Algebra and Logic, Vol. 38, No. 5, 1999

SHMEL'KIN EMBEDDINGS FOR ABSTRACT AND PROFINITE GROUPS

N. S. Romanovskii*

UDC 512.5

The Magnus embedding is well known: given a group $A = F/R$, where F is a free group, the group $F/[R,R]$ can be represented as a subgroup of a semidirect product AT, where T is an additive group of a free ZA-module. Shmel'kin generalized this construction and found an embedding for $F/\mathcal{V}(R)$, where $\mathcal{V}(R)$ is the verbal subgroup of R corresponding to a variety $\mathcal{V}$. Later, he treated F as a free product of arbitrary groups, and on condition that R is contained in a Cartesian subgroup of the product, pointed out an embedding for $F/\mathcal{V}(R)$. Here, we combine both these Shmel'kin embeddings and weaken the condition on R, by assuming that F is a free product of groups A_i ($i \in I$) and a free group X, and that its normal subgroup R has trivial intersection with each factor A_i. Subject to these conditions, an embedding for $F/\mathcal{V}(R)$ is found; we call it the generalized Shmel'kin embedding. For the case where $\mathcal{V}$ is an Abelian variety of groups, a criterion is specified determining whether elements of AT belong to an embedded group $F/\mathcal{V}(R)$. Similar results are proved also for profinite groups.

INTRODUCTION

The Magnus embedding is well known; see [1]. Given a group $A = F/R$, where F is free, that embedding allows the group $F/[R, R]$ to be represented as a subgroup of a semidirect product AT, where T is an additive group of a free ZA-module. In [2, 3], Shmel'kin generalized this construction and found an embedding for a group $F/\mathcal{V}(R)$, where $\mathcal{V}(R)$ is the verbal subgroup of R corresponding to a variety $\mathcal{V}$. In [4], Shmel'kin treated F as a free product of arbitrary groups, and with the requirement that R is contained in a Cartesian subgroup of the product, pointed out an embedding for $F/\mathcal{V}(R)$.

In Sec. 1 (Thm. 1), we combine these two Shmel'kin embeddings and weaken the requirement on R, by assuming that F is a free product of groups A_i ($i \in I$) and a free group X, and that its normal subgroup R intersects triviality with each of the factors A_i. Subject to these conditions, an embedding for $F/\mathcal{V}(R)$ is found, which we call the *generalized Shmel'kin embedding.* Such a generalization seems important by reason of the fact that in dealing with the group $F/\mathcal{V}(R)$, it is often necessary to treat some one of its subgroups $H/\mathcal{V}(R)$, where $H \geq R$. By the Kurosh theorem, the group H, in our case, factors into a free product of groups conjugate to subgroups in A_i and a free group. Obviously, w.r.t. this factorization, the subgroup R again satisfies the condition above, which permits us to apply the embedding construction also to $H/\mathcal{V}(R)$. In Theorem 2, we specify the criterion determining whether elements of AT belong to an embedded group $F/\mathcal{V}(R)$, with $\mathcal{V}$ an Abelian variety of groups. This generalizes the relevant criteria obtained in [5] and [6] for the factor group of a free group, and in [7] for the factor group of a free product.

*Supported by RFFR grant No. 99-01-00567.

Translated from *Algebra i Logika*, Vol. 38, No. 5, pp. 598-612, September-October, 1999. Original article submitted October 20, 1998.

In Secs. 2 and 3, we are concerned with the generalized Shmel'kin embedding in a class of profinite groups. Here, F is taken to be a free product of profinite groups A_i ($i \in I$) and a free profinite group X, and $\mathcal{V}$ is an arbitrary variety of profinite groups. As in the case of abstract groups, we point out an embedding for $F/\mathcal{V}(R)$ (Thm. 3), and for $\mathcal{V}$ an Abelian variety, specify the criterion determining whether elements of AT belong to $F/\mathcal{V}(R)$ (Thm. 4). Previously, the Magnus embedding for the factor group of a free profinite group was studied in [8], and the Shmel'kin embedding — in [9].

