Siberian Mathematical Journal, Vol. 39, No. 5, 1998

ON QUASIELLIPTIC OPERATORS IN R_n [†)]

G. V. Demidenko

UDC 517.953+517.983

In the present article, we consider one class of matrix quasielliptic operators

$$\mathcal{L}(D_x) = \big(l_{k,j}(D_x)\big) \tag{0.1}$$

on the whole R_n. For these operators we establish some isomorphism properties and prove unique solvability of the systems

$$\mathcal{L}(D_x)U = F(x), \quad x \in R_n, \tag{0.2}$$

in the special weighted Sobolev spaces $W^r_{p,\sigma}(R_n)$. We exemplify the application of the obtained results to the theory of equations that are not solved with respect to the higher order derivative.

§ 1. Statement of the Main Results

We indicate the conditions to be imposed on the operator (0.1). Denote by $l_{k,j}(i\xi)$ the entries of the symbol $\mathcal{L}(i\xi)$ of the operator.

CONDITION 1. Let m be the order of the matrix $\mathcal{L}(i\xi)$. Suppose that there is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, where $1/\alpha_i$ are natural numbers, such that

$$l_{k,j}(c^\alpha i\xi) = c\, l_{k,j}(i\xi), \quad c > 0.$$

CONDITION 2. The equality $\det \mathcal{L}(i\xi) = 0$, $\xi \in R_n$, holds if and only if $\xi = 0$.

Conditions 1 and 2 are enjoyed, for instance, by Petrovskiĭ elliptic and parabolic operators, parabolic operators with "opposite times directions," etc. (see [1]).

Studying quasielliptic equations, the author [2, 3] introduced the weighted Sobolev spaces

$$W^r_{p,\sigma}(R_n), \quad r = (1/\alpha_1, \dots, 1/\alpha_n), \quad 1 < p < \infty, \quad \sigma \in R_1,$$

with the norm

$$\|u(x), W^r_{p,\sigma}(R_n)\| = \sum_{0\le\beta\alpha\le1} \|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)} D^\beta_x u(x), L_p(R_n)\|, \quad \langle x\rangle^2 = \sum_{i=1}^n x_i^{2/\alpha_i},$$

and the space $\overset{\circ}{W}{}^r_{p,\sigma}(R_n)$ that is the completion of $C_0^\infty(R_n)$ with respect to this norm. In particular, it was proven in [3] that these spaces coincide for $\sigma \le 1$. Henceforth we suppose that $0 \le \sigma \le 1$. Observe that in the isotropic case it was L. D. Kudryavtsev [4] who introduced such spaces with $\sigma = 1$ (see also the survey [5]).

Denote by $L_{p,\gamma}(R_n)$ the space of summable functions with the norm

$$\|u(x), L_{p,\gamma}(R_n)\| = \|(1+\langle x\rangle)^{-\gamma} u(x), L_p(R_n)\|.$$

We have the following results:

†) The research was financially supported by the Russian Foundation for Basic Research (Grant 95-01-01176).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 39, No. 5, pp. 1028–1037, September–October, 1998. Original article submitted June 18, 1997.

Theorem 1. *Suppose that*

$$|\alpha| > 1, \quad \frac{|\alpha|}{p} > \sigma > \frac{|\alpha|}{p} - (|\alpha| - 1). \tag{1.1}$$

Then for every vector-function $F(x) \in L_{p,\sigma-1}(R_n)$ the system (0.2) *has a unique solution $U(x) \in W^{r}_{p,\sigma}(R_n)$; moreover,*

$$\|U(x), W^{r}_{p,\sigma}(R_n)\| \le c\,\|F(x), L_{p,\sigma-1}(R_n)\| \tag{1.2}$$

with some constant $c > 0$ independent of $F(x)$.

Theorem 2. *Suppose that the conditions* (1.1) *are satisfied. Then the following estimate holds for every vector-function $U(x) \in C_0^{\infty}(R_n)$:*

$$\begin{aligned} &\|\langle x\rangle^{-\sigma(1-\beta\alpha)} D_x^{\beta} U(x), L_p(R_n)\| \\ &\le c\,\|\langle x\rangle^{(1-\sigma)(1-\beta\alpha)} \mathcal{L}(D_x) U(x), L_p(R_n)\|, \quad 0 \le \beta\alpha \le 1, \end{aligned} \tag{1.3}$$

with some constant $c > 0$ independent of $U(x)$.

Theorem 3. *Suppose that $|\alpha|/p > 1$. Then the operator $\mathcal{L}(D_x) : W^{r}_{p,1}(R_n) \to L_p(R_n)$ is an isomorphism.*

REMARK 1. Theorem 1 improves the corresponding results of [2] on unconditional solvability of quasielliptic equations ($m = 1$).

