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Limits of Sequences of Operators on Spaces of Vector Valued Functions

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ABSTRACT. *We generalize the celebrated theorem of Stein on the maximal operator of a sequence of translation invariant operators, from the scalar case to vector valued functions.*

1. Preliminaries

We begin with some probabilistic prerequisites.

The following theorem, proved in [3], will be used in this article.

Theorem 1.

Let $X = \{X_j\}$ be a sequence of independent mean–zero random variables defined on $(\Omega, \sum, \nu)$, and B a Banach space. Assume that X satisfies the Khinchin–Kahane inequality in B, as generalized in [3],

$$\left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_p \le A_p \left\| u_0 + \sum_{j=1}^{N} X_j u_j \right\|_1 , \text{ for some } p > 1, \forall N > 0 ,$$

where $u_j \in B$. There exist constants $\alpha > 0$, $\beta_q > 0$ where $0 < q \le p$ s.t. if $\sum_{j=1}^{\infty} X_j u_j$ converges a.e. then $\forall E \in \sum$, $\mu(E) > 0$, $\exists n = n(E)$ s.t.

$$\mu \left\{ w \in E : \left\| u_0 + \sum_{j=1}^{\infty} X_j(w) u_j \right\| \ge \beta_q \left\| \sum_{j=n+1}^{\infty} X_j u_j \right\|_q \right\} \ge \alpha \cdot \mu(E) .$$

For a scalar version of this theorem, see [1].

Corollary 1.

Let $X = \{X_j\}$ be a sequence of independent mean–zero random variables satisfying the conditions in Theorem 1. There is a constant $\beta_p > 0$ s.t. $\forall E \in \sum$, $\exists n = n(E)$, if $\{u_j, j \ge 0\}$ are vectors in the Banach space, and $\sum_{j=1}^{\infty} X_j u_j$ converges a.e. on E, then

$$\beta_p \left\| \sum_{j=n}^{\infty} X_j u_j \right\|_p \le esssup_{w \in E} \left\| u_0 + \sum_{j=1}^{\infty} X_j u_j \right\| .$$

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We prove a principle of contraction in Orlicz spaces.

Theorem 2.

Let $(\Omega, \sum, \mu)$ be a probability space, and $\{X_j, j \geq 1\}$ be a sequence of independent mean-zero random variables in L^ϕ. Let $0 \leq \lambda_j \leq 1, \forall j$. Let $\{u_j, j \geq 1\}$ be vectors in a Banach space. Then for any $N \geq 1$,

$$\left\| \sum_{j=1}^{N} \lambda_j X_j u_j \right\|_{L^\phi} \leq \left\| \sum_{j=1}^{N} X_j u_j \right\|_{L^\phi} .$$

Proof. It is enough to show that for any $\gamma > \| \sum_{j=1}^{N} X_j u_j \|_{L^\phi}$,

$$\int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} \lambda_j X_j u_j \right\| \right) d\mu \leq \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} X_j u_j \right\| \right) d\mu .$$

Let us consider first $\lambda_j = 0, 1$. We may assume that $\lambda_j = 1, 1 \leq j \leq k$; $\lambda_j = 0, k+1 \leq j \leq N$. Then by Jessen's inequality

$$\int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} \lambda_j X_j u_j \right\| \right) d\mu = \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{k} X_j(w) u_j \right\| \right) d\mu(w)$$

$$= \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{k} X_j(w) u_j + \int_\Omega \sum_{j=k+1}^{N} X_j(w') u_j \, d\mu(w') \right\| \right) d\mu(w)$$

$$\leq \int_\Omega \phi \left(\frac{1}{\gamma} \int_\Omega \left\| \sum_{j=1}^{k} X_j(w) u_j + \sum_{j=k+1}^{N} X_j(w') u_j \right\| d\mu(w') \right) d\mu(w)$$

$$\leq \int_\Omega \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{k} X_j(w) u_j + \sum_{j=k+1}^{N} X_j(w') u_j \right\| \right) d\mu(w') \, d\mu(w)$$

$$= \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} X_j(w) u_j \right\| \right) d\mu(w) .$$

