to the fact that every linear factor of P_i and Q_i depends on s_i. Lemma 1 is proven.

We now describe the process of constructing a function $\varphi(s)$ which is a solution to (6), provided that the agreement conditions (8) are satisfied. Suppose that

$$\frac{P(s)}{Q(s)} = \frac{\prod_{i=1}^{p}(a_{i1}s_1 + \dots + a_{in}s_n + c_i)}{\prod_{j=1}^{q}(b_{j1}s_1 + \dots + b_{jn}s_n + d_j)}$$

is a rational function; moreover, P and Q have no common divisors. Denote the fraction

$$\frac{\prod_{i=1}^{p}\Gamma(a_{i1}s_1 + \dots + a_{in}s_n + c_i)}{\prod_{j=1}^{q}\Gamma(b_{j1}s_1 + \dots + b_{jn}s_n + d_j)}$$

by $\Gamma(\frac{P}{Q})$ and call it the Γ-*fraction*. The rational function $\frac{P(s)}{Q(s)}$ written down in the form

$$a\frac{\prod_{i=1}^{p}(\frac{a_{i1}}{a_{ik}}s_1+\ldots+\frac{a_{in}}{a_{ik}}s_n+\frac{c_i}{a_{ik}})}{\prod_{j=1}^{q}(\frac{b_{j1}}{b_{jk}}s_1+\ldots+\frac{b_{jn}}{b_{jk}}s_n+\frac{d_j}{b_{jk}})},$$

where $a=\frac{a_{1k}\ldots a_{pk}}{b_{1k}\ldots b_{qk}}$ is the normalization constant, is said to be *normalized with respect to* s_k.

Thus, the scheme of constructing a function $\varphi(s)$ which satisfies (6) is as follows:

STEP 1. Let $a_1\frac{\widetilde{P}_1(s)}{\widetilde{Q}_1(s)}$ be the right-hand side of the first equation in (6) normalized with respect to s_1. Put

$$\varphi_1(s)=a_1^{s_1}\Gamma\left(\frac{\widetilde{P}_1}{\widetilde{Q}_1}\right). \tag{16}$$

By the remark on the general form of a solution to the difference equation (7), the function $\varphi_1(s)$ satisfies the first equation in (6).

STEP 2. Let $a_2\frac{\widetilde{P}_2(s)}{\widetilde{Q}_2(s)}$ be the right-hand side of the second equation in (6); moreover, $\widetilde{P}_2(s)=\widetilde{P}_{21}(s)\widetilde{P}_{22}(s)$ and $\widetilde{Q}_2(s)=\widetilde{Q}_{21}(s)\widetilde{Q}_{22}(s)$, where $\widetilde{P}_{21}(s)$ and $\widetilde{Q}_{21}(s)$ comprise only those linear factors of $\widetilde{P}_2(s)$ and $\widetilde{Q}_2(s)$ that depend on s_1 (by Lemma 1, this implies that $\widetilde{P}_{22}(s)$ and $\widetilde{Q}_{22}(s)$ are the products of only those linear factors of $\widetilde{P}_2(s)$ and $\widetilde{Q}_2(s)$ that are independent of s_1 but depend on s_2, since every linear factor on the right-hand side of the second equation in (6) depends on s_2). Moreover, $\widetilde{P}_{22}(s)$ and $\widetilde{Q}_{22}(s)$ are normalized with respect to s_2 and a_2 is the corresponding normalization constant. Put

$$\varphi_2(s)=\varphi_1(s)a_2^{s_2}\Gamma\left(\frac{\widetilde{P}_{22}}{\widetilde{Q}_{22}}\right)=a_1^{s_1}a_2^{s_2}\Gamma\left(\frac{\widetilde{P}_1}{\widetilde{Q}_1}\right)\Gamma\left(\frac{\widetilde{P}_{22}}{\widetilde{Q}_{22}}\right).$$

