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# THE RELATION BETWEEN INTUITIONISTIC AND CLASSICAL MODAL LOGICS

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*Intuitionistic propositional logic* Int *and its extensions, known as intermediate or superintuitionistic logics, in many respects can be regarded as just fragments of classical modal logics containing* S4. *The main aim of this paper is to construct a similar correspondence between intermediate logics augmented with modal operators — we call them intuitionistic modal logics — and classical polymodal logics. We study the class of intuitionistic polymodal logics in which modal operators satisfy only the congruence rules and so may be treated as various sorts of* □ *and* ◇.

Intuitionistic propositional logic Int and its extensions, known as intermediate or superintuitionistic logics, in many respects can be regarded as just fragments of classical modal logics containing S4. At the syntactical level, the Gödel translation $t$ embeds every intermediate logic $L = \mathrm{Int} + \Gamma$ in modal logics in the interval $\rho^{-1}L = [\tau L = \mathrm{S4} \oplus t(\Gamma), \sigma L = \mathrm{Grz} \oplus t(\Gamma)]$. Semantically this is reflected by the fact that Heyting algebras are precisely the algebras of open elements of topological Boolean algebras. From the lattice-theoretic standpoint, the map $\rho$ is a homomorphism of the lattice of logics containing S4 onto the lattice of intermediate logics, where $\sigma$, according to the Blok–Esakia theorem, is an isomorphism of the latter onto the lattice of extensions of the Grzegorczyk system Grz. At the philosophical level, the Gödel translation provides a classical interpretation of the intuitionistic connectives. And from the technical point of view this embedding is a powerful tool for transferring various kinds of results from intermediate logics to modal ones and back via preservation theorems (see [6]). Both classical modal logic and the theory of intermediate logics have gained from this correspondence.

The main aim of this paper is to construct a similar correspondence between intermediate logics enriched with modal operators — we call them intuitionistic modal logics — and classical polymodal logics. That the Gödel translation can be extended to an embedding of at least a few particular intuitionistic modal systems in some classical polymodal logics was observed by several authors (cf. [5, 13, 25, 26]). Fischer Servi [13, 15] used a version of that translation to define "true" intuitionistic analogs of a number of classical modal systems. In [27] we exploited the translation proposed by Shehtman [25] to embed intuitionistic modal logics with the single necessity operator □ of K in bimodal logics above $\mathrm{S4} \otimes \mathrm{K}$. However, like $\forall$ and $\exists$, the necessity and possibility operators □ and ◇ are not supposed to be dual under the intuitionistic laws.

Here we consider a much more extensive class of intuitionistic polymodal logics (first brought in sight by Sotirov [26]), in which modal operators satisfy only the congruence rules and so may be regarded as various sorts of independent □ and ◇. These logics are defined in Sec. 1. Sec. 2 introduces algebraic and (quasi-)

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relational semantics for the... and develops a duality theory and a little bit of correspondence theory for logics with normal □-like and ◇-like operators. In Sec. 3 we bridge the semantics for intuitionistic and classical modal logics and show that the translation prefixing the S4-necessity to all subformulas of intuitionistic modal formulas embeds the intuitionistic modal logics under consideration in classical polymodal logics. Moreover, we prove an analog of the Blok–Esakia theorem by establishing that the lattice of intuitionistic modal logics is isomorphic to a principal filter in the lattice of classical modal logics. We show that the embedding reflects decidability, the finite model property, and tabularity, and then use this result, along with preservation theorems of [28] and [12], to prove that the finite model property of an intermediate logic is inherited under adding to it a modal operator satisfying some simple axioms and inference rules. In the final Sec. 4 we study the embedding of normal intuitionistic modal logics.

Note that all the results obtained in this paper can be extended in the straightforward way to intuitionistic modal logics with polyadic operators.

Intuitionistic modal logics have never been considered in such a general setting as were classical ones. Much is still to be done to obtain results comparable, say, to Fine's or Sahlqvist's completeness theorems. We hope that this paper will serve as a basis for further systematic studies in this branch of modal logic.

## 1. LOGICS

All the logics considered in this paper are formulated in the propositional modal language $\mathcal{LM}_n$ with the standard connectives $\to$, $\wedge$, $\vee$, $\bot$ ($\neg\varphi$ is defined as $\varphi \to \bot$ and $\top$ as $\bot \to \bot$) and the modal operators $\bigcirc_i$, for $i = 1, \ldots, n$. An *intuitionistic modal logic* in the language $\mathcal{LM}_n$ (IM-*logic* for short) is a set of $\mathcal{LM}_n$-formulas which contains an intuitionistic logic **Int** in the language $\mathcal{LM}_0$ (with only the first four connectives given above) and is closed under substitution, modus ponens, and the congruence rules $\varphi \leftrightarrow \psi / \bigcirc_i \varphi \leftrightarrow \bigcirc_i \psi$, for all $i = 1, \ldots, n$. The smallest monomodal IM-logic is denoted by **IntC** (**C** stands for "congruential" in accordance with Segerberg's nomenclature in [24]). For a set of formulas $\Gamma$ and an IM-logic $L$, we denote by $L \oplus \Gamma$ the smallest IM-logic containing $\Gamma$ and $L$. Several kinds of IM-logics have been considered in the literature, and all of them are covered by our definition, which is similar to one in [26]. Here are a few basic monomodal and bimodal systems.

A monomodal IM-logic $L$ (in the language $\mathcal{LM}_1$ with $\bigcirc = \bigcirc_1$) is said to be *regular* if it is closed under the regularity rule $\varphi \to \psi / \bigcirc \varphi \to \bigcirc \psi$. Equivalently, $L$ is regular iff it contains $\bigcirc(p \wedge q) \to \bigcirc p$. The smallest regular IM-logic is denoted by **IntR**. A regular IM-logic $L$ is said to be □-*normal* if it contains $\bigcirc(p \wedge q) \leftrightarrow \bigcirc p \wedge \bigcirc q$ and $\bigcirc\top$. In such a case we write □ instead of $\bigcirc$ and call it the *necessity operator*. Every □-normal logic is closed under necessitation $\varphi / \Box\varphi$. The smallest □-normal IM-logic is denoted by $\mathbf{IntK}_\Box$. A regular IM-logic $L$ is called ◇-*normal* if $\bigcirc(p \vee q) \leftrightarrow \bigcirc p \vee \bigcirc q$ and $\neg \bigcirc \bot$ belong to it. In this case we write ◇ instead of $\bigcirc$ and call it the *possibility operator*. Every ◇-normal logic is closed under $\neg\varphi / \neg\Diamond\varphi$. The smallest ◇-normal logic is denoted by $\mathbf{IntK}_\Diamond$. Some particular □-normal IM-systems were investigated in [5, 21, 26]; general results on the finite model property of such logics can be found in [27]. ◇-Normal systems were considered in [5, 26].

As in the classical modal logic, given a □-normal IM-logic $L$, we can define the dual operator ◇ by setting $\Diamond\varphi = \neg\Box\neg\varphi$. Likewise, in a ◇-normal logic $L'$ we can take $\neg\Diamond\neg\varphi$ as a definition of $\Box\varphi$. However, $L$ and $L'$ are not necessarily ◇-normal and □-normal with respect to the operators defined. (This will certainly be the case if their underlying nonmodal logic is classical.) On the other hand, the dual definition of □ and ◇ is not consistent with intuitionistic principles (according to which $\forall$ and $\exists$ are not dual).

To construct IM-logics with absolutely independent modal operators, we can take IM-logics $L_1$ and $L_2$, formulated in languages with disjoint sets of modal operators, and then form their *fusion* $L_1 \otimes L_2$, the smallest IM-logic in the joined language containing $L_1 \cup L_2$. In this way we can define the bimodal logic $\mathbf{IntK}_{\Box\Diamond} = \mathbf{IntK}_{\Box} \otimes \mathbf{IntK}_{\Diamond}$. Its extensions are called $\Box\Diamond$-IM-*logics*. There is no connection between $\Diamond$ and $\Box$ in $\mathbf{IntK}_{\Box\Diamond}$, the latter being both $\Box$- and $\Diamond$-normal. Extensions of $\mathbf{IntK}_{\Box}$ ($\mathbf{IntK}_{\Diamond}$) can clearly be identified with extensions of $\mathbf{IntK}_{\Box\Diamond} \oplus \Diamond p \leftrightarrow p$ (respectively, $\mathbf{IntK}_{\Box\Diamond} \oplus \Box p \leftrightarrow p$).

The well-motivated $\Box\Diamond$-IM-logic

$$\mathbf{FS} = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond(p \to q) \to (\Box p \to \Diamond q) \oplus (\Diamond p \to \Box q) \to \Box(p \to q)$$

was constructed by Fischer Servi in [14, 15]. Extensions of **FS** will be called **FS**-*logics*. Some of them were studied in [15, 1, 10].

By adding to a consistent IM-logic in the language $\mathcal{LM}_n$ the Law of the Excluded Middle $p \vee \neg p$, we obtain a classical logic with $n$ modal operators. Denote by **C**, **R**, and **K** the monomodal logics $\mathbf{IntC} \oplus p \vee \neg p$, $\mathbf{IntR} \oplus p \vee \neg p$, and $\mathbf{IntK}_{\Box} \oplus p \vee \neg p = \mathbf{IntK}_{\Diamond} \oplus p \vee \neg p$, respectively.

## 2. SEMANTICS AND DUALITY

The logics introduced above correspond to varieties (equational classes) of Heyting (or pseudo-Boolean) algebras with operators. More precisely, given a language $\mathcal{LM}_n$, we consider algebras of the form

$$\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \bigcirc_1, \dots, \bigcirc_n \rangle,$$

where $\langle A, \to, \wedge, \vee, \top \rangle$ is a Heyting algebra with unit element $\top$, and $\bigcirc_i$ for $1 \leq i \leq n$ are unary operators on $A$. Such algebras will be called IM-*algebras*. A *valuation* $\mathfrak{V}$ of $\mathcal{LM}_n$ in $\mathfrak{A}$ is a homomorphism of the algebra of $\mathcal{LM}_n$-formulas into $\mathfrak{A}$. A formula $\varphi$ is *true* in $\mathfrak{A}$ under $\mathfrak{V}$ if $\mathfrak{V}(\varphi) = \top$; $\varphi$ is *valid* in $\mathfrak{A}$, written $\mathfrak{A} \models \varphi$, if it is true under any valuation.

An IM-logic $L$ is *characterized* by a class $\mathcal{C}$ of IM-algebras if $L = \{\varphi : \forall \mathfrak{A} \in \mathcal{C}\ \mathfrak{A} \models \varphi\}$. In the standard way one can show that the class of IM-algebras, validating all the formulas in an IM-logic $L$, forms a variety characterizing $L$.

The relational semantics is usually derived from the algebraic one using the Stone–Jónsson–Tarski representation of Heyting and modal algebras. Since the logics under consideration are rather weak, we need, first, introduce some intermediate structures combining a relational intuitionistic component and an algebraic modal one.