1. GENERALIZED SHMEL'KIN EMBEDDING FOR ABSTRACT GROUPS

1.1. If a group G acts on a group H, we say that H is a *G-module*. For a given variety $\mathcal{V}$ of abstract groups, there exists a free G-module with basis $\{x_i \mid i \in I\}$ in $\mathcal{V}$. As a group, this module is free in $\mathcal{V}$ with basis $\{x_i^g \mid i \in I,\ g \in G\}$; the action of G is defined in the obvious manner. Below, for the group elements a and b, b^a and b^{a-1} stand for $a^{-1}ba$ and b^ab^{-1}, respectively.

LEMMA 1. Let $G \geq A$ be groups; $\mathcal{V}$ a variety of groups; T a free G-module in $\mathcal{V}$, with one free generating element t; $\varphi\colon G \to H$ an epimorphism; S some H-module of $\mathcal{V}$ (treated also as a G-module); $L = HS$ the respective semidirect product. Suppose that the homomorphism $\psi\colon A \to L$ is given so that

$$a\psi = a\varphi \cdot s_a,\ s_a \in S\ (a \in A).$$

Then the map

$$t^{a-1} \to s_a\ (a \in A)$$

extends to an homomorphism of the G-module $\widehat{T}$, generated by elements t^{a-1} ($a \in A$), onto a G-module $\widehat{S}$ generated by s_a ($a \in A$). The first of these modules being a group is a retract of the group T free in $\mathcal{V}$.

Proof. Let t^A be an A-submodule of T generated by an element t. Then T being a group is a $\mathcal{V}$-free product of groups t^A, $(t^A)^{g_i}$ ($i \in I$), free in $\mathcal{V}$, where $\{g_i \mid i \in I\}$ is a system of representatives of the right cosets of G w.r.t. A other than A. Let $\widehat{T}_A$ be an A-module generated by elements t^{a-1} ($a \in A$). In view of the formula

$$(t^{a-1})^{a'} = t^{aa'-1} \cdot (t^{a'-1})^{-1}, \tag{1}$$

$\widehat{T}_A$ being a group is generated by the set $U = \{t^{a-1} \mid 1 \neq a \in A\}$, and it is a retract of the group t^A free in $\mathcal{V}$, for the basis of the latter group is obtained by adding to U one element t. The group $\widehat{T}$, too, is a retract of T, and its basis is constituted by the set

$$\{t^{a-1},\ t^{(a-1)g_i} \mid 1 \neq a \in A,\ i \in I\},$$

which is complemented to a basis of T by the elements t, t^{g_i} ($i \in I$).

Consider in S an A-submodule $\widehat{S}_A$ generated by elements s_a ($1 \neq a \in A$). Since

$$(a\varphi \cdot s_a) \cdot (a'\varphi \cdot s_{a'}) = (aa')\varphi \cdot s_{aa'},$$

we have

$$s_a^{a'} = s_{aa'} \cdot s_{a'}^{-1}. \tag{2}$$

This formula implies that $\widehat{S}_A$ is generated as a group by elements s_a ($1 \neq a \in A$), and $\widehat{S}$ is generated by s_a, $s_a^{g_i}$ ($1 \neq a \in A$, $i \in I$). Since $\widehat{T}$ is free in $\mathcal{V}$, the mapping of its basis

$$t^{a-1} \to s_a,\ (t^{a-1})^{g_i} \to s_a^{g_i}\ (1 \neq a \in A,\ i \in I)$$

extends to a group homomorphism, which is a module homomorphism in view of (1) and (2). The lemma is proved.

THEOREM 1. Let $F = (\underset{i\in I}{*} A_i) * X$ be a free product of nontrivial groups A_i $(i \in I)$ and a free group X with basis $\{x_j \mid j \in J\}$ (we do not exclude the case with A_i or X missing); R is a normal subgroup of F such that $R \cap A_i = 1$ $(i \in I)$; $A = F/R$. Identify the groups A_i $(i \in I)$ with their canonical images in A; denote by $\overline{x}$ the canonical image of an element $x \in X$ in A. Let $\mathcal{V}$ be a variety of groups and T be an A-module free in $\mathcal{V}$ with basis $\{t_k \mid k \in I \cup J\}$. Consider a semidirect product $M = AT$ and an homomorphism $\tau\colon F \to M$ given by the map

$$a_i \to t_i a_i t_i^{-1} = a_i \cdot t_i^{a_i - 1} \ (a_i \in A_i,\ i \in I),\ x_j \to \overline{x}_j \cdot t_j \ (j \in J).$$

Then the kernel of τ coincides with $\mathcal{V}(R)$, that is, τ yields an embedding of $F/\mathcal{V}(R)$ in M.