STEP i. Suppose that

$$\frac{P_i}{Q_i}=a_i\frac{\widetilde{P}_{i1}\widetilde{P}_{i2}\ldots\widetilde{P}_{ii}}{\widetilde{Q}_{i1}\widetilde{Q}_{i2}\ldots\widetilde{Q}_{ii}}, \tag{17}$$

where $\widetilde{P}_{ij}(s)$ and $\widetilde{Q}_{ij}(s)$ comprise only those linear factors of P_i and Q_i that depend on s_j but are independent of $s_1,\ldots s_{j-1}$, $j=1,\ldots,i$. Moreover, $\widetilde{P}_{ij}(s)$ and $\widetilde{Q}_{ij}(s)$ are normalized with respect to s_j and a_j is the corresponding normalization constant. By Lemma 1, we can represent $\frac{P_i}{Q_i}$ in the form (17) for every $i=1,\ldots,n$, since every linear factor on the right-hand side of the ith equation of (6) depends on s_i. Put

$$\varphi_i(s)=\varphi_{i-1}(s)a_i^{s_i}\Gamma\left(\frac{\widetilde{P}_{ii}}{\widetilde{Q}_{ii}}\right).$$

Prove that the function $\varphi_n(s)$ satisfies (6). From the recurrent representation for this function we can easily conclude that

$$\varphi_n(s)=a_1^{s_1}\ldots a_n^{s_n}\Gamma\left(\frac{\widetilde{P}_1}{\widetilde{Q}_1}\right)\Gamma\left(\frac{\widetilde{P}_{22}}{\widetilde{Q}_{22}}\right)\Gamma\left(\frac{\widetilde{P}_{33}}{\widetilde{Q}_{33}}\right)\ldots\Gamma\left(\frac{\widetilde{P}_{nn}}{\widetilde{Q}_{nn}}\right), \tag{18}$$

where $\widetilde{P}_{ii}$ and $\widetilde{Q}_{ii}$ are independent of $s_1,\ldots.s_{i-1}$. In particular, $\widetilde{P}_{ii}$ and $\widetilde{Q}_{ii}$ are independent of s_1; therefore,

$$\frac{\varphi_n(s+e_1)}{\varphi_n(s)}=a_1\frac{\Gamma(\frac{\widetilde{P}_1(s+e_1)}{\widetilde{Q}_1(s+e_1)})}{\Gamma(\frac{\widetilde{P}_1(s)}{\widetilde{Q}_1(s)})}.$$

Recalling that $\widetilde{P}_1$ and $\widetilde{Q}_1$ are normalized and using the identity $\Gamma(z+1) = z\Gamma(z)$, we obtain

$$\frac{\varphi_n(s+e_1)}{\varphi_n(s)} = a_1\Gamma\left(\frac{\widetilde{P}_1(s+e_1)\widetilde{Q}_1(s)}{\widetilde{Q}_1(s+e_1)\widetilde{P}_1(s)}\right) = a_1\frac{\widetilde{P}_1(s)}{\widetilde{Q}_1(s)};$$

i.e., $\varphi_n(s)$ satisfies the first equation of (6).

Prove that $\varphi_n(s)$ satisfies the ith equation of (6), $i = 2, \dots, n$. In accord with the above notations and in view of the fact that $\widetilde{P}_{ii}$ and $\widetilde{Q}_{ii}$ are normalized with respect to s_i, we have

$$\frac{\Gamma\left(\frac{\widetilde{P}_{ii}(s+e_i)}{\widetilde{Q}_{ii}(s+e_i)}\right)}{\Gamma\left(\frac{\widetilde{P}_{ii}(s)}{\widetilde{Q}_{ii}(s)}\right)} = \frac{\widetilde{P}_{ii}(s)}{\widetilde{Q}_{ii}(s)}. \tag{19}$$