We remind the reader that an *intuitionistic frame* (or Int-*frame* for short) is a structure of the form $\mathfrak{F} = \langle W, R, P \rangle$, where $R$ is a partial order on a nonempty set $W$ and $P$ is a collection of cones (i.e., upward closed sets) in $W$ with respect to $R$ which contain $\varnothing$ and are closed under $\cap$, $\cup$, and the operation

$$X \supset Y = \{x \in W : \forall y \in W\ (xRy \wedge y \in X \Rightarrow y \in Y)\}.$$

If $P$ contains all the cones in $W$, then we call $\mathfrak{F}$ a *full* (or *Kripke*) *frame* and write $\langle W, R \rangle$ instead of $\langle W, R, P \rangle$. The underlying full frame of $\mathfrak{F}$ is denoted by $\kappa\mathfrak{F}$.

Now we define a *quasi*-IM-*frame* as a structure $\mathfrak{F} = \langle W, R, \bigcirc_1, \dots, \bigcirc_n, P \rangle$ such that $\langle W, R, P \rangle$ is an Int-frame and the $\bigcirc_i$, $i = 1, \dots, n$, are just operations on $P$. Every quasi-IM-frame gives rise to the IM-algebra $\mathfrak{F}^{\dagger} = \langle P, \supset, \cap, \cup, W, \bigcirc_1, \dots, \bigcirc_n \rangle$, called the *dual* of $\mathfrak{F}$. Writing $\mathfrak{F} \models \varphi$ means that $\mathfrak{F}^{\dagger} \models \varphi$. All the other semantic notions above can be translated to quasi-frames in the same way. A model on $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{V}$ is a valuation in $\mathfrak{F}$ (= in $\mathfrak{F}^\dagger$). If $x \in \mathfrak{V}(\varphi)$ then we write $(\mathfrak{M}, x) \models \varphi$, or simply $x \models \varphi$ if this is understood, and say that $\varphi$ is *true* at $x$ (under $\mathfrak{V}$). It is clear that $\mathfrak{V}(\varphi)$ is a cone for every formula $\varphi$.

Conversely, with each IM-algebra $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \bigcirc_1, \dots, \bigcirc_n \rangle$ we can associate its *dual*, the quasi-IM-frame $\mathfrak{A}_\dagger = \langle W, R, \bigcirc'_1, \dots, \bigcirc'_n, P \rangle$ in which $W$ is the set of prime filters in $\mathfrak{A}$, and for every $x, y \in W$ and $a \in A$,

$$xRy \text{ iff } x \subseteq y,$$

$$P(a) = \{x \in W : a \in x\},$$

$$P = \{P(a) : a \in A\},$$

$$\bigcirc'_i(P(a)) = P(\bigcirc_i(a)), \ 1 \leq i \leq n.$$

Using the well-known correspondence between Int-frames and Heyting algebras (see, e.g., [7]), one can readily see that every IM-algebra $\mathfrak{A}$ is isomorphic to its bidual, written $\mathfrak{A} \simeq (\mathfrak{A}_\dagger)^\dagger$. A quasi-IM-frame $\mathfrak{F}$ is called *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^\dagger)_\dagger$. Every quasi-IM-frame of the form $\mathfrak{A}_\dagger$ is clearly descriptive. Hence, we have

**Proposition 1.** Each IM-logic is characterised by a suitable class of descriptive quasi-IM-frames.

Another sort of adequate relational semantics for IM-logics — neighborhood frames — was introduced in [26]. For $\Box\Diamond$-IM-logics, the algebraic modal component in quasi-IM-frames can also be replaced with a relational one.

We say that an IM-algebra $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \Box, \Diamond \rangle$ is a $\Box\Diamond$-IM-*algebra* if the following identities hold in it:

$$\Box\top = \top, \ \Box(a \wedge b) = \Box a \wedge \Box b, \ \neg\Diamond\bot = \top, \ \Diamond(a \vee b) = \Diamond a \vee \Diamond b.$$

All $\Box\Diamond$-IM-logics are clearly characterised by varieties of $\Box\Diamond$-IM-algebras.

Given a $\Box\Diamond$-IM-algebra $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \Box, \Diamond \rangle$, we define its *dual* $\mathfrak{A}_+$ to be the structure $\langle W, R, R_\Box, R_\Diamond, P \rangle$, where $\langle W, R, P \rangle$ is the dual of the Heyting algebra $\langle A, \to, \wedge, \vee, \top \rangle$, and for every $x, y \in W$,

$$xR_\Box y \ \text{ iff } \ \forall a \in A \ (\Box a \in x \Rightarrow a \in y),$$

$$xR_\Diamond y \ \text{ iff } \ \forall a \in A \ (a \in y \Rightarrow \Diamond a \in x).$$

It follows immediately from the definition that, for all $x, u, v, y \in W$,

$$xRu \wedge uR_\Box v \wedge vRy \Rightarrow xR_\Box y,$$

$$xRu \wedge vR_\Diamond u \wedge vRy \Rightarrow yR_\Diamond x$$

or, equivalently,

$$R \circ R_\Box \circ R \subseteq R_\Box, \tag{1}$$

$$R \circ R_\Diamond^{-1} \circ R \subseteq R_\Diamond^{-1}. \tag{2}$$

(Here $\circ$ denotes the composition of relations.)

Structures of the form $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$, where $\langle W, R, P \rangle$ is an Int-frame, $R_\Box$, $R_\Diamond$ are binary relations on $W$ satisfying (1) and (2), and $P$ is closed under the operations $\Box$ and $\Diamond$ defined by

$$\Box X = \{x \in W : \forall y \in X \ (xR_\Box y \Rightarrow y \in X)\},$$

$$\Diamond X = \{x \in W : \exists y \in X \ xR_\Diamond y\},$$

will be called $\Box\Diamond$-IM-*frames*. The dual of a $\Box\Diamond$-IM-frame $\mathfrak{F}$ is then the algebra $\mathfrak{F}^+ = \langle P, \supset, \cap, \cup, W, \Box, \Diamond \rangle$. It is not hard to check that $\mathfrak{F}^+$ is a $\Box\Diamond$-IM-algebra and that again $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$ for every $\Box\Diamond$-IM-algebra $\mathfrak{A}$. We say that a $\Box\Diamond$-IM-frame $\mathfrak{F}$ is *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$. Since frames of the form $\mathfrak{A}_+$ are descriptive, we have

**Proposition 2.** Every $\Box\Diamond$-IM-logic is characterized by a suitable class of descriptive $\Box\Diamond$-IM-frames.

The following internal characterisation of descriptive $\Box\Diamond$-IM-frames is obtained by the straightforward combination of corresponding characterisations of descriptive modal and intuitionistic frames. For details, consult [17, 7].

**Proposition 3.** A $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$ is descriptive iff $\mathfrak{F}$ is tight$_R$, tight$_{R_\Box}$, and tight$_{R_\Diamond}$, i.e.,

$$xRy \text{ iff } \forall X \in P\ (x \in X \Rightarrow y \in X);$$

$$xR_\Box y \text{ iff } \forall X \in P\ (x \in \Box X \Rightarrow y \in X);$$

$$xR_\Diamond y \text{ iff } \forall X \in P\ (y \in X \Rightarrow x \in \Diamond X),$$

and compact, i.e., for any $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \{W - X : X \in P\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \varnothing$.

A $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$ is a *full* (or *Kripke*) $\Box\Diamond$-IM-frame if $\langle W, R, P \rangle$ is a full Int-frame. (As far as we know full $\Box\Diamond$-IM-frames were first introduced in [26].) A $\Box\Diamond$-IM-logic is called *complete* if it is characterised by a class of full $\Box\Diamond$-IM-frames. The underlying full frame of a $\Box\Diamond$-IM-frame $\mathfrak{F}$ is denoted by $\kappa\mathfrak{F}$. A $\Box\Diamond$-IM-logic $L$ is said to be d-*persistent* if $\kappa\mathfrak{F} \models L$ whenever $\mathfrak{F}$ is a descriptive frame validating $L$. All d-persistent logics are clearly complete. Another useful property of d-persistence is its being preserved under sums, i.e., if logics $L_1$ and $L_2$ are d-persistent then so is $L_1 \oplus L_2$. (In general, however, completeness as well as many other important properties are not preserved under sums of logics.) We give some examples of d-persistent $\Box\Diamond$-IM-logics. To this end we need the following well-known lemma on the existence of prime filters (see [22]).

**LEMMA 4.** Suppose that $\mathfrak{A} = \langle A, \rightarrow, \wedge, \vee, \top \rangle$ is a Heyting algebra and $B$, $C$ are nonempty subsets of $A$ such that (i) $b_1 \wedge \ldots \wedge b_n \not\leq c$ for any $b_1, \ldots, b_n \in B$, $c \in C$, and (ii) for every $c_1, c_2 \in C$, there is $c \in C$ for which $c_1 \vee c_2 \leq c$. Then there exists a prime filter $\nabla$ in $\mathfrak{A}$ such that $B \subseteq \nabla$ and $C \cap \nabla = \varnothing$.

Here $\leq$ is the lattice partial order on $A$ defined by $a \leq b$ iff $a \wedge b = a$.

**Proposition 5.** FS is d-persistent.

**Proof.** It suffices to show that any $\Box\Diamond$-IM-frame satisfying the conditions

$$xR_\Diamond y \Rightarrow \exists z\ (yRz \wedge xR_\Box z \wedge xR_\Diamond z), \tag{3}$$

$$xR_\Box y \Rightarrow \exists z\ (xRz \wedge zR_\Box y \wedge zR_\Diamond y) \tag{4}$$

validates FS and that (3) and (4) hold in any descriptive frame for FS.

To prove the former claim, suppose that a $\Box\Diamond$-IM-frame $\mathfrak{F}$ satisfies (3) but $\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$ is refuted in $\mathfrak{F}$ under some valuation. Then $x \models \Diamond(p \rightarrow q)$, $x \models \Box p$, and $x \not\models \Diamond q$, for some $x$ in $\mathfrak{F}$, and so there is $y$ such that $xR_\Diamond y$ and $y \models p \rightarrow q$. By (3), we have $yRz$, $xR_\Box z$, and $xR_\Diamond z$ for some point $z$. Then $z \models p \rightarrow q$ (since the truth-set of any formula is a cone), $z \models p$, and $z \not\models q$, which is impossible. The second axiom of FS is treated analogously using (4) and (1).

