Cybernetics and Systems Analysis, Vol. 33, No. 5, 1997

NECESSARY AND SUFFICIENT CONDITIONS OF EQUIVALENCE OF PROBABILITY MEASURES CORRESPONDING TO HOMOGENEOUS RANDOM FIELDS WITH SPECTRAL DENSITY

S. M. Krasnitskii

UDC 519.21

Let R^N be an N-dimensional real space, C^N an N-dimensional complex space, and $\mathcal{E}, \mathcal{D}, \mathcal{S}$ respectively spaces of infinitely differentiable functions, infinitely differentiable functions with a compact support, and rapidly decreasing infinitely differentiable functions on R^N; these spaces are endowed with the topologies accepted in the theory of generalized functions [1, 2]. Denote by $\mathcal{E}'$, $\mathcal{D}'$, $\mathcal{S}'$ the corresponding spaces of generalized functions. We refer to $\mathcal{E}'$ by its the accepted name, i.e., the space of generalized functions with a compact support. In what follows, $\mathcal{A}$ stands for the space $\mathcal{D}$ or $\mathcal{S}$. Let $(\Omega, \Sigma, P_1, P_2)$ be a statistical structure, $\xi: \Omega \times A \to C^1$ – a Gaussian real homogeneous field on $\mathcal{A}$ [3-6]. We assume that the field ξ has zero expectations over each of the measures P_1, P_2. We also assume that spectral densities corresponding to P_1, P_2 exist. We denote these densities by f_1, f_2, respectively (or $f_1(\lambda), f_2(\lambda)$, $\lambda \in R^N$).

For $T \subset R^N$ denote by Φ the set $C_0^\infty(T)$ (i.e., Φ is the set of infinitely differentiable functions with a compact support concentrated in T), by $\Sigma_\xi^\Phi \subset \Sigma$ the σ-algebra generated by the values of ξ on Φ, and by P_j^Φ the restriction of the measure P_j to Σ_ξ^Φ, $j = 1, 2$.

It is well known [6] that criteria of equivalence of P_1^Φ and P_2^Φ ($P_1^\Phi \sim P_2^\Phi$) and criteria of their orthogonality ($P_1^\Phi \perp P_2^\Phi$) play an essential role in the statistics of stochastic processes and random fields. In this article we present a number of such criteria in terms of the mutual behavior of spectral densities f_1, f_2. Although many such results have been published in the literature [6-8], necessary and sufficient conditions formulated in terms of f_1, f_2 are not easy to find [9].

AUXILIARY INFORMATION

Let us state some helpful results. We use the following notation: $\tilde{\cdot}$ is the Fourier transform, $\hat{\cdot}$ is the inverse Fourier transform, $\bar{\cdot}$ is the complex conjugate. For nonnegative x, y, we write $x << y$ for $x \le cy$, $c = \text{const} \ne 0, \infty$; $x .=. y$ stands for $x << y$, $y << x$. Always $\int = \int\limits_{R^N}$, and l_N is the Lebesgue measure in R^N. For $s, t \in R^N$: $s = (s_1, \ldots, s_N)$, $t = (t_1, \ldots, t_N)$, the scalar product $s_1t_1 + \ldots + s_Nt_N$ is denoted by (s, t).

THEOREM 1 [10]. Let T be a bounded set in R^N and assume that the function f_1 satisfies the inequality

$$f_1(\lambda) >> |\tilde{k}_0|^2(\lambda), \tag{1}$$

where $k_0 \in \mathcal{E}'$ (i.e., k_0 has a compact support) and is not identically zero. If for some $p \in [1, 2]$ we have

$$(f_1 - f_2)|\tilde{k}_0|^{-2} \in L_p(R^N), \tag{2}$$

then $P_1^\Phi \sim P_2^\Phi$.

Translated from Kibernetika i Sistemnyi Analiz, No. 5, pp. 36-44, September-October, 1997. Original article submitted August 1, 1997.