In general, for $0 \leq \lambda_j \leq 1$, let

$$\lambda_j = \sum_{k=1}^{\infty} 2^{-k} \lambda_{jk}, \ \lambda_{jk} = 0, 1, \ 1 \leq j \leq N ,$$

we have

$$\sum_{j=1}^{N} \lambda_j X_j u_j = \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{N} \lambda_{jk} X_j u_j .$$

Since ϕ is convex,

$$\int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} \lambda_j X_j u_j \right\| \right) d\mu$$

$$\leq \sum_{k=1}^{\infty} 2^{-k} \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} \lambda_{jk} X_j u_j \right\| \right) d\mu$$

$$\leq \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=1}^{N} X_j u_j \right\| \right) d\mu . \qquad \square$$

For an L^p version of Theorem 2 with $\{X_j\}$ an independent sequence of symmetric random variables, see [2, 4].

Corollary 2.

If ϕ is a strictly increasing Young function, then the convergence of $\sum_{j=1}^{\infty} X_j u_j$ in L^ϕ–norm implies the a.e. convergence of $\sum_{j=1}^{\infty} \lambda_j X_j u_j$ for any bounded sequence $\{\lambda_j\}$.

Proof. Let $\gamma > \| \sum_{j=1}^{\infty} X_j u_j \|_{L^\phi}$. We can assume that $\lambda_j \in \mathcal{R}$ and $|\lambda_j| \leq 1$.

Since ϕ is strictly increasing, $\forall \alpha > 0$, $\phi(\alpha\gamma^{-1}) > 0$. Thus,

$$\mu \left\{ w : \left\| \sum_{j=n}^{M} \lambda_j X_j(w) u_j \right\| > \alpha \right\} = \mu \left\{ w : \phi \left(\frac{1}{\gamma} \left\| \sum_{j=n}^{M} \lambda_j X_j(w) u_j \right\| \right) > \phi \left(\frac{\alpha}{\gamma} \right) \right\}$$

$$\leq \frac{1}{\phi(\alpha\gamma^{-1})} \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=n}^{M} \lambda_j X_j u_j \right\| \right) d\mu$$

$$\leq \frac{1}{\phi(\alpha\gamma^{-1})} \int_\Omega \phi \left(\frac{1}{\gamma} \left\| \sum_{j=n}^{M} X_j u_j \right\| \right) d\mu \to 0, \quad \text{as } n, M \to \infty .$$

This shows that $S_n = \sum_{j=1}^{n} \lambda_j X_j(w) u_j$ is Cauchy in measure, and hence converges in measure. Since $\{X_j\}$ is an independent sequence, we get that $S_n = \sum_{j=1}^{n} \lambda_j X_j(w) u_j$ also converges a.e. $\square$

Theorem 3.

Let $X = \{X_j\}$ be a sequence of random variables as in Theorem 2, and B a Banach space. Let $S_n = \sum_{j=1}^{n} X_j u_j$, $n = 1, 2, \ldots$ where $u_j \in B$. Let ϕ be a Young function. If a subsequence $\{S_{n_k}\}$ converges in L^Φ, then $\{S_n\}$ converges in L^Φ.

Proof. Given $\epsilon > 0$, $\exists N = N(\epsilon)$ s.t. whenever $n_r, n_q > N$,

$$\| S_{n_q} - S_{n_r} \|_{L^\phi} = \left\| \sum_{j=n_r+1}^{n_q} X_j u_j \right\|_{L^\phi} < \epsilon .$$

Now let $N < n_r < n < m < n_q$, applying Theorem 2, we get

$$\| S_m - S_n \|_{L^\phi} = \left\| \sum_{j=n+1}^{m} X_j u_j \right\|_{L^\phi}$$

$$\leq \left\| \sum_{j=n_r+1}^{n_q} X_j u_j \right\|_{L^\phi} = \| S_{n_q} - S_{n_r} \|_{L^\phi} < \epsilon .$$

Thus, S_n converges in L^Φ. $\square$

2. Limits of Sequences of Operators on Spaces of Vector Valued Functions

In [5] Stein proved that with some minimal conditions, the maximal operator defined by a sequence of linear operators on $L^\Phi(M)$ which commute with group action on M, is of weak type

(Φ, Φ) for Φ in a certain class of Young functions.