Let $j \in \{1, \dots, n\}$ be an arbitrary index. Since

$$\frac{P_i}{Q_i} = a_i\frac{\widetilde{P}_{i1}\dots\widetilde{P}_{ij}\dots\widetilde{P}_{ii}}{\widetilde{Q}_{i1}\dots\widetilde{Q}_{ij}\dots\widetilde{Q}_{ii}}, \quad \frac{P_j}{Q_j} = a_j\frac{\widetilde{P}_{j1}\dots\widetilde{P}_{jj}}{\widetilde{Q}_{j1}\dots\widetilde{Q}_{jj}}$$

in accord with the above notations, the agreement condition in (8) corresponding to the indices i and j takes the form

$$\begin{aligned}
&\frac{\widetilde{P}_{j1}(s+e_i)\dots\widetilde{P}_{jj}(s+e_i)\widetilde{Q}_{j1}(s)\dots\widetilde{Q}_{jj}(s)}{\widetilde{Q}_{j1}(s+e_i)\dots\widetilde{Q}_{jj}(s+e_i)\widetilde{P}_{j1}(s)\dots\widetilde{P}_{jj}(s)}\\
&= \frac{\widetilde{P}_{i1}(s+e_j)\dots\widetilde{P}_{ij}(s+e_j)\dots\widetilde{P}_{ii}(s+e_j)\widetilde{Q}_{i1}(s)\dots\widetilde{Q}_{ij}(s)\dots\widetilde{Q}_{ii}(s)}{\widetilde{Q}_{i1}(s+e_j)\dots\widetilde{Q}_{ij}(s+e_j)\dots\widetilde{Q}_{ii}(s+e_j)\widetilde{P}_{i1}(s)\dots\widetilde{P}_{ij}(s)\dots\widetilde{P}_{ii}(s)}.
\end{aligned} \tag{20}$$

Now, (20) is an equality between two rational functions; moreover, the numerator and the denominator of each of them is representable as the product of linear factors. Hence, the rational function composed of only those linear factors on the right-hand side of (20) which are independent of $s_1, \dots, s_{j-1}$ but depend on s_j must equal the rational function that is composed of only those linear factors on the left-hand side which are independent of $s_1, \dots, s_{j-1}$ but depend on s_j. However, in (20), it is only the polynomials $\widetilde{P}_{jj}$, $\widetilde{Q}_{jj}$, $\widetilde{P}_{ij}$, and $\widetilde{Q}_{ij}$ (and only they) that consist of such factors; therefore,

$$\frac{\widetilde{P}_{jj}(s+e_i)\widetilde{Q}_{jj}(s)}{\widetilde{Q}_{jj}(s+e_i)\widetilde{P}_{jj}(s)} = \frac{\widetilde{P}_{ij}(s+e_j)\widetilde{Q}_{ij}(s)}{\widetilde{Q}_{ij}(s+e_j)\widetilde{P}_{ij}(s)}. \tag{21}$$

Since the polynomials $\widetilde{P}_{ij}$ and $\widetilde{Q}_{ij}$ are normalized with respect to s_j, (21) implies that

$$\frac{\Gamma\left(\frac{\widetilde{P}_{jj}(s+e_i)}{\widetilde{Q}_{jj}(s+e_i)}\right)}{\Gamma\left(\frac{\widetilde{P}_{jj}(s)}{\widetilde{Q}_{jj}(s)}\right)} \equiv \Gamma\left(\frac{\widetilde{P}_{jj}(s+e_i)\widetilde{Q}_{jj}(s)}{\widetilde{Q}_{jj}(s+e_i)\widetilde{P}_{jj}(s)}\right) = \Gamma\left(\frac{\widetilde{P}_{ij}(s+e_j)\widetilde{Q}_{ij}(s)}{\widetilde{Q}_{ij}(s+e_j)\widetilde{P}_{ij}(s)}\right) \equiv \frac{\Gamma\left(\frac{\widetilde{P}_{ij}(s+e_j)}{\widetilde{Q}_{ij}(s+e_j)}\right)}{\Gamma\left(\frac{\widetilde{P}_{ij}(s)}{\widetilde{Q}_{ij}(s)}\right)} = \frac{\widetilde{P}_{ij}(s)}{\widetilde{Q}_{ij}(s)}. \tag{22}$$