THEOREM 2 [10]. Assume that the conditions of Theorem 1 are satisfied. If for some open $V \subset R^N$ we have the inequality

$$\iint \nu(\lambda)\nu(\mu) \left| \int e^{i(\mu - \lambda, t)} I_V(t) dt \right|^2 d\lambda d\mu < +\infty, \tag{3}$$

where $\nu(\lambda) = (f_1 - f_2)(\lambda) \cdot |\tilde{k}_0|^{-2}(\lambda)$, and I_V is the indicator function of the set V, then $P_1^\Phi - P_2^\Phi$.

THEOREM 3 [8, 11]. Let T be an arbitrary subset of R^N, assume that the function f_1 satisfies the inequality

$$f_1(\lambda) << |\tilde{s}_0|^{-2}(\lambda), \tag{4}$$

where $s_0 \in \mathcal{E}'$ and moreover the difference $f_1 - f_2$ does not change its sign. If there exists a bounded open $S \subset R^N$ such that $S_0 + S \subset T$ (an algebraic sum), where S_0 is the support of s_0, then for equivalent P_1^Φ, P_2^Φ we have the inequality

$$\iint u(\lambda) u(\lambda) \left| \int_S e^{-i(\lambda - \mu, t)} dt \right|^2 d\lambda d\mu < \infty,$$

where $u(\lambda) = |\tilde{s}_0|^2(\lambda)(f_1 - f_2)(\lambda)$.

LEMMA 1. Let f be an arbitrary measurable function on R^N, T and V measurable bounded subsets of R^N, $V \subset T$. Then convergence of

$$\iint f(\lambda) f(\lambda) \left| \int_T e^{-i(\lambda - \mu, t)} dt \right|^2 d\lambda du \tag{5}$$

implies convergence of

$$\iint f(\lambda) f(\lambda) \cdot \left| \int_V e^{-i(\lambda - \mu, t)} dt \right|^2 d\lambda du.$$

Proof. In the usual sense we mean absolute convergence, and it is thus sufficient to consider a nonnegative function f. Denoting the integral (5) by $J(T)$, we will show that $J(T)$ is a monotone increasing function of T, i.e., $J(V) \leq J(T)$ for $V \subset T$. Since f is nonnegative, this will complete the proof.

For $n = 1, 2, \ldots$, let

$$f_n(\lambda) = \begin{cases} f(\lambda), & \text{if } f(\lambda) \leqslant n, \ |\lambda| \leqslant n, \\ n, & \text{if } f(\lambda) > n, \ |\lambda| \leqslant n, \\ 0, & \text{if } |\lambda| \leqslant n. \end{cases}$$

We have

$$J(T) = \lim_{n \to \infty} \iint f_n(\lambda) f_n(\mu) \left| \int_T e^{-i(\lambda - \mu, t)} dt \right|^2 d\lambda du = \lim_{n \to \infty} \iint f_n(\lambda) f_n(\mu) \int_T e^{-i(\lambda - \mu, t)} dt \int_T e^{i(\lambda - \mu, s)} ds d\lambda du .$$

Denoting $r_n(t) = \int e^{i(\lambda, t)} f_n(\lambda) d\lambda$ and noting that the function $f_n(\lambda) f_n(\mu) e^{-i(\lambda - \mu, t)} e^{i(\lambda - \mu, s)}$ is absolutely integrable in the space $R^N \times R^N \times T \times T$, we change the order of integration under the sign of the limit and obtain

$$J(T) = \lim_{n \to \infty} \int_T \int_T |r_n(t - s)^2| dt ds.$$

The double integral under lim in the last equality is monotone in T. Therefore, $J(T)$ is also monotone in T. Q.E.D.

LEMMA 2. If the integral (5) converges for $T = T_\varepsilon = \{t: t = (t_1, \ldots, t_N), -\varepsilon < t_j < \varepsilon, j = 1, \ldots, N\}$ for some $\varepsilon > 0$, then it also converges for $T = T_{2\varepsilon}$.