Now, letting $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$ be a descriptive frame for FS, we show that it satisfies (4). Without loss of generality, we may assume that $\mathfrak{F} \simeq \mathfrak{A}_+$ for some $\Box\Diamond$-IM-algebra $\mathfrak{A} \models$ FS. Thus, points in $\mathfrak{F}$ are prime filters in $\mathfrak{A}$. Let $x, y \in W$ and $xR_\Box y$. Putting $B = x \cup \{\Diamond b : b \in y\}$ and $C = \{\Box c : c \notin y\}$, we show that $B$ and $C$ satisfy (i) and (ii) in Lemma 4. Suppose $a \wedge \Diamond b_1 \wedge \ldots \wedge \Diamond b_n \leq \Box c$ for some $a \in x$ ($x$ is closed under $\wedge$), $b_1, \ldots, b_n \in y$, and $c \notin y$. Then $a \wedge \Diamond b_1 \wedge \ldots \wedge \Diamond b_n \rightarrow \Box c = \top$ in $\mathfrak{A}$, from which by the second axiom of FS we obtain $a \rightarrow \Box(b_1 \wedge \ldots \wedge b_n \rightarrow c) = \top$. It follows that $\Box(b \rightarrow c) \in x$ for some $b \in y$ and $c \notin y$. Since $xR_\Box y$, we have $b \rightarrow c \in y$ and $c \in y$, which is a contradiction. Therefore, (i) holds. To derive (ii), assume $c_1, c_2 \notin y$. Since $y$ is prime, $c_1 \vee c_2 \notin y$, and so $\Box(c_1 \vee c_2) \in C$ and $\Box c_1 \vee \Box c_2 \leq \Box(c_1 \vee c_2)$.

By Lemma 4, there is a prime filter $z \in W$ such that $B \subseteq z$ and $C \cap z = \varnothing$. This means that $xRz$, $zR_\Box y$, and $zR_\Diamond y$, as is required by (4).

In the same way, using Lemma 4 and the first axiom of FS, we can show that $\mathfrak{F}$ satisfies (3).

Applying the same sort of technique, it is not hard to prove the following proposition in which $\Box^n$ and $\Diamond^n$ are strings of $n$ boxes and diamonds, respectively.

Proposition 6. For all $k, l, m, n \geq 0$, the logic

$$\mathbf{L}(k,l,m,n) = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond^k \Box^l p \rightarrow \Box^m \Diamond^n p$$

is d-persistent, with every descriptive $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ for $\mathbf{L}(k,l,m,n)$ satisfying the condition

$$xR_\Diamond^k y \wedge xR_\Box^m z \Rightarrow \exists u\ (yR_\Box^l u \wedge zR_\Diamond^n u).$$

In fact, we face an analog of the result by Lemmon and Scott [18] which was the starting point for the development of correspondence theory in the classical modal logic and which ultimately led to Sahlqvist's theorem [23].

The correspondence between $\Box\Diamond$-IM-algebras and $\Box\Diamond$-IM-frames being established, we extend it to algebraic operators of forming subalgebras, homomorphic images, and direct products. As to the latter operator, its relational analog is the standard *disjoint union* of frames defined in exactly the same way as in the purely intuitionistic or classical modal case (see [17, 7]). However, the duals of the notions of a homomorphism and a subalgebra of $\Box\Diamond$-IM-algebras are not direct translations of the standard definitions.

Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ be a $\Box\Diamond$-IM-frame and $V$ a nonempty subset of $W$ satisfying the following two conditions:

$$\forall x \in V \forall y \in W\ (xRy \vee xR_\Box y \Rightarrow y \in V) \tag{5}$$

and

$$\forall x \in V \forall y \in W\ (xR_\Diamond y \Rightarrow \exists z \in V\ (xR_\Diamond z \wedge yRz)). \tag{6}$$

Then it is easy to see that the structure

$$\mathfrak{G} = \langle V, R{\restriction}V, R_\Box{\restriction}V, R_\Diamond{\restriction}V, \{X \cap V : X \in P\}\rangle$$

is also a $\Box\Diamond$-IM-frame. It is called a *generated subframe* of $\mathfrak{F}$. Condition (5) is standard: it requires $V$ to be upward closed with respect to both $R$ and $R_\Box$. However, according to (6), $V$ is not necessarily upward closed with respect to $R_\Diamond$. This is illustrated by Fig. 1 in which $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$, although the set $\{x, z\}$ is not upward closed in $\mathfrak{F}$ with respect to $R_\Diamond$.

THEOREM 7. (i) If $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$ is a generated subframe of a $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, then the map $h$ defined by

$$h(X) = X \cap V \text{ for every } X \in P$$

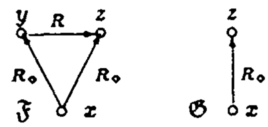

Fig. 1

is a homomorphism of $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.

(ii) If $h$ is a homomorphism of a $\Box\Diamond$-IM-algebra $\mathfrak{A}$ onto a $\Box\Diamond$-IM-algebra $\mathfrak{B}$, then the map $h_+$ defined by

$$h_+(\nabla) = h^{-1}(\nabla) \text{ for every prime filter } \nabla \text{ in } \mathfrak{B}$$

is an isomorphism of $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.

**Proof.** (i) That $h$ is a surjection preserving $\supset$, $\cap$, $\cup$, and $\Box$ is proved in the usual way (see [17, 7]). We show that $h$ preserves $\Diamond$, i.e., that $h(\Diamond X) = \Diamond h(X)$ for every $X \in P$.

Suppose $x \in \Diamond X \cap V$ in $\mathfrak{F}$. Then there is $y \in X$ such that $xR_\Diamond y$, and so (6) yields $z \in V$ with $xR_\Diamond z$ and $yRz$. Since $X$ is a cone, it follows that $z \in X$; hence $x \in \Diamond(X \cap V)$ in $\mathfrak{G}$. Thus, $h(\Diamond X) \subseteq \Diamond h(X)$, the reverse being trivial.

(ii) Let $\mathfrak{A}_+ = \langle W, R, R_\Box, R_\Diamond, P\rangle$ and $\mathfrak{B}_+ = \langle U, S, S_\Box, S_\Diamond, Q\rangle$. Put

$$V = \{\nabla \in W : h^{-1}(\top) \subseteq \nabla\}.$$

It is shown in [17, 7] that $V$ is upward closed in $\mathfrak{A}_+$ with respect to $R$ and $R_\Box$, $h$ is a bijection of $V$ onto $U$, and $h_+$ is an isomorphism of $\mathfrak{B}_+$ onto the subframe of $\mathfrak{A}_+$ generated by $V$ if $R$ and $R_\Box$ coincide. So it remains to show that $V$ satisfies (6) and that $\nabla_1 S_\Diamond \nabla_2$ iff $h_+(\nabla_1) R_\Diamond h_+(\nabla_2)$ for each of the $\nabla_1, \nabla_2 \in U$.

Assume that $h^{-1}(\top) \subseteq \nabla$ (i.e., $\nabla \in V$) and $\nabla R_\Diamond \nabla'$ for some $\nabla' \in W$. Putting $B = \nabla' \cup h^{-1}(\top)$, $C = \{a \in \mathfrak{A} : \Diamond a \notin \nabla\}$, we show that $B$ and $C$ satisfy the conditions of Lemma 4. Suppose $b \wedge c \leq a$ for some $b \in \nabla'$, $c \in h^{-1}(\top)$, and $\Diamond a \notin \nabla$. Then $h(b \wedge c) = h(b) \leq h(a)$, and so $h(\Diamond b) \leq h(\Diamond a)$. Since $\Diamond b \in \nabla$, $h(\nabla)$ is a filter in $\mathfrak{B}$ and $h^{-1}(h(\nabla)) = \nabla$, we must have $\Diamond a \in \nabla$, which is a contradiction. Now suppose $\Diamond a_1, \Diamond a_2 \notin \nabla$. Since $\nabla$ is prime, $\Diamond a_1 \vee \Diamond a_2 \notin \nabla$, whence $\Diamond(a_1 \vee a_2) = \Diamond a_1 \vee \Diamond a_2 \notin \nabla$.

Let $\nabla_1$ be a prime filter in $\mathfrak{A}$ such that $B \subseteq \nabla_1$ and $C \cap \nabla_1 = \varnothing$. Then clearly $\nabla_1 \in V$, $\nabla R_\Diamond \nabla_1$, and $\nabla' \subseteq \nabla_1$. Thus $V$ satisfies (6).

Suppose that $\nabla_1 S_\Diamond \nabla_2$, i.e., $\Diamond b \in \nabla_1$ whenever $b \in \nabla_2$, and that $a \in h_+(\nabla_2)$ for some $a$ in $\mathfrak{A}$. Then $h(a) \in \nabla_2$, $h(\Diamond a) = \Diamond h(a) \in \nabla_1$, and so $\Diamond a \in h_+(\nabla_1)$. Conversely, assume that $h_+(\nabla_1) R_\Diamond h_+(\nabla_2)$. Then, for all $a$ in $\mathfrak{A}$, $a \in h_+(\nabla_2)$ implies $\Diamond a \in h_+(\nabla_1)$. Since $h$ is a bijection of $V$ onto $U$, we see that if $b \in \nabla_2$ then $b = h(a)$ for some $a \in h_+(\nabla_2)$. So $\Diamond a \in h_+(\nabla_1)$ and $\Diamond b = \Diamond h(a) = h(\Diamond a) \in \nabla_1$.

Given $\Box\Diamond$-IM-frames $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ and $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$, we say that the map $f$ from $W$ onto $V$ is a *reduction* (or p-*morphism*) of $\mathfrak{F}$ to $\mathfrak{G}$ if, for all $x, y \in W$, $u \in V$, and $X \in Q$, the following four conditions hold:

$$xR_\bullet y \Rightarrow f(x)S_\bullet f(y), \ \bullet \in \{\text{blank}, \Box, \Diamond\}, \tag{7}$$

$$f(x)S_\bullet u \Rightarrow \exists z \in f^{-1}(u)\ xR_\bullet z, \ \bullet \in \{\text{blank}, \Box\}, \tag{8}$$

$$f(x)S_\Diamond u \Rightarrow \exists z \in W\ (xR_\Diamond z \wedge uSf(z)), \tag{9}$$

$$f^{-1}(X) \in P. \tag{10}$$


Fig. 2

For example, the map gluing the points 0 and 1 in the frame $\mathfrak{F}$ in Fig. 2 is a reduction of $\mathfrak{F}$ to $\mathfrak{G}$ in the same figure. Notice that if we consider these frames as classical bimodal frames we see that $\mathfrak{F}$ is not reducible to $\mathfrak{G}$ because the points 2 and 3 as well as 2 and 4 are connected by $S_\Diamond$-arrows. If we remove these arrows, then the modified $\mathfrak{G}$ will not be a $\Box\Diamond$-IM-frame, since condition (2) does not hold.

THEOREM 8. (i) If $f$ is a reduction of a $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ to a $\Box\Diamond$-IM-frame $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$, then the map $f^+$ defined by

$$f^+(X) = f^{-1}(X) \text{ for every } X \in Q$$

is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.

(ii) If $\mathfrak{B}$ is a subalgebra of a $\Box\Diamond$-IM-algebra $\mathfrak{A}$, then the map $f$ defined by

$$f(\nabla) = \nabla \cap \mathfrak{B} \text{ for every prime filter } \nabla \text{ in } \mathfrak{A}$$

is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.

Proof. (i) It is known (cf. [17, 7]) that $f^+$ is an injection preserving $\to$, $\wedge$, $\vee$, $\Box$. So we need only show that $f^{-1}(\Diamond X) = \Diamond f^{-1}(X)$ for every $X \in Q$. Suppose that $x \in f^{-1}(\Diamond X)$, i.e., $f(x) S_\Diamond u$ for some $u \in X$. By (9), we then have a $z \in W$ such that $x R_\Diamond z$ and $u S f(z)$. Since $X$ is a cone, $f(z) \in X$, and so $x \in \Diamond f^{-1}(X)$. Thus, $f^{-1}(\Diamond X) \subseteq \Diamond f^{-1}(X)$. The reverse inclusion follows from (7).