REMARK 2. In the isotropic case the inequality (1.3) was obtained for $\sigma = 1$ in [6].

REMARK 3. Theorem 3 is also new for scalar operators. Some analogs of this theorem have been known only for elliptic operators (see, for instance, [7–9]).

REMARK 4. The results of the present article were announced in [10].

Theorem 2 in particular yields the following estimate for the Laplace operator

$$\||x|^{-2\sigma} u(x), L_p(R_n)\| \le c\,\||x|^{2(1-\sigma)} \Delta u(x), L_p(R_n)\|,$$

$$n \ge 3, \quad \frac{n}{n - 2(1-\sigma)} < p < \frac{n}{2\sigma},$$

and the heat operator

$$\|(|x_1| + |x'|^2)^{-\sigma} u(x), L_p(R_n)\| \le c\,\|(|x_1| + |x'|^2)^{(1-\sigma)} (D_{x_1} - \Delta') u(x), L_p(R_n)\|,$$

$$x' = (x_2, \dots, x_n), \quad \Delta' = D^2_{x_2} + \dots + D^2_{x_n}, \quad \frac{n+1}{n-1+2\sigma} < p < \frac{n+1}{2\sigma}.$$

Theorem 3 implies in particular that the Laplace operator Δ is an isomorphism of $W^2_{p,1}(R_n)$ onto $L_p(R_n)$ for $n \ge 3$ and $p \in (1, n/2)$, and the heat operator $D_{x_1} - \Delta'$ is an isomorphism of the space $W^{r}_{p,1}(R_n)$, $r = (1, 2, \dots, 2)$, onto $L_p(R_n)$ for $p \in (1, (n+1)/2)$. In § 4, we exemplify the application of this theorem to the theory of Sobolev-type operator equations.

§ 2. Approximate Solutions to Quasielliptic Systems

To prove solvability of the system (0.2), we use the idea of construction of approximate solutions to quasielliptic equations $L(D_x)u = f(x)$ on the whole space R_n which was proposed by S. V. Uspenskiĭ [11, 12] and the technique of L_p-estimates for solutions which was developed by the author (see, for instance, [2]).

Consider the family of the integral operators $P_{j,h}$, $j = 1, \dots, m$, $0 < h < 1$:

$$P_{j,h}F(x) = (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|} \int_{R_n}\int_{R_n} \exp\left(i\frac{(x-y)\xi}{v^\alpha}\right) G(\xi) \left(\sum_{k=1}^m l^{j,k}(i\xi)F_k(y)\right) d\xi dy dv, \tag{2.1}$$

where $l^{j,k}(i\xi)$ are the entries of the inverse matrix $(\mathcal{L}(i\xi))^{-1}$,

$$G(\xi) = 2\kappa\langle\xi\rangle^{2\kappa}\exp(-\langle\xi\rangle^{2\kappa}), \quad \langle\xi\rangle^2 = \sum_{i=1}^n \xi_i^{2/\alpha_i}, \tag{2.2}$$

and κ is a natural number.

It follows from the definition and Conditions 1 and 2 that the functions $P_{j,h}F(x)$ are infinitely differentiable and

$$\sum_{j=1}^m l_{k,j}(D_x)P_{j,h}F(x) = F_{k,h}(x),$$

$$F_{k,h}(x) = (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|-1} \int_{R_n}\int_{R_n} \exp\left(i\frac{(x-y)\xi}{v^\alpha}\right) G(\xi)F_k(y)\, d\xi dy dv;$$

moreover, by [11]

$$\|F_{k,h}(x) - F_k(x),\, L_p(R_n)\| \to 0, \quad h \to 0.$$

Consequently, we can consider the vector-function $U_h(x)$ with components $P_{j,h}F(x)$, $j = 1, \dots, m$, as an approximate solution to the system (0.2).

While considering quasielliptic equations ($m = 1$), the author [2] proved some properties of the integral operators $P_{1,h}$ which helped him to establish conditions for solvability of these equations in the spaces $W^r_{p,\sigma}(R_n)$ and distinguish the cases of correct solvability. In Lemmas 1 and 3, we present analogs of these properties for $m \geq 1$. In particular, Lemma 1 is a generalization of Lemma 1 of [2] and Lemma 3 improves Lemma 2 of [2].

Lemma 1. *If a vector-function $F(x)$ belongs to $L_p(R_n)$ then the following estimate holds:*

$$\left\|D_x^\beta P_{j,h}F(x),\, L_p(R_n)\right\| \leq c\|F(x),\, L_p(R_n)\|, \quad \beta\alpha = 1, \quad j = 1, \dots, m, \tag{2.3}$$

with some constant $c > 0$ independent of $F(x)$ and h.