The proof is elementary, using the equality [8]

$$\int_{T_\varepsilon} e^{-i(\lambda-\mu,t)}dt = \prod_{j=1}^{N} \frac{\sin 2\varepsilon(\lambda_j - \mu_j)}{\lambda_j - \mu_j}.$$

Combining Lemmas 1 and 2, we easily obtain the following corollary.

COROLLARY 1. If the integral (5) converges for a bounded $T \subset R^N$ with at least one interior point, then it converges for any open bounded T.

LEMMA 3. Let Λ be a compact subset of R^N and for some functional $k_0 \in \mathcal{E}'$ we have the inequality

$$f_1(\lambda) >> |\tilde{k}_0|^2(\lambda),\ \lambda \in R^N \setminus \Lambda, \tag{6}$$

where

$$|\tilde{k}_0|^2(\lambda) \geq \text{const} > 0, \lambda \in \Lambda. \tag{7}$$

Then for any open bounded set $T \subset R^N$ and $\Phi = C_0^\infty(T)$, given that

$$\int_{R^N \setminus \Lambda} \left| \frac{f_1 - f_2}{\tilde{k}_0^2} \right|^p d\lambda < +\infty \tag{8}$$

with some $p \in [1, 2]$, we have $P_1^\Phi \sim P_2^\Phi$.

Proof. Let c_1 be a constant such that $f_1(\lambda) \geq c_1 |\tilde{k}_0|^2(\lambda)$ for $\lambda \in R^N \setminus \Lambda$. Define the spectral density $f_1'(\lambda)$:

$$f'_1(\lambda) = \begin{cases} f_2(\lambda) + c_1 |\tilde{k}_0|^2(\lambda), \lambda \in \Lambda, \\ f_1(\lambda),\ \lambda \in R^N \setminus \Lambda. \end{cases}$$

The Gaussian measure $P_1'^\Phi$ corresponding to f_1' is equivalent to P_1^Φ. Indeed, noting that f_1' satisfies inequality (1), we apply Theorem 1 for $p = 1$. We have

$$\int \left| \frac{f'_1 - f_1}{\tilde{k}_0^2}(\lambda) \right| d\lambda = \int_\Lambda \left| \frac{f_2 + c_1 |\tilde{k}_0|^2 - f_1}{\tilde{k}_0^2} \right| (\lambda) d\lambda \leq c_1 l^N(\Lambda) + \text{const} \int_\Lambda |f_2 - f_1| d\lambda < +\infty.$$

It remains to show that given condition (8) we have $P'^\Phi_1 \sim P_2^\Phi$. This follows directly from Theorem 1:

$$\int \left| \frac{f'_1 - f_2}{\tilde{k}_0^2} \right|^p (\lambda) d\lambda = c_1 l^N(\Lambda) + \int_{R^N \setminus \Lambda} \left| \frac{f_1 - f_2}{\tilde{k}_0^2} \right|^p (\lambda) d\lambda < +\infty.$$

COROLLARY 2. Given conditions (6), (7), the measure P_1^Φ is equivalent to the measure Q_1^Φ corresponding to the spectral density g:

$$g(\lambda) = \begin{cases} f(\lambda),\ \lambda \in \Lambda, \\ f_1(\lambda), \lambda \in R^N \setminus \Lambda, \end{cases}$$

where f is any nonnegative summable function on Λ.

