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JORDAN BIALGEBRAS AND THEIR RELATION TO LIE BIALGEBRAS

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We develop the notion of Jordan bialgebras and study the way in which such are related to Lie bialgebras. In particular, it is shown that if a Lie algebra $L(J)$ obtained from a Jordan algebra J by applying the Kantor–Koecher–Tits construction admits the structure of a Lie bialgebra, under some natural constraints, then, J permits the structure of a Jordan algebra.

Hopf algebras [1] that exemplify associative bialgebras take on considerable importance in the theory of associative algebras. Those are associative algebras on which the structure of an associative coalgebra is given so as to fit in well with the initial multiplication. Hopf algebras are closely connected with objects such as quantum groups and Lie bialgebras. The latter were introduced in [2], where they aimed at a study of solutions for the classical Yang–Baxter equation. Like associative bialgebras, Lie bialgebras are Lie algebras with Lie comultiplication which is a 1-cocycle. The notion of a Lie coalgebra was defined in [3]. The definition of a coalgebra related to a certain variety of algebras was given in [4]; in particular, a Jordan coalgebra was defined as one whose dual is a Jordan algebra. In [5], based on the analogy with the known Kantor–Koecher–Tits construction as it applies to ordinary algebras, we established the relationship between Jordan coalgebras and Lie coalgebras.

In the present article, we define Jordan bialgebras and study the way in which they are related to Lie bialgebras. In particular, it will be shown that if a Lie algebra $L(J)$ obtained from a Jordan algebra J by applying the KKT construction admits the structure of a Lie bialgebra, then, under some natural constraints, J affords the structure of a Jordan bialgebra.

1. ASSOCIATIVE AND JORDAN BIALGEBRAS ACCORDING TO DRINFELD

Let Φ be a field. For linear spaces U and V over Φ, denote by $U \otimes V$ their tensor product over Φ. On the space $V \otimes V$, define a linear map τ by setting $\tau(\sum_i a_i \otimes b_i) = (\sum_i b_i \otimes a_i)$. As usual, id is the identity map of V, and we write $V^\star$ for the space of all linear functionals on V. For elements $f \in V^\star$ and $v \in V$, $\langle f, v\rangle$ expresses a value of the linear functional f at v. Let $\rho : V^\star \otimes V^\star \longrightarrow (V \otimes V)^\star$ be a linear map defined by $\langle \rho(f \otimes g), \sum_{i,j} a_i \otimes b_j\rangle = \sum_{i,j}\langle f, a_i\rangle\langle g, b_j\rangle$. Similarly we specify the linear map $\rho_1 : V^\star \otimes V^\star \otimes V^\star \longrightarrow (V \otimes V \otimes V)^\star$.

Definition. A pair (A, Δ), where A is a linear space over Φ and $\Delta : A \longrightarrow A \otimes A$ is a linear map, is called a *coalgebra*.

If the pair (A, Δ) is a coalgebra, the map Δ is referred to as *comultiplication*. For an element a in A, if $\Delta(a) = \sum a_{1i} \otimes a_{2i}$, we write $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$, following Sweedler (see [1]).

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Consider the space $A^\star$ and define multiplication on $A^\star$ by setting

$$\langle fg, a\rangle = \sum_a \langle f, a_{(1)}\rangle\langle g, a_{(2)}\rangle,$$

where $f, g \in A^\star$, $a \in A$, and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$. The space $A^\star$ with the assigned multiplication is then an ordinary algebra over Φ, which we call the *dual* of the coalgebra (A, Δ). Obviously, $\langle fg, a\rangle = \langle \rho(f \otimes g), \Delta(a)\rangle$.

The dual $A^\star$ of (A, Δ) determines the bimodule action $(\cdot)$ on A, defined as follows: $f \cdot a = \sum_a a_{(1)}\langle f, a_{(2)}\rangle$ and $a \cdot f = \sum_a \langle f, a_{(1)}\rangle a_{(2)}$, where $f \in A^\star$ and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$. For any $f, g \in A^\star$ and $a \in A$, $\langle fg, a\rangle = \langle f, g \cdot a\rangle = \langle g, a \cdot f\rangle$ holds.

Now let A be some algebra with comultiplication Δ and let $A^\star$ be its dual. The algebra A induces the bimodule action $(\bullet)$ on the space $A^\star$, defined by $\langle f \bullet a, b\rangle = \langle f, ab\rangle$ and $\langle b \bullet f, a\rangle = \langle f, ab\rangle$. Consider the space $B(A, A^\star) = A \oplus A^\star$, on which we define multiplication $(\star)$ by setting

$$(a + f) \star (b + g) = (ab + f \cdot b + a \cdot g) + (fg + f \bullet b + a \bullet g).$$

Then $B(A, A^\star)$ is an ordinary algebra over Φ.

Suppose that M is some variety of Φ-algebras and A is an algebra in M, with comultiplication Δ. A pair (A, Δ) is then called an *M-bialgebra* — in the sense of Drinfeld — if the algebra $B(A, A^\star)$ belongs to M. In this case, (A, Δ) is an M-coalgebra (see [4]).

In what follows, associative and Jordan bialgebras as are defined by Drinfeld will be called D-*bialgebras*.

Let A be an associative Φ-algebra. On the space $A \otimes A$, define the action of A by setting $a(b \otimes c) = (b \otimes ac)$ and $(b \otimes c)a = (ba \otimes c)$. Under that action, $A \otimes A$ is an associative A-bimodule. If $a \in A$ and $b \in A \otimes A$, we put $[a, b] = ab - ba$.

THEOREM 1. Let A be an associative Φ-algebra with comultiplication Δ. Then the pair (A, Δ) is an associative D-bialgebra if and only if (A, Δ) is an associative coalgebra and Δ satisfies the equalities

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \tag{1}$$

$$[a, \tau\Delta(b)] = -\tau([b, \tau\Delta(a)]). \tag{2}$$

Proof. Let (A, Δ) be an associative D-bialgebra. Consider the algebra $B(A, A^\star)$. Since it is associative, for any elements a, b in A and f in $A^\star$ we have $(f \star a) \star b = f \star (ab)$. It follows that $(f \cdot a)b + (f \bullet a) \cdot b = f \cdot (ab)$. Consequently, if $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = \sum_b b_{(1)} \otimes b_{(2)}$, then $f \cdot (ab) = \sum_a a_{(1)}b\langle f, a_{(2)}\rangle + \sum_b b_{(1)}\langle f, ab_{(2)}\rangle$.

Let g be an arbitrary element in $A^\star$. In view of the latter equality, then, we obtain

$$\langle \rho(g \otimes f), \Delta(ab)\rangle = \langle g, f \cdot ab\rangle = \langle \rho(g \otimes f), \sum_a a_{(1)}b \otimes a_{(2)} + \sum_b b_{(1)} \otimes ab_{(2)}\rangle.$$

Since the space $\rho(A^\star \otimes A^\star)$ is dense in $(A \otimes A)^\star$, it is clear that $\Delta(ab) = \Delta(a)b + a\Delta(b)$, and the first equality is thereby proved.

The second equality is obtained by applying a similar argument to $(a \star f) \star b = a \star (f \star b)$.

Conversely, assume that (A, Δ) is an associative coalgebra and that Δ satisfies (1) and (2). We show that $B(A, A^\star)$ is an associative algebra. Let f, g and a, b be arbitrary elements in $A^\star$ and A, respectively. Suppose $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = \sum_b b_{(1)} \otimes b_{(2)}$. In view of (1), then, $f \cdot (ab) = \sum_a a_{(1)}b\langle f, a_{(2)}\rangle +$

$\sum_b b_{(1)}\langle f, ab_{(2)}\rangle$. It follows that $f\cdot(ab) = (f\cdot a)b + (f\bullet a)\cdot b$. Therefore, $f\star(a\star b) = (f\star a)\star b$. The equality $(a\star b)\star f = a\star(b\star f)$ is derived similarly. Since

$$\langle g, (a\cdot f)b + (a\bullet f)\cdot b\rangle = \langle\rho(f\otimes g), \sum_a a_{(1)}\otimes a_{(2)}b + \sum_b b_{(2)}a\otimes b_{(1)}\rangle$$

and

$$\langle g, a(f\cdot b) + a\cdot(f\bullet b)\rangle = \langle\rho(f\otimes g), \sum_a ba_{(1)}\otimes a_{(2)} + \sum_b b_{(2)}\otimes ab_{(1)}\rangle,$$

by (2) we obtain $(a\cdot f)b + (a\bullet f)\cdot b = a(f\cdot b) + a\cdot(f\bullet b)$. This gives $(a\star f)\star b = a\star(f\star b)$.

The equalities $(f\star g)\star a = f\star(g\star b)$, $(a\star f)\star g = a\star(f\star g)$, and $(f\star a)\star g = f\star(a\star g)$ can be proved similarly.