PROOF. Obviously, it suffices to prove (2.3) for vector-functions $F(x) \in C_0^\infty(R_n)$. For $\beta\alpha = 1$ we have

$$D_x^\beta P_{j,h}F(x) = (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|-1} \int_{R_n}\int_{R_n} \exp\left(i\frac{(x-y)\xi}{v^\alpha}\right) G(\xi)(i\xi)^\beta \left(\sum_{k=1}^m l^{j,k}(i\xi)F_k(y)\right) d\xi dy dv.$$

Executing the change of variables $s_k = \xi_k v^{-\alpha_k}$, $k = 1, \dots, n$, and using the properties of the Fourier transform, we can rewrite this equality as

$$D_x^\beta P_{j,h}F(x) = (2\pi)^{-3n/2} \int_h^{h^{-1}} v^{-1} \int_{R_n} \left(\int_{R_n} \exp(i(x-y)s)G(sv^\alpha)ds\right)$$
$$\times\left(\int_{R_n} \exp(iy\xi)(i\xi)^\beta \left(\sum_{k=1}^m l^{j,k}(i\xi)\hat{F}_k(\xi)\right) d\xi\right) dy dv. \tag{2.4}$$

Denote

$$f_\beta(y) = (2\pi)^{-n/2} \int\limits_{R_n} \exp(iy\xi)(i\xi)^\beta \left(\sum_{k=1}^{m} l^{j,k}(i\xi)\hat{F}_k(\xi) \right) d\xi. \tag{2.5}$$

Using Conditions 1 and 2 and Lizorkin's theorem on multipliers [13], we arrive at the inequality

$$\|f_\beta(x), L_p(R_n)\| \le c_\beta \|F(x), L_p(R_n)\| \tag{2.6}$$

with some constant c_β independent of $F(x)$. Repeating similar arguments of the proof of Lemma 1 in [2], we come to (2.3). The lemma is proven.

To obtain estimates for the derivatives $D_x^\beta P_{j,h}F(x)$ for $0 \le \beta\alpha < 1$, consider the functions

$$\mathcal{K}_{\beta,h}^{j,k}(z) = (2\pi)^{-n} \int\limits_{h}^{h^{-1}} v^{-|\alpha|-\beta\alpha} \int\limits_{R_n} \exp\left(i\frac{z\xi}{v^\alpha} \right) G(\xi)(i\xi)^\beta l^{j,k}(i\xi)\, d\xi dv. \tag{2.7}$$

Lemma 2. *Suppose that* $|\alpha| > 1 - \beta\alpha$. *Then there is* κ_0 *such that, for* $\kappa \ge \kappa_0$ *in the definition* (2.2), *the following estimate is valid:*

$$\langle z\rangle^{|\alpha|+\beta\alpha-1} |\mathcal{K}_{\beta,h}^{j,k}(z)| \le c, \quad z \in R_n, \tag{2.8}$$

with some constant $c > 0$ *independent of* h.

PROOF. Observe that from the definition (2.7) we obtain

$$\mathcal{K}_{\beta,h}^{j,k}(z) = \lambda^{|\alpha|+\beta\alpha-1} \mathcal{K}_{\beta,\lambda h}^{j,k}(\lambda^\alpha z), \quad \lambda > 0. \tag{2.9}$$

Demonstrate that there exists κ_0 such that, for $\kappa \ge \kappa_0$, the following estimate holds:

$$\max_{\langle z\rangle=1} |\mathcal{K}_{\beta,h}^{j,k}(z)| \le c \tag{2.10}$$

with some constant $c > 0$ independent of h.

It follows from the definition of the function $G(\xi)$ and Condition 2 that the integral

$$K_\beta^{j,k}(z) = \int\limits_{R_n} \exp(iz\xi) G(\xi)(i\xi)^\beta l^{j,k}(i\xi)\, d\xi \tag{2.11}$$

is infinitely differentiable and for every $m > 0$ there obviously exists κ_0 such that the uniform estimate

$$|K_\beta^{j,k}(z)| \le c_0(1 + \langle z\rangle)^{-m}, \quad z \in R_n,$$

is valid for $\kappa \ge \kappa_0$. Consequently, for $0 < h < 1$ and $\langle z\rangle = 1$ we obtain

$$|\mathcal{K}_{\beta,h}^{j,k}(z)| \le (2\pi)^{-n} \int\limits_{h}^{h^{-1}} v^{-|\alpha|-\beta\alpha} \left| K_\beta^{j,k}\left(\frac{z}{v^\alpha}\right)\right| dv \le c_0 \int\limits_{h}^{1} v^{m-|\alpha|-\beta\alpha}\, dv + c_0 \int\limits_{1}^{h^{-1}} v^{-|\alpha|-\beta\alpha}\, dv.$$

Therefore, using the condition $|\alpha| > 1 - \beta\alpha$ and choosing the corresponding κ_0 for $m > |\alpha|$, we arrive at the inequality (2.10) for $\kappa \ge \kappa_0$.