Since $\widetilde{P}_{kk}$ and $\widetilde{Q}_{kk}$ are independent of s_i for $k > i$, using (19) and applying (22) $i-1$ times, we obtain

$$\frac{\varphi(s+e_i)}{\varphi(s)} = a_i \frac{\Gamma\left(\frac{\widetilde{P}_1(s+e_i)}{\widetilde{Q}_1(s+e_i)}\right)\Gamma\left(\frac{\widetilde{P}_{22}(s+e_i)}{\widetilde{Q}_{22}(s+e_i)}\right)\dots\Gamma\left(\frac{\widetilde{P}_{nn}(s+e_i)}{\widetilde{Q}_{nn}(s+e_i)}\right)}{\Gamma\left(\frac{\widetilde{P}_1(s)}{\widetilde{Q}_1(s)}\right)\Gamma\left(\frac{\widetilde{P}_{22}(s)}{\widetilde{Q}_{22}(s)}\right)\dots\Gamma\left(\frac{\widetilde{P}_{nn}(s)}{\widetilde{Q}_{nn}(s)}\right)}$$

$$= a_i \frac{\Gamma\left(\frac{\widetilde{P}_1(s+e_i)}{\widetilde{Q}_1(s+e_i)}\right)\Gamma\left(\frac{\widetilde{P}_{22}(s+e_i)}{\widetilde{Q}_{22}(s+e_i)}\right)\dots\Gamma\left(\frac{\widetilde{P}_{ii}(s+e_i)}{\widetilde{Q}_{ii}(s+e_i)}\right)}{\Gamma\left(\frac{\widetilde{P}_1(s)}{\widetilde{Q}_1(s)}\right)\Gamma\left(\frac{\widetilde{P}_{22}(s)}{\widetilde{Q}_{22}(s)}\right)\dots\Gamma\left(\frac{\widetilde{P}_{ii}(s)}{\widetilde{Q}_{ii}(s)}\right)}$$

$$= a_i \frac{\Gamma\left(\frac{\widetilde{P}_1(s+e_i)}{\widetilde{Q}_1(s+e_i)}\right)\Gamma\left(\frac{\widetilde{P}_{22}(s+e_i)}{\widetilde{Q}_{22}(s+e_i)}\right)\dots\Gamma\left(\frac{\widetilde{P}_{i-1i-1}(s+e_i)}{\widetilde{Q}_{i-1i-1}(s+e_i)}\right)}{\Gamma\left(\frac{\widetilde{P}_1(s)}{\widetilde{Q}_1(s)}\right)\Gamma\left(\frac{\widetilde{P}_{22}(s)}{\widetilde{Q}_{22}(s)}\right)\dots\Gamma\left(\frac{\widetilde{P}_{i-1i-1}(s)}{\widetilde{Q}_{i-1i-1}(s)}\right)} \frac{\widetilde{P}_{ii}(s)}{\widetilde{Q}_{ii}(s)}$$

$$= a_i \frac{\widetilde{P}_{i1}(s)\widetilde{P}_{i2}(s)\dots\widetilde{P}_{ii-1}(s)\widetilde{P}_{ii}(s)}{\widetilde{Q}_{i1}(s)\widetilde{Q}_{i2}(s)\dots\widetilde{Q}_{ii-1}(s)\widetilde{Q}_{ii}(s)} = \frac{P_i(s)}{Q_i(s)};$$

i.e., we have obtained the right-hand side of the ith equation of (6). This means that $\varphi_n(s)$ satisfies the ith of the equation of the system. In view of the arbitrariness of $i = 2, \dots, n$ and the fact that $\varphi_n(s)$ satisfies the first equation of (6), the function $\varphi_n(s)$ is a solution to the system. We have thus proven sufficiency of the agreement conditions for existence of a solution to (6).