(ii) It was proved in [17] and [7] that $f$ satisfies (7), (8), and (10). To derive (9), assume that $(\nabla_1 \cap \mathfrak{B}) S_\Diamond \nabla_2$ for some prime filters $\nabla_1$ on $\mathfrak{A}$ and $\nabla_2$ on $\mathfrak{B}$. Putting $B = \nabla_2$, $C = \{a \in \mathfrak{A} : \Diamond a \notin \nabla_1\}$, we show that $B$ and $C$ satisfy the conditions of Lemma 4. That (ii) holds was established in the proof of Theorem 7. Suppose that (i) does not hold. Then there exist $b \in \nabla_2$ and $\Diamond a \notin \nabla_1$ such that $b \leq a$. It follows that $\Diamond b \leq \Diamond a$. Since $(\nabla_1 \cap \mathfrak{B}) S_\Diamond \nabla_2$, we have $\Diamond b \in \nabla_1 \cap B$, and so $\Diamond a \in \nabla_1$, which is a contradiction. Let $\nabla$ be a prime filter in $\mathfrak{A}$ for which $B \subseteq \nabla$ and $C \cap \nabla = \varnothing$. Then clearly $\nabla_1 R_\Diamond \nabla$ and $\nabla_2 R(\nabla \cap \mathfrak{B})$.

In exactly the same way as in the classical modal logic (cf. [17, 2, 3]), we can use the duality results above to prove the following definability theorems.

THEOREM 9. A class $\mathcal{C}$ of $\Box\Diamond$-IM-frames is definable by $\mathcal{LM}_2$-formulas (in the sense that there exists a set $\Gamma$ of $\mathcal{LM}_2$-formulas such that $\mathcal{C} = \{\mathfrak{F} : \mathfrak{F} \models \Gamma\}$) iff $\mathcal{C}$ is closed under forming generated subframes, reducts, disjoint unions, and both $\mathcal{C}$ and its complement (in the class of all $\Box\Diamond$-IM-frames) are closed under the operator $\mathfrak{F} \mapsto (\mathfrak{F}^+)_+$.

For a full $\Box\Diamond$-IM-frame $\mathfrak{F}$, the frame $\kappa(\mathfrak{F}^+)_+$ is called the *prime filter extension* of $\mathfrak{F}$. This concept is an intuitionistic counterpart of the notion of an ultrafilter extension in the classical modal logic, introduced in [2].

THEOREM 10. A class $\mathcal{C}$ of full $\Box\Diamond$-IM-frames coincides with the class of all full $\Box\Diamond$-IM-frames validating a d-persistent $\Box\Diamond$-IM-logic $L$ iff $\mathcal{C}$ is closed under the formation of generated subframes, reducts, disjoint unions, and both $C$ and its complement (in the class of all full $\Box\Diamond$-IM-frames) are closed under forming prime filter extensions.

THEOREM 11. If a $\Box\Diamond$-IM-logic $L$ is characterized by the class of full $\Box\Diamond$-IM-frames which is closed under elementary equivalence (in the first-order language with predicates $=$, $R$, $R_\Box$, and $R_\Diamond$), then $L$ is d-persistent.

We conclude this section with a few remarks concerning other semantics for $\Box\Diamond$-IM-logics. Note, first, that conditions (1) and (2) can be made considerably weaker. We say that a structure $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is a *weak* $\Box\Diamond$-IM-*frame* if $\langle W, R, P\rangle$ is an Int-frame, $R_\Box$ is an arbitrary binary relation, $R_\Diamond$ is a binary relation such that, for every $x, y \in W$,

$$xRy \wedge xR_\Diamond z \Rightarrow \exists u \in W\ (yR_\Diamond u \wedge zRu), \tag{11}$$

and $P$ is closed under the operations

$$\Box X = \{x \in W : \forall y, z\ (xRyR_\Box z \Rightarrow z \in X)\},$$

$$\Diamond X = \{x \in W : \exists y \in X\ xR_\Diamond y\}.$$

For instance, both $\Box\Diamond$-IM-frames and frames from [5] are weak $\Box\Diamond$-IM-frames. One can readily check that if $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is a weak $\Box\Diamond$-IM-frame then the structure $\mathfrak{G} = \langle W, R, \Box, \Diamond, P\rangle$ is a quasi-IM-frame validating $\mathrm{IntK}_{\Box\Diamond}$. It follows that $\mathfrak{G}^\dagger$ is a $\Box\Diamond$-IM-algebra, which we denote by $\mathfrak{F}^+$. The set of cones (with respect to $R$) in $\mathfrak{F}$ is closed under $\Box$ and $\Diamond$. If $P$ contains all such cones, then $\mathfrak{F}$ is called *full*. A $\Box\Diamond$-IM-logic is *weakly complete* if it is characterized by a class of full weak $\Box\Diamond$-IM-frames.

One can argue as to which conditions on $R_\Box$ and $R_\Diamond$ are more natural: (11) or (1) and (2), or something in-between them, for instance, Ono's frames from [20] or those of Božić and Došen in [5]. From the technical point of view, however, this gives us nothing new. Indeed, with every weak $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ we can associate the $\Box\Diamond$-IM-frame $\mathfrak{F}^\circ = \langle W, R, R \circ R_\Box \circ R, R^{-1} \circ R_\Diamond \circ R^{-1}, P\rangle$. And then we have

Proposition 12. For every weak $\Box\Diamond$-IM-frame $\mathfrak{F}$ and every formula $\varphi$, $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}^\circ \models \varphi$.

Proof. We can either show that $\mathfrak{F}^+ = (\mathfrak{F}^\circ)^+$ or simply use a straightforward induction on the complexity of $\varphi$.

In particular, we obtain

COROLLARY 13. A $\Box\Diamond$-IM-logic is complete iff it is weakly complete.

Fischer Servi's birelational frames for FS, introduced in [14], can also be derived from weak $\Box\Diamond$-IM-frames. We say that a $\Box\Diamond$-IM-frame is an FS-*frame* if it satisfies conditions (3) and (4). Given an FS-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, define the relation $S = R_\Box \cap R_\Diamond$. It follows from (1)-(4) that $S$ satisfies (11), i.e., $xRy$ and $xSz$ imply $ySu$ and $zRu$ for some $u \in W$, and

$$xSyRz \Rightarrow \exists u\ xRuSz. \tag{12}$$

Denote the weak $\Box\Diamond$-IM-frame $\langle W, R, S, S, P\rangle$ by $\mathfrak{F}^\bullet$.

We say that a weak $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, S, S, P\rangle$ is a *birelational* FS-*frame* if it satisfies (12). One can easily verify that every birelational FS-frame validates FS and that $\mathfrak{F}^+ = (\mathfrak{F}^\bullet)^+$ for every FS-frame $\mathfrak{F}$. Therefore, we have

Proposition 14. Every FS-logic is characterized by a class of birelational FS-frames.

Since $\mathfrak{F}^\circ$ is an FS-frame whenever $\mathfrak{F}$ is a birelational FS-frame, we have also

**Proposition 15.** An FS-logic is complete iff it is characterised by full birelational FS-frames.

## 3. EMBEDDING

Gödel [16] embedded Int in S4 via the translation $t$ prefixing $\Box$ to all subformulas of intuitionistic formulas.* Dummett and Lemmon [8] extended Gödel's embedding to all intermediate logics. Maksimova and Rybakov [19], Blok [4], and Esakia [9] started a systematic investigation into the structure of "modal companions" of intermediate logics.

In [27] we used the natural generalization of Gödel's translation, which embeds extensions of $\mathbf{IntK}_\Box$ in classical bimodal logics containing $\mathbf{S4} \otimes \mathbf{K}$, to obtain a number of general completeness results for intuitionistic modal logics. Our aim here is to study the embedding of (not necessarily normal or regular) IM-logics, in an arbitrary language $\mathcal{LM}_n$, in classical logics with $n+1$ modal operators.

Given a language $\mathcal{LM}_n$, we define its extension $\mathcal{LM}'_n$ with one more modal operator $\Box_I$ and consider classical $n+1$-modal logics in $\mathcal{LM}'_n$ (CM-*logics* for short) containing the S4-axioms for $\Box_I$:

$$\Box_I(p \wedge q) \leftrightarrow \Box_I p \wedge \Box_I q, \quad \Box_I \top, \quad \Box_I p \rightarrow p, \quad \Box_I p \rightarrow \Box_I \Box_I p.$$

These logics can be interpreted by *quasi-CM-frames* which are structures of the form $\mathfrak{F} = \langle W, R_I, \bigcirc_1, \ldots, \bigcirc_n, P\rangle$, where $R_I$ is a quasi-order on $W \neq \varnothing$, $\bigcirc_i$ is an arbitrary operation on $P$, and $P \subseteq 2^W$ contains $\varnothing$ and is closed under Boolean operations and the operation $\Box_I$ defined by

$$\Box_I X = \{x \in W : \forall y\ (xR_I y \Rightarrow y \in X)\}.$$

The dual of $\mathfrak{F}$, i.e., the modal algebra $\langle P, \cap, -, \top, \Box_I, \bigcirc_1, \ldots, \bigcirc_n\rangle$, is denoted by $\mathfrak{F}^\dagger$. Conversely, for a topological Boolean algebra with $n$ operators $\mathfrak{A} = \langle A, \wedge, -, \top, \Box_I, \bigcirc_1, \ldots, \bigcirc_n\rangle$ (which validates the S4-axioms), we define its dual $\mathfrak{A}_\dagger = \langle W, R_I, \bigcirc'_1, \ldots, \bigcirc'_n, P\rangle$ in almost the same way as in Sec. 2; the only difference is that now

$$xR_I y \text{ iff } \forall a \in A\ (\Box_I a \in x \Rightarrow a \in y).$$

Again we have $\mathfrak{A} \simeq (\mathfrak{A}_\dagger)^\dagger$ and call a quasi-CM-frame $\mathfrak{F}$ *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^\dagger)_\dagger$. It should be clear that all CM-logics are characterized by corresponding varieties of topological Boolean algebras with operators and, hence, by suitable classes of (descriptive) quasi-CM-frames.

Let $t$ be the translation of $\mathcal{LM}_n$ into $\mathcal{LM}'_n$ which prefixes $\Box_I$ to every subformula of a given $\mathcal{LM}_n$-formula. To show that $t$ is an embedding of IM-logics (in $\mathcal{LM}_n$) in CM-logics (in $\mathcal{LM}'_n$), we need operators transforming quasi-IM-frames to quasi-CM-frames and back. Those are generalizations of the operators $\sigma$ and $\rho$ defined in [27]. Since the number of modal operators is not essential, for simplicity we will be considering the monomodal language $\mathcal{LM}$ with operator $\bigcirc$.