In view of (2.9), from (2.10) we obtain (2.8). The lemma is proven.

Henceforth we assume that $\kappa \ge \kappa_0$ in (2.2).

Lemma 3. *Suppose that* $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta\alpha < 1$, *and*

$$\frac{|\alpha|}{p} > \sigma(1 - \beta\alpha) > 1 - \beta\alpha - \frac{|\alpha|}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Then the following estimate holds for every vector-function $F(x) \in L_{p,\sigma-1}(R_n)$:

$$\begin{aligned} &\left\| \langle x \rangle^{\sigma(\beta\alpha-1)} D_x^\beta P_{j,h} F(x), L_p(R_n) \right\| \\ &\le c \| \langle x \rangle^{(\sigma-1)(\beta\alpha-1)} F(x), L_p(R_n) \|, \quad j = 1, \dots, m, \end{aligned} \tag{2.12}$$

with some constant $c > 0$ *independent of* $F(x)$ *and* h.

PROOF. Since the set of compactly-supported functions is dense in $L_{p,\gamma}(R_n)$, it suffices to establish (2.12) for compactly-supported $F(x)$. Recalling the definitions (2.1), (2.2), and (2.7), we can represent the function $D_x^\beta P_{j,h} F(x)$ as

$$D_x^\beta P_{j,h} F(x) = \sum_{k=1}^m \int_{R_n} \mathcal{K}_{\beta,h}^{j,k}(x - y) F_k(y)\, dy.$$

By Lemma 2, we have the inequality

$$\begin{aligned} &\left\| \langle x \rangle^{\sigma(\beta\alpha-1)} D_x^\beta P_{j,h} F(x), L_p(R_n) \right\| \\ &\le c \sum_{k=1}^m \left\| \langle x \rangle^{\sigma(\beta\alpha-1)} \int_{R_n} \langle x - y \rangle^{-|\alpha|-\beta\alpha+1} |F_k(y)|\, dy, L_p(R_n) \right\| \end{aligned}$$

and, recalling that $\beta\alpha - 1 < 0$, $-|\alpha| - \beta\alpha + 1 < 0$, and $\sigma \le 1$, we obtain

$$\begin{aligned} &\left\| \langle x \rangle^{\sigma(\beta\alpha-1)} D_x^\beta P_{j,h} F(x), L_p(R_n) \right\| \\ &\le c' \sum_{k=1}^m \left\| \int_{R_n} \prod_{i=1}^n |x_i|^{\sigma(\beta\alpha-1)/|\alpha|} |x_i - y_i|^{(1-\beta\alpha)/|\alpha|-1} |y_i|^{(1-\sigma)(\beta\alpha-1)/|\alpha|} \right. \\ &\qquad \left. \times \langle y \rangle^{(\sigma-1)(\beta\alpha-1)} |F_k(y)|\, dy, L_p(R_n) \right\|. \end{aligned}$$

Using the conditions of the lemma and the Hardy–Littlewood inequality [14], we obtain (2.12). The lemma is proven.

§ 3. Solvability of Quasielliptic Systems

Grounding on the above-obtained estimates for approximate solutions, we prove Theorems 1–3. To this end, we need the following lemma:

Lemma 4. *Suppose that the conditions of Theorem 1 are satisfied. Then the following convergence holds for every compactly-supported vector-function* $F(x) \in L_p(R_n)$:

$$\left\| P_{j,h_1} F(x) - P_{j,h_2} F(x), W_{p,\sigma}^r(R_n) \right\| \to 0, \quad h_1, h_2 \to 0, \quad j = 1, \dots, m. \tag{3.1}$$

PROOF. Consider (2.4) for $\beta\alpha = 1$. In view of (2.5) and (2.6), from [11] we obtain

$$\left\| D_x^\beta P_{j,h} F(x) - f_\beta(x), L_p(R_n) \right\| \to 0, \quad h \to 0.$$

Hence,

$$\left\| D_x^\beta P_{j,h_1} F(x) - D_x^\beta P_{j,h_2} F(x), L_p(R_n) \right\| \to 0, \quad h_1, h_2 \to 0, \quad j = 1, \dots, m. \tag{3.2}$$

Establish the convergence

$$\left\| (1 + \langle x \rangle)^{\sigma(\beta\alpha - 1)} (D_x^\beta P_{j,h_1} F(x) - D_x^\beta P_{j,h_2} F(x)), L_p(R_n) \right\| \to 0, \\ h_1, h_2 \to 0, \quad j = 1, \dots, m, \tag{3.3}$$

for $\beta\alpha < 1$.