Now, it is rather obvious that $B(A, A^*)$ is associative. Consequently, (A, Δ) is an associative D-bialgebra.

It should be observed that Theorem 1 is an associative analog of Theorem 2 in [2] and that the equality $[a, \tau(\Delta(b))] = -\tau[b, \tau\Delta(a)]$ is equivalent to

$$\sum_a a_{(1)}\otimes a_{(2)}b - ba_{(1)}\otimes a_{(2)} = \sum_b b_{(2)}\otimes ab_{(1)} - b_{(2)}a\otimes b_{(1)}, \tag{3}$$

where $\Delta(a) = \sum_a a_{(1)}\otimes a_{(2)}$ and $\Delta(b) = \sum_b b_{(1)}\otimes b_{(2)}$.

We give examples of associative D-algebras. Consider the ring $\Phi[x]$ of polynomials in one variable. On the space $\Phi[x]$, define Δ by setting $\Delta(1) = 0$ and $\Delta(x^n) = \sum_{i=1}^n x^i\otimes x^{n+1-i}$. Then

$$\Delta(x^n)x^m + x^n\Delta(x^m) = \sum_{i=1}^n x^{i+m}\otimes x^{n+1-i} + \sum_{i=1}^m x^i\otimes x^{n+m+1-i} =$$

$$\sum_{i=m+1}^{n+m} x^i\otimes x^{n+m+1-i} + \sum_{i=1}^m x^i\otimes x^{n+m+1-i} = \Delta(x^{n+m}).$$

It is not hard to infer, then, that the pair $(\Phi[x], \Delta)$ is an associative coalgebra. If $m\geq n$, then

$$\sum_{i=1}^m x^{m+1-i}\otimes x^{n+i} - x^{m+n+1-i}\otimes x^i = \sum_{i=1}^n x^i\otimes x^{m+n+1-i} +$$

$$\sum_{i=n+1}^m x^i\otimes x^{m+n+1-i} - \sum_{i=1}^n x^{m+n+1-i}\otimes x^i - \sum_{i=n+1}^m x^{m+n+1-i}\otimes x^i =$$

$$\sum_{i=1}^n x^i\otimes x^{m+n+1-i} - x^{m+i}\otimes x^{n+1-i}.$$

Therefore, $(\Phi[x], \Delta)$ is an associative D-bialgebra by Theorem 1.

Let A be an associative algebra and x and y be linearly independent elements of A. Assume $x^2 = y^2 = xy = yx = 0$. Define comultiplication Δ on the space A by setting $\Delta(a) = [r, a]$, where $a\in A$ and $r = x\otimes y - y\otimes x$. Then $\Delta(ab) = \Delta(a)b + a\Delta(b)$. Verification of the equality $[a, \tau\Delta(b)] = -\tau([b, \tau\Delta(a)])$ is overt. Comultiplication Δ is coassociative, that is, $(\Delta\otimes\mathrm{id} - \mathrm{id}\otimes\Delta)\Delta(a) = 0$ for all $a\in A$. Indeed, the condition on x and y yields $\Delta(x) = \Delta(y) = 0$, and so

$$(\Delta\otimes\mathrm{id} - \mathrm{id}\otimes\Delta)\Delta(a) =$$

$$\Delta(xa)\otimes y - \Delta(ya)\otimes x + x\otimes\Delta(ay) - y\otimes\Delta(ax) =$$

$$-x \otimes xay \otimes y + y \otimes xax \otimes y + x \otimes yay \otimes x - y \otimes yax \otimes x +$$

$$x \otimes xay \otimes y - x \otimes yay \otimes x - y \otimes xax \otimes y + y \otimes yax \otimes x = 0.$$

This implies that, for $n \geq 3$, the algebra Φ_n of $n \times n$-matrices can be endowed with the structure of an associative D-bialgebra. If $Z(\Phi_n)$ is the center of Φ_n, then $\Delta(Z(\Phi_n)) = 0$.

In the above-given examples of associative D-bialgebras, comultiplication Δ satisfies $\Delta(z) = \tau(\Delta(z))$ for any central element z. If the latter equality holds for some D-bialgebra, we say that its comultiplication is *cocommutative on its center.*

We bring in a definition of a Lie bialgebra in terms of comultiplication. Let L be a Lie algebra over Φ. Consider the linear space $L \otimes L$. The space $L \otimes L$, as is known, is a Lie L-bimodule provided that the action of L on $L \otimes L$ is defined by $[l_1 \otimes l_2, l] = [l_1, l] \otimes l_2 + l_1 \otimes [l_2, l]$. Suppose that on L, comultiplication Δ is given. The pair (L, Δ) is a Lie bialgebra if and only if (L, Δ) is a Lie coalgebra and Δ is a 1-cocycle, that is, it satisfies $\Delta([a, b]) = [\Delta(a), b] + [a, \Delta(b)]$ (see [2]).

For Jordan D-bialgebras over a field Φ of characteristic not 2, 3, an analog of Theorem 1 is the following:

THEOREM 2. Let J be a Jordan Φ-algebra with comultiplication Δ. Then the pair (J, Δ) is a Jordan D-bialgebra if and only if (J, Δ) is a Jordan coalgebra and Δ satisfies the following equalities:

$$\begin{aligned}(\Delta \otimes \mathrm{id} - \mathrm{id} \otimes \Delta)\Delta(a^2) = 2(1 \otimes a \otimes 1)(\Delta \otimes \mathrm{id} - \mathrm{id} \otimes \Delta)\Delta(a) +\\ 2(1 \otimes 1 \otimes a - a \otimes 1 \otimes 1)(\mathrm{id} \otimes \tau)(\Delta \otimes \mathrm{id})\Delta(a) +\\ 2(\Delta(a) \otimes 1 - 1 \otimes \Delta(a))(\mathrm{id} \otimes \tau)(\Delta(a) \otimes 1),\end{aligned} \tag{4}$$

$$\begin{aligned}(\Delta \otimes \mathrm{id} + \mathrm{id} \otimes \Delta + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \Delta))(1 \otimes a + a \otimes 1)\Delta(a) =\\ 2((1 \otimes a \otimes 1)(\mathrm{id} \otimes \Delta)\Delta(a) + (a \otimes 1 \otimes 1)(\mathrm{id} \otimes \tau)(\Delta \otimes \mathrm{id})\Delta(a) +\\ (1 \otimes \Delta(a))(\mathrm{id} \otimes \tau)(\Delta(a) \otimes 1)) + (\Delta \otimes \mathrm{id})\Delta(a^2),\end{aligned} \tag{5}$$

$$\begin{aligned}\Delta(a^2 b) - \Delta(a^2)(b \otimes 1) - \Delta(b)(1 \otimes a^2) + 2\Delta(b)(a \otimes a) -\\ 2\Delta(ab)(a \otimes 1) + 2(\Delta(a)(b \otimes 1))(a \otimes 1) +\\ 2(\Delta(a)(1 \otimes b))(1 \otimes a) - 2\Delta(a)(1 \otimes ab) = 0.\end{aligned} \tag{6}$$

Proof. We follow the same line of argument as was used to prove Theorem 1. If 1 is unity in J, then (6) yields $\Delta(1) = 0$.

Let (A, Δ) be an associative D-bialgebra. Consider an adjoint Lie algebra $A^{(-)}$ (an adjoint Jordan algebra $A^{(+)}$, $1/2 \in \Phi$). On the space $A^{(-)}$ ($A^{(+)}$), define comultiplication $\Delta^{(-)}$ ($\Delta^{(+)}$) by setting $\Delta^{(-)}(a) = \Delta(a) - \tau(\Delta(a))$ ($\Delta^{(+)}(a) = \frac{1}{2}(\Delta(a) + \tau(\Delta(a)))$. Then the pair $(A^{(-)}, \Delta^{(-)})$ is a Lie bialgebra ($(A^{(+)}, \Delta^{(+)})$ is a Jordan D-bialgebra).

2. JORDAN D-BIALGEBRAS OF TYPE $A^{(+)}$

Let Φ be a field of characteristic $\neq 2$ and let (A, Δ) be an associative D-bialgebra. Our present goal is to show that a Lie algebra obtained by applying the Kantor–Koecher–Tits construction to $A^{(+)}$ admits the structure of a Lie bialgebra.

We recall how the KKT process goes for Jordan algebras. Let J be a Jordan Φ-algebra with unity 1. For an element $a \in J$, denote by a' an operator of right multiplication by a. The linear subspace of

$\mathrm{End}_\Phi(J)$ generated by operators a' is denoted $R(J)$. The linear map $d : J \longrightarrow J$ is called a *derivation* of J if $(ab)d = adb + a(bd)$ for any elements a and b in J. If $a, b \in J$, put $[a', b'] = a'b' - b'a'$ and $a \bigtriangledown b = (ab)' - [a', b']$. It is well known that a linear combination of operators of the form $[a', b']$ is a derivation of J. Such derivations are called *inner*.