Given a quasi-IM-frame $\mathfrak{F} = \langle W, R, \bigcirc, P\rangle$, we construct a quasi-CM-frame $\sigma\mathfrak{F} = \langle W, R_I, \sigma\bigcirc, \sigma P\rangle$ by taking $R_I = R$, $\sigma P$ to be the Boolean closure of $P$ and $\sigma \bigcirc X = \bigcirc \Box_I X$, for every $X \in \sigma P$. It is well known [27] that $\Box_I X \in P$ for every $X \in \sigma P$ (see [22]). Therefore, $\sigma P$ is closed under $\sigma\bigcirc$ and so $\sigma\mathfrak{F}$ is a quasi-CM-frame indeed. Moreover, $\Box_I \sigma \bigcirc \Box_I X = \Box_I \bigcirc \Box_I \Box_I X = \bigcirc \Box_I X = \sigma \bigcirc X$ for all $X \in \sigma P$. It follows that the formula

$$Mix = \Box_I \bigcirc \Box_I p \leftrightarrow \bigcirc p$$

*Actually, Gödel used a somewhat different translation, but it is equivalent to $t$ as far as only S4 and its normal extensions are concerned.

is valid in $\sigma\mathfrak{F}$. We also know that $\langle W, R_I, \sigma P\rangle$ validates the monomodal Grzegorczyk logic Grz $=$ S4 $\oplus$ $\Box_I(\Box_I(p \to \Box_I p) \to p) \to p$. To sum up, we obtain

LEMMA 16. If $\mathfrak{F}$ is a quasi-IM-frame, then $\sigma\mathfrak{F}$ is a quasi-CM-frame validating $Mix$ and Grz.

Conversely, let $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ be a quasi-CM-frame. From it, we construct a quasi-IM-frame $\rho\mathfrak{F}$ by, first, modifying $\bigcirc$ so that the resulting frame $\mathfrak{F}^*$ would validate $Mix$ (and the same $t$-translations of IM-formula as $\mathfrak{F}$), and then collapsing clusters in $\mathfrak{F}^*$ into single points and converting the result to a quasi-IM-frame in the standard way (see [22]).

Define an operation $\bigcirc^*$ on $P$ by setting $\bigcirc^* X = \Box_I \bigcirc \Box_I X$ for every $X \in P$, and put $\mathfrak{F}^* = \langle W, R_I, \bigcirc^*, P\rangle$.

LEMMA 17. If $\mathfrak{F}$ is a quasi-CM-frame, then

(i) $\mathfrak{F}^*$ is a quasi-CM-frame also;

(ii) $\mathfrak{F}^* \models Mix$;

(iii) for every $\mathcal{CM}$-formula $\varphi$, $\mathfrak{F}^* \models t(\varphi)$ iff $\mathfrak{F} \models t(\varphi)$.

Proof. Clauses (i) and (ii) are trivial; (iii) is proved by a straightforward induction on the complexity of $\varphi$.

Now assume that a quasi-CM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ validates $Mix$. Denote by $[x]$ the cluster containing $x$, i.e., $[x] = \{y \in W : xR_Iy \text{ and } yR_Ix\}$, and put

$$[X] = \{[x] : x \in X\},$$

$$[x][R_I][y] \text{ iff } xR_Iy,$$

$$[P] = \{[X] : \bigcup[X] \in P\},$$

$$[\bigcirc][X] = \{[x] : x \in \bigcirc(\bigcup[X])\}.$$

The structure $[\mathfrak{F}] = \langle [W], [R_I], [\bigcirc], [P]\rangle$ is called the *skeleton* of $\mathfrak{F}$.

LEMMA 18. If $\mathfrak{F}$ is a quasi-CM-frame validating $Mix$, then

(i) $[\mathfrak{F}]$ is also a quasi-CM-frame, with $[\mathfrak{F}]^\dagger$ a subalgebra of $\mathfrak{F}^\dagger$;

(ii) $[R_I]$ is a partial order on $[W]$;

(iii) for every $\mathcal{CM}$-formula $\varphi$,

$$\mathfrak{F} \models t(\varphi) \text{ iff } [\mathfrak{F}] \models t(\varphi).$$

Proof. (i) It is well known that the map $x \mapsto [x]$ is a p-morphism of the S4-frame $\langle W, R_I, P\rangle$ onto $\langle [W], [R_I], [P]\rangle$. Therefore, the map $f : [X] \mapsto \bigcup[X]$ is an injection of $[P]$ into $P$ preserving $\Box_I$. So it remains to show that $f$ preserves the second modal operator. Obviously, we have $f([\bigcirc][X]) = \bigcup\{[x] : x \in \bigcirc(\bigcup[X])\} \supseteq \bigcirc(\bigcup[X]) = \bigcirc(f([X]))$. And the reverse inclusion follows from $Mix$. Indeed, $\bigcirc(\bigcup[X]) = \Box_I \bigcirc \Box_I(\bigcup[X])$, and so the whole cluster $[x]$ lies in $\bigcirc(\bigcup[X])$ whenever one of its points does.

Clause (ii) is obvious; (iii) is established by induction.

Finally, given an arbitrary quasi-CM-frame $\mathfrak{F}$, we first form the frame $[\mathfrak{F}^*] = \langle W, R_I, \bigcirc, P\rangle$, and then transform it to a quasi-IM-frame $\rho\mathfrak{F} = \langle W, R, \bigcirc, \rho P\rangle$ by taking $R = R_I$ and $\rho P = \{\Box_I X : X \in P\}$. If we drop $\bigcirc$, $\rho$ will be just the standard operator converting S4-frames to Int-frames. By $Mix$, $\bigcirc$ maps cones to cones, and so $\rho\mathfrak{F}$ is a quasi-IM-frame. Using induction on the complexity of $\varphi$ and Lemmas 17, 18, it is easy to prove the following:

LEMMA 19. For every $\mathcal{CM}$-formula $\varphi$ and every quasi-CM-frame $\mathfrak{F}$,

$$\mathfrak{F} \models t(\varphi) \text{ iff } \rho\mathfrak{F} \models \varphi.$$

We also have

LEMMA 20. $\mathfrak{F} \simeq \rho\sigma\mathfrak{F}$ for every quasi-IM-frame $\mathfrak{F}$.

Now we are in a position to embed IM-logics $L$ in extensions of $S4 \otimes C$, where S4 is treated in the language with $\Box_I$ and C is treated in the language with $\bigcirc$ (with the modal operators of $L$, to be more exact). We say that a CM-logic $M$ is a CM-*companion* of $L$ and $L$ is the IM-*fragment* of $M$ if, for all $\mathcal{CM}$-formulas $\varphi$,

$$\varphi \in L \text{ iff } t(\varphi) \in M.$$

It is easy to see that, for every extension $M$ of $S4 \otimes C$ (in $\mathcal{CM}'$), the set

$$\rho M = \{\varphi \in \mathcal{CM} : t(\varphi) \in M\}$$

is the IM-fragment (in $\mathcal{CM}$) of $M$, and that $\rho$ is a homomorphism of the lattice of CM-logics onto the lattice of IM-logics.

Proposition 21. If a CM-logic M is characterized by a class $\mathcal{C}$ of quasi-CM-frames, the $\rho M$ is characterized by the class $\rho\mathcal{C} = \{\rho\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.

The proof follows from Lemma 19.

The theorem given below describes an (infinite) family of CM-companions of each consistent IM-logic.

THEOREM 22. Every logic $M$ in the interval

$$[(S4 \otimes C) \oplus t(\Gamma), (Grz \otimes C) \oplus t(\Gamma) \oplus Mix]$$

is a CM-companion of the IM-logic $L = \mathrm{IntC} \oplus \Gamma$, where $\Gamma$ is a set of $\mathcal{CM}$-formulas.

Proof. Suppose $\varphi \notin L$. Then there is a quasi-IM-frame $\mathfrak{F}$ for $L$ refuting $\varphi$. By Lemmas 19 and 20, we have $\sigma\mathfrak{F} \not\models t(\varphi)$ and $\sigma\mathfrak{F} \models t(\Gamma)$. By Lemma 16, $\sigma\mathfrak{F} \models Grz$ and $\sigma\mathfrak{F} \models Mix$. Thus, we obtain $\sigma\mathfrak{F} \models M$ and $\sigma\mathfrak{F} \not\models t(\varphi)$, whence $\varphi \notin \rho M$.

Conversely, if $\varphi \notin \rho M$, then $t(\varphi) \notin M$, and so there is a quasi-CM-frame $\mathfrak{F}$ for $M$ refuting $t(\varphi)$. By Lemma 19, $\rho\mathfrak{F} \not\models \varphi$ and $\rho\mathfrak{F} \models \Gamma$. So $\varphi \notin L$.

Example 23. 1. If an extension $M$ of S4 is a modal companion of the intermediate logic $\mathrm{Int} + \Gamma$, then $M \otimes C$ is a CM-companion of $\mathrm{IntC} \oplus \Gamma$.* (For we have $M = M' \oplus t(\Gamma)$ for some $M'$ in the interval [S4, Grz], and so $M \otimes C = (M' \otimes C) \oplus t(\Gamma)$.) In particular, $S4 \otimes C$, $S4.1 \otimes C$, and $Grz \otimes C$ are CM-companions of IntC.

2. $S4 \otimes (C \oplus \bigcirc\top)$ is a CM-companion of $\mathrm{IntC} \oplus \bigcirc\top$. This follows from the inclusions

$$(S4 \otimes C) \oplus t(\bigcirc\top) \subseteq S4 \otimes (C \oplus \bigcirc\top) \subseteq (S4 \oplus C) \oplus Mix \oplus t(\bigcirc\top).$$

3. $S4 \otimes (C \oplus \bigcirc p \to p)$ is a CM-companion of $\mathrm{IntC} \oplus \bigcirc p \to p$. The proof is analogous.

4. Each IM-logic $L = \mathrm{IntR} \oplus \Gamma$ is embeddable via $t$ in any logic in the interval $[(S4 \otimes R) \oplus t(\Gamma), (Grz \otimes R) \oplus Mix \oplus t(\Gamma)]$. Indeed, let $\phi = \bigcirc(p \wedge q) \to \bigcirc p$. Then the claim follows from the inclusions

$$(S4 \otimes C) \oplus t(\Gamma) \oplus t(\phi) \subseteq (S4 \otimes R) \oplus t(\Gamma) \subseteq (S4 \oplus C) \oplus Mix \oplus t(\Gamma) \oplus t(\phi),$$

which are established by a simple syntactical argument.

5. Each IM-logic $L = \mathrm{IntK}_\Box \oplus \Gamma$ is embeddable via $t$ in any logic in the interval $[(S4 \otimes K) \oplus t(\Gamma), (Grz \otimes K) \oplus Mix \oplus t(\Gamma)]$. The proof is similar (for details, consult [27]).

*Here + presupposes taking the closure only under modus ponens and substitution.