From (2.1), (2.11), and Minkowski's inequality we obtain

$$\left\| (1 + \langle x \rangle)^{\sigma(\beta\alpha - 1)} (D_x^\beta P_{j,h_1} F(x) - D_x^\beta P_{j,h_2} F(x)), L_p(R_n) \right\|$$
$$\leq \sum_{k=1}^m \int_{h_1}^{h_2} v^{-|\alpha| - \beta\alpha} \left\| (1 + \langle x \rangle)^{\sigma(\beta\alpha - 1)} \int_{R_n} K_\beta^{j,k} \left(\frac{x - y}{v^\alpha} \right) F_k(y)\, dy\, L_p(R_n) \right\| dv$$
$$+ \sum_{k=1}^m \int_{h_1^{-1}}^{h_2^{-1}} v^{-|\alpha| - \beta\alpha} \left\| (1 + \langle x \rangle)^{\sigma(\beta\alpha - 1)} \int_{R_n} K_\beta^{j,k} \left(\frac{x - y}{v^\alpha} \right) F_k(y)\, dy\, L_p(R_n) \right\| dv$$
$$= \sum_{k=1}^m I_{\beta,1}^{j,k}(h_1, h_2) + \sum_{k=1}^m I_{\beta,2}^{j,k}(h_1, h_2).$$

Consider the summands $I_{\beta,1}^{j,k}(h_1, h_2)$. Since $\sigma(\beta\alpha - 1) \leq 0$, applying Minkowski's inequality and Young's inequality, we find that

$$I_{\beta,1}^{j,k}(h_1, h_2) \leq \int_{h_1}^{h_2} v^{-|\alpha| - \beta\alpha} \left\| K_\beta^{j,k} \left(\frac{x}{v^\alpha} \right), L_1(R_n) \right\| dv \| F_k(y) L_p(R_n) \|$$
$$= \int_{h_1}^{h_2} v^{-\beta\alpha} dv \left\| K_\beta^{j,k}(x), L_1(R_n) \right\| \| F_k(y) L_p(R_n) \|.$$

Since $\beta\alpha < 1$, we conclude that

$$I_{\beta,1}^{j,k}(h_1, h_2) \to 0, \quad h_1, h_2 \to 0.$$

Now, consider the summands $I_{\beta,2}^{j,k}(h_1, h_2)$. Since $\sigma(\beta\alpha - 1) \leq 0$; applying Minkowski's inequality, Young's inequality, and the estimate

$$\langle x - y \rangle (1 + \langle x \rangle)^{-1} \leq a(1 + \langle y \rangle),$$

we obtain

$$I_{\beta,2}^{j,k}(h_1, h_2) = \int_{h_1^{-1}}^{h_2^{-1}} v^{-1} \left\| (1 + \langle x \rangle)^{\sigma(\beta\alpha - 1)} \right.$$

$$\times \int_{R_n}\int_{R_n} \exp(i(x-y)\xi)G(\xi v^{\alpha})(i\xi)^{\beta} l^{j,k}(i\xi)F_k(y)\,d\xi dy, L_p(R_n)\Big\|\,dv$$

$$\le c\int_{h_1^{-1}}^{h_2^{-1}} v^{-1}\Big\|\int_{R_n}\langle x-y\rangle^{\sigma(\beta\alpha-1)}\Big|\int_{R_n}\exp(i(x-y)\xi)G(\xi v^{\alpha})(i\xi)^{\beta}l^{j,k}(i\xi)\,d\xi\Big|$$

$$\times(1+\langle y\rangle)^{-\sigma(\beta\alpha-1)}|F_k(y)|dy, L_p(R_n)\Big\|\,dv$$

$$\le c\int_{h_1^{-1}}^{h_2^{-1}} v^{-1}\Big\|\langle x\rangle^{\sigma(\beta\alpha-1)}\int_{R_n}\exp(ix\xi)G(\xi v^{\alpha})(i\xi)^{\beta}l^{j,k}(i\xi)d\xi, L_p(R_n)\Big\|\,dv$$

$$\times\|(1+\langle y\rangle)^{-\sigma(\beta\alpha-1)}F_k(y), L_1(R_n)\|.$$

Executing the change of variables $s=\xi v^{\alpha}$, $z=xv^{-\alpha}$, we arrive at

$$I_{\beta,2}^{j,k}(h_1,h_2)\le c\int_{h_1^{-1}}^{h_2^{-1}} v^{-|\alpha|/p'-\beta\alpha+\sigma(\beta\alpha-1)}\,dv$$

$$\times\Big\|\langle z\rangle^{\sigma(\beta\alpha-1)}\int_{R_n}\exp(izs)G(s)(is)^{\beta}l^{j,k}(is)ds, L_p(R_n)\Big\|\|(1+\langle y\rangle)^{-\sigma(\beta\alpha-1)}F_k(y), L_1(R_n)\|.$$

Recalling the conditions of the lemma and the definition of the function $G(s)$, we obtain

$$I_{\beta,2}^{j,k}(h_1,h_2)\to 0,\quad h_1,h_2\to 0.$$

The above arguments immediately yield (3.3) for $\beta\alpha<1$.