Proof. Since $R\tau \leq T$ and $T \in \mathcal{V}$, we have $\ker\tau \geq \mathcal{V}(R)$. We need to prove the inverse inclusion.

Consider a group $F/\mathcal{V}(R)$ and the Kaloujnin–Krasner embedding ψ of this group in a Cartesian wreath product $R/\mathcal{V}(R) \wr A$, which is represented as a semidirect product AS, where S is a basis subgroup of the wreath product (cf. [10] and [11, Thm. 6.28]). We have

$$a_i\psi = a_i \cdot s_{a_i},\ s_{a_i} \in S\ (a_i \in A_i,\ i \in I);\ x_j\psi = \overline{x}_j \cdot s_j,\ s_j \in S\ (j \in J).$$

Notice that $S \in \mathcal{V}$. For $k \in I \cup J$, denote by T_k an A-submodule of T generated by an element t_k. For $i \in I$, denote by $\widehat{T}_i$ an A-submodule generated by elements $t_i^{a_i - 1}$ $(a_i \in A_i)$. The subgroup of T generated by all $\widehat{T}_i$ $(i \in I)$ and all T_j $(j \in J)$ is denoted $\widehat{T}$. The free group T of $\mathcal{V}$ factors into a $\mathcal{V}$-free product of its $\mathcal{V}$-free subgroups T_k $(k \in I \cup J)$. By Lemma 1, the group $\widehat{T}_i$ is a retract of T_i. For this reason, $\widehat{T}$ factors into a $\mathcal{V}$-free product of its $\mathcal{V}$-free subgroups $\widehat{T}_i$ $(i \in I)$ and T_j $(j \in J)$, each of which is an A-submodule. Again by Lemma 1, for every $i \in I$, the map $t_i^{a_i - 1} \to s_{a_i}$ $(a_i \in A_i)$ determines an homomorphism $\sigma_i : \widehat{T}_i \to S$ of A-modules and the map $t_j \to s_j$ determines an homomorphism $\sigma_j : T_j \to S$ of A-modules, for every $j \in J$. Therefore, the homomorphisms σ_k $(k \in I \cup J)$ extend to an homomorphism σ of the group $\widehat{T}$ into S, which is also an A-module homomorphism. The homomorphism σ and the identity map $A \to A$ yield an homomorphism $\gamma : A\widehat{T} \to AS$ of semidirect products. By construction, $\tau\gamma = \psi$. In particular, $\ker\tau \leq \ker\psi = \mathcal{V}(R)$. The theorem is proved.

1.2. Here, we deal with a more specific case where $\mathcal{V}$ is an Abelian variety, that is, $\mathcal{V}$ is one of the varieties $\mathcal{A}_m$; $\mathcal{A}_0$ is a variety of all Abelian groups and $\mathcal{A}_m$ $(m \geqslant 1)$ a variety of Abelian groups of period m. Then T is the usual module over a group ring $(Z/mZ)A$. The group M, in this case, is identified with a group of matrices $\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$. A proof of the next theorem follows essentially the same line as Proposition 1 in [12], but with substantial deviations in some places.

THEOREM 2. Suppose that the conditions of Theorem 1 are satisfied and $\mathcal{V} = \mathcal{A}_m$. The matrix $\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$ in M, where

$$t = \sum_{i\in I} u_i t_i + \sum_{j\in J} v_j t_j,$$

belongs to $F\tau$ if and only if the following hold:

$$u_i \in (A_i - 1)\cdot(Z/mZ)A\ (i \in I),\ \ \sum_{i\in I} u_i + \sum_{j\in J}(\overline{x}_j - 1)v_j = a - 1. \tag{3}$$

Proof. A check that conditions (3) distinguish a subgroup in M is straightforward. Since the generating elements

$$\begin{pmatrix} a_i & 0 \\ t_i(a_i-1) & 1 \end{pmatrix} \; (a_i \in A_i,\; i \in I), \; \begin{pmatrix} \overline{x}_j & 0 \\ t_j & 1 \end{pmatrix} \; (j \in J)$$

for the group $F\tau$ satisfy (3), all elements of $F\tau$ also satisfy these. We need to prove the inverse: an element of M satisfying (3) belongs to $F\tau$. First we define some new objects.