Uniqueness: Suppose that $\varphi(s)$ and $\widetilde{\varphi}(s)$ are solutions to (6). Then $\widehat{\varphi}(s) = \frac{\varphi(s)}{\widetilde{\varphi}(s)}$ satisfies the system

$$\frac{\widehat{\varphi}(s+e_i)}{\widehat{\varphi}(s)} = 1, \quad i = 1, \dots, n;$$

i.e., $\widehat{\varphi}(s)$ is a periodic function in $s_1, \dots, s_n$. Theorem 1 is proven.

Theorem 1 enables us to find a solution to an arbitrary system of the form (6) in the class of meromorphic functions in many complex variables $s_1, \dots, s_n$ or establish insolvability of the system (6). The concrete construction (18) for a solution to (6) allows us to state the following result:

Theorem 2. *If the system* (6) *is solvable then its general solution has the form*

$$\varphi(s) = a^s \Gamma\left(\frac{P(s)}{Q(s)}\right)\phi(s),$$

where P and Q are polynomials representable as products of linear factors and $\phi(s)$ is an arbitrary function with the property $\phi(s+e_i) \equiv \phi(s)$, $i = 1, \dots, n$.

The integral (2) in which $\varphi(s)$ is a solution to (6) is a formal solution to the system of hypergeometric differential equations under consideration. For this solution to meet (1), it is necessary that the integration contour C satisfy the conditions A and B. The choice of the contour is one of the steps on construction of a solution to (1).

§ 3. Conditions for Existence of the Integral Transformation

Suppose that the system (1) satisfies the agreement condition (8). By Theorem 2, the general solution to the auxiliary system (6) of difference equations has the form

$$\varphi(s) = a^s \Gamma\left(\frac{P}{Q}\right)\phi(s). \tag{23}$$

Here $a^s = a_1^{s_1} \dots a_n^{s_n}$, P and Q are some polynomials that expand into linear factors, and $\phi(s)$ is an arbitrary function with the periodicity properties $\phi(s+e_i) \equiv \phi(s)$, $i = 1, \dots, n$.

Suppose that P and Q have the following expansions into linear factors:

$$P(s_1,\dots,s_n)=\prod_{i=1}^{p}(\langle A_i,s\rangle-c_i),\quad Q(s_1,\dots,s_n)=\prod_{j=1}^{q}(\langle B_j,s\rangle-d_j),$$

where $A_i, B_j \in \mathbb{C}^n$, $i\in 1,\dots,p$, $j\in 1,\dots,q$, and $\langle\,,\rangle$ stands for the inner product.

The only singularities of the expression $\Gamma\binom{P}{Q}$ in (23) are the poles on the complex hyperplanes $\langle A_j,s\rangle-c_j=-m$, where $j\in\{1,\dots,p\}$ and $m\in\mathbb{N}_0\equiv\{0,1,2,\dots\}$. Denote the whole collection of these hyperplanes by $\mathfrak{m}$. Our goal is to find n-dimensional contours in $\mathbb{C}^n\setminus\mathfrak{m}$ that do not change in the homology group $H_n(\mathbb{C}^n\setminus\mathfrak{m})$ under translation by an arbitrary vector $-e_i$, i.e., to find n-cycles homologous to their translations.

If we cannot choose n linearly independent vectors from the collection of the vectors $A_1,\dots,A_p$ then every n-dimensional contour in $\mathbb{C}^n\setminus\mathfrak{m}$ is homologous to zero; i.e., the n-dimensional homology group of $\mathbb{C}^n\setminus\mathfrak{m}$ is trivial. This assertion is a consequence of Serre's theorem, de Rham's theorem, and the singularity separation theorem and is well known in multidimensional complex analysis [9, Proposition 19.7].

It follows from the above that the integral (2) with the function $\varphi(s)$ defined by (23) yields only the trivial (identically zero) solution if the periodic function $\phi(s)$ is entire and $\mathfrak{m}$ includes less than n linearly independent hyperplanes. First we settle the case in which $\phi(s)\equiv 1$ and $\mathfrak{m}$ includes at least one collection of linearly independent hyperplanes. Next, in § 4, we show by using an example how the function $\phi(s)$ with poles can be chosen for finding some solutions in other cases.