Let $\mathrm{Der}\,(J)$ ($\mathrm{Intder}\,(J)$) be a linear space of all derivations (inner derivations) of J. Then the space $\mathrm{Der}\,(J)$ is a Lie algebra under commutation and, moreover, $\mathrm{Intder}\,(J)$ is an ideal in $\mathrm{Der}\,(J)$. Now let D be a Lie subalgebra of $\mathrm{Der}\,(J)$ containing $\mathrm{Intder}\,(J)$ and let $\overline{J}$ be an isomorphic copy of the space J. Consider the linear space

$$L(J) = J \oplus R(J) \oplus D \oplus \overline{J}.$$

First, on the space $L(J)$, define the linear map ε by the rule $\varepsilon(a + b' + d + \overline{c}) = c - b' + d + \overline{a}$, where $a, b, c \in J$ and $d \in D$.

Next, define multiplication $[\, ,\,]_L$ on $L(J)$ by setting

$$[a_1 + V_1 + \overline{b}_1, a_2 + V_2 + \overline{b}_2]_L = a_1 V_2 - a_2 V_1 + \overline{b_1 \varepsilon(V_2)} - \overline{b_2 \varepsilon(V_1)} +$$

$$a_1 \bigtriangledown b_2 - a_2 \bigtriangledown b_1 + V_1 V_2 - V_2 V_1,$$

where $a_i, b_i \in J$ and $V_i \in R(J) \oplus D$, $i = 1, 2$. Then the space $L(J)$ with the assigned multiplication will be a $-1,1$-graded Lie algebra. Denote by L_i components of $L(J)$, where $i = -1, 0, 1$. Write U for a simple three-dimensional subalgebra of $L(J)$ generated by elements 1, $1'$, and $\overline{1}$. The map $\varepsilon : L(J) \longrightarrow L(J)$ is an automorphism of $L(J)$.

Assume that, on the space $L(J)$, comultiplication Δ_L is defined. We say that Δ_L *agrees* with the grading of the algebra $L(J)$ if $\Delta(L_i) \subset \sum_{j+k=i} L_j \otimes L_k$. Let Δ be comultiplication given on the space J. We say that Δ_L is *associated* with Δ if, for any element $a \in J$, $\Delta_L(a) \in \sum_a a_{(1)} \otimes a'_{(2)} - a_{(2)} \otimes a'_{(1)} + L_1 \otimes D + D \otimes L_1$, where $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$.

Let A be an associative algebra with unity 1 and let $A^{(+)}$ be its adjoint Jordan algebra. Denote by D_a an inner derivation of A given by a. The Lie algebra of all inner derivations of A is denoted $\mathrm{Intder}\,(A)$. Any element in $\mathrm{Intder}\,(A)$ is a derivation of $A^{(+)}$, and $\mathrm{Intder}\,(A^{(+)}) \subseteq \mathrm{Intder}\,(A)$. Suppose that $L(A^{(+)})$ is a Lie algebra obtained by applying the KKT construction to $A^{(+)}$ and to $\mathrm{Intder}\,(A)$. Write $[\, ,\,]_L$ to denote the bimodule action of the algebra $L(A^{(+)})$ on the space $L(A^{(+)}) \otimes L(A^{(+)})$; multiplication in $A^{(+)}$ is denoted $(\circ)$. As usual, $[\, ,\,]$ stands for multiplication in $A^{(-)}$. The following is then valid:

THEOREM 3. Let (A, Δ) be an associative D-bialgebra and let $(A^{(+)}, \Delta^{(+)})$ be an adjoint Jordan D-bialgebra. Assume that comultiplication Δ is cocommutative on the center of A. Then the algebra $L(A^{(+)})$ can be endowed with the structure of a Lie bialgebra with comultiplication Δ_L, such that Δ_L agrees with the grading of $L(A^{(+)})$, Δ_L is associated with $\Delta^{(+)}$, and $\Delta_L(U) = 0$.

Proof. By R $[D]$ we denote a linear map of the space A into $R(A^{(+)})$ ($\mathrm{Intder}\,(A)$), given by $R(a) = a'$ ($D(a) = D_a$). Let $a, b \in A$, $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$, and $\Delta(b) = \sum_b b_{(1)} \otimes b_{(2)}$. In view of (3), then, the space $L(A^{(+)}) \otimes L(A^{(+)})$ satisfies the following equalities:

$$\sum_a a'_{(1)} \otimes (a_{(2)} b)' - (b a_{(1)})' \otimes a'_{(2)} = \sum_b b'_{(2)} \otimes (a b_{(1)})' - (b_{(2)} a)' \otimes b'_{(1)}, \tag{7}$$

$$\sum_a D_{a_{(1)}} \otimes D_{a_{(2)} b} - D_{b a_{(1)}} \otimes D_{a_{(2)}} = \sum_b D_{b_{(2)}} \otimes D_{a b_{(1)}} - D_{b_{(2)} a} \otimes D_{b_{(1)}}, \tag{8}$$

$$\sum_a a_{(1)} \otimes (a_{(2)}b)' - ba_{(1)} \otimes a'_{(2)} = \sum_b b_{(2)} \otimes (ab_{(1)})' - b_{(2)}a \otimes b'_{(1)}, \tag{9}$$

$$\sum_a a'_{(1)} \otimes a_{(2)}b - (ba_{(1)})' \otimes a_{(2)} = \sum_b b'_{(2)} \otimes ab_{(1)} - (b_{(2)}a)' \otimes b_{(1)}, \tag{10}$$

$$\sum_a a_{(1)} \otimes D_{a_{(2)}b} - ba_{(1)} \otimes D_{a_{(2)}} = \sum_b b_{(2)} \otimes D_{ab_{(1)}} - b_{(2)}a \otimes D_{b_{(1)}}, \tag{11}$$

$$\sum_a D_{a_{(1)}} \otimes a_{(2)}b - D_{ba_{(1)}} \otimes a_{(2)} = \sum_b D_{b_{(2)}} \otimes ab_{(1)} - D_{b_{(2)}a} \otimes b_{(1)}, \tag{12}$$

$$\sum_a \bar{a}_{(1)} \otimes a_{(2)}b - \overline{ba}_{(1)} \otimes a_{(2)} = \sum_b \bar{b}_{(2)} \otimes ab_{(1)} - \overline{b_{(2)}a} \otimes b_{(1)}, \tag{13}$$

$$\sum_a a_{(1)} \otimes \overline{a_{(2)}b} - ba_{(1)} \otimes \bar{a}_{(2)} = \sum_b b_{(2)} \otimes \overline{ab}_{(1)} - b_{(2)}a \otimes \bar{b}_{(1)}. \tag{14}$$

These were obtained by applying to both parts of (3) the operators $R \otimes R$, $D \otimes D$, $\mathrm{id} \otimes R$, $R \otimes \mathrm{id}$, $\mathrm{id} \otimes D$, $D \otimes \mathrm{id}$, $\varepsilon \otimes \mathrm{id}$, and $\mathrm{id} \otimes \varepsilon$, respectively.

If $x, y \in L(A^{(+)})$, we put $\{x, y\} = x \otimes y - y \otimes x$. Obviously, $[\{x,y\}, z]_L = \{[x,z]_L, y\} + \{x, [y,z]_L\}$. Now let $a \in A$ and $\Delta^{(+)}(a) = \frac{1}{2}\sum_a a_{(1)} \otimes a_{(2)} + a_{(2)} \otimes a_{(1)}$. Then comultiplication Δ_L is defined by setting

$$2\Delta_L(a) = \sum_a (\{a_{(1)}, a'_{(2)}\} + \{a_{(2)}, a'_{(1)}\}) + \frac{1}{2}\sum_a (\{a_{(2)}, D_{a_{(1)}}\} + \{D_{a_{(2)}}, a_{(1)}\}),$$

$$2\Delta_L(a') = \sum_a (\{a_{(1)}, \bar{a}_{(2)}\} + \{a_{(2)}, \bar{a}_{(1)}\}) + \frac{1}{2}\sum_a (\{a'_{(2)}, D_{a_{(1)}}\} + \{D_{a_{(2)}}, a'_{(1)}\}),$$

$$2\Delta_L(\bar{a}) = \sum_a (\{a'_{(2)}, \bar{a}_{(1)}\} + \{a'_{(1)}, \bar{a}_{(2)}\}) + \frac{1}{2}\sum_a (\{\bar{a}_{(2)}, D_{a_{(1)}}\} + \{D_{a_{(2)}}, \bar{a}_{(1)}\}),$$

$$\Delta_L(D_a) = \sum_a (\{a_{(1)}, \bar{a}_{(2)}\} - \{a_{(2)}, \bar{a}_{(1)}\} - \{a'_{(1)}, a'_{(2)}\} - \frac{1}{4}\{D_{a_{(1)}}, D_{a_{(2)}}\}).$$

Since Δ is cocommutative on the center of A, the map Δ_L is well defined. The map Δ_L agrees with the grading of the algebra $L(A^{(+)})$, Δ_L is associated with $\Delta^{(+)}$, and $\Delta_L(U) = 0$. For any element $a \in L(A^{(+)})$, the equality $(\varepsilon \otimes \varepsilon)(\Delta_L(a)) = \Delta_L(\varepsilon(a))$ holds.