Consider a right free $(Z/mZ)F$-module S with basis $\{s_k \mid k \in I \cup J\}$, a matrix group $L = \begin{pmatrix} F & 0 \\ S & 1 \end{pmatrix}$, and an embedding $\sigma \colon F \to L$ given by the map

$$a_i \to \begin{pmatrix} a_i & 0 \\ s_i(a_i-1) & 1 \end{pmatrix} (a_i \in A_i,\; i \in I), \;\; x_j \to \begin{pmatrix} x_j & 0 \\ t_j & 1 \end{pmatrix} (j \in J).$$

The embedding σ extends to an embedding (also denoted σ) of the ring $(Z/mZ)F$ in the ring

$$L = \begin{pmatrix} (Z/mZ)F & 0 \\ S & Z/mZ \end{pmatrix}.$$

Let $v \in ZF$ and

$$v\sigma = \begin{pmatrix} v & 0 \\ \sum s_k \cdot D_k(v) & \alpha \end{pmatrix}.$$

We call the function

$$D_k \colon v \to D_k(v) \; (k \in I \cup J)$$

a *partial Fox derivative*. Notice that $\alpha = \varepsilon(v)$, where

$$\varepsilon \colon (Z/mZ)F \to Z/mZ,$$

is a trivialization map, and the following formulas hold:

$$\begin{gathered} \text{if } a_i \in A_i \; (i \in I), \text{ then } D_i(a_i) = a_i - 1,\; D_k(a_i) = 0 \text{ for } k \neq i; \\ D_j(x_j) = 1 \; (j \in J),\; D_k(x_j) = 0 \text{ for } k \neq j; \\ \text{if } u, v \in (Z/mZ)F,\; f \in F, \text{ then } D_k(u+v) = D_k(u) + D_k(v), \\ D_k(uv) = D_k(u)v + \varepsilon(u)D_k(v),\; D_k(f^{-1}) = -D_k(f)f^{-1}. \end{gathered} \tag{4}$$

Furthermore, for $i \in I$, we have $D_i(v) \in (A_i - 1) \cdot (Z/mZ)F$. A fundamental ideal Δ of the group ring $(Z/mZ)F$ being a right $(Z/mZ)F$-module decomposes into

$$\sum_{i \in I} (A_i - 1) \cdot (Z/mZ)F + \sum_{j \in J} (x_j - 1) \cdot (Z/mZ)F.$$

This sum is direct. Using (4), we obtain the following:

if $v \in (A_i - 1) \cdot (Z/mZ)F$, then $D_i(v) = v$ and $D_k(v) = 0$ for $k \neq i$;

if $v = (x_j - 1)u \in (x_j - 1) \cdot (Z/mZ)F$, then $D_j(v) = u$ and $D_k(v) = 0$ for $k \neq j$.

For an arbitrary element $v \in \Delta$, therefore, its projection onto

$$(A_i - 1) \cdot (Z/mZ)F \; (i \in I)$$

coincides with $D_i(v)$, and one onto

$$(x_j - 1) \cdot (Z/mZ)F \; (j \in J)$$

coincides with $(x_j - 1) \cdot D_j(v)$. Hence, if $v \in (Z/mZ)F$, then

$$v - \varepsilon(v) = \sum_{i \in I} D_i(v) + \sum_{j \in J} (x_j - 1) \cdot D_j(v).$$

The canonical epimorphism $F \to A$ yields the ring epimorphism

$$(Z/mZ)F \to (Z/mZ)A,$$

and together with the map

$$s_k \to t_k \ (k \in I \cup J),$$

these determine a module epimorphism $S \to T$ and a matrix group epimorphism $L \to M$. We make the convention that all of these epimorphisms are denoted by one letter — γ. Consider Fox derivatives $d_k = D_k\gamma$ from $(Z/mZ)F$ into $(Z/mZ)A$. Since $\tau = \sigma \cdot \gamma$, for $f \in F$ we have

$$f\tau = \begin{pmatrix} f\gamma & 0 \\ \sum t_k \cdot d_k(f) & 1 \end{pmatrix}.$$