We extend Stein's result by considering operators that are defined on function spaces whose elements take values in Banach spaces of type p, $1 \le p \le 2$, and map them to functions with values in an arbitrary Banach space. We also consider a somewhat larger class of operators than that of translation invariant ones.

Definition 1.

An operator $T : L^{\phi}_{B}(M) \to L^{\phi}_{E}(M)$ is of type (ϕ, ϕ) if there is a constant A' so that

$$\|Tf\|_{L^{\phi}_{E}} \le A' \, \|f\|_{L^{\phi}_{B}}$$

holds for every $f \in L^{\phi}_{B}(M)$ where ϕ is a Young function and B, E are Banach spaces.

Definition 2.

Let G be a group and M a homogeneous space of G. Assume that G acts on M transitively. Let f be defined on M. We define $(\tau_g f)(x) = f(g^{-1}(x))$, $\forall g \in G, \forall x \in M$ and call τ_g a translation by g.

Let T map B–valued functions on M to E–valued functions on M, where B and E are two Banach spaces. If there exist linear operators $\Lambda(g, f, T) : E \to E$ so that $\|\Lambda(g, f, T)\, u\| \ge c\, \|u\|$, $\forall u \in E$, for some $c = c(T) > 0$, and so that

$$T(\tau_g f)(x) = \Lambda(g, f, T)\, \tau_g (Tf)(x)\,,$$

then we say that T is a quasi–translation–invariant (QTI) operator.

We say $\{T_n\}$ is uniformly QTI if $c(T_n) \ge c > 0$.

Let Φ denote a Young function on $[0, \infty)$ so that $\Phi(t^{\frac{1}{p}})$ is concave and $\Phi(0) = 0$. It follows that for all $\alpha \ge 1$, $\Phi(\alpha t) \le \alpha^p \Phi(t)$, and this implies that $f \in L^{\Phi}_{B}(M)$ iff f is strongly measurable and $\int_M \Phi(\|f\|)dm < \infty$. Hence, for functions Φ as above, T is of type (Φ, Φ) iff

$$\int_M \Phi(\|Tf\|_E)dm \le \int_M \Phi(A\|f\|_B)dm$$

for some constant A.

Theorem 4.

Let B be a Banach space of Rademacher type p, $1 < p \le 2$. Let Φ be a strictly monotone Young function, and assume also that $\Phi(t^{\frac{1}{p}})$ is concave. Let T_n be a sequence of linear operators which are uniformly QTI, and each of which is of type (Φ, Φ). Let G be a compact group and m the G–invariant measure on M s.t. $m(M) = 1$. If $\forall f \in L^{\Phi}_{B}(M)$, $T^*(f)(x) := \sup_{n\ge 1} \|(T_n f)(x)\|_E < \infty$ on some set of positive measure then there exists a constant A (independent of f and α) s.t.

$$m\{x : T^* f(x) > \alpha\} \le \int_M \Phi\left(\frac{A}{\alpha}\, \|f\|_B\right) dm, \ \forall \alpha > 0\,.$$

Proof. We will omit subscripts E, B for the norms $\|.\|_E$ and $\|.\|_B$ in the following proof.

Suppose that the theorem does not hold. Then one can find $\{f_j\}$, a sequence of functions in $L^{\Phi}_{B}(M)$ so that

$$\int_M \Phi(j^2 \, \|f_j\|)dm \le m\left\{x : T^* f_j(x) > 1\right\} \le 1\,.$$

Since $\Phi(0) = 0$ and Φ is convex, we have

$$j^2 \cdot \int_M \Phi(\|f_j\|)dm \le \int_M \Phi(j^2 \, \|f_j\|)dm\,.$$

Let k_j be positive integers s.t.

$$\frac{1}{2j^2} < k_j \cdot \int_M \Phi(\|f_j\|)dm \le \frac{1}{j^2}, \quad j \ge 1 .$$

Define

$$F_{ji} = f_j, \; i = 1, 2, \ldots, k_j ,$$
$$E_{ji} = \{x : T^* F_{ji} > 1\} = \{x : T^* f_j(x) > 1\}, \quad j \ge 1 .$$