From (3.2) and (3.3) we derive the convergence (3.1). The lemma is proven.

PROOF OF THEOREM 1. From Lemmas 1 and 3 for $F(x)\in L_{p,\sigma-1}(R_n)$ we obtain the inequality

$$\big\|P_{j,h}F(x), W_{p,\sigma}^{r}(R_n)\big\|\le c\|F(x), L_{p,\sigma-1}(R_n)\|,\quad 0<h<1,\quad j=1,\dots,m, \tag{3.4}$$

with some constant $c>0$ independent of $F(x)$ and h. By Lemma 4, we have the convergence (3.1) for every compactly-supported $F(x)\in L_p(R_n)$. Then, by completeness of the space $W_{p,\sigma}^{r}(R_n)$, there exist continuous linear operators

$$P_j:\ L_{p,\sigma-1}(\dot{R}_n)\to W_{p,\sigma}^{r}(R_n)$$

defined for compactly-supported vector-functions; moreover,

$$\big\|P_jF(x), W_{p,\sigma}^{r}(R_n)\big\|\le c\|F(x), L_p(R_n)\|,$$

$$\big\|P_jF(x)-P_{j,h}F(x), W_{p,\sigma}^{r}(R_n)\big\|\to 0,\quad h\to 0.$$

By denseness of the set of compactly-supported functions in $L_{p,\sigma-1}(R_n)$ and the classical theorem of "extension by continuity," we can uniquely extend the operators P_j to the whole space $L_{p,\sigma-1}(R_n)$ so as to preserve the norm. We use the same notations P_j for the extensions.

It follows from (3.4) that the linear operator

$$P_{j,h}:\ L_{p,\sigma-1}(R_n)\to W^r_{p,\sigma}(R_n)$$

is continuous and $\|P_{j,h}\|\le c$. Consequently, by the Banach–Steinhaus theorem, the convergence

$$\|P_jF(x)-P_{j,h}F(x),W^r_{p,\sigma}(R_n)\|\to 0,\quad h\to 0,$$

holds for every vector-function $F(x)\in L_{p,\sigma-1}(R_n)$.

The above arguments imply existence of a solution $U(x)\in W^r_{p,\sigma}(R_n)$ to (0.2) for every $F(x)\in L_{p,\sigma-1}(R_n)$; moreover, $U_j(x)=P_jF(x)$ and the solution satisfies (1.2).

Prove uniqueness of a solution to (0.2) in the space $W^r_{p,\sigma}(R_n)$. First observe that, by Condition 2, for every vector-function $U(x)\in C_0^\infty(R_n)$ we have

$$(i\xi)^\beta\hat U(\xi)=(i\xi)^\beta(\mathcal{L}(i\xi))^{-1}(\widehat{\mathcal{L}(D_x)U})(\xi),\quad \xi\in R_n\backslash\{0\}.$$

Then, for $\beta\alpha=1$, from Condition 1 and the theorem on multipliers [13] we obtain the estimate

$$\|D_x^\beta U(x),L_p(R_n)\|\le c\|\mathcal{L}(D_x)U(x),L_p(R_n)\|$$

with some constant $c>0$ independent of $U(x)$. Since $W^r_{p,\sigma}(R_n)=\overset{\circ}{W}{}^r_{p,\sigma}(R_n)$ for $\sigma\le 1$, this estimate is valid for every $U(x)\in W^r_{p,\sigma}(R_n)$. Hence, the kernel of the operator $\mathcal{L}(D_x)$ may contain only polynomials. Consequently, for $|\alpha|/p>\sigma$ a solution to (0.2) in $W^r_{p,\sigma}(R_n)$ is determined uniquely. The theorem is proven.

Proof of Theorem 2. The estimate (1.3) for $\beta\alpha=1$ has been already proven. Consider the case of $\beta\alpha<1$. Introduce the notation

$$\omega_\varepsilon=\{x\in R_n:\langle x\rangle<\varepsilon\}.$$

Prove that the following estimate holds for every $\varepsilon>0$:

$$\begin{aligned}&\|\langle x\rangle^{-\sigma(1-\beta\alpha)}D_x^\beta U(x),L_p(R_n\backslash\omega_\varepsilon)\|\\&\le c\|\langle x\rangle^{(1-\sigma)(1-\beta\alpha)}\mathcal{L}(D_x)U(x),L_p(R_n)\|,\quad 0\le\beta\alpha<1,\end{aligned}\tag{3.5}$$

with some constant $c>0$ independent of $U(x)$ and ε.