MAIN RESULTS

THEOREM 4. Let T be a bounded open subset of R^N, $k_0 \in \mathcal{E}'$, $s_0 \in \mathcal{E}'$. Assume that the following relationships hold for some $r \geq 0$:

$$f_1(\lambda) .=. |\tilde{k}_0|^2(\lambda) \quad \text{for} \quad |\lambda| \geq r, \tag{9}$$

$$f_1(\lambda) .=. |\tilde{s}_0|^{-2}(\lambda) \quad \text{for} \quad |\lambda| \geq r, \tag{10}$$

$$f_1(\lambda) - f_2(\lambda) \leq 0 \quad \text{or} \quad f_1(\lambda) - f_2(\lambda) \geq 0 \quad \text{for} \quad |\lambda| \geq r. \tag{11}$$

$$|\tilde{k}_0(\lambda)| \geq \text{const} > 0 \quad \text{for} \quad |\lambda| \leq r. \tag{12}$$

Then a necessary condition for equivalence of P_1^Φ, P_2^Φ is that for every bounded open subset $V \subset R^N$ we have

$$\int_{|\lambda| \geq r} \int_{|\mu| \geq r} \frac{f_1 - f_2}{f_1}(\lambda) \frac{f_1 - f_2}{f_1}(\mu) \left| \int_V e^{-i(\mu - \lambda, t)} dt \right|^2 d\lambda d\mu < +\infty. \tag{13}$$

A sufficient condition for equivalence of P_1^Φ, P_2^Φ is that inequality (13) is satisfied for some open $V \subset R^N$. In particular, it is necessary and sufficient that the integral converges.

$$\int_{|\lambda| \geq r} \int_{|\mu| \geq r} \frac{f_1 - f_2}{f_1}(\lambda) \frac{f_1 - f_2}{f_1}(\mu) \prod_{j=1}^{N} \frac{\sin^2(\lambda_j - \mu_j)}{(\lambda_j - \mu_j)^2} d\lambda d\mu. \tag{14}$$

Sufficiency. Note that we need to consider only the first inequality in (11). Indeed, if the second inequality holds, then we can define the spectral density $f_2' = 2f_1 - f_2$ and the corresponding measure P'^Φ_2. Noting that $f_1 - f'_2 = -(f_1 - f_2)$, we see that if $P_1^\Phi \sim P_2'^\Phi$, then $P_1^\Phi \sim P_2^\Phi$ (the last follows directly also from general conditions of equivalence of P_1^Φ, P_2^Φ; see, e.g., [11, Theorem 1]). In this case, $f_2' \geq f_1$ at least for $\lambda \in R^N$ such that $f_1(\lambda) \geq f_2(\lambda)$. Moreover, the integrand in (13) is invariant to substitution of f_2' for f_2. We thus assume that $f_2(\lambda) \geq f_1(\lambda)$, $|\lambda| \geq r$. In this case it follows from (9) that for $|\lambda| \leq r$ the values of both f_2 and f_1 can be replaced with the values of any locally summable nonnegative functions (Corollary 2). We therefore assume that for these λ we have $f_1(\lambda) = f_2(\lambda) = c_1 |\tilde{k}_0|^2(\lambda)$, where c_1 is such that $f_1 \geq c_1 |\tilde{k}_0|^2(\lambda)$ for $|\lambda| \geq r$ (condition (9)). We may thus assume that $f_1(\lambda)$ satisfies (1) for all $\lambda \in R^N$ and apply Theorem 2. The integral in (3) is equal to the integral in (13); (14) is obtained when V is identified with the rectangle $\{t: t = (t_1, \ldots, t_N), -1 \leq t_j \leq 1, j = 1, \ldots, N\}$.

Necessity. First assume that (11) holds for all $\lambda \in R^N$, and for definiteness again consider the first inequality. Define the spectral density

$$f'_1(\lambda) = \begin{cases} d_1 |\tilde{s}_0|^{-2}(\lambda), & |\lambda| \leq r, \\ f_1, & |\lambda| > r, \end{cases}$$