We embark on the proof of the theorem. Suppose that the matrix

$$c = \begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix} \in M,$$

where $t = \sum t_k u_k$, satisfies (3). In the group F, there exists an element f such that

$$f\tau = \begin{pmatrix} a & 0 \\ * & 1 \end{pmatrix}.$$

Replace c by $c(f\tau)^{-1}$. We have thereby reduced our problem to the case where c is a unitriangular matrix, and

$$\sum_{i \in I} u_i + \sum_{j \in J} (\overline{x}_j - 1)u_j = 0.$$

For an element u_i $(i \in I)$, in the ring $(Z/mZ)F$ we choose a preimage v_i that belongs to $(A_i - 1) \cdot (Z/mZ)F$, and for u_j $(j \in J)$, choose an arbitrary preimage v_j. Let

$$v = \sum_{i \in I} v_i + \sum_{j \in J} (x_j - 1)v_j.$$

Since $v\gamma = 0$, we have $v \in \ker\gamma = (R - 1) \cdot (Z/mZ)F$, and v is represented thus:

$$v = \sum_l (r_l - 1)f_l,$$

where $r_l \in R$ and $f_l \in F$. Using formulas (4) yields the equality $d_k((r_l - 1)f_l) = d_k(f_l^{-1} r_l f_l)$. Furthermore,

$$d_k\left(\sum_l (r_l - 1)f_l\right) = \sum_l d_k((r_l - 1)f_l), \ d_k\left(\prod_l (f_l^{-1} r_l f_l)\right) = \sum_l d_k(f_l^{-1} r_l f_l).$$

Therefore, if

$$f = \prod_l (f_l^{-1} r_l f_l),$$

then

$$d_k(f) = d_k(v) = d_k(v_k) = v_k\gamma = u_k.$$

Hence $f\tau = c$. Theorem 2 is proved.

2. ABOUT PROFINITE GROUPS

Recall that a *profinite group* is a topological group represented as a projective limit of finite groups. From the topological standpoint, profinite groups are characterized as ones that are compact and totally disconnected. (For information about profinite groups, see [13-16].), Below, when we speak about profinite groups, the terms a "subgroup," an "homomorphism," etc., are meant to bear connotations of the category of topological groups, that is, respectively, a "closed subgroup," a "continuous homomorphism," etc.

Suppose that a profinite group A is represented as a projective limit of finite groups A_λ $(\lambda \in \Lambda)$, and K is a compact topological ring. Then $KA = \varprojlim KA_\lambda$ is called a *group algebra* over K of the profinite A.

In each profinite group, there exists a system of generating elements such that every neighborhood of unity contains almost all, that is, all but finitely many, elements of that system. A *free profinite group* X with basis $\{x_j \mid j \in J\}$ is a completion of an abstract free group with a basis $\{x_j \mid j \in J\}$ in the profinite topology defined by subgroups of finite index containing almost all elements of the basis. A *free product of profinite groups* A_i $(i \in I)$ is a completion of an abstract free product of these groups in the profinite topology defined by subgroups U of finite index such that U contains almost all groups A_i and $U \cap A_i$ is an open subgroup in A_i $(i \in I)$.

A *variety of profinite groups* is a class of profinite groups closed under subgroups, homomorphic images, and direct (in the category of topological groups) products. The variety of profinite groups is uniquely assigned a class K of finite groups closed under subgroups, homomorphic images, and direct (in the category of abstract groups) products. A corresponding variety of profinite groups consists of pro-K-groups only. As is the case with abstract groups, the variety of profinite ones can be defined via identities. An *identity*, in this case, is an element of the free profinite group X_∞ with a countable basis. The identity $v \in X_\infty$ is satisfied on a profinite group G if, for any homomorphism $X_\infty \to G$, the image of v (*value* of v) equals 1. As distinct from the abstract case, every variety of profinite groups is defined by one identity.

Let $\mathcal{V}$ be a variety of profinite groups, v its defining identity, and G a profinite group. The subgroup in G generated by all values of v is called a *verbal subgroup* and is denoted by $\mathcal{V}(G)$. If X is a free profinite group with basis $\{x_j \mid j \in J\}$ and $\mathcal{V}$ a nontrivial variety, then the factor group $X/\mathcal{V}(X)$ in which is the set $\{x_j \mid j \in J\}$ embedded is a free group with basis $\{x_j \mid j \in J\}$ in $\mathcal{V}$. A *free $\mathcal{V}$-product* of groups A_i $(i \in I)$ in the variety $\mathcal{V}$ is the factor group of a free product of these groups w.r.t. a verbal subgroup corresponding to $\mathcal{V}$. Its *Cartesian subgroup* is the kernel of the canonical homomorphism onto a direct product of A_i $(i \in I)$.