Then

$$\begin{aligned}\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} m(E_{ji}) &= \sum_{j=1}^{\infty} k_j m \{x : T^* f_j(x) > 1\} \\ &\ge \sum_{j=1}^{\infty} k_j \int_M \Phi(j^2 \|f_j\|)dm \\ &\ge \sum_{j=1}^{\infty} j^2 k_j \int_M \Phi(\|f_j\|)dm = \infty .\end{aligned}$$

Let us recall Lemma 1 in [5]:

If $\{E_n\}$ is a sequence of sets in M, with the property that $\sum m(E_n) = \infty$, then there exists a sequence of elements in G, $\{g_n\}$, s.t. almost every point in M belongs to infinitely many sets $g_n(E_n) = \{g_n(x) : x \in E_n\}$, i.e.,

$$m\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} g_n(E_n)\right) = 1 .$$

Applying this lemma, we have that there exists $\{g_{ji}\}$, a sequence of elements in G, s.t.

$$m\left(\bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} \bigcup_{i=1}^{k_j} g_{ji}(E_{ji})\right) = 1 \tag{2.1}$$

Since $\int_M \Phi(\|\tau_{g_{ji}} F_{ji}\|)\, dm = \int_M \Phi(\|F_{ji}\|)\, dm$, we have

$$\begin{aligned}&\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} \int_M \Phi\left(\left\| j^{\frac{1}{4}} \tau_{g_{ji}} F_{ji} \right\|\right) dm \\ &= \int_M \sum_{j=1}^{\infty} k_j \Phi\left(\left\| j^{\frac{1}{4}} f_j \right\|\right) dm \\ &\le \sum_{j=1}^{\infty} j^{\frac{p}{4}} k_j \int_M \Phi(\|f_j\|)\, dm \\ &\le \sum_{j=1}^{\infty} j^{-2+\frac{p}{4}} < \infty .\end{aligned}$$

Let $\{r_{ji}\}$ be an independent sequence of Rademacher functions on $I = [0, 1)$, and let μ be Lebesgue measure on I. Since B is of Rademacher type p, we get

$$\int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, \tau_{g_{ji}} F_{ji}(x) \right\|^p d\mu \leq C^p \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{p}{4}} \left\| \tau_{g_{ji}} F_{ji}(x) \right\|^p .$$

Let $\psi(t) = \phi(t^{1/p})$. Since $\Psi \geq 0$ is concave it is subadditive and so

$$\begin{aligned}
&\int_M \int_I \Phi\left(\left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji} \tau_{g_{ji}} F_{ji} \right\| \right) d\mu\, dm \\
&= \int_M \int_I \Psi\left(\left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji} \tau_{g_{ji}} F_{ji} \right\|^p \right) d\mu\, dm \\
&\leq \int_M \Psi\left(\int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji} \tau_{g_{ji}} F_{ji} \right\|^p d\mu \right) dm \\
&\leq \int_M \Psi\left(C^p \sum_{j=1}^{\infty} j^{\frac{p}{4}} \sum_{i=1}^{k_j} \left\| \tau_{g_{ji}} F_{ji} \right\|^p \right) dm \\
&\leq C^p \sum_{j=1}^{\infty} j^{\frac{p}{4}} k_j \int_M \Phi(\|f_j\|)\, dm < \infty .
\end{aligned}$$

This implies that $\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, \tau_{g_{ji}} F_{ji}(x)$ converges in $L_B^{\Phi}(I \times M)$. We define

$$G(t, x) = \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, \tau_{g_{ji}} F_{ji}(x) .$$