where d_1 is a constant such that $f_1(\lambda) \geq d_1 |\tilde{s}_0|^{-2}(\lambda)$ for $|\lambda| \geq r$. By Lemma 3, the measure $P_1'^\Phi$, corresponding to $f_1'(\lambda)$ is equivalent to P_1^Φ, and everywhere in R^N we have $f_1'(\lambda) \leq f_2(\lambda)$, and also $f_1' .=. |\tilde{s}_0|^{-2}(\lambda)$. Therefore Theorem 3 and Lemma 3 are applicable to the pair f_1', f_2. Finally, if (11) is satisfied only for $|\lambda| \geq r$, then define the spectral density

$$f''_1(\lambda) = \begin{cases} f_2(\lambda), & |\lambda| \leq r, \\ f_1(\lambda), & |\lambda| > r, \end{cases}$$

and the corresponding measure P''^Φ_1, which by Corollary 2 is equivalent to the measure P_1^Φ, and $f_1'' \leq f_2$ everywhere in R^N. We have thus reduced this case to the previously considered particular case. Q.E.D.

We give one corollary of Theorem 4, which holds under an additional assumption about the behavior of the spectral densities f_1 and f_2. To this end we define the function W:

$$W(\lambda) = \begin{cases} 0, & |\lambda| \leq r, \\ \dfrac{f_1 - f_2}{f_1}(\lambda), & |\lambda| > r. \end{cases}$$

The function W is defined for almost all $\lambda \in R^N$, because from (9) it follows that the set of zeros of the function f_1 for $|\lambda| > r$ is of Lebesgue measure zero.

COROLLARY 3. Assume that conditions (9)-(12) are satisfied and, moreover, there exists a set $H \subset R^N$ of positive Lebesgue measure such that for any $h \in H$,

$$0 < \underline{\lim} \frac{W(\lambda + h)}{W(\lambda)} < +\infty \quad (|\lambda| \to \infty). \tag{15}$$

Then for equivalence of $P_1{}^\Phi$, $P_2{}^\Phi$ it is necessary and sufficient that

$$\int_{|\lambda| > R} W^2(\lambda) d\lambda < +\infty \tag{16}$$

for some $R \geq 0$.

Sufficiency. It is easy to see that if (16) holds, then integral (14) converges.

Necessity. Convergence of (14) implies convergence of

$$\iint W(\lambda) W(\mu) \prod_{j=1}^{N} \frac{\sin^2(\lambda_j - \mu_j)}{(\lambda_j - \mu_j)^2} d\lambda d\mu.$$

Rewriting the last integral as a repeated integral and making the change of variables $\lambda - \mu = h$, we obtain convergence of

$$\iint W(\mu) W(\mu + h) \prod_{j=1}^{N} \frac{\sin^2 h_j}{h_j^2} d\mu dh.$$

Changing the order of integration, we obtain the convergent integral

$$\int \prod_{j=1}^{N} \frac{\sin^2 h_j}{h_j^2} \left(\int W(\mu) W(\mu + h) d\mu \right) dh.$$

In particular, $\int W(\mu) W(\mu + h) d\mu$ should converge for almost every $h \in R^N$.

Condition (15) at the same time implies that for some such h there exist constants $R \geq 0$ and $c > 0$ for which

$$\operatorname{vrai} \inf_{|\mu| > R} \frac{W(\mu + h)}{W(\mu)} \geqslant c.$$

Thus, applying also (11), we obtain

$$+\infty > \int_{|\mu| > R} W(\mu) W(\mu + h) d\mu = \int_{|\mu| > R} W^2(\mu) \left| \frac{W(\mu + h)}{W(\mu)} \right| d\mu \geqslant c \int_{|\mu| > R} W^2(\mu) d\mu.$$