The whole homology group $H_n(\mathbb{C}^n\setminus\mathfrak{m})$ is implemented by cycles localized in neighborhoods of the intersection points of at least n linearly independent hyperplanes in $\mathfrak{m}$. If exactly n hyperplanes with linearly independent normals $A_{i_1},\dots,A_{i_n}$ pass through some point then these cycles are given as follows [9, § 19]:

$$|\langle A_{i_1},s\rangle-c_{i_1}+m_1|=\varepsilon,\dots,|\langle A_{i_n},s\rangle-c_{i_n}+m_n|=\varepsilon. \tag{24}$$

Suppose that $\mathcal{I}=(i_1,\dots,i_n)$ is a collection of the numbers $1,\dots,p$ such that the corresponding collection $A_{i_1},\dots,A_{i_n}$ of vectors is linearly independent. Consider the system of the equations

$$\begin{aligned}\langle A_{i_1},s\rangle-c_{i_1}&=-m_1,\\ \dots\dots&\dots\dots\\ \langle A_{i_n},s\rangle-c_{i_n}&=-m_n,\end{aligned} \tag{25}$$

where $m_1,\dots,m_n$ vary independently in $\mathbb{N}_0$. Denote the set of solutions to (25) by X. The elements of X are the points of $\mathbb{C}^n$ in whose neighborhoods the nontrivial n-dimensional contours in $\mathbb{C}^n\setminus\mathfrak{m}_{\mathcal{I}}$ are localized, where $\mathfrak{m}_{\mathcal{I}}$ is the polar set of the function $\Gamma(\langle A_{i_1},s\rangle-c_{i_1})\dots\Gamma(\langle A_{i_n},s\rangle-c_{i_n})$.

Suppose that the contour C is given in the form

$$C=\sum_{m\in\mathbb{N}_0^n}\tau(m),$$

where the contours $\tau(m)$ are defined by (24).

Suppose that all entries of the vectors $A_{i_1},\dots,A_{i_n}$ are integers. In this case the translation of the contour C by the vector $-e_i$ is localized in a neighborhood of some solutions to the system

$$\begin{aligned}\langle A_{i_1},s\rangle-c_{i_1}&=-\tilde m_1,\\ \dots\dots&\dots\dots\\ \langle A_{i_n},s\rangle-c_{i_n}&=-\tilde m_n,\end{aligned} \tag{26}$$

where $\widetilde m_1,\dots,\widetilde m_n$ vary independently in $\mathbb{Z}$.

The condition that the contour C is homologous to its translation by the vector $-e_i$ amounts to the fact that the hyperplane $\langle A_k,s\rangle-c_k=-m_k$ contains no points which are solutions to (26) for every $k\in\{1,\dots,p\}\setminus\{i_1,\dots,i_n\}$.

We have thus obtained the following proposition:

Proposition 1. *If the vectors $A_{i_1}, \dots, A_{i_n}$ are linearly independent for some collection $\mathcal{I} = (i_1, \dots, i_n)$ of indices and, for all $k \notin \mathcal{I}$, the system*

$$\begin{aligned} \langle A_{i_1}, s\rangle - c_{i_1} &= -m_1, \\ &\dots\dots\dots\dots \\ \langle A_{i_n}, s\rangle - c_{i_n} &= -m_n, \\ \langle A_k, s\rangle - c_k &= -m_k \end{aligned}$$

is inconsistent for $m_j \in \mathbb{Z}$, $j = 1, \dots, n$, and $m_k \in \mathbb{N}_0$ then the contour

$$C = \sum_{m \in \mathbb{N}_0^n} \tau(m)$$

does not change in the homology group $H_n(\mathbb{C}^n \setminus \mathfrak{m})$ under the translation by the vector $-e_i$ for all $i \in 1, \dots, n$.

Uniform convergence of the integral (2) is examined by analogy with [10, § 3]. A sufficient condition for uniform convergence of the integral is the condition

$$\operatorname{Re}(\langle \Delta, s^0\rangle) < 0,$$

where $\Delta = \sum_{i=1}^p A_i - \sum_{j=1}^q B_j$ and s^0 is an arbitrary vector decomposable with positive coefficients in the basis of the octant including the set of solutions to (25).