It is worth noting that every CM-companion $M$ of an IM-logic $L$ can be reduced, in a sense, to a CM-companion of $L$ containing $Mix$. We say that a CM-logic $M'$ is a *Mix-reduct* of a CM-logic $M$ if $Mix \in M'$, and for every formula $\varphi$, $\varphi \in M'$ iff $r(\varphi) \in M$, where $r$ replaces each occurrence of $\bigcirc$ in $\varphi$ with $\Box_I \bigcirc \Box_I$. Then, by Lemma 17, for each CM-companion $M$ of an IM-logic $L$, there exists a $Mix$-reduct $M'$ of $M$ such that $\rho M' = L$ (if $M$ is characterised by a frame $\mathfrak{F}$ then $M'$ can be defined as a logic of $\mathfrak{F}^*$).

As far as CM-companions with $Mix$ are concerned, we can get a correspondence similar to one between intermediate logics and their modal companions above S4 (see [6]). Indeed, the logic $(\mathbf{S4} \otimes \mathbf{C}) \oplus t(\Gamma) \oplus Mix$ is clearly the smallest CM-companion with $Mix$ for an IM-logic $L = \mathbf{IntC} \oplus \Gamma$; we denote it by $\tau L$. Now we want to show that the greatest CM-companion of $L$ containing $Mix$ is the logic $\sigma L = (\mathbf{Grz} \otimes \mathbf{C}) \oplus t(\Gamma) \oplus Mix$. To this end we need the following lemma concerning monomodal frames for Grz in the language with $\Box$.

LEMMA 24. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a model based upon a partially ordered frame $\mathfrak{F} = \langle W, R, P \rangle$ for Grz and let $\Gamma$ be a finite set of formulas closed under subformulas. Then there is a model $\mathfrak{N} = \langle \sigma\rho\mathfrak{F}, \mathfrak{U} \rangle$ (based upon the frame $\sigma\rho\mathfrak{F} = \langle W, R, \sigma\rho P \rangle$) such that, for every $\varphi \in \Gamma$, $\mathfrak{V}(\Box\varphi) = \mathfrak{U}(\Box\varphi)$.

Proof. It is enough to show that there exists a valuation $\mathfrak{U}$ in $\sigma\rho\mathfrak{F}$ such that $\mathfrak{V}(\Diamond\varphi) = \mathfrak{U}(\Diamond\varphi)$ for all $\varphi \in \Gamma$. To construct it, we first apply to $\mathfrak{M}$ and $\Gamma$ the selection procedure introduced in [29]. As a result we obtain a finite model $\mathfrak{M}^* = \langle \mathfrak{F}^*, \mathfrak{V}^* \rangle$ and a cofinal subreduction $f$ of $\mathfrak{F}$ to $\mathfrak{F}^* = \langle W^*, R^* \rangle$ satisfying the following properties:

(i) $\mathfrak{F}^*$ is a partial order (since $\mathfrak{F} \models \mathbf{Grz}$);

(ii) $\forall x \in \mathrm{dom} f\ \forall \varphi \in \Gamma\ (x \in \mathfrak{V}(\varphi) \Leftrightarrow f(x) \in \mathfrak{V}^*(\varphi))$;

(iii) $\forall x \in W - \mathrm{dom} f\ \exists y \in \mathrm{dom} f\ (xRy \wedge x \sim_\Gamma y)$* (from which it follows that $f$ satisfies the closed domain condition for the set $\mathfrak{D}^*$ of closed domains in $\mathfrak{M}^*$).

With each point $v \in W^*$ we associate the set

$$X_v = \Diamond f^{-1}(v) - \bigcup_{\neg v R^* u} \Diamond f^{-1}(u).$$

Since $f^{-1}(v) \in P$, it follows immediately from the definition that $X_v \in \sigma\rho P$, $f^{-1}(v) \subseteq X_v$, and $f^{-1}(v)$ is a cover for $X_v$. Then, for every $x \in W$, we put

$$g(x) = \begin{cases} v & \text{if } x \in X_v,\ v \in W^*; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One can readily check (see [28] or [7]) that $g$ is a cofinal subreduction of $\sigma\rho\mathfrak{F}$ to $\mathfrak{F}^*$ satisfying the closed domain condition for $\mathfrak{D}^*$.

Now we define a valuation $\mathfrak{U}$ in $\sigma\rho\mathfrak{F}$ in the same way as was done in the proof of Proposition 9 in [29]. Namely, for every $x \in \mathrm{dom} g$ and every variable $p$, put

$$x \in \mathfrak{U}(p) \ \text{ iff } \ g(x) \in \mathfrak{V}^*(p).$$

And if $x \notin \mathrm{dom} g$, then by (iii), there is $y \in \mathrm{dom} g$ for which $xRy$ and $x \sim_\Gamma y$. For every $z \notin \mathrm{dom} g$ such that $g(\{u : xRu\}) = g(\{u : zRu\})$, we then put

$$x \in \mathfrak{U}(p) \ \text{ iff } \ g(y) \in \mathfrak{V}^*(p).$$

Let $\mathfrak{N} = \langle \sigma\rho\mathfrak{F}, \mathfrak{U} \rangle$. By Proposition 9 in [29], for every $\varphi \in \Gamma$ we have

*Here $x \sim_\Gamma y$ means that the same formulas in $\Gamma$ are true at $x$ and $y$ in $\mathfrak{M}$.

if $x \in \mathrm{dom}g$, then $x \in \mathfrak{U}(\varphi)$ iff $g(x) \in \mathfrak{V}^*(\varphi)$;

if $x \notin \mathrm{dom}g$, then there is $y \in \mathrm{dom}g$ such that $xRy$ and $x \in \mathfrak{U}(\varphi)$ iff $y \in \mathfrak{U}(\varphi)$.

The claim of our lemma follows immediately from these properties, and from (ii) and (iii).

Now we prove the following:

LEMMA 25. Let $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ be a quasi-CM-frame for Grz in the language with $\Box_I$ such that $R_I$ is a partial order and $\mathfrak{F} \models Mix$. Then, for all $\varphi \in \mathcal{LM}'$, we have

$$\mathfrak{F} \models \varphi \text{ iff } \sigma\rho\mathfrak{F} \models \varphi.$$

Proof. The implication $\mathfrak{F} \models \varphi \Rightarrow \sigma\rho\mathfrak{F} \models \varphi$ follows from $\sigma\rho P \subseteq P$.

Conversely, assume that $\mathfrak{F}$ refutes $\varphi$. For each subformula $\bigcirc\psi$ of $\varphi$, we fix a new variable $q(\bigcirc\psi)$ and put

$$\begin{array}{rcl} \chi^q & = & \chi, \ \ \chi \text{ is atomic}, \\ (\chi_1 \rightarrow \chi_2)^q & = & \chi_1^q \rightarrow \chi_2^q, \\ (\chi_1 \wedge \chi_2)^q & = & \chi_1^q \wedge \chi_2^q, \\ (\chi_1 \vee \chi_2)^q & = & \chi_1^q \vee \chi_2^q, \\ (\Box_I \chi)^q & = & \Box_I \chi^q, \\ (\bigcirc\chi)^q & = & \Box_I q(\bigcirc\chi). \end{array}$$

Let $\Gamma = \{\psi^q : \psi \in \mathrm{Sub}\varphi\}$, where $\mathrm{Sub}\varphi$ is the set of $\varphi$'s subformulas, and let $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$ be a model refuting $\varphi$. Define a valuation $\mathfrak{U}$ of the extended language in $\mathfrak{F}$ by setting

$$\mathfrak{U}(p) = \mathfrak{V}(p) \text{ for every variable } p \in \mathrm{Sub}\varphi,$$

$$\mathfrak{U}(q(\bigcirc\psi)) = \mathfrak{V}(\bigcirc\psi) \text{ for } \bigcirc\psi \in \mathrm{Sub}\varphi.$$

By $Mix$, $\mathfrak{V}(\bigcirc\psi)$ is a cone with respect to $R_I$, and so for $\psi \in \mathrm{Sub}\varphi$ and $x \in W$, we clearly have

$$\mathfrak{V}(\psi) = \mathfrak{U}(\psi) = \mathfrak{U}(\psi^q). \quad (13)$$

By Lemma 24, there exists a valuation $\mathfrak{U}'$ in $\sigma\rho\mathfrak{F}$ such that, for all $\psi \in \Gamma$,

$$\mathfrak{U}'(\Box_I\psi) = \mathfrak{U}(\Box_I\psi). \quad (14)$$

Now we use induction to prove that for all $\psi \in \mathrm{Sub}\varphi$,

$$\mathfrak{U}'(\psi^q) = \mathfrak{U}'(\psi). \quad (15)$$

The only nontrivial case is with $\bigcirc\psi$. We have

$$\begin{array}{rcll} \mathfrak{U}'((\bigcirc\psi)^q) & = & \mathfrak{U}((\bigcirc\psi)^q) & \text{(by (14))} \\ & = & \mathfrak{U}(\Box_I \bigcirc \Box_I \psi^q) & \text{(by (13) and } Mix) \\ & = & \mathfrak{U}'(\Box_I \bigcirc \Box_I \psi^q) & \text{(by (14))} \\ & = & \mathfrak{U}'(\Box_I \bigcirc \Box_I \psi) & \text{(by IH)} \\ & = & \mathfrak{U}'(\bigcirc\psi) & \text{(by } Mix). \end{array}$$

It follows from (13), (14), and (15) that $\varphi$ is refuted in $\sigma\rho\mathfrak{F}$.

**LEMMA 26.** Each CM-logic $L$ containing $(\mathrm{Grz} \otimes \mathrm{C}) \oplus \mathit{Mix}$ is characterized by a quasi-CM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$, where $R_I$ is a partial order.

**Proof.** Consider a descriptive quasi-CM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ which determines $L$. We say that a point $x \in W$ is *eliminable* in $\mathfrak{F}$ if it has a proper $R_I$-successor in every set $X \in P$ containing $x$. Put $W' = \{x \in W : x \text{ is noneliminable in } \mathfrak{F}\}$ and $P' = \{X \cap W' : X \in P\}$. One can readily check that the structure $\mathfrak{F}' = \langle W', R_I \restriction W', \bigcirc', P'\rangle$, where $\bigcirc'(X \cap W') = W' \cap \bigcirc X$, is a quasi-CM-frame such that $\mathfrak{F}^{\dagger} \simeq \mathfrak{F}'^{\dagger}$ and $R_I \restriction W'$ is a partial order (for details, see [11] or [7]).

Now we are in a position to prove an analog of the Blok–Esakia theorem for IM-logics and their CM-companions containing $\mathit{Mix}$.

**THEOREM 27.** A CM-logic $M$ containing $\mathit{Mix}$ is a CM-companion of an IM-logic $L$ iff $\tau L \subseteq M \subseteq \sigma L$.

**Proof.** ($\Leftarrow$) Follows from Theorem 22.

($\Rightarrow$) It suffices to show that $M \subseteq \sigma L$. First we prove that

$$\{\rho\mathfrak{F} : \mathfrak{F} \models M\} = \{\mathfrak{G} : \mathfrak{G} \models L\} \tag{16}$$

(of course, we do not distinguish between isomorphic frames). We will need Birkhoff's characterization of varieties, and therefore it will be more convenient to consider frames as algebras, that is, establish the equality $\{(\rho\mathfrak{F})^{\dagger} : \mathfrak{F} \models M\} = \{\mathfrak{G}^{\dagger} : \mathfrak{G} \models L\}$.