We give a number of technical lemmas which help us prove Theorem 3.

LEMMA 1. For any a, b in A,

$$\Delta_L([a,b]_L) = [\Delta_L(a), b]_L + [a, \Delta_L(b)]_L$$

and

$$\Delta_L([\bar{a}, \bar{b}]_L) = [\Delta_L(\bar{a}), \bar{b}]_L + [\bar{a}, \Delta_L(\bar{b})]_L.$$

Proof. Let us derive the first equality. Suppose $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ and $\Delta(b) = \sum_b b_{(1)} \otimes b_{(2)}$. We state, first, that a D-bialgebra (A, Δ) satisfies the equality

$$4\Delta^{(+)}(a \circ b) = 4(\Delta^{(+)}(a) \circ (b \otimes 1) + \Delta^{(+)}(b) \circ (1 \otimes a)) - \Delta^{(-)}(a)(\mathrm{id} \otimes D_b) + \Delta^{(-)}(b)(D_a \otimes \mathrm{id}).$$

Indeed, by the definition of $\Delta^{(+)}$, $4\Delta^{(+)}(a\circ b)=\Delta(ab)+\Delta(ba)+\tau(\Delta(ab))+\tau(\Delta(ba))$. Therefore,

$$4\Delta^{(+)}(a\circ b)=\sum_a a_{(1)}b\otimes a_{(2)}+\sum_b b_{(1)}\otimes ab_{(2)}+\sum_b b_{(1)}a\otimes b_{(2)}+\sum_a a_{(1)}\otimes ba_{(2)}+$$

$$\sum_a a_{(2)}\otimes a_{(1)}b+\sum_b ab_{(2)}\otimes b_{(1)}+\sum_b b_{(2)}\otimes b_{(1)}a+\sum_a ba_{(2)}\otimes a_{(1)}.$$

By (3), we obtain

$$\sum_a a_{(2)}\otimes a_{(1)}b=\sum_a a_{(2)}\otimes [a_{(1)},b]+\sum_a a_{(2)}\otimes ba_{(1)}=$$

$$\sum_a a_{(2)}\otimes [a_{(1)},b]+\sum_a a_{(2)}b\otimes a_{(1)}+\sum_b b_{(1)}\otimes b_{(2)}a-\sum_b ab_{(1)}\otimes b_{(2)}.$$

Similarly,

$$\sum_a a_{(1)}\otimes ba_{(2)}=\sum_a a_{(1)}\otimes [b,a_{(2)}]+\sum_b b_{(2)}\otimes ab_{(1)}-\sum_b b_{(2)}a\otimes b_{(1)}+\sum_a ba_{(1)}\otimes a_{(2)}.$$

This gives the desired equality.

Now, take elements $[\Delta_L(a),b]_L$ and $[a,\Delta_L(b)]_L$. Since

$$[2\Delta_L(a),b]_L=-\sum_a(\{a_{(1)},a_{(2)}\circ b\}+\{a_{(2)},a_{(1)}\circ b\})-\frac{1}{2}\sum_a(\{a_{(2)},[b,a_{(1)}]\}+\{[b,a_{(2)}],a_{(1)}\})$$

and

$$[a,2\Delta_L(b)]_L=\sum_b(\{b_{(1)},b_{(2)}\circ a\}+\{b_{(2)},b_{(1)}\circ a\})+\frac{1}{2}\sum_b(\{b_{(2)},[a,b_{(1)}]\}+\{[a,b_{(2)}],b_{(1)}\}),$$

the above equality yields $[\Delta_L(a),b]_L+[a,\Delta_L(b)]_L=\Delta^{(+)}(a\circ b)-\Delta^{(+)}(b\circ a)=0$. The first equality is proved because $[a,b]_L=0$. The second equality is easily obtained from the first by applying the map $\varepsilon\otimes\varepsilon$.

LEMMA 2. For any a and b in A, we have

$$\Delta_L([a,b']_L)=[\Delta_L(a),b']_L+[a,\Delta_L(b')]_L$$

and

$$\Delta_L([\bar{a},b']_L)=[\Delta_L(\bar{a}),b']_L+[\bar{a},\Delta_L(b')]_L.$$

Proof. To prove the first equality, let $\Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}$ and $\Delta(b)=\sum_b b_{(1)}\otimes b_{(2)}$. It is then clear that

$$4\Delta_L(a\circ b)=\sum_a(\{a_{(1)}b,a'_{(2)}\}+\{a_{(2)},(a_{(1)}b)'\}+\{a_{(1)},(ba_{(2)})'\}+\{ba_{(2)},a'_{(1)}\})+$$

$$\frac{1}{2}\sum_a(\{a_{(2)},D_{a_{(1)}b}\}+\{D_{a_{(2)}},a_{(1)}b\}+\{ba_{(2)},D_{a_{(1)}}\}+\{D_{ba_{(2)}},a_{(1)}\})+$$

$$\sum_b(\{b_{(1)}a,b'_{(2)}\}+\{b_{(2)},(b_{(1)}a)'\}+\{b_{(1)},(ab_{(2)})'\}+\{ab_{(2)},b'_{(1)}\})+$$

$$\frac{1}{2}\sum_b(\{b_{(2)},D_{b_{(1)}a}\}+\{D_{b_{(2)}},ab_{(1)}\}+\{ab_{(2)},D_{b_{(1)}}\}+\{D_{ab_{(2)}},b_{(1)}\}).$$

Since $[a,b']_L=a\circ b$, and for any elements x and y in A, $4[x',y']_L=D_{[x,y]}$ holds, it is routine to verify that

$$4(\Delta_L([a,b']_L)-[\Delta_L(a),b']_L-[a,\Delta_L(b')]_L)=$$

$$\sum_a(\{a_{(2)},(ba_{(1)})'\}-\{ba_{(1)},a'_{(2)}\}+\{a_{(1)},(a_{(2)}b)'\}-\{a_{(2)}b,a'_{(1)}\})+$$

$$\frac{1}{2}\sum_a(\{a_{(2)},D_{ba_{(1)}}\}-\{D_{a_{(2)}},ba_{(1)}\}+\{D_{a_{(2)}b},a_{(1)}\}-\{a_{(2)}b,D_{a_{(1)}}\})+$$

$$\sum_b(\{ab_{(1)},b'_{(2)}\}-\{b_{(2)},(ab_{(1)})'\}+\{b_{(2)}a,b'_{(1)}\}-\{b_{(1)},(b_{(2)}a)'\})+$$

$$\frac{1}{2}\sum_b(\{b_{(2)},D_{ab_{(1)}}\}-\{D_{b_{(2)}},b_{(1)}a\}+\{D_{b_{(2)}a},b_{(1)}\}-\{b_{(2)}a,D_{b_{(1)}}\}).$$

Equalities (9)-(12) imply

$$\Delta_L([a,b']_L)=[\Delta_L(a),b']_L+[a,\Delta_L(b')]_L.$$

Applying the map $\varepsilon\otimes\varepsilon$ to both sides of the latter yields the second equality.