There exists a subsequence of partial sums of $\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, \tau_{g_{ji}} F_{ji}(x)$ which converges in $L_B^{\Phi}(M)$ for a.e. $t \in I$. Since T_n is a linear operator of type (Φ, Φ), we have that for a.e. $t \in I$,

$$T_n G(t, x) = \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, T_n \left(\tau_{g_{ji}} F_{ji} \right)(x)$$

where a fixed subsequence of partial sums of right-hand side converges in $L_E^{\Phi}(M)$ to $T_n G(t, x)$. Similarly,

$$\begin{aligned}
&\int_I \int_M \Phi(\|T_n G(t, x)\|)\, dm\, d\mu \\
&\leq \int_I \int_M \Phi(A_n \|G(t, x)\|)\, dm\, d\mu \\
&\leq A_n^p \int_I \int_M \Phi(\|G(t, x)\|)\, dm\, d\mu
\end{aligned}$$

$$= A_n^p \int_M \int_I \Phi\left(\left\|\sum_{j=1}^{\infty}\sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}\, \tau_{g_{ji}} F_{ji}\right\|\right) d\mu\, dm$$

$$\leq A_n^p C^p \sum_{j=1}^{\infty} j^{\frac{p}{4}} k_j \int_M \Phi(\|f_j\|)\, dm < \infty .$$

This implies that

$$T_n G(t,x) = \sum_{j=1}^{\infty}\sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, T_n\left(\tau_{g_{ji}} F_{ji}\right)(x)$$

is also a $L_E^{\Phi}(I \times M)$ function. Hence, there exists a subsequence of partial sums of $\sum_{j=1}^{\infty}\sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, T_n\, \tau_{g_{ji}} F_{ji}(x)$ which converges in $L_E^{\Phi}(I)$ for a.e. $x \in M$. By Theorem 3, we have for a.e. $x \in M$,

$$T_n G(t,x) = \sum_{j=1}^{\infty}\sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, T_n\left(\tau_{g_{ji}} F_{ji}\right)(x) \quad \text{converges in} \quad L_E^{\Phi}(I),$$

and by Corollary 2 the series converges a.e. in $I \times M$.

It follows from (2.1) that for a.e. x there exist infinitely many indices (j, i) s.t. $g_{ji}^{-1}x \in E_{ji}$. Therefore, for infinitely many (j, i), we have

$$j^{\frac{1}{4}} T^* F_{ji}\left(g_{ji}^{-1}x\right) > j^{\frac{1}{4}} .$$

Let us show that this implies $T^*G(t,x) = \infty$ a.e. on $I \times M$.

If $T^*G(t,x) < \infty$ on a set of positive measure in $I \times M$, then there exists a constant $C > 0$ and a set $S \subset I \times M$ s.t.

$$\mu \bigotimes m(S) > 0, \quad \text{and} \quad T^*G(t,x) \leq C, \ (t,x) \in S .$$

This implies that $\|T_n G(t,x)\| \leq C$ on S, for all n.

For $x \in M$ denote $S_x = \{t \in I : (t,x) \in S\}$. $m\{x \in M : \mu(S_x) > 0\} > 0$, and so from (2.1), $\{x \in M : \mu(S_x) > 0\} \bigcap \left(\bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} \bigcup_{i=1}^{k_j} g_{ji}(E_{ji})\right)$ is not empty. Let x be a point in this intersection.

By Corollary 1, there exists $N = N(x)$ s.t. for all n,

$$\int_I \left\|\sum_{j=N}^{\infty}\sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}\, T_n\left(\tau_{g_{ji}} F_{ji}\right)(x)\right\| d\mu$$

$$\leq \beta_1^{-1} esssup_{t \in S_x} \left\|\sum_{j=1}^{\infty}\sum_{i=1}^{k_j} j^{\frac{1}{4}} r_{ji}(t)\, T_n\left(\tau_{g_{ji}} F_{ji}\right)(x)\right\| \leq \beta_1^{-1} C .$$

Applying the principle of contraction to the left-hand side of the inequality above, we have

$$j^{\frac{1}{4}} \left\|T_n\left(\tau_{g_{ji}} F_{ji}\right)(x)\right\| \leq \beta_1^{-1} C, \ \forall j \geq N, \ 1 \leq i \leq k_j .$$

Since $T_n(\tau_{g_{ji}} F_{ji})(x) = \Lambda(g_{ji}, F_{ji}, T_n)\, \tau_{g_{ji}}(T_n F_{ji})(x)$, and since T_n satisfy the QTI condition uniformly, we have

$$\frac{1}{c} j^{\frac{1}{4}} < \frac{1}{c} j^{\frac{1}{4}} T^* F_{ji} \left(g_{ji}^{-1} x \right) \leq \beta_1^{-1} C, \; j \geq N, \; 1 \leq i \leq k_j ,$$

a contradiction.