Using Minkowski's inequality and Lemma 3, we obtain

$$\begin{aligned}&\|\langle x\rangle^{-\sigma(1-\beta\alpha)}D_x^\beta U_j(x),L_p(R_n\backslash\omega_\varepsilon)\|\\&\le\|\langle x\rangle^{-\sigma(1-\beta\alpha)}D_x^\beta P_{j,h}\mathcal{L}(D_x)U(x),L_p(R_n\backslash\omega_\varepsilon)\|\\&+\|\langle x\rangle^{-\sigma(1-\beta\alpha)}(D_x^\beta P_{j,h}\mathcal{L}(D_x)U(x)-D_x^\beta U_j(x)),L_p(R_n\backslash\omega_\varepsilon)\|\\&\le\|\langle x\rangle^{-\sigma(1-\beta\alpha)}D_x^\beta P_{j,h}\mathcal{L}(D_x)U(x),L_p(R_n)\|\\&+c(\varepsilon)\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}(D_x^\beta P_{j,h}\mathcal{L}(D_x)U(x)-D_x^\beta U_j(x)),L_p(R_n\backslash\omega_\varepsilon)\|\\&\le c\|\langle x\rangle^{(1-\sigma)(1-\beta\alpha)}\mathcal{L}(D_x)U(x),L_p(R_n)\|\\&+c(\varepsilon)\|(1+\langle x\rangle)^{-\sigma(1-\beta\alpha)}(D_x^\beta P_{j,h}\mathcal{L}(D_x)U(x)-D_x^\beta U_j(x)),L_p(R_n)\|,\end{aligned}$$

where the constant $c>0$ is independent of $U(x)$, h, and ε and the constant $c(\varepsilon)>0$ depends only on ε. The proof of the preceding theorem implies that

$$\|(1+\langle x\rangle)^{\sigma(\beta\alpha-1)}(D_x^\beta P_{j,h}\mathcal{L}(D_x)U(x)-D_x^\beta U_j(x)),L_p(R_n)\|\to 0,\quad h\to 0.$$

Passing to the limit as $h\to 0$, we now obtain (3.5). In view of the arbitrariness of ε, we arrive at (1.3). The theorem is proven.

PROOF OF THEOREM 3. Obviously, for $|\alpha|/p > 1$ and $\sigma = 1$ the conditions of Theorem 1 are satisfied. Thereby the system (0.2) is uniquely solvable in $W^r_{p,1}(R_n)$ for every $F(x) \in L_p(R_n)$ and

$$U_j(x) = P_j F(x), \quad \|U(x), W^r_{p,1}(R_n)\| \le c\|F(x), L_p(R_n)\|.$$

It follows from Condition 1 that

$$\mathcal{L}(D_x) : W^r_{p,1}(R_n) \to L_p(R_n) \tag{3.6}$$

is a continuous linear operator. From the above we infer that the range $R(\mathcal{L}(D_x))$ coincides with the whole space $L_p(R_n)$; moreover,

$$\|U(x), W^r_{p,1}(R_n)\| \le c\|\mathcal{L}(D_x)U(x), L_p(R_n)\|.$$

Then there is an inverse operator

$$(\mathcal{L}(D_x))^{-1} : L_p(R_n) \to W^r_{p,1}(R_n)$$

which is a continuous linear operator. Consequently, the quasielliptic operator (3.6) is an isomorphism. The theorem is proven.

§ 4. Sobolev-Type Equations

In conclusion, we exemplify the application of Theorem 3. Consider the following Cauchy problem for the system that is not solved with respect to the higher order time derivative

$$\begin{aligned} &\mathcal{L}_0(D_x)D_t^l U + \sum_{k=0}^{l-1} \mathcal{L}_{l-k}(D_x)D_t^k U = F(t,x), \quad t > 0, \ x \in R_n, \\ &D_t^k U\big|_{t=0} = 0, \quad k = 0, \dots, l-1, \end{aligned} \tag{4.1}$$

where $\mathcal{L}_0(D_x)$ is a quasielliptic operator satisfying Conditions 1 and 2. We suppose that the matrix differential operators $\mathcal{L}_{l-k}(D_x)$ satisfy Condition 1.

The conditions on the operator $\mathcal{L}_0(D_x)$ imply that for solvability of the problem (4.1) in the Sobolev spaces, we have to impose some additional constraints on the right-hand side $F(t,x)$ like

$$\int_{R_n} x^\beta F(t,x)\, dx = 0;$$

moreover, the number of orthogonality conditions depends on the order of the differential operators $\mathcal{L}_k(D_x)$, the dimension n, and the exponent p (see [15, 16]). Using the spaces like $W^r_{p,\sigma}(R_n)$ and Theorem 3, we can extend the class of Sobolev-type equations for which the Cauchy problem is unconditionally solvable.