LEMMA 3. For any a and b in A, the following equalities are satisfied:

$$\Delta_L([a,D_b]_L)=[\Delta_L(a),D_b]_L+[a,\Delta_L(D_b)]_L,$$

$$\Delta_L([\overline{a},D_b]_L)=[\Delta_L(\overline{a}),D_b]_L+[\overline{a},\Delta_L(D_b)]_L,$$

$$\Delta_L([a',D_b]_L)=[\Delta_L(a'),D_b]_L+[a',\Delta_L(D_b)]_L,$$

$$\Delta_L([a',b']_L)=[\Delta_L(a'),b']_L+[a',\Delta_L(b')]_L,$$

$$\Delta_L([D_a,D_b]_L)=[\Delta_L(D_a),D_b]_L+[D_a,\Delta_L(D_b)]_L.$$

Proof. Let $\Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}$ and $\Delta(b)=\sum_b b_{(1)}\otimes b_{(2)}$. Then $2\Delta^{(+)}([a,b])=\sum_a(a_{(1)}b\otimes a_{(2)}+a_{(2)}\otimes a_{(1)}b-a_{(1)}\otimes ba_{(2)}-ba_{(2)}\otimes a_{(1)})+\sum_b(b_{(1)}\otimes ab_{(2)}+ab_{(2)}\otimes b_{(1)}-b_{(1)}a\otimes b_{(2)}-b_{(2)}\otimes b_{(1)}a)$. Therefore,

$$2\Delta_L([a,b])=\sum_a(\{a_{(1)}b,a'_{(2)}\}+\{a_{(2)},(a_{(1)}b)'\}-\{a_{(1)},(ba_{(2)})'\}-\{ba_{(2)},a'_{(1)}\})+$$

$$\frac{1}{2}\sum_a(\{a_{(2)},D_{a_{(1)}b}\}+\{D_{a_{(2)}},a_{(1)}b\}-\{D_{ba_{(2)}},a_{(1)}\}-\{ba_{(2)},D_{a_{(1)}}\})+$$

$$\sum_b(\{b_{(1)},(ab_{(2)})'\}+\{ab_{(2)},b'_{(1)}\}-\{b_{(1)}a,b'_{(2)}\}-\{b_{(2)},(b_{(1)}a)'\})+$$

$$\frac{1}{2}\sum_b(\{ab_{(2)},D_{b_{(1)}}\}+\{D_{ab_{(2)}},b_{(1)}\}-\{b_{(2)},D_{b_{(1)}a}\}-\{D_{b_{(2)}},(b_{(1)}a)\}).$$

Since $[a,D_b]_L=[a,b]$, and for any x and y in A, $4[x',y']_L=D_{[x,y]}$, we can verify it overtly that

$$2(\Delta_L([a,D_b]_L)-[\Delta_L(a),D_b]_L-[a,\Delta_L(D_b)]_L)=$$

$$\sum_a(\{ba_{(1)},a'_{(2)}\}+\{a_{(2)},(ba_{(1)})'\}-\{a_{(1)},(a_{(2)}b)'\}-\{ba_{(2)},a'_{(1)}\})+$$

$$\frac{1}{2}\sum_a(\{a_{(2)},D_{ba_{(1)}}\}+\{D_{a_{(2)}},ba_{(1)}\}-\{D_{a_{(2)}b},a_{(1)}\}-\{a_{(2)}b,D_{a_{(1)}}\})+$$

$$\sum_b(-\{b_{(1)},(b_{(2)}a)'\}-\{b_{(2)}a,b'_{(1)}\}+\{ab_{(1)},b'_{(2)}\}+\{b_{(2)},(ab_{(1)})'\})+$$

$$\frac{1}{2}\sum_b(\{b_{(2)}a,D_{b_{(1)}}\}+\{D_{b_{(2)}a},b_{(1)}\}-\{b_{(2)},D_{ab_{(1)}}\}-\{D_{b_{(2)}},ab_{(1)}\}).$$

From (9)-(12), we obtain the equality

$$\Delta_L([a, D_b]_L) = [\Delta_L(a), D_b]_L + [a, \Delta_L(D_b)]_L.$$

If, now, we apply the map $\varepsilon \otimes \varepsilon$ to both sides of the latter we arrive at the second equality envisaged above. Proofs of the last three equalities follow the same route as is one for the first, in which case use must be made of (7), (8), (13), and (14).

LEMMA 4. For any a and b in A, the following equality is satisfied:

$$\Delta_L([a, \bar{b}]_L) = [\Delta_L(a), \bar{b}]_L + [a, \Delta_L(\bar{b})]_L.$$

Proof. Let $a, b \in A$. Then $[a, \bar{b}]_L = (a \circ b)' - [a', b']_L$. For any $c \in A$, we have $\Delta_L([c, \bar{1}]_L) = [\Delta_L(c), \bar{1}]_L$ and $\Delta_L([c', \bar{1}]_L) = [\Delta_L(c'), \bar{1}]_L$. Since $(a \circ b)' = [(a \circ b), \bar{1}]_L$ and $a \circ b = [a, b']_L$, it follows that $\Delta_L((a \circ b)') = [\Delta_L([a, b']_L), \bar{1}]_L$. In view of Lemma 2, $\Delta_L((a \circ b)') = [[\Delta_L(a), b']_L, \bar{1}]_L + [[a, \Delta_L(b')]_L, \bar{1}]_L$, and by the Jacobi identity, we obtain

$$\Delta_L((a \circ b)') = [[\Delta_L(a), \bar{1}]_L, b']_L + [\Delta(a), [b', \bar{1}]_L]_L + [[a, \bar{1}]_L, \Delta_L(b')]_L + [a, [\Delta(b'), \bar{1}]_L]_L.$$

Consequently,

$$\Delta_L((a \circ b)') = [\Delta_L(a'), b']_L + [\Delta(a), \bar{b}]_L + [a', \Delta_L(b')]_L + [a, \Delta(\bar{b})]_L,$$

and by Lemma 3,

$$\Delta_L((a \circ b)') = \Delta_L([a', b']_L) + [\Delta(a), \bar{b}]_L + [a, \Delta(\bar{b})]_L.$$

This gives the desired equality.

LEMMA 5. A pair $(L(A^{(+)}), \Delta_L)$ is a Lie coalgebra.

Proof. Denote by $A^\star$ and $L^\star$ the duals of coalgebras (A, Δ) and $(L(A^{(+)}), \Delta_L)$, respectively. Clearly, $L^\star$ is a $-1,1$-graded algebra. Let $L^\star_{-1}$, $L^\star_{\bar{0}}$, and $L^\star_{\bar{1}}$ be components of $L^\star$. It is not hard to see, then, that $L^\star_{\bar{0}} = V \oplus W$, where the spaces V and W are isomorphic to, respectively, $A^\star$ and $\mathrm{Intder}\,(A)^\star$. It is also evident that $L^\star_{-1}$ and $L^\star_{\bar{1}}$ are isomorphic to the space $A^\star$. For each functional w in W, define the functional $\widehat{w}$ in $A^\star$ by setting $\langle \widehat{w}, a \rangle = \langle w, D_a \rangle$.

We show that $L^\star$ is a Lie algebra. Denote multiplication in $L^\star$ $((A^\star)^{(-)})$ by $[\,,]_\star$ ($[\,,]$). By the definition of Δ_L, $L^\star$ is anticommutative, and the following relations hold: $[L^\star_{-1}, L^\star_{\bar{1}}]_\star \subseteq L^\star_{\bar{0}}$, $[L^\star_{\bar{i}}, L^\star_{\bar{i}}]_\star = 0$, and $[L^\star_{\bar{j}}, L^\star_{\bar{0}}]_\star \subseteq L^\star_{\bar{j}}$, where $i = -1, 1$ and $j = -1, 0, 1$. The inclusions $[V, W]_\star \subseteq V$, $[V, V]_\star \subseteq W$, and $[W, W]_\star \subseteq W$ are also obvious.

Let $w_1, w_2, w_3 \in W$, $a \in A$, and $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$. Then $\langle [[w_1, w_2]_\star, w_3]_\star, D_a \rangle = -\frac{1}{4} \sum_a (\langle [w_1, w_2]_\star, D_{a_{(1)}} \rangle \langle w_3, D_{a_{(2)}} \rangle - \langle [w_1, w_2]_\star, D_{a_{(2)}} \rangle \langle w_3, D_{a_{(1)}} \rangle)$. If $\Delta(a_{(1)}) = \sum_{a_{(1)}} a_{(1)(1)} \otimes a_{(1)(2)}$ and $\Delta(a_{(2)}) = \sum_{a_{(2)}} a_{(2)(1)} \otimes a_{(2)(2)}$, then $\langle [[w_1, w_2]_\star, w_3]_\star, D_a \rangle = \frac{1}{16} \sum_a (\langle \widehat{w_1}, a_{(1)(1)} \rangle \langle \widehat{w_2}, a_{(1)(2)} \rangle \langle \widehat{w_3}, a_2 \rangle - \langle \widehat{w_1}, a_{(1)(2)} \rangle \langle \widehat{w_2}, a_{(1)(1)} \rangle \langle \widehat{w_3}, a_2 \rangle - \langle \widehat{w_1}, a_{(2)(1)} \rangle \langle \widehat{w_2}, a_{(2)(2)} \rangle \langle \widehat{w_3}, a_1 \rangle + \langle \widehat{w_1}, a_{(2)(2)} \rangle \langle \widehat{w_2}, a_{(2)(1)} \rangle \langle \widehat{w_3}, a_1 \rangle)$. Consequently, $16 \langle [[w_1, w_2]_\star, w_3]_\star, D_a \rangle = \langle [[\widehat{w_1}, \widehat{w_2}], \widehat{w_3}], a \rangle$. Therefore,

$$16 \langle J(w_1, w_2, w_3), D_a \rangle = \langle J(\widehat{w_1}, \widehat{w_2}, \widehat{w_3}), a \rangle,$$

where $J(w_1, w_2, w_3)$ and $J(\widehat{w_1}, \widehat{w_2}, \widehat{w_3})$ are Jacobians in the algebras $L^\star$ and $(A^\star)^{(-)}$, respectively. Since $J(\widehat{w_1}, \widehat{w_2}, \widehat{w_3}) = 0$, we have $J(w_1, w_2, w_3) = 0$.