Since $G(t, x)$ is a $L_B^{\Phi}(M)$ function for a.e. $t \in I$ and $T^* G(t, x) = \infty$ a.e. on $I \times M$, we have a contradiction. □

Remark. When $E = B = C$, then $p = 2$. Taking $\Lambda(g, f, T_n)$ to be the identity, we obtain Theorem 3 in [5]. □

Theorem 5.

Let B be a Banach space of Rademacher type p, $1 < p \leq 2$. Let T_n be a sequence of uniformly QTI bounded operators from $L_B^q(M)$ to $L_E^r(M)$ where $1 \leq q < \infty$, $0 < r < \infty$. Let $T^*(f)(x) = \sup_{n \geq 1} \|(T_n f)(x)\|_E$, $\forall f \in L_B^q(M)$, and denote $s = \min(p, q)$. If $\forall f \in L_B^q(M)$, $T^*(f)(x) < \infty$ on a set of positive measure then T^* is of weak type (q, s), i.e., there exists a constant A (independent of f and α) s.t.

$$m \left\{ x : T^* f(x) > \alpha \right\} \leq A \alpha^{-s} \|f\|_q^s, \; \forall \alpha > 0 .$$

Proof. The case $q = 1$ follows from Theorem 6 below. Let us consider, therefore, $1 < q < \infty$. It is enough to prove the case $q \geq p$, since if $q < p$, then B is also of type q, and the same proof applies.

Although the proof is similar to that of Theorem 4, there are some important differences, in particular if $\Phi(t) = t^q$ then $\Phi(t^{\frac{1}{p}})$ is not concave. We therefore give a full proof of this theorem.

If the theorem does not hold, then one can find $\{f_j\}$, a sequence of functions in L_B^q so that

$$j^3 \left\| f_j \right\|_q^p \leq m \left\{ x : T^* f_j(x) > 1 \right\} \leq 1 .$$

Let $\{k_j\}$ be positive integers s.t.

$$\frac{1}{2 j^3} < k_j \left\| f_j \right\|_q^p \leq \frac{1}{j^3} .$$

Define

$$F_{ji} = f_j, 1 \leq i \leq k_j ,$$
$$E_{ji} = \left\{ x : T^* F_{ji} > 1 \right\} ,$$

then

$$\sum_{j=1}^{\infty} j \sum_{i=1}^{k_j} \left\| \tau_{g_{ji}} F_{ji} \right\|_q^p = \sum_{j=1}^{\infty} j k_j \left\| f_j \right\|_q^p \leq \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty ,$$

and

$$\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} m(E_{ji})$$
$$= \sum_{j=1}^{\infty} k_j m \left\{ x : T^* f_j(x) > 1 \right\} = \infty .$$

As before, this implies that there exists a sequence of elements $\{g_{ji}\}$ in G s.t.

$$m \left(\bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} \bigcup_{i=1}^{k_j} g_{ji}(E_{ji}) \right) = 1 .$$

Since B is of type p, we have

$$\int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t)\, \tau_{g_{ji}} F_{ji}(x) \right\|^p d\mu$$
$$\leq C^p \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j \left\| \tau_{g_{ji}} F_{ji}(x) \right\|^p .$$

Moreover, for a.e. $x \in M$, by the Khinchin–Kahane inequality for Rademacher series,

$$\int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t)\ \tau_{g_{ji}} F_{ji}(x) \right\|^q d\mu$$
$$\leq C(p,q) \left(\int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t)\ \tau_{g_{ji}} F_{ji}(x) \right\|^p d\mu \right)^{\frac{q}{p}}$$
$$\leq C(p,q) \left(\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j \left\| \tau_{g_{ji}} F_{ji}(x) \right\|^p \right)^{\frac{q}{p}} .$$