Of course, this condition can be satisfied only in the case of $\Delta \neq 0$. A sufficient condition for uniform convergence of the integral for $\Delta = 0$ is given in [10].

Thus, under the conditions of Proposition 1 and the condition $\operatorname{Re}(\langle \Delta, s^0\rangle) < 0$, the conditions A and B are satisfied and the collection $A_{i_1}, \dots, A_{i_n}$ of linearly independent vectors determines a nontrivial solution to (1). Moreover, the function $\varphi(s)$ is determined from the system of difference equations which has a solution, provided that the conditions of Theorem 1 are satisfied.

§ 4. Representation of a Solution to the System (1) as a Multiple Series (the Case of Simple Singularities)

Suppose that the conditions A and B on the integration contour C are satisfied and the function $\varphi(s)$ is defined by (23) with $\phi(s) \equiv 1$. Then the function $y(x)$ defined by (2) satisfies the system (1). Suppose that at most n hyperplanes in $\mathfrak{m}$ pass through each point of $\mathbb{C}^n$ (in this case we say that φ has simple singularities).

Consider the contour C given as the sum of elements of the basis for the n-dimensional homologies of the set $\mathbb{C}^n \setminus \mathfrak{m}_{\mathcal{I}}$ (where $\mathfrak{m}_{\mathcal{I}}$ is the polar set of the function $\Gamma(\langle A_{i_1}, s\rangle - c_{i_1}) \dots \Gamma(\langle A_{i_n}, s\rangle - c_{i_n})$) with unit coefficients.

Taking the elements of the basis to be the n-multiple intersections of cylindrical hypersurfaces surrounding the polar planes, we can write

$$C = \sum_{m \in \mathbb{N}_0^n} \tau(m),$$

where $\tau(m)$ is the contour determined by (24). Therefore,

$$y(x) = \sum_{m \in \mathbb{N}_0^n} \int_{\tau(m)} (a_1 x_1)^{s_1} \dots (a_n x_n)^{s_n} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s\rangle - c_i)}{\prod_{j=1}^{q} \Gamma(\langle B_j, s\rangle - d_j)}\, ds. \tag{27}$$

Applying the multidimensional Cauchy integral formula [11, § 5] in (27), we obtain the following theorem:

Theorem 3. *If a collection $A_{i_1}, \dots, A_{i_n}$ satisfies the conditions of Proposition 1 (i.e., the contour is invariant under the translations by the vectors $-e_i, i \in 1, \dots, n$) and the octant of solutions to (25) lies in the half-space defined by the condition $\mathrm{Re}(\langle\Delta, s^0\rangle) < 0$ (i.e., the condition B is satisfied) then the series*

$$y(x) = \sum_{m \in \mathbb{N}_0^n} \frac{(-1)^{m_{i_1}+\dots+m_{i_n}}}{m_1! \dots m_n! \det(A_{i_1}, \dots, A_{i_n})} \times (a_1 x_1)^{s_1(A,m)} \dots (a_n x_n)^{s_n(A,m)} \frac{\prod\limits_{k \notin \mathcal{I}} \Gamma(\langle A_k, s(A,m)\rangle - c_k)}{\prod\limits_{j=1}^{q} \Gamma(\langle B_j, s(A,m)\rangle - d_j)} \tag{28}$$

satisfies the system (1); here $s_j(A,m) = \frac{\det A^{(j)}}{\det(A_{i_1}, \dots, A_{i_n})}$ $A^{(j)}$ denotes the matrix that results from replacing the jth column of the matrix A with rows $A_{i_1}, \dots, A_{i_n}$ by the column

$$\begin{pmatrix} c_{i_1} - m_1 \\ \dots \\ c_{i_n} - m_n \end{pmatrix}.$$

The series (28), characterized by the fact that its coefficients represent the ratio of products of Γ-functions whose arguments depend linearly on the summation variables $m_1, \dots, m_n$, is referred to as a *Horn series.* Thus, Theorem 3 claims that in the case of simple singularities the integral representation (2) yields a solution which is a Horn series.