We show how Abelian varieties of profinite groups are structured. Each such variety [denoted $\mathcal{A}(\Omega)$] is uniquely assigned the set

$$\Omega = \{p^{\alpha(p)} \mid p \in \pi\},$$

where π is the set of all primes and $\alpha(p)$ either is a nonnegative integer or equals ∞. A class of profinite groups corresponding to this variety consists of finite Abelian groups the periods of primary components of which have the form p^β, where $\beta \leqslant \alpha(p)$ $(p \in \pi)$. A free one-generated group of the variety in question is an additive group of the ring Z_Ω, which is a direct (topological) sum of the rings of p-adic integers Z_p for $\alpha(p) = \infty$ and the residue rings $Z/p^{\alpha(p)}Z$ for a nonnegative integer $\alpha(p)$ $(p \in \pi)$. Free groups in a bigger rank are delivered as additive groups of direct sums of copies of the ring Z_Ω.

If a profinite group A acts continuously on a profinite B, then the group B is called an *A-module*. In this event we can form a direct product of the groups A and B, which is also a profinite group; see [16, Lemma 1.3.6].

Let $\mathcal{V}$ be some variety of profinite groups and A a fixed profinite group. Consider a class of A-modules contained in $\mathcal{V}$. This class hosts free objects. A free profinite A-group with basis $\{x_j \mid j \in J\}$ in $\mathcal{V}$ is constructed as follows. Represent A as a projective limit of finite groups A_λ ($\lambda \in \Lambda$), and for every $\lambda \in \Lambda$, consider a free group X_λ with basis $\{x_j^{a_\lambda} \mid j \in J,\ a_\lambda \in A_\lambda\}$ in $\mathcal{V}$. On X_λ, the canonical action of the finite group A_λ is defined, which we can translate into the continuous action of A. Then $X = \varprojlim X_\lambda$ is a free A-group with basis $\{x_j \mid j \in J\}$ in $\mathcal{V}$. (The set $\{x_j \mid j \in J\}$ is embedded in X in the obvious way.) If $\mathcal{V}$ is an Abelian variety of profinite groups that coincides with $\mathcal{A}(\Omega)$, then X is the usual topological module with basis $\{x_j \mid j \in J\}$ over the group algebra $Z_\Omega A$.

Remark. Lemma 1, without any changes, can be brought to bear on profinite groups, provided that G is assumed finite in its formulation.

3. GENERALIZED SHMEL'KIN EMBEDDING FOR PROFINITE GROUPS

THEOREM 3. Let $F = (\underset{i\in I}{*} A_i) * X$ be a free product of nontrivial profinite groups A_i ($i \in I$) and a free profinite X with basis $\{x_j \mid j \in J\}$ (we do not exclude the case with A_i or X missing); R is a normal subgroup of F such that $R \cap A_i = 1$ ($i \in I$); $A = F/R$. Identify A_i ($i \in I$) with their canonical images in A, and write $\overline{x}$ for the canonical image of $x \in X$ in A. Let $\mathcal{V}$ be a variety of profinite groups and T a free A-module with basis $\{t_k \mid k \in I \cup J\}$ in $\mathcal{V}$. Consider a semidirect product $M = AT$ and an homomorphism $\tau\colon F \to M$ given by the map

$$a_i \to t_i a_i t_i^{-1} = a_i \cdot t_i^{a_i - 1}\ (a_i \in A_i,\ i \in I),\ x_j \to \overline{x}_j \cdot t_j\ (j \in J).$$

Then the kernel of τ coincides with $\mathcal{V}(R)$, that is, τ yields an embedding of $F/\mathcal{V}(R)$ in M.

Proof. It is a simple matter to verify that $\ker\tau \le \mathcal{V}(R)$. To prove the inverse inclusion, it suffices to establish that if $\varphi\colon F \to G$ is an homomorphism into a finite group for which $\ker\varphi \ge \mathcal{V}(R)$, then it goes through τ. Note that $\ker\varphi$ contains almost all subgroups A_i ($i \in I$) and almost all elements x_j ($j \in J$). If we allow the application of the Kaloujnin–Krasner embedding to the wreath product we reduce our problem to the case where $G = BC$ is a semidirect product (C a normal subgroup), $C \in \mathcal{V}$, $R\varphi \le C$, and if $\pi\colon G \to B$ is a canonical projection then the homomorphism $\varphi\pi\colon F \to B$ is surjective. Since $\ker\varphi \cap A_i = A_i$ for almost all $i \in I$, the group F contains an open normal subgroup H such that

$$\ker\varphi\pi \ge H \ge R,\ H \cap A_i \le \ker\varphi \cap A_i\ (i \in I).$$

Therefore, $A' = F/H$ is a finite group, and since $H \ge R$, there exists a canonical epimorphism $\psi\colon A \to A'$.