Integrating both sides and applying Minkowski's inequality, we have

$$\int_M \int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}\ \tau_{g_{ji}} F_{ji} \right\|^q d\mu dm$$
$$\leq \int_M \left(C(p,q) \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j \left\| \tau_{g_{ji}} F_{ji}(x) \right\|^p \right)^{\frac{q}{p}} dm$$
$$\leq \left(C(p,q) \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j \left\| \tau_{g_{ji}} F_{ji} \right\|_q^p \right)^{\frac{q}{p}} < \infty .$$

This shows that

$$G(t,x) = \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t)\ \tau_{g_{ji}} F_{ji}(x)$$

is an $L^q_B(I \times M)$ function, and moreover the series converges in $L^q_B(I \times M)$. Thus, there exists a subsequence of partial sums of $\sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t)\ \tau_{g_{ji}} F_{ji}(x)$ which converges in $L^q(M)$ for a.e. $t \in I$. Since T_n is a bounded linear operator from $L^q_B(M)$ to $L^r_E(M)$, we have that for a.e. $t \in I$,

$$T_n G(t,x) = \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t) T_n \left(\tau_{g_{ji}} F_{ji}\right)(x)$$

where a subsequence of partial sums of right-hand side converges in $L^r_E(M)$ to $T_n G(t,x)$ for a.e. t. We may assume that this subsequence converges a.e. in x for a.e. t. Thus, it also converges a.e. in t for a.e. x.

It follows that

$$\int_I \int_M \|T_n G(t,x)\|^r \, dm d\mu \le A_n^r \int_I \left(\int_M \|G(t,x)\|^q dm\right)^{\frac{r}{q}} d\mu \, .$$

If $r \ge q$, then by the Minkowski's inequality and the Khinchin–Kahane inequality

$$\int_I \left(\int_M \|G(t,x)\|^q dm\right)^{\frac{r}{q}} d\mu \le \left(\int_M \left(\int_I \|G(t,x)\|^r d\mu\right)^{\frac{q}{r}} dm\right)^{\frac{r}{q}}$$

$$\le C(q,r) \left(\int_M \int_I \left\| \sum_{j=1}^{\infty} \sum_{i=1}^{k_j} j^{\frac{1}{p}} r_{ji}(t) \tau_{g_{ji}} F_{ji}(x) \right\|^q d\mu dm\right)^{\frac{r}{q}} < \infty$$

as shown before. If $r < q$, then

$$\int_I \left(\int_M \|G(t,x)\|^q dm\right)^{\frac{r}{q}} d\mu \le \left(\int_M \int_I \|G(t,x)\|^q d\mu dm\right)^{\frac{r}{q}} < \infty \, .$$

Thus, we conclude that $T_n G(t,x)$ is an $L_E^r(I \times M)$ function.

Notice that $\forall x \in \bigcap_{l=1}^{\infty} \bigcup_{j=l}^{\infty} \bigcup_{i=1}^{k_j} g_{ji}(E_{ji})$, there exist infinitely many indices (j,i) s.t. $g_{ji}^{-1} x \in E_{ji}$. Therefore, for infinitely many (j,i), we have

$$j^{\frac{1}{p}} T^* \, F_{ji}\left(g_{ji}^{-1} x\right) > j^{\frac{1}{p}} \, .$$

This implies $T^* G(t,x) = \infty$ a.e. on $I \times M$.

The rest of the proof is the same as in Theorem 4. □

Remark. When $E = B = C$, then $p = 2$. Taking $q = r$ and $\Lambda(g, f, T_n)$ to be the identity, we obtain Theorem 1 in [5] from the case $q \le 2$, and Theorem 11 in [5] from the case $q \ge 2$.

In view of Theorem 5 and the fact that $B \subset L^p$ is of Rademacher type $\min(2, p)$, we immediately have

Corollary 3.

Let $p \ge 1$ and $B \subset L^p$ be a Banach subspace. Let T_n be a sequence of operators as in Theorem 5. Suppose that for every $f \in L_B^q(M)$ with $q \ge 1$, $T^*(f)(x) < \infty$ on a set of positive measure. Let $s = \min\{p, q, 2\}$. Then T^* is of weak type (q, s).

Theorem 6.