For $v_1, v_2, v_3 \in V$ and $u \in L^\star_{-1} \bigcup L^\star_{\bar{1}}$, the equalities $J(v_1, v_2, v_3) = 0$, $J(v_1, v_2, w_3) = 0$, $J(v_1, w_2, w_3) = 0$, $J(u, v_2, v_3) = 0$, and $J(u, w_2, w_3) = 0$ are to be proved similarly.

Let $v \in L^\star_{-1}$, $u \in L^\star_1$, $w \in L^\star_0$, and $a \in A$. Then $J(v,u,w) = 0$. Indeed, by the definition of Δ_L,

$$\langle [[v,u]_\star, w]_\star, a' \rangle = \frac{1}{4}\sum_a (\langle [v,u]_\star, a'_{(2)}\rangle \langle w, D_{a_{(1)}}\rangle - \langle [v,u]_\star, D_{a_{(1)}}\rangle \langle w, a'_{(2)}\rangle +$$

$$\langle [v,u]_\star, D_{a_{(2)}}\rangle \langle w, a'_{(1)}\rangle - \langle [v,u]_\star, a'_{(1)}\rangle \langle w, D_{a_{(2)}}\rangle).$$

If $w \in W$, repeating the above argument, we have

$$\langle [[v,u]_\star, w]_\star, a' \rangle =$$

$$\frac{1}{8}\sum_a (\langle v, a_{(2)(1)}\rangle \langle u, a_{(2)(2)}\rangle \langle \widehat{w}, a_1\rangle + \langle v, a_{(2)(2)}\rangle \langle u, a_{(2)(1)}\rangle \langle \widehat{w}, a_1\rangle -$$

$$\langle v, a_{(1)(1)}\rangle \langle u, a_{(1)(2)}\rangle \langle \widehat{w}, a_2\rangle - \langle v, a_{(1)(2)}\rangle \langle u, a_{(1)(1)}\rangle \langle \widehat{w}, a_2\rangle).$$

Therefore,

$$\langle [[v,u]_\star, w]_\star, a' \rangle = \frac{1}{4}\sum_a (\langle v \circ u, a_{(2)}\rangle \langle \widehat{w}, a_1\rangle - \langle v \circ u, a_{(1)}\rangle \langle \widehat{w}, a_2\rangle) = -\frac{1}{4}\langle [v \circ u, \widehat{w}], a\rangle,$$

where $2v \circ u = vu + uv$. Likewise we obtain $\langle [[v,w]_\star, u]_\star, a'\rangle = -\frac{1}{4}\langle [v,\widehat{w}] \circ u, a\rangle$ and $\langle [v,[u,w]_\star]_\star, a'\rangle = -\frac{1}{4}\langle v \circ [u,\widehat{w}], a\rangle$. Then $\langle J(v,u,w), a'\rangle = 0$. The case $w \in V$ can be proved similarly. Repeating the argument, we infer that $\langle J(v,u,w), D_a\rangle = 0$. Thus $J(v,u,w) = 0$. The equality $J(v,u,w) = 0$, where $v \in L^\star_{-1} \bigcup L^\star_1$, $u \in V$, and $w \in W$, is derived in a similar way.

Let $v, u \in L^\star_1$ and $w \in L^\star_{-1}$. Then $[[v,w]_\star, u]_\star \in L^\star_1$. Reasoning as in the two cases envisaged above, we have

$$\langle [[v,w]_\star, u]_\star, a\rangle = \langle \frac{1}{4}[u,[v,w]] - u \circ (v \circ w), a\rangle$$

for any $a \in A$. Consequently,

$$\langle J(v,w,u), a\rangle =$$

$$\langle \frac{1}{4}[u,[v,w]] - u \circ (v \circ w) - \frac{1}{4}[v,[u,w]] + v \circ (u \circ w), a\rangle =$$

$$\langle \frac{1}{4}[[u,v],w] - u \circ (v \circ w) + v \circ (u \circ w), a\rangle.$$

Since $\frac{1}{4}[[u,v],w] - u \circ (v \circ w) + v \circ (u \circ w) = 0$, it follows that $J(v,w,u) = 0$. The equality $J(v,w,u) = 0$, where $v, u \in L^\star_{-1}$ and $w \in L^\star_1$, can be proved similarly.

Finally, since $[L^\star_i, L^\star_i]_\star = 0$, where $i = -1, 1$, we have $J(v,u,w)_\star = 0$ for any $v, u \in L^\star_i$ and $w \in L^\star_0$. Consequently, $L^\star$ is a Lie algebra. Therefore, the pair $(L(A^{(+)}), \Delta_L)$ is a Lie coalgebra. That $(L(A^{(+)}), \Delta_L)$ is a Lie bialgebra is implied by Lemmas 1-5. Theorem 3 is thus proved.

3. LIE BIALGEBRAS ASSOCIATED WITH JORDAN ALGEBRAS

Let Φ be a field of characteristic $\neq 2$ and J a Jordan Φ-algebra with unity 1. Suppose that D is some algebra of derivations of J containing $\mathrm{Intder}(J)$, and L is a Lie algebra obtained from J and D following the KKT process. Denote by U a subalgebra of L generated by 1, $1'$, and $\bar{1}$. Multiplication in L is denoted $[\,,\,]$. Suppose that a pair (L, Δ_L) is a Lie bialgebra. Then the following is valid:

THEOREM 4. Assume that the characteristic of Φ is not 2, 3, comultiplication Δ_L agrees with the grading of L, and $\Delta(U) = 0$. Then J admits the structure of a Jordan D-bialgebra with comultiplication Δ and, moreover, Δ_L is associated with Δ.

First we argue for some properties that characterize Δ_L. If $x, y \in L$, then, as above, put $\{x,y\} = x \otimes y - y \otimes x$.

LEMMA 6. Let comultiplication Δ_L satisfy the conditions of Theorem 4. Then, for any a in J and d in D, the following equalities are true:

$$\Delta_L(a) = \sum_i \{a_i, b_i'\} + \sum_j \{c_j, d_j\},$$

$$\Delta_L(a') = \sum_i \{a_i, \bar{b}_i\} + \sum_j \{c_j', d_j\},$$

$$\Delta_L(\bar{a}) = \sum_i \{b_i', \bar{a}_i\} + \sum_j \{\bar{c}_j, d_j\},$$

$$\Delta_L(d) = \sum_i (\{x_i, \bar{y}_i\} - x_i' \otimes y_i') + \sum_j d_{1j} \otimes d_{2j},$$

where $a_i, b_i, c_j, x_i, y_i \in J$ and $d_j, d_{1j}, d_{2j} \in D$.

Proof. Let $a \in J$. Because comultiplication Δ_L agrees with the grading of L, we have

$$\Delta_L(a) = \sum_i a_i \otimes b_i' + \sum_k e_k' \otimes f_k + \sum_j c_j \otimes d_j + \sum_l h_l \otimes g_l,$$

where all $a_i, b_i, e_k, f_k, c_j, g_l \in J$ and $d_j, h_l \in D$. Since $\Delta_L(a) = -\tau(\Delta_L(a))$, it is clear that $\Delta_L(a) = \sum_i \{a_i, b_i'\} + \sum_j \{c_j, d_j\}$. The equalities $[a, 1] = 0$ and $\Delta_L(1) = 0$ give us $\Delta_L([a,1]) = [\Delta_L(a), 1] + [a, \Delta_L(1)] = 0$. This yields

$$\sum_i a_i \otimes b_i = \sum_i b_i \otimes a_i. \qquad (*)$$

To both parts of this equality we apply the map $R \otimes R$ to obtain $\sum_i a_i' \otimes b_i' = \sum_i b_i' \otimes a_i'$.

Now we take an element a'. Since $a' = [a, \bar{1}]$ and $\Delta_L(\bar{1}) = 0$, we have $\Delta_L(a') = [\Delta_L(a), \bar{1}] + [a, \Delta_L(\bar{1})] = \sum_i \{a_i, \bar{b}_i\} + \sum_i \{a_i', b_i'\} + \sum_j \{c_j', d_j\} = \sum_i \{a_i, \bar{b}_i\} + \sum_j \{c_j', d_j\}$. The third equality can be proved similarly.