Consider a free A'-module S with basis $\{s_k \mid k \in I \cup J\}$ in $\mathcal{V}$. There is an homomorphism $\gamma\colon AT \to A'S$ determined by the homomorphism $\psi\colon A \to A'$ and by the embedding $t_k \to s_k$ ($k \in I \cup J$). Let

$$a_i\varphi = a_i\varphi\pi \cdot c_{a_i\varphi},\ c_{a_i\varphi} \in C\ (a_i \in A_i,\ i \in I),\ x_j\varphi = x_j\varphi\pi \cdot c_j,\ c_j \in C\ (j \in J).$$

Keeping in mind the remark in the preceding section and taking into account that, for any $i \in I$, the map $a_i\psi \to a_i\varphi$ ($a_i \in A_i$) is an homomorphism of the group $A_i\psi$ onto $A_i\varphi$, we can assert that the map

$$a_i\psi \to a_i\varphi\pi,\ s_i^{a_i\psi - 1} \to c_{a_i\varphi}\ (a_i \in A_i\ i \in I),\ x_j\psi \to x_j\varphi\pi,\ s_j \to c_j\ (j \in J)$$

extends to an homomorphism $\sigma\colon A'\widehat{S} \to G$, where $\widehat{S}$ is an A'-submodule of S generated by the elements

$$s_i^{a_i\psi - 1}\ (a_i \in A_i,\ i \in I),\ s_j\ (j \in J).$$

By construction, $\varphi = \tau\gamma\sigma$; so, φ goes through τ. The theorem is proved.

Let $\mathcal{V} = \mathcal{A}(\Omega)$ be an Abelian variety of profinite groups. Then, as indicated in Sec. 2, T is a right free topological module (in the classical sense) with basis $\{t_k \mid k \in I \cup J\}$ over the ring $Z_\Omega A$. The semidirect product AT can be identified with a matrix group $M = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$; the homomorphism τ is given by the map

$$a_i \to \begin{pmatrix} a_i & 0 \\ t_i(a_i - 1) & 1 \end{pmatrix} \ (a_i \in A_i,\ i \in I),\ x_j \to \begin{pmatrix} \overline{x}_j & 0 \\ t_j & 1 \end{pmatrix} \ (j \in J).$$

We establish the criterion for elements in M to belong to an embedded group $F\tau$. A similar criterion was stated in [8] for the case where $F = X$, $A = F/R$ is a pro-p-group, and $\mathcal{V} = \mathcal{A}(p^\infty)$ is a variety of all Abelian pro-p-groups.

THEOREM 4. Assume that the conditions of Theorem 3 are satisfied, and $\mathcal{V} = \mathcal{A}(\Omega)$. The matrix $\begin{pmatrix} a & 0 \\ t & 1 \end{pmatrix}$ in M, where

$$t = \sum_{i \in I} u_i t_i + \sum_{j \in J} v_j t_j,$$

belongs to $F\tau$ if and only if the following hold:

$$u_i \in (A_i - 1) \cdot Z_\Omega A \ (i \in I), \ \sum_{i \in I} u_i + \sum_{j \in J} (\overline{x}_j - 1) v_j = a - 1. \tag{5}$$

Proof. As in the proof of Theorem 2, we note that conditions (5) will distinguish a subgroup in M, denoted H, and that $F\tau \leq H$.