As an application, consider the case with asymptotic power isotropic spectral densities. We use the notation

$$g(\lambda) \approx \sum_{k=1}^{n} g_k(\lambda), \ \lambda \to a, \ n \geqslant 1,$$

in the sense that for every m, $1 \leq m \leq n$,

$$g(\lambda) = \sum_{k=1}^{m} g_k(\lambda) + o(g_m(\lambda)), \lambda \to a.$$

Example. Let T be a bounded open subset of R^N and for $|\lambda| \to \infty$,

$$f_1(\lambda) \approx \sum_{k=1}^{n} a_k |\lambda|^{\alpha_k}, \tag{17}$$

$$f_2(\lambda) = \sum_{k=1}^{n} b_k |\lambda|^{\beta_k}, \tag{18}$$

where a_k, b_k, α_k, β_k are real numbers, $a_k \neq 0$, $b_k \neq 0$ $(k = 1, \ldots, n)$, $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, $\beta_1 > \beta_2 > \ldots > \beta_n$.

Application of Corollary 3 leads to the following criterion: given conditions (17), (18), P_1^Φ, P_2^Φ are equivalent only if for all $k = \{1, \ldots, n\}$ such that

$$\alpha_1 - \alpha_k \leq \frac{N}{2} \quad \text{or} \quad \alpha_1 - \beta_k \leq \frac{N}{2}, \tag{19}$$

we have

$$a_k = b_k, \; \alpha_k = \beta_k. \tag{20}$$

If the greatest k for which (19) holds is less than n, then (20) is also sufficient for equivalence of P_1^Φ, P_2^Φ.

Remark. This example and equivalence criterion constitute a generalization and a certain refinement of the well-known examples and criteria from [13] ($N = 1$, $n = 1$) and [14] ($N = 1$, $n = 1, 2$).

REFERENCES

1. V. S. Vladimirov, Generalized Functions in Mathematical Physics [in Russian], Nauka, Moscow (1976).
2. L. Hoermander, Analysis of Linear Partial Differential Operators [Russian translation], Vol. 1, Nauka, Moscow (1986).
3. A. M. Yaglom, "Some classes of random fields in *n*-dimensional space related to stationary stochastic processes," Teor. Veroyatn. Ee Primen., **2**, No. 3, 292-338 (1957).
4. I. M. Gel'fand and N. Ya. Vilenkin, Generalized Functions [in Russian], Vol. 4, Fizmatgiz, Moscow (1961).
5. I. I. Gikhman, A. V. Skorokhod, and M. I. Yadrenko, Probability Theory and Mathematical Statistics [in Russian], Vishcha Shkola, Kiev (1979).
6. Handbook of Probability Theory and Mathematical Statistics [in Russian], Nauka, Moscow (1975).
7. Yu. A. Rozanov, Gaussian Infinite-Dimensional Distributions [in Russian], Trudy Mat. Inst. im. V. M. Steklova, Nauka, Moscow (1968).
8. M. I. Yadrenko, Spectral Theory of Random Fields [in Russian], Vishcha Shkola, Kiev (1980).
9. S. M. Krasnitsky (Krasnitskii), "On some conditions for Gaussian measures equivalence," Abstracts First Ukr.-Scand. Conf. on Stochastic Dynamical Systems TB i MC Sci. Publ., Kiev (1995).
10. S. M. Krasnitskii, "A class of spectral equivalence conditions of Gaussian measures corresponding to homogeneous random fields," Dop. AN Ukr., No. 3, 37-39 (1993).
11. S. M. Krasnitskii, "Necessary conditions of equivalence of Gaussian measures corresponding to homogeneous random fields," Ukr. Mat. Zh., **46**, No. 6, 708-719 (1994).
12. S. M. Krasnitskii, "Equivalence conditions for distributions of Gaussian homogeneous fields," Teor. Veroyatn. Ee Primen., **34**, No. 4, 786-791 (1989).
13. S. G. Gladyshev, "A new limit theorem for stochastic processes with Gaussian increments," Teor. Veroyatn. Ee Primen., **6**, No. 1, 57-66 (1961).
14. V. G. Alekseev, "Conditional mutual singularity of Gaussian measures corresponding to two stochastic processes," Teor. Veroyatn. Ee Primene., **8**, No. 3, 304-308 (1963).