Let us derive the fourth equality. Let $d \in D$. In the same way as above, we obtain

$$\Delta_L(d) = \sum_i \{x_i, \bar{y}_i\} + \sum_k v_k' \otimes u_k' + \sum_l \{w_l', d_l\} + \sum_j d_{1j} \otimes d_{2j},$$

where $x_i, y_i, v_k, u_k, w_l \in J$ and $d_l, d_{1j}, d_{2j} \in D$. The pair (L, Δ_L) is a Lie bialgebra and $[1, d] = 0$, and so $[1, \Delta_L(d)] = 0$. Hence,

$$\sum_i \{x_i, y_i'\} + \sum_k (v_k \otimes u_k' + v_k' \otimes u_k) + \sum_l \{w_l, d_l\} = 0.$$

Consequently, $\sum_i x_i \otimes y_i' + \sum_k v_k \otimes u_k' = 0$ and $\sum_l \{w_l, d_l\} = 0$. If, now, to both parts of the first equality we apply the map $R \otimes \mathrm{id}$ we obtain $\sum_i x_i' \otimes y_i' = -\sum_k v_k' \otimes u_k'$. Since $\sum_l \{w_l', d_l\} = [\sum_l \{w_l, d_l\}, \bar{1}]$, it follows that $\sum_l \{w_l', d_l\} = 0$, whence the desired equality.

Now, define comultiplication Δ on the algebra J. Let $a \in J$. In view of Lemma 6, $\Delta_L(a) = \sum_i \{a_i, b_i'\} + \sum_j \{c_j, d_j\}$. We then put $\Delta(a) = \sum_i a_i \otimes b_i$. By $(*)$, comultiplication Δ is cocommutative. Let $J^\star$ be the dual of the coalgebra (J, Δ). The algebra $J^\star$ is commutative. Let $L^\star$ be dual to the coalgebra (L, Δ). There is no loss of generality in assuming that $L^\star = J^\star \oplus R(J)^\star \oplus D^\star \oplus \overline{J^\star}$, where $\overline{J^\star}$ is an isomorphic copy of the space $J^\star$. It is easy to state that there exists an isomorphism between spaces $J^\star$ and $R(J)^\star$, for which the image of an element f from J^* is denoted f'. Consider the algebra $B = B(L, L^*)$, introduced in Sec. 1. Obviously, L and L^* are subalgebras of B, in which case multiplications on L and L^*, induced by multiplication in B, coincide with the initial ones. Denote multiplication in B by $[\,,\,]$. Since the pair (L, Δ_L) is a Lie bialgebra, B is a Lie algebra. Write π for the projection of the space B onto L^*. By the definition of multiplication in B, we have $\langle \pi([l^*, l]), l_1\rangle = \langle l^*, [l, l_1]\rangle$ for any elements $l^* \in L^*$ and $l, l_1 \in L$.

LEMMA 7. Let $f, g, h \in J^*$ and $a \in J$. Then $\langle [h', [f', g']], a'\rangle = \langle (f, h, g), a\rangle$, where (f, h, g) is an associator of elements f, h, g in the algebra J^*.

Proof. Let $a \in J$, $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$, $\Delta(a_{(1)}) = \sum_a a_{(1)(1)} \otimes a_{(1)(2)}$, and $\Delta(a_{(2)}) = \sum_a a_{(2)(1)} \otimes a_{(2)(2)}$. By the definition of Δ and Lemma 6, then, $\Delta_L(a') = \sum_a \{a_{(1)}, \overline{a}_{(2)}\} + \sum_i \{c_i', d_i\}$, where $c_i \in J$, $d_i \in D$, and $\Delta_L(d_i) = \sum_j (\{x_{ij}, \overline{y}_{ij}\} - x_{ij}' \otimes y_{ij}') + v$ for $x_{ij}, y_{ij} \in J$ and $v \in D \otimes D$. Consequently,

$$(\mathrm{id} \otimes \Delta_L)\Delta_L(a') = \sum_a a_{(1)} \otimes \{a'_{(2)(2)}, \overline{a}_{(2)(1)}\} - \sum_a \overline{a}_{(2)} \otimes \{a_{(1)(1)}, a'_{(1)(2)}\} +$$

$$\sum_{i,j} c_i' \otimes \{x_{ij}, \overline{y}_{ij}\} + u,$$

where $u \in R(J) \otimes R(J) \otimes R(J) + D \otimes L \otimes L + L \otimes D \otimes L + L \otimes L \otimes D$.

Induce a linear map ξ on the tensor cube of the space L, putting $\xi(x \otimes y \otimes z) = y \otimes z \otimes x$. Since (L, Δ_L) is a Lie coalgebra, we have $(\mathrm{id} + \xi + \xi^2)(\mathrm{id} \otimes \Delta_L)\Delta_L(a') = 0$ (see [3]). From this equality, we obtain

$$\sum_{i,j} c_i' \otimes x_{ij} \otimes \overline{y}_{ij} = \sum_a (a'_{(2)(2)} \otimes a_{(1)} \otimes \overline{a}_{(2)(1)} - a'_{(1)(2)} \otimes a_{(1)(1)} \otimes \overline{a}_{(2)}).$$

If to the latter we apply the map $\mathrm{id} \otimes R \otimes R\varepsilon$ we see that

$$\sum_{i,j} c_i' \otimes x_{ij}' \otimes y_{ij}' = \sum_a (a'_{(2)(2)} \otimes a'_{(1)} \otimes a'_{(2)(1)} - a'_{(1)(2)} \otimes a'_{(1)(1)} \otimes a'_{(2)}).$$

Therefore,

$$\langle [h', [f', g']], a'\rangle = -\sum_{i,j} \langle \rho_1(h' \otimes f' \otimes g'), c_i' \otimes x_{ij}' \otimes y_{ij}'\rangle =$$

$$\sum_a \langle \rho_1(h \otimes f \otimes g), a_{(1)(2)} \otimes a_{(1)(1)} \otimes a_{(2)} - a_{(2)(2)} \otimes a_{(1)} \otimes a_{(2)(1)}\rangle = \langle (f, h, g), a\rangle.$$

Consider the space $C = J \oplus \overline{J^*}$ with multiplication $(\star)$ given by

$$(a + \overline{f}) \star (b + \overline{g}) = (ab + f \cdot b + g \cdot a) + \overline{(fg + f \bullet b + g \bullet a)},$$

where $a, b \in J$, $f, g \in J^*$, $\langle f \bullet b, a\rangle = \langle f, ba\rangle$, and $\langle f, a \cdot g\rangle = \langle fg, a\rangle$. For elements $x, y, z \in C$, (x, y, z) is an associator in the algebra C. Denote by λ the projection of the space C onto $\overline{J^*}$.

LEMMA 8. Let $a, b \in J$ and $f, g \in J^*$. Then the algebra B satisfies the following equalities:

$$[f, \overline{g}] = (fg)' - [f', g'], \tag{15}$$

$$[f, g'] = fg, \tag{16}$$

$$[\overline{f}, g'] = -\overline{fg}, \tag{17}$$

$$[a,f]+[\overline{a},\overline{f}]=2[a',f'], \tag{18}$$

$$(\overline{f},b,a)=[b,[a',f']], \tag{19}$$

$$(\overline{f},\overline{g},a)=[[f',a'],\overline{g}], \tag{20}$$

$$(\overline{f},a,\overline{g})=[a,[f',g']], \tag{21}$$

$$(a,\overline{f},b)=[\overline{f},[a',b']]. \tag{22}$$

Proof. Let $\Delta(a)=\sum_a a_{(1)}\otimes a_{(2)}$ and $\Delta(b)=\sum_b b_{(1)}\otimes b_{(2)}$. By Lemma 6, we have $\Delta_L(a)=\sum_a\{a_{(1)},a'_{(2)}\}+\sum_i\{c_i,d_i\}$, $\Delta_L(a')=\sum_a\{a_{(1)},\overline{a}_{(2)}\}+\sum_i\{c'_i,d_i\}$, $\Delta_L(b)=\sum_b\{b_{(1)},b'_{(2)}\}+\sum_j\{e_j,D_j\}$, and $\Delta_L(b')=\sum_b\{b_{(1)},\overline{b}_{(2)}\}+\sum_i\{e'_j,D_j\}$, where $c_i,e_j\in J$ and $d_i,D_j\in D$.

We prove (15). Take an element $u=[f,\overline{g}]-(fg)'+[f',g']$. Then

$$\langle u,a'\rangle=\langle[f,\overline{g}],a'\rangle-\langle(fg)',a'\rangle=\sum_a\langle f,a_{(1)}\rangle\langle g,a_{(2)}\rangle-\langle fg,a\rangle=0.$$

Likewise we obtain $\langle u,d\rangle=0$ for any $d\in D$. Therefore, $[f,\overline{g}]=(fg)'-[f',g']$. Equalities (16) and (17) are to be proved similarly.