By Proposition 21, $L$ is characterized by the class $C = \{(\rho\mathfrak{F})^{\dagger} : \mathfrak{F} \models M\}$, and so it suffices to show that $C$ is closed under forming direct products, subalgebras, and homomorphic images. That $C$ is closed under the first two operations can be shown in the same way as in [19]. To prove the closure under homomorphisms, assume that $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ is a quasi-CM-frame for $M$ and $h$ is a homomorphism from $(\rho\mathfrak{F})^{\dagger}$ onto $\mathfrak{H}^{\dagger}$. Since $(\sigma\rho\mathfrak{F})^{\dagger}$ is a subalgebra of $\mathfrak{F}^{\dagger}$, it follows that $(\sigma\rho\mathfrak{F})^{\dagger} \models M$. Besides, by Lemma 20 we have $(\rho\sigma\mathfrak{H})^{\dagger} \simeq \mathfrak{H}^{\dagger}$. These facts are all that we need to construct a homomorphism $g$ from $(\sigma\rho\mathfrak{F})^{\dagger}$ onto $(\sigma\mathfrak{H})^{\dagger}$, and then we shall have $\sigma\mathfrak{H} \models M$. Every set $X \in \sigma\rho P$ can be represented as

$$X = \bigcap_{i=1}^{n}(-Y_i \cup Z_i)$$

for some $Y_i, Z_i \in \rho P$. Define $g$ by setting

$$g(X) = \bigcap_{i=1}^{n}(-h(Y_i) \cup h(Z_i)).$$

Clearly, $g(X)$ is an element in $(\sigma\mathfrak{H})^{\dagger}$ which coincides with $h(X)$ for every $X \in \rho P$. It was shown in [19] that $g$ is a surjection that preserves the Boolean operations and $\Box_I$. We prove that it preserves $\bigcirc$ as well. Using $\mathit{Mix}$ we have

$$\begin{aligned} g(\bigcirc X) &= g(\bigcirc\Box_I \bigcap_{i=1}^{n}(-Y_i \cup Z_i)) \\ &= g(\bigcirc \bigcap_{i=1}^{n} \Box_I(-Y_i \cup Z_i)) \\ &= g(\bigcirc \bigcap_{i=1}^{n}(Y_i \supset Z_i)) \\ &= h(\bigcirc \bigcap_{i=1}^{n}(Y_i \supset Z_i)) \end{aligned}$$

$$= \bigcirc\bigcap_{i=1}^{n}(h(Y_i) \supset h(Z_i))$$
$$= \bigcirc g(X).$$

Now, to prove that $M \subseteq \sigma L$, it suffices to show that a characteristic frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ for $\sigma L$ is also a frame for $M$. By Lemma 26, without loss of generality we may assume that $R_I$ is a partial order. In view of $\rho\mathfrak{F} \models L$ and (16), there is a frame $\mathfrak{F}'$ for $M$ for which $\rho\mathfrak{F} \simeq \rho\mathfrak{F}'$, and so $\sigma\rho\mathfrak{F} \simeq \sigma\rho\mathfrak{F}'$. Clearly, $\sigma\rho\mathfrak{F}' \models M$. Therefore, $\sigma\rho\mathfrak{F} \models M$, and by Lemma 25, $\mathfrak{F} \models M$.

COROLLARY 28. The map $\sigma$ is an isomorphism from the lattice of IM-logics onto the lattice of CM-logics containing $(\mathbf{Grz} \otimes \mathbf{C}) \oplus Mix$.

**Remark.** It is worth noting that the analogy with the Blok–Esakia theorem is not complete if we consider CM-logics without $Mix$. For, as has been observed by C. Grefe, there is an IM-logic $L$ and its CM-companion $M$ (without $Mix$) such that $L \neq \rho(M \oplus Mix)$. This means that there are at least two maximal logics in $\rho^{-1}L$.

Proposition 29. If an IM-logic $L$ is characterized by a class $\mathcal{C}$ of quasi-IM-frames, then $\sigma L$ is characterized by the class $\sigma\mathcal{C} = \{\sigma\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.

Proof. If $\mathfrak{F} \models L$ then $\sigma\mathfrak{F} \models t(L)$ by Lemmas 19 and 20, and $\sigma\mathfrak{F}$, as is known, validates the Grzegorczyk formula in the monomodal language with $\Box_I$. Hence $\mathfrak{F} \models \sigma L$. Now assume that $\varphi \notin \sigma L$ and consider the logic $\sigma L \oplus \varphi$. By Theorem 27, $\rho(\sigma L \oplus \varphi)$ is a proper extension of $L$, and so there is a formula $\psi \notin L$ for which $\sigma L \oplus \varphi = \sigma L \oplus t(\psi)$. Take any frame $\mathfrak{F} \in \mathcal{C}$ separating $\psi$ from $L$. Then, by Lemmas 19 and 20, $\sigma\mathfrak{F}$ will separate $t(\psi)$ and, hence, $\varphi$ from $\sigma L$.

THEOREM 30. The map $\rho$ preserves decidability, the finite model property, and tabularity. The map $\sigma$ preserves the finite model property and tabularity.

Proof. That $\rho$ preserves decidability follows directly from the definition of $\rho$, and the rest — from Propositions 21, 29 and the fact that $\rho\mathfrak{F}$ is a finite IM-frame if $\mathfrak{F}$ is a finite CM-frame and $\sigma\mathfrak{F}$ is finite whenever $\mathfrak{F}$ is finite.

This preservation result provides us with a tool for establishing the finite model property (FMP for short) of IM-logics by means of proving it for suitable CM-companions. For example, we have

THEOREM 31. Suppose that an intermediate logic $\mathbf{Int} + \Gamma$ has FMP. Then FMP is shared by the following IM-logics:

$\mathbf{IntC} \oplus \Gamma$, $\mathbf{IntC} \oplus \Gamma \oplus \bigcirc\top$, $\mathbf{IntC} \oplus \Gamma \oplus \bigcirc p \to p$;

$\mathbf{IntR} \oplus \Gamma$, $\mathbf{IntR} \oplus \Gamma \oplus \bigcirc\top$, $\mathbf{IntR} \oplus \Gamma \oplus \bigcirc p \to p$;

$\mathbf{IntK_\Box} \oplus \Gamma$, $\mathbf{IntK_\Box} \oplus \Gamma \oplus \bigcirc p \to p$.

Proof. By Theorem 30, it suffices to present CM-companions of these logics with FMP. Example 23 shows that the logics under consideration have CM-companions of the form $(\mathbf{S4} \oplus t(\Gamma)) \otimes L$, where $L$ is a monomodal classical logic in the list

$$\{\mathbf{C}, \mathbf{C} \oplus \bigcirc\top, \mathbf{C} \oplus \bigcirc p \to p, \mathbf{R}, \mathbf{R} \oplus \bigcirc\top, \mathbf{R} \oplus \bigcirc p \to p, \mathbf{K}, \mathbf{K} \oplus \bigcirc p \to p\}.$$

All the above-listed logics are known to have the global FMP in the sense that for any formulas $\varphi$ and $\psi$, if there is a model $\mathfrak{M}$ based on a frame for $L$ such that $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \not\models \psi$, then there is a finite model with the same properties. The claim of the theorem now follows from the two preservation results obtained in [28] and [12]. Namely, (i) if $\mathbf{Int} + \Gamma$ has FMP then $\mathbf{S4} \oplus t(\Gamma)$ has the global FMP, and (ii) if two classical monomodal logics $L_1$ and $L_2$ have the global FMP then $L_1 \otimes L_2$ has it as well.

For further results on the finite model property of IM-logics, see [27].

## 4. CM-COMPANIONS OF $\Box\Diamond$-IM-LOGICS

Now we focus attention on CM-companions of extensions of $\mathbf{IntK}_{\Box\Diamond}$. According to Example 23, all extensions of $\mathbf{IntK}_{\Box}$ are embedded via $t$ in normal bimodal logics. However, nothing guarantees that extensions of $\mathbf{IntK}_{\Diamond}$ and, more generally, arbitrary $\Box\Diamond$-IM-logics can be embedded in normal CM-logics. The reason is that although the $t$-translation of $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$ is deductively equal to itself in $(\mathbf{S4} \otimes \mathbf{C}) \oplus Mix$, this is not the case for the $t$-translation of $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$, which will be denoted by $t\Diamond$: modulo $Mix$, it is deductively equal only to $\Diamond(\Box_I p \vee \Box_I q) \leftrightarrow \Diamond p \vee \Diamond q$. This is another important difference between $\Diamond$-like and $\Box$-like operators in intuitionistic modal logic, which reflects the nonstandard behavior of generated subframes and p-morphisms.

We proceed to formulate a version of the Blok–Esakia theorem for $\Box\Diamond$-IM-logics. Put

$$\Phi_1 = \{\Diamond(\Box_I p \vee \Box_I q) \leftrightarrow \Diamond p \vee \Diamond q, \neg\Diamond\bot\},$$

$$\Phi_2 = \{\Box p \leftrightarrow \Box_I \Box \Box_I p, \Diamond p \leftrightarrow \Box_I \Diamond \Box_I p\},$$

$$\Phi = \Phi_1 \cup \Phi_2.$$

As a consequence of Theorem 22, Example 23, and Corollary 28, we obtain

**THEOREM 32.** Each $\Box\Diamond$-IM-logic $\mathbf{IntK}_{\Box\Diamond} \oplus \Gamma$ is embeddable via $t$ in any logic in the interval

$$[(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus t\Diamond \oplus t(\neg\Diamond\bot) \oplus t(\Gamma), (\mathbf{Grz} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi \oplus t(\Gamma)],$$

where $\mathbf{S4}$ is formulated in the language with $\Box_I$, $\mathbf{K}$ in the language with $\Box$, and $\mathbf{R}$ in the language with $\Diamond$. The map $\sigma$, restricted to the lattice of $\Box\Diamond$-IM-logics, is an isomorphism of that lattice onto the lattice of extensions of $(\mathbf{Grz} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi$.

**Example 33.** Using $\Phi_2$, one can easily show that for all $k, l, m, n \geq 0$, the logic $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi \oplus \Diamond^k \Box^l p \to \Box^m \Diamond^n p$ is a CM-companion of $\mathbf{IntK}_{\Box\Diamond} \oplus \Diamond^k \Box^l p \to \Box^m \Diamond^n p$.

It is worth mentioning that although logics containing $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi$ are not necessarily normal (in fact, these are normal only if $\Diamond$ is almost trivial), they have a rather natural relational semantics with a nonstandard truth-condition for $\Diamond$, viz., frames of the form $\mathfrak{F} = \langle W, R_I, R_\Box, R_\Diamond, P\rangle$ such that $\langle W, R_I, P\rangle$ is an S4-frame, $R_\Box$ and $R_\Diamond$ satisfy conditions (1) and (2), respectively, and $P$ is closed under the usual $\Box$ and the unusual $\Diamond$:

$$\Diamond X = \{x \in W : \exists y \in \Box_I X \; x R_\Diamond y\}.$$

By adapting the Stone–Jónsson–Tarski argument to this case, we can show that the defined semantics is adequate for logics containing $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi$.