Theorem 4. *Suppose that $|\alpha|/p > 1$. Then for every vector-function*

$$F(t,x) \in C([0,T]; L_p(R_n))$$

the problem (4.1) *has a unique solution*

$$U(t,x) \in C^l([0,T]; W^r_{p,1}(R_n)).$$

PROOF. By Theorem 3, the linear operators

$$(\mathcal{L}_0(D_x))^{-1}\mathcal{L}_{l-k}(D_x) : \ W^r_{p,1}(R_n) \to W^r_{p,1}(R_n), \quad k = 0, \dots, l-1,$$

are bounded and, since

$$(\mathcal{L}_0(D_x))^{-1}F(t,x) \in C([0,T]; W^{r}_{p,1}(R_n)),$$

correct solvability of the Cauchy problem (4.1) follows if we rewrite the problem in the equivalent form

$$D_t^l U + \sum_{k=0}^{l-1} (\mathcal{L}_0(D_x))^{-1}\mathcal{L}_{l-k}(D_x) D_t^k U = (\mathcal{L}_0(D_x))^{-1}F(t,x), \quad t > 0,$$

$$D_t^k U\big|_{t=0} = 0, \quad k = 0, \ldots, l-1.$$

The theorem is proven.

References

1. L. R. Volevich, "Local properties of solutions to quasielliptic systems," Mat. Sb., **59**, No. 3, 3–52 (1962).
2. G. V. Demidenko, "Integral operators determined by quasielliptic equations. I," Sibirsk. Mat. Zh., **34**, No. 6, 52–67 (1993).
3. G. V. Demidenko, "On weighted Sobolev spaces and integral operators determined by quasielliptic equations," Dokl. Akad. Nauk, **334**, No. 4, 420–423 (1994).
4. L. D. Kudryavtsev, "Embedding theorems for classes of functions determined in the whole space or in the half-space," I: Mat. Sb., **69**, No. 4, 616–639 (1966); II: Mat. Sb., **70**, No. 1, 3–35 (1966).
5. L. D. Kudryavtsev and S. M. Nikol'skiĭ, "Spaces of differentiable functions in several variables and embedding theorems," in: Contemporary Problems of Mathematics. Fundamental Trends [in Russian], VINITI, Moscow, 1988, **26**, pp. 5–157. (Itogi Nauki i Tekhniki.)
6. L. Nirenberg and H. F. Walker, "The null spaces of elliptic partial differential operators in R^n," J. Math. Anal. Appl., **42**, No. 2, 271–301 (1973).
7. M. Cantor, "Elliptic operators and decomposition of tensor fields," Bull. Amer. Math. Soc., **5**, No. 3, 235–262 (1981).
8. Y. Choquet-Bruhat and D. Christodoulou, "Elliptic systems in $H_{s,\sigma}$ spaces on manifolds which are Euclidean at infinity," Acta Math., **146**, No. 1–2, 129–150 (1981).
9. R. B. Lockhart and R. C. McOwen, "Elliptic differential operators on noncompact manifolds," Ann. Scuola Norm. Sup. Pisa Cl. Sci., **12**, No. 3, 409–447 (1985).
10. G. V. Demidenko, "On properties of quasielliptic operators," in: The Second Siberian Congress on Applied and Industrial Mathematics, Inst. Mat. (Novosibirsk), Novosibirsk, 1996, p. 92.
11. S. V. Uspenskiĭ, "On representation of functions determined by a certain class of hypoelliptic operators," Trudy Mat. Inst. Steklov., **117**, 292–299 (1972).
12. S. V. Uspenskiĭ, "On differential properties at infinity of solutions to a certain class of pseudodifferential equations," I: Sibirsk. Mat. Zh., **13**, No. 3, 665–678 (1972); II: Sibirsk. Mat. Zh., **13**, No. 4, 903–909 (1972).
13. P. I. Lizorkin, "Generalized Liouville differentiation and the multiplier method in the theory of embeddings of classes of differentiable functions and its applications," Trudy Mat. Inst. Steklov., **105**, 89–167 (1969).
14. G. H. Hardy, D. E. Littlewood, and G. Pólya, Inequalities [Russian translation], Izdat. Inostr. Lit., Moscow (1948).
15. S. V. Uspenskiĭ, G. V. Demidenko, and V. G. Perepëlkin, Embedding Theorems and Applications to Differential Equations [in Russian], Nauka, Novosibirsk (1984).
16. G. V. Demidenko, "L_p-theory of boundary value problems for Sobolev type equations," in: Partial Differential Equations, Banach Center Publications, Warszawa, 1992, **27**, Part I, pp. 101–109.