Let K be a Banach space whose elements are measurable functions defined on M. Assume that K is translation invariant, i.e., $\forall g \in G, \forall f \in K, \|\tau_g f\|_K = \|f\|_K$. Let $\{T_n\}$ be a sequence of uniformly QTI operators from K to $L_E^r(M)$ where E is a Banach space and $0 < r < \infty$. Let $T^*(f)(x) = \sup_{n \ge 1} \|(T_n f)(x)\|_E$, $\forall f \in K$. If $\forall f \in K$, $T^*(f)(x) < \infty$ a.e. on a set of positive measure then there exists a constant A (independent of f and α) s.t.

$$m\left\{x : T^* f(x) > \alpha\right\} \le \frac{A}{\alpha} \|f\|_K, \ \forall \alpha > 0 \, .$$

Proof. If the conclusion does not hold, we have, as in the proof of Theorem 4, sequences $\{f_j\}$, $\{k_j\}$ s.t.

$$j^2 \, \left\|f_j\right\|_K \le m\left\{x : T^* f_j(x) > 1\right\} \le 1 \, ,$$
$$\tfrac{1}{2\, j^2} < k_j \left\|f_j\right\|_K \le \tfrac{1}{j^2}, \ \ j = 1, 2, \ldots$$

Define as before

$$F_{ji} = f_j,\ 1 \le i \le k_j\ ,$$
$$E_{ji} = \left\{x : T^* F_{ji} > 1\right\}\ ,$$

we then have

$$\sum_{j=1}^{\infty}\sum_{i=1}^{k_j} m\left(E_{ji}\right) = \sum_{j=1}^{\infty} k_j m\left\{x : T^* f_j(x) > 1\right\} = \infty\ ,$$

and so there exists a sequence of elements $\{g_{ji}\}$ in G s.t.

$$m\left(\bigcap_{l=1}^{\infty}\bigcup_{j=l}^{\infty}\bigcup_{i=1}^{k_j} g_{ji}(E_{ji})\right) = 1\ .$$

Since $\|\tau_{g_{ji}} F_{ji}\|_K = \|F_{ji}\|_K$, we have

$$\sum_{j=1}^{\infty}\sum_{i=1}^{k_j}\left\|j^{\frac{1}{2}}\,\tau_{g_{ji}} F_{ji}\right\|_K = \sum_{j=1}^{\infty} j^{\frac{1}{2}}\, k_j\, \|f_j\|_K$$
$$\le \sum_{j=1}^{\infty} j^{-\frac{3}{2}} < \infty\ .$$

Let $\{r_{ji}\}$ be a sequence of independent Rademacher functions, then

$$G(t,\cdot) = \sum_{j=1}^{\infty}\sum_{i=1}^{k_j} r_{ji}(t)\ j^{\frac{1}{2}}\,\tau_{g_{ji}} F_{ji}(\cdot) \quad \text{converges in} \quad K\ \ \forall t \in I, \quad \text{and}$$

$$\|G(t,\cdot)\|_K \le \sum_{j=1}^{\infty} j^{-\frac{3}{2}} < \infty\ .$$

Since T_n is a bounded operator from K to $L_E^r(M)$, we have that for a.e. $t \in I$,

$$T_n G(t,x) = \sum_{j=1}^{\infty}\sum_{i=1}^{k_j} r_{ji}(t)\ j^{\frac{1}{2}}\, T_n\left(\tau_{g_{ji}} F_{ji}\right)(x)$$

where the series converges in $L_E^r(M)$.

Moreover, since $\|T_n G(t,;)\|_r \le \|T_n\|\ \|G(t,\cdot)\|_K$, we have that

$$\int_I \int_M \|T_n G(t,x)\|^r\, dm\, d\mu < \infty\ ,$$

i.e., $T_n G(t,x)$ is an $L_E^r(I \times M)$ function. We get $T^* G(t,x) = \infty$ a.e. on $I \times M$, and the proof proceeds as in Theorem 4. □

Remark. When $E = \mathcal{C}, r = 1, K \subset L^1(M)$, and $\Lambda(g, f, T_n)$ is the identity, we obtain Theorem 10 in [5]. □

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