We do not know whether all $\Box\Diamond$-IM-logics have normal (with respect to $\Diamond$) CM-companions. (We conjecture that this is not the case.) But complete logics do have them.

Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond\rangle$ be a full weak $\Box\Diamond$-IM-frame. We can also treat it as a frame for the language with three modal operators $\Box_I$, $\Box$, and $\Diamond$ satisfying the classical truth-conditions. The classical modal logic with these operators, characterized by the class $\mathcal{C}$ of such frames, is denoted by $\mathrm{Th}\mathcal{C}$.

By (11), every full weak $\Box\Diamond$-IM-frame $\mathfrak{F}$ validates $\Diamond\Box_I p \to \Box_I \Diamond p$. We also have

LEMMA 34. For every full weak $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_{\Box}, R_{\Diamond}\rangle$ and for every $\varphi$ in the language with $\Box$ and $\Diamond$,

$$\mathfrak{F} \models \varphi \text{ iff } \mathfrak{F} \models t(\varphi),$$

where the former $\models$ is intuitionistic, while the latter is classical.

Proof. ($\Rightarrow$) Let $\mathfrak{V}$ be an intuitionistic valuation in $\mathfrak{F}$. We can treat it also as a classical valuation, denoted $\mathfrak{U}$. By induction on the complexity of $\varphi$, we show that $\mathfrak{V}(\varphi) = \mathfrak{U}(t(\varphi))$.

The nontrivial cases are $\varphi = \Box\psi$ and $\varphi = \Diamond\psi$. Suppose $x \notin \mathfrak{V}(\Box\psi)$. Then there are $y \in W$ and $z \notin \mathfrak{V}(\psi)$ such that $xRyR_{\Box}z$. By the induction hypothesis, $z \notin \mathfrak{U}(t(\psi))$, and so $x \notin \mathfrak{U}(t(\Box\psi))$ because $t(\Box\psi) = \Box_I\Box t(\psi)$. Let $x \notin \mathfrak{U}(\Box_I\Box t(\psi))$. Then $xRyR_{\Box}z$ for some $y \in W$ and $z \notin \mathfrak{U}(t(\psi))$, whence $x \notin \mathfrak{V}(\Box\psi)$.

Now let $x \in \mathfrak{V}(\Diamond\psi)$, i.e., $xR_{\Diamond}y$ for some $y \in \mathfrak{V}(\psi)$, and let $x \notin \mathfrak{U}(\Box_I\Diamond t(\psi))$, i.e., for some $z$, $xRz$ holds, and no $R_{\Diamond}$-successor of $z$ is in $\mathfrak{U}(t(\psi))$. By (11), we have a point $u$ for which $yRu$ and $zR_{\Diamond}u$. Then $u \in \mathfrak{V}(\psi)$ (since $\mathfrak{V}(\psi)$ is a cone) and $u \notin \mathfrak{U}(t(\psi))$, contrary to the induction hypothesis. Conversely, if $x \in \mathfrak{U}(\Box_I\Diamond t(\psi))$, in view of $xR_Ix$, then, there is $y \in \mathfrak{U}(t(\psi))$ such that $xR_{\Diamond}y$; hence, $x \in \mathfrak{V}(\Diamond\psi)$.

($\Leftarrow$) Given a classical valuation $\mathfrak{U}$ in $\mathfrak{F}$, we define an intuitionistic valuation $\mathfrak{V}$ by setting $\mathfrak{V}(p) = \mathfrak{U}(\Box_I p)$ for every variable $p$, and in exactly the same way as above, prove that $\mathfrak{V}(\varphi) = \mathfrak{U}(t(\varphi))$.

THEOREM 35. Suppose that a $\Box\Diamond$-IM-logic $L = \mathbf{IntK}_{\Box\Diamond} \oplus \Gamma$ is characterized by a class $C$ of full weak $\Box\Diamond$-IM-frames. Then $L$ is embedded via $t$ in every logic $M$ in the interval

$$[(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{K}) \oplus \Diamond\Box_I p \to \Box_I\Diamond p \oplus t(\Gamma), \mathrm{Th}C].$$

Proof. Let $\varphi \notin L$. Then there is $\mathfrak{F} \in C$ separating $\varphi$ from $L$. By Lemma 34, $\mathfrak{F} \not\models t(\varphi)$, and so $t(\varphi) \notin \mathrm{Th}C$. On the other hand, it is readily checked that the $t$-translations of the axioms of $\mathbf{IntK}_{\Box\Diamond}$ are in $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{K}) \oplus \Diamond\Box_I p \to \Box_I\Diamond p$, and hence $t(\varphi) \in M$ whenever $\varphi \in L$.

Some consequences of Theorem 35 for FS-logics are worth noting. Fischer Servi [14, 15] proposed a somewhat different embedding $t'$ of a few complete FS-logics in bimodal classical logics (in the language with $\Box_I$ and $\Box$) containing $\Box_I\Box p \to \Box\Box_I p$ and $\Diamond'\Box_I p \to \Box_I\Diamond' p$, where $\Diamond'$ is dual to $\Box$, i.e., $\Diamond'\varphi = \neg\Box\neg\varphi$. Namely, she defined

$$t'(\Box\varphi) = \Box_I\Box t'(\varphi), \quad t'(\Diamond\varphi) = \Diamond' t'(\varphi).$$

It turns out, however, that in fact $t'$ is a special case of $t$ in the framework of complete FS-logics. Indeed, let $L = \mathbf{FS} \oplus \Gamma$ be a complete FS-logic. By Proposition 15, it is characterized by a class of full birelational frames (in which the relations for $\Box$ and $\Diamond$ coincide). It follows from Theorem 35 and (12) that $L$ is embedded via $t$ in the logic

$$(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{K}) \oplus \Box_I\Box p \to \Box\Box_I p \oplus \Diamond\Box_I p \to \Box_I\Diamond p \oplus \Diamond' p \leftrightarrow \Diamond p \oplus t(\Gamma),$$

where again $\Diamond'$ is dual to $\Box$. Identifying $\Diamond'$ and $\Diamond$, we then obtain

COROLLARY 36. Each complete FS-logic $L = \mathbf{FS} \oplus \Gamma$ is embedded via $t'$ in $(\mathbf{S4} \otimes \mathbf{K}) \oplus \Box_I\Box p \to \Box\Box_I p \oplus \Diamond'\Box_I p \to \Box_I\Diamond' p \oplus t'(\Gamma)$.

However, it is not clear whether all FS-logics are embedded via $t'$ in bimodal logics of this type.

## REFERENCES


1. G. Amati and F. Pirri, "A uniform tableau method for intuitionistic modal logics. I," *Studia Logica*, 53, 29-60 (1994).

2. J. F. van Benthem, "Canonical modal logics and ultrafilter extensions," *J. Symb. Log.*, 44, 1-8 (1979).

3. J. F. van Benthem, *Modal Logic and Classical Logic*, Napoli (1983).

4. W. J. Blok, Varieties of Interior Algebras, University of Amsterdam (1976), Dissertation.

5. M. Božić and K. Došen, "Models for normal intuitionistic logics," *Studia Logica*, 43, 217-245 (1984).

6. A. V. Chagrov and M. V. Zakharyaschev, "Modal companions of intermediate propositional logics," *Studia Logica*, 51, 49-82 (1992).

7. A. V. Chagrov and M. V. Zakharyaschev, *Modal and Superintuitionistic Logics*, Oxford Univ. Press (1996).

8. M. A. E. Dummett and E. J. Lemmon, "Modal logics between S4 and S5," *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 5, 250-264 (1959).

9. L. L. Esakia, "On varieties of Grzegorczyk algebras," in *Studies in Non-Classical Logics and Set Theory*, Nauka, Moscow (1979), pp. 257-287.

10. W. B. Ewald, "Intuitionistic tense and modal logic," *J. Symb. Log.*, 51, 166-179 (1986).

11. K. Fine, "Logics containing K4. II," *J. Symb. Log.*, 50, 619-651 (1985).

12. K. Fine and G. Schurz, "Transfer theorems for stratified multimodal logics," in *Logic and Reality. Essays in Pure and Applied Logic*, J. Copeland (ed.), Oxford Univ. Press (1996), pp. 169-213.

13. G. Fischer Servi, "On modal logics with an intuitionistic base," *Studia Logica*, 36, 141-149 (1977).

14. G. Fischer Servi, "Semantics for a class of intuitionistic modal calculi," in *Italian Studies in the Philosophy of Science*, M. L. Dalla Chiara (ed.), Reidel, Dordrecht (1980), pp. 59-72.

15. G. Fischer Servi, "Axiomatizations for some intuitionistic modal logics," *Rend. Sem. Mat. Univ. Polit.*, 42, 179-194 (1984).

16. K. Gödel, "Eine Interpretation des intuitionistischen Aussagenkalkuls," *Ergebnisse eines mathematischen Kolloquiums*, 6, 39-40 (1933).

17. R. I. Goldblatt, "Metamathematics of modal logic," *Reports on Mathematical Logic*, 6, 41-77 (1976).

18. E. J. Lemmon and D. S. Scott, *An Introduction to Modal Logic*, Blackwell, Oxford (1977).

19. L. Maksimova and V. Rybakov, "Lattices of modal logics," *Algebra Logika*, 13, No. 2, 188-216 (1974).

20. H. Ono, "Some results on the intermediate logics," *Publ. Research Inst. Math. Science*, 8, 117-130 (1972).

21. H. Ono, "On some intuitionistic modal logics," *Publ. Research Inst. Math. Science*, 13, 55-67 (1977).

22. H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, Polish Scientific Publishers (1963).

23. H. Sahlqvist, "Completeness and correspondence in the first and second order semantics for modal logic," in *Proc. Third Scandinavian Logic Symposium*, S. Kanger (ed.), North-Holland, Amsterdam (1975), pp. 110-143.

24. K. Segerberg, *Classical Propositional Operators*, Clarendon, Oxford (1982).

25. V. B. Shehtman, "Kripke type semantics for propositional modal logics with intuitionistic base," in *Modal and Tense Logics* [In Russian], V. A. Smirnov (ed.), Institute of Philosophy, AN SSSR (1979), pp. 108-112.

26. V. H. Sotirov, "Modal theories with intuitionistic logic," in *Proc. Conf. Math. Logic, Sofia, 1980*, Bulgarian Academy of Sciences (1984), pp. 139-171.

27. F. Wolter and M. Zakharyaschev, "Intuitionistic modal logics as fragments of classical bimodal logics," to appear in *Studia Logica.*

28. M. V. Zakharyaschev, "Syntax and semantics of intermediate logics," *Algebra Logika*, 28, No. 4, 402-429 (1989).

29. M. V. Zakharyaschev, "Canonical formulas for K4. I: Basic results," *J. Symb. Log.*, 57, 1377-1402 (1992).