We derive (18). By the definition of multiplication in B, we have $[a,f]=(a\cdot f)'+\sum_i\langle f,c_i\rangle d_i+\pi([a,f])$ and $[\overline{a},\overline{f}]=-(a\cdot f)'+\sum_i\langle f,c_i\rangle d_i+\pi([\overline{a},\overline{f}])$. Let $u=\pi([a,f])+(\pi[\overline{a},\overline{f}])$. Then $u\in R(J)^\star+D^\star$. If $v\in R(J)^\star$ and $d\in D$, then $\langle u.v+d\rangle=-2\langle f,ad\rangle=2\langle\pi([a',f']),v+d\rangle$. Therefore, $u=2\pi([a',f'])$. On the other hand, $[a',f']=\sum_i\langle f,c_i\rangle d_i+\pi([a',f'])$. Consequently, $[a,f]+[\overline{a},\overline{f}]=2[a',f']$.

Take an element $[b,[a',f']]$. It is clear that

$$[b,[a',f']]=\sum_i\langle f,c_i\rangle bd_i+\sum_j\langle f,aD_j\rangle e_j+\pi([b,\pi[a',f']]),$$

and for any $c\in J$, we have $\langle\pi[b,\pi[a',f']],\overline{c}\rangle=\langle f',[[\overline{c},b],a']\rangle=\langle\lambda((\overline{f},b,a)),\overline{c}\rangle$. Therefore, $[b,[a',f']]=\sum_i\langle f,c_i\rangle bd_i+\sum_j\langle f,aD_j\rangle e_j+\lambda((\overline{f},b,a))$.

Alternatively, since $ba=[b,a']$ and the pair (L,Δ_L) is a Lie bialgebra, we have

$$\Delta_L(ba)=\sum_b\{b_{(1)},(b_{(2)}a)'\}+\sum_a\{a_{(1)}b,a'_{(2)}\}+\sum_j\{e'_j,aD_j\}-\sum_i\{c_i,(bd_i)'\}+u,$$

where $u\in R(J)\otimes D+D\otimes R(J)$. Therefore,

$$\Delta(ba)=\sum_a a_{(1)}b\otimes a_{(2)}+\sum_b b_{(1)}\otimes ab_{(2)}-\sum_j aD_j\otimes e_j-\sum_i c_i\otimes bd_i.$$

Hence,

$$(\overline{f},b,a)=\sum_i\langle f,c_i\rangle bd_i+\sum_j\langle f,aD_j\rangle e_j+\lambda((\overline{f},b,a))=[b,[a',f']].$$

Equality (19) is thus proved.

We prove (20) and (21). By the definition of multiplication in B, $[\overline{f},a']=f\cdot a+\pi([\overline{f},a'])$. Therefore $\pi([\overline{f},a'])\in\overline{J^\star}$ and, moreover, $\pi([\overline{f},a'])=\lambda(\overline{f}\star a)$. Consequently, $[\overline{f},a']=f\cdot a+\lambda(\overline{f}\star a)$. Similarly, $[f',a]=f\cdot a+\lambda(\overline{f}\star a)$. Using (17), it is not difficult to show that

$$[\overline{fg},a']-[f',[\overline{g},a']]=(fg)\cdot a+\lambda((\overline{fg}\star a))-f\cdot(g\cdot a)-\lambda(\overline{f}\star(g\cdot a))-\overline{f\lambda(\overline{g}\star a)}.$$

The expression that fills the right part of this equality is equal to $(\overline{f},\overline{g},a)$. Likewise we obtain $(\overline{f},\overline{g},a) = [(fg)',a]+[f',[g',a]]$. In view of (17) and the Jacobi identity, we have $[\overline{fg},a'] - [f'[\overline{g},a']] = [[f',a'],\overline{g}]$. Consequently, $(\overline{f},\overline{g},a) = [[f',a'],\overline{g}]$. Since the algebra C is commutative, $(\overline{f},a,\overline{g}) + (a,\overline{g},\overline{f}) + (\overline{g},\overline{f},a) = 0$. Therefore,

$$(\overline{f},a,\overline{g}) = [(gf)',a] - [f',[g',a]] - [(fg)',a] + [g',[f',a]] = [a,[f',g']].$$

Equalities (20) and (21) are thus proved.

And, finally, we derive (22). Since the pair (L,Δ_L) is a Lie bialgebra, we have $\Delta_L([a',b']) = \sum_a(\{ba_{(1)},\overline{a_{(2)}}\} - \{a_{(1)},\overline{ba_{(2)}}\}) - \sum_b\{ab_{(1)}a,\overline{b_{(2)}}\} - \{b_{(1)},\overline{ab_{(2)}}\}) + u$, where $u \in R(J)\otimes R(J) + D\otimes D$. Therefore $[\overline{f},[a',b']] = \sum_a\langle f,a_{(2)}\rangle ba_{(1)} - \langle f,ba_{(2)}\rangle a_{(1)} - \langle f,b_{(2)}\rangle ab_{(1)} + \langle f,ab_{(2)}\rangle b_{(1)} + \pi([\overline{f},[a',b']])$. Since $\pi([\overline{f},[a',b']]) \in \overline{J^\star}$, we have $\langle\pi([\overline{f},[a',b']]),\overline{c}\rangle = \langle\lambda((a,\overline{f},b)),\overline{c}\rangle$. Using the last two equalities, we arrive at (22).

Proof of Theorem 4. Comultiplication Δ_L is associated with Δ. We show that $B(J,J^\star)$ is a Jordan algebra. It is not difficult to state that there exists an isomorphism of algebras C and $B(J,J^\star)$, under which the image of the subalgebra $\overline{J^\star}$ is $J^\star$. We prove, first, that $J^\star$ is a Jordan algebra. In view of Lemma 7, it suffices to show that, for any $f \in J^\star$,

$$[(f^2)',f'] = 0. \tag{**}$$

From (15), (16), (17) and the Jacobi identity, we obtain $[(f^2)',f'] = [[f,\overline{f}],f'] = [[f,f'],\overline{f}]+[f,[\overline{f},f']] = [f^2,\overline{f}] - [f,\overline{f^2}] = -2[(f^2)',f']$. Consequently, $3[(f^2)',f'] = 0$, and so $[(f^2)',f'] = 0$. Thus, $J^\star$ is a Jordan algebra. Therefore, $\overline{J^\star}$ is a Jordan subalgebra in C.

Now we show that C is a Jordan algebra. To do this, it suffices to prove the validity of the following:

$$(\overline{f^2},a,\overline{f}) = 0,$$

$$(a^2,\overline{f},a) = 0,$$

$$(\overline{f^2},b,a) + 2(\overline{f}\star a,b,\overline{f}) = 0,$$

$$(a^2,\overline{g},\overline{f}) + 2(\overline{f}\star a,\overline{g},a) = 0,$$

$$(a^2,b,\overline{f}) + 2(\overline{f}\star a,b,a) = 0,$$

where $a,b \in J$ and $f,g \in J^\star$. The first equality follows from (21) and (**). Since J is a Jordan algebra, the second equality is implied by (22).

Let us prove the third equality. In view of (19), $(\overline{f^2},b,a) = [b,[a',(f^2)']]$. By (15) and the Jacobi identity, we obtain $[a',(f^2)'] = [a',[f,\overline{f}]] = [[a',f],\overline{f}] + [f,[a',\overline{f}]]$. By the definition of multiplication in B, we have $[[a',f],\overline{f}] = [\overline{a\cdot f},\overline{f}]+[[\pi([a',f]),\overline{f}]$ and $[f,[a',\overline{f}]] = -[f,a\cdot f]+[f,[\pi([a',\overline{f}])]$. It is easy to see that $\pi([a',f]) = f\bullet a$ and $\pi([a',\overline{f}]) = -\overline{a\bullet f}$. Therefore, by (18) and (15), $[a',(f^2)'] = 2[(a\cdot f)',f'] + [f\bullet a,\overline{f}] - [f,\overline{f\bullet a}] = 2[(a\cdot f)',f'] - 2[(f\bullet a)',f']$. Consequently, $[b,[a',(f^2)']] = 2[b,[(a\cdot f)',f']] - 2[b,[(f\bullet a)',f']]$. But, by (19) and (21), we then have

$$[b,[a',(f^2)']] = -2(a\cdot f,b,\overline{f}) - 2(\overline{f\bullet a},b,\overline{f}) = -2(\overline{f}\star a,b,\overline{f}).$$

The fourth and the fifth equalities can be proved similarly. In this way C is a Jordan algebra, and so too is $B(J,J^\star)$. Consequently, the pair (J,Δ) is a Jordan D-bialgebra. Theorem 4 is thus proved.

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