We argue for the inverse inclusion. Assume, to the contrary, that there exists a matrix $c \in H \setminus F\tau$. Standard manipulations will help us reduce the problem to the case where I and J are finite sets and all the groups A_i $(i \in I)$ are finite. We may also assert that there exists a canonical epimorphism of the ring Z_Ω onto some residue ring Z/mZ and there exists an epimorphism φ of the group A onto a finite A'; moreover, if T' is a free $(Z/mZ)A'$-module with basis $\{t'_k \mid k \in I \cup J\}$, and

$$\psi \colon M \to L = \begin{pmatrix} A' & 0 \\ T' & 1 \end{pmatrix}$$

is a group epimorphism determined by the epimorphism $\varphi \colon A \to A'$ and by the map $t_k \to t'_k$ $(k \in I \cup J)$, then $c\psi \notin F\tau\psi$. We also assume that the kernel S of the through homomorphism $F \to A \to A'$ satisfies the following condition: $S \cap A_i = 1$ $(i \in I)$. Let $\widehat{F}$ be an abstract free product of groups A_i $(i \in I)$ and X. The group $\widehat{F}$ is contained as an abstract subgroup in F and is dense in it; therefore, $F\tau\psi = \widehat{F}\tau\psi$. Let

$$\widehat{\tau} = \tau\psi\,|_{\widehat{F}},\ \widehat{R} = S \cap \widehat{F}.$$

By Theorem 1, the map $\widehat{\tau}$ yields an embedding for the group $\widehat{F}/(\widehat{R}^m[\widehat{R}, \widehat{R}])$. The criterion determining whether elements of a corresponding matrix group belong to $\widehat{F}/(\widehat{R}^m[\widehat{R}, \widehat{R}])$ was stated in Theorem 2. We can therefore say that the matrix

$$\begin{pmatrix} a' & 0 \\ \sum t'_k u'_k & 1 \end{pmatrix}$$

of L lies in $\widehat{F}\widehat{\tau}$ iff conditions (3) are satisfied. The matrix $c\psi$ satisfies these, which is a contradiction with $c\psi \notin F\tau\psi$. The theorem is proved.

REFERENCES

1. W. Magnus, "On a theorem of Marshall Hall," *Ann. Math.*, 40, No. 4, 764-768 (1939).
2. A. L. Shmel'kin, "Wreath products and varieties of groups," *Izv. Akad. Nauk SSSR, Ser. Mat.*, 29, No. 1, 149-170 (1965).
3. A. L. Shmel'kin, "Note on the 'Wreath products and varieties of groups'," *Izv. Akad. Nauk SSSR, Ser. Mat.*, 31, No. 2, 443-444 (1967).
4. A. L. Shmel'kin, "Free products of groups," *Mat. Sb.*, 79, No. 4, 616-620 (1969).
5. N. Blackburn, "Note on a theorem of Magnus," *J. Austr. Math. Soc.*, 10, Nos. 3/4, 469-474 (1969).
6. V. N. Remeslennikov and V. G. Sokolov, "Some properties of a Magnus embedding," *Algebra Logika*, 9, No. 5, 566-578 (1970).
7. A. L. Shmel'kin, "Some factor groups in free products," *Proc. I. G. Petrovskii Seminar*, No. 5, 209-216 (1979).
8. V. N. Remeslennikov, "Embedding theorems for profinite groups," *Izv. Akad. Nauk SSSR, Ser. Mat.*, 43, No. 2, 399-417 (1979).
9. C. Gupta and N. S. Romanovskii, "Normal automorphisms of a free pro-p-group in the variety $\mathcal{N}_2\mathcal{A}$," *Algebra Logika*, 35, No. 3, 249-267 (1996).
10. M. Krasner and L. Kaloujnin, "Produit complet des groupes de permutation et le probleme d'extension de groups, III," *Acta Sci. Math., Szeged*, 14, 69-82 (1951).
11. M. I. Kargapolov and Yu. I. Merzlyakov, *Fundamentals of Group Theory* [in Russian], 2nd edn., Nauka, Moscow (1977).
12. N. S. Romanovskii, "A Freiheitssatz for products of groups," *Algebra Logika*, 38, No. 3, 354-367 (1999).
13. H. Koch, *Galoissche Theorie der p-Erweiterungen*, VEB Deutscher Verlag der Wissenschaften, Berlin (1970).
14. J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic Pro-p-Groups, London Math. Soc., Lect. Note Ser.*, Vol. 157, Cambridge Univ., Cambridge (1991).
15. L. Ribes, *Introduction to Profinite Groups and Galois Cohomology, Queen's Papers Pure Appl. Math.*, Vol. 24, Queen's Univ., Kingston, Ontario (1970).
16. J. S. Wilson, *Profinite Groups, London Math. Soc. Mon., New Ser.*, Vol. 19, Clarendon, Oxford (1998).