We give some simple examples.

1. Consider the system of differential equations

$$x_1^{-1} y = \left(x_1 \frac{\partial}{\partial x_1} + 1\right) y, \quad x_2^{-1} y = \left(x_2 \frac{\partial}{\partial x_2} + 1\right) y.$$

To this system there corresponds the system of the difference equations

$$\frac{\varphi(s + e_1)}{\varphi(s)} = s_1 + 1, \quad \frac{\varphi(s + e_2)}{\varphi(s)} = s_2 + 1.$$

The general solution to the system of difference equations is the function

$$\varphi(s) = \Gamma(s_1 + 1)\Gamma(s_2 + 1)\phi(s),$$

where $\phi(s)$ is the periodic part; we set it equal to 1.

Take the integration contour C to be the sum of double intersections of cylindrical hypersurfaces localized in a neighborhood of solutions to the system

$$s_1 + 1 = -m_1, \quad s_2 + 1 = -m_2,$$

where $m_1, m_2 \in \mathbb{N}_0$. The translation invariance conditions of C are satisfied, since in our case we have a unique collection of linearly independent hyperplanes. The condition B of convergence of the integral is satisfied, for in this case $\Delta = (1,1)$ and the integration contour C lies in the negative octant. By Theorem 3, we conclude that the function

$$y(x) = \sum_{m_1, m_2 \geq 0} \frac{(-1)^{m_1+m_2}}{m_1! m_2!} x_1^{-m_1-1} x_2^{-m_2-1} = \frac{1}{x_1 x_2} e^{-\frac{1}{x_1} - \frac{1}{x_2}}$$

must be a solution to the system under consideration. We immediately verify that this function is indeed a solution to the system.

2. Consider the system of the differential equations

$$x_1 \frac{\partial}{\partial x_1} x_1^{-1} y = y, \quad x_2 \frac{\partial}{\partial x_2} x_2^{-1} y = y.$$

To this system there corresponds the system of the difference equations

$$\frac{\varphi(s+e_1)}{\varphi(s)} = \frac{1}{s_1}, \quad \frac{\varphi(s+e_2)}{\varphi(s)} = \frac{1}{s_2}.$$

The general solution to this system of difference equations is the function

$$\varphi(s) = \frac{1}{\Gamma(s_1)\Gamma(s_2)} \phi(s),$$

where $\phi(s)$ is the periodic part. The function $\frac{1}{\Gamma(s_1)\Gamma(s_2)}$ is entire and if $\phi(s)$ is also an entire function then the integral representation (2) enables us to obtain only the trivial solution to the system. To find a nontrivial solution to the system, we take $\phi(s)$ in the form

$$\phi(s) = (-1)^{s_1+s_2} \frac{\pi^2}{\sin(\pi s_1)\sin(\pi s_2)}.$$

From the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ we obtain

$$\varphi(s) = (-1)^{s_1+s_2} \Gamma(1-s_1)\Gamma(1-s_2).$$

Take the integration contour C to be the sum of double intersections of cylindrical hypersurfaces localized in a neighborhood of solutions to the system

$$1 - s_1 = -m_1, \quad 1 - s_2 = -m_2,$$

where $m_1, m_2 \in \mathbb{N}_0$. As in the preceding case the contour C is invariant under the translations by the vectors $-e_i$, $i \in 1, \dots, n$. The condition B of convergence of the integral is satisfied, since in this case $\Delta = (-1, -1)$ and the integration contour C lies in the positive octant. From Theorem 3 we conclude that the function

$$y(x) = \sum_{m_1, m_2 \geq 0} \frac{1}{m_1! m_2!} x_1^{m_1+1} x_2^{m_2+1} = x_1 x_2 e^{x_1+x_2}$$

must be a solution of the system under consideration. We can immediately verify that this function is indeed a solution to the system.

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