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One-Sided Littlewood-Paley Theory

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ABSTRACT. In this article we develop the theory of one-sided versions of the g function of Littlewood and Paley, the area function S of Lusin and the g_λ^* that admit weighted norm estimates with weights belonging to the classes A_p^+ of Sawyer. In Sections 1 and 2 we give definitions and some lemmas that shall be needed. Section 3 is devoted to the study of the one-sided version of the functions g and S. In Section 4 we obtain a good λ estimate for the one-sided g_λ^* function, and in Sections 5 and 6 we apply the results already obtained to fractional integrals and multiplier operators.

1. Notations and Definitions

As usual, $\mathcal{S}(\mathbb{R})$ denotes the class of all those C^∞–functions φ defined on $\mathbb{R}$ such that

$$\sup_{x\in R} |x^\alpha (D^\beta \varphi)(x)| < \infty\,,$$

for all non–negative integers α and β. Let B be a Banach space and let r be a positive integer. We shall consider the space $C_0^r(B)$ of all B–valued functions φ defined on $\mathbb{R}$, with compact support and such that its derivatives $D^\beta\varphi$, $1 \le \beta \le r$, are continuous. If $B = \mathbb{R}$, we simply write C_0^r. Given a Lebesgue measurable set $E \subseteq \mathbb{R}$, we denote its Lebesgue measure by $|E|$ and the characteristic function of E by χ_E. Let f be a measurable function defined on $\mathbb{R}$, the one–sided Hardy–Littlewood maximal functions $M^- f$ and $M^+ f$, are given by

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|\, dt \text{ and } M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)|\, dt\,.$$

For $0 < \alpha < 1$, the one–sided fractional integrals of f are defined as

$$I_\alpha^- f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-\alpha}}\, dy \text{ and } I_\alpha^+ f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-\alpha}}\, dy\,.$$

We extend these definitions to the case $\alpha = 0$, setting $I_0^- f(x) = I_0^+ f(x) = f(x)$.

As usual, a weight w is a measurable and non–negative function. If $E \subseteq \mathbb{R}$ is a Lebesgue measurable set, we denote its w–measure by $w(E) = \int_E w(t)\, dt$. A weight w belongs to the class

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A_p^+, $1 < p < \infty$, see [12], if there exists a constant C such that

$$\sup_{h>0}\left(\frac{1}{h}\int_{x-h}^{x} w(t)\,dt\right)\left(\frac{1}{h}\int_{x}^{x+h} w(t)^{-\frac{1}{p-1}}\,dt\right)^{p-1} \le C\,,$$

for all real number x, and w belongs to A_1^+ if $M^-w(x) \le Cw(x)$ holds for almost every x. Given w belonging to A_p^+, $1 \le p < \infty$, we can define $x_{-\infty} \ge -\infty$ and $x_\infty \le \infty$ such that

$$\begin{cases} (i) & w(x) \equiv 0 \qquad \text{in } (-\infty, x_{-\infty})\,, \\ (ii) & w(x) \equiv \infty \qquad \text{in } (x_\infty, \infty), \text{ and} \\ (iii) & 0 < w(x) < \infty \quad \text{for almost every } x \in (x_{-\infty}, x_\infty)\,. \end{cases} \tag{1.1}$$

We always have $x_{-\infty} \le x_\infty$. In order to avoid the non–interesting case of $x_{-\infty} = x_\infty$, we assume that there exists a measurable set E satisfying $0 < w(E) < \infty$.

If $(B, \|\cdot\|_B)$ is a Banach space, we shall consider the Bochner–Lebesgue space $L_B^p(w)$, $1 \le p < \infty$, consisting of all strongly measurable functions $f : \mathbb{R} \to B$ for which

$$\|f\|_{L_B^p(w)} = \left(\int_{-\infty}^{\infty} \|f(x)\|_B^p\, w(x)\,dx\right)^{1/p}\,,$$

is finite. If $B = \mathbb{R}$, the space $L_B^p(w)$ shall be denoted by $L^p(w)$.

Given two Banach spaces, $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$, we denote by $\mathcal{L}(A, B)$ the space of all bounded linear operators T from A into B with the norm $\|T\|_{\mathcal{L}(A,B)} = \sup_{\|x\|_A=1} \|T(x)\|_B$. The Hilbert space $H = L^2\left(\mathbb{R}^+, \frac{dt}{t}\right)$ consists of all measurable functions f defined on $\mathbb{R}^+ = (0, \infty)$ such that $\|f\|_H = \left(\int_0^\infty f(t)^2 \frac{dt}{t}\right)^{1/2}$ is finite. Let c be a real number, we shall say that f belongs to $L^1_{loc}(c, \infty)$ if $\int_a^b |f(x)|\,dx < \infty$ for every $c < a < b < \infty$.

2. Basic Lemmas

The next lemma contains the results about fractional integrals that shall be needed in the sequel.

Lemma 1.

Let σ be a continuous function defined on the real line and $\gamma \ge 1$ such that,

(a) $supp(\sigma) \subset (-\infty, 0]$, $\lim_{x\to-\infty} D^s\sigma(x) = 0$, for every s, $0 \le s < \gamma$, and

(b) there exists β, $0 < \beta \le 1$ such that $|D^\gamma\sigma(x)| \le c/(1-x)^{\gamma+1+\beta}$ holds for $x < 0$.

These conditions on σ imply that $|D^s\sigma(x)| \le c/(1-x)^{s+1+\beta}$ for $x < 0$, and $0 \le s < \gamma$. In addition to (a) and (b) we ask σ to satisfy

(c) $\int_{-\infty}^0 \sigma(x)\,dx = 0$.

Then, given $0 < \alpha < \beta \le 1$ and any β', $\alpha < \beta' < \beta$, we have that $I_\alpha^+(\sigma)$ satisfies:

(i) $I_\alpha^+(\sigma)(x) = 0$ if $x > 0$ and $\lim_{x\to-\infty} D^s I_\alpha^+(\sigma)(x) = 0$, for every s, $0 \le s < \gamma$,

(ii) $|D^s I_\alpha^+(\sigma)(x)| \le c/(1-x)^{s+1+\beta'-\alpha}$ for $x < 0$, and $0 \le s \le \gamma$, and

(iii) $\int_{-\infty}^0 I_\alpha^+(\sigma)(x)\,dx = 0$.

Proof. We observe that if ρ is a bounded and integrable function, then the fractional integral $I_\alpha^+(\rho)$ is a bounded and continuous function tending to zero at infinity.

Let us estimate $I_\alpha^+(D^s\sigma)$, $1 \le s \le \gamma$. Let $x < -2$. Since $\text{supp}(\sigma) \subset (-\infty, 0]$, we have

$$I_\alpha^+(D^s\sigma)(x) = \int_0^{-x/2} \frac{D^s\sigma(x+y)}{y^{1-\alpha}}\,dy + \int_{-x/2}^{-x} \frac{D^s\sigma(x+y)}{y^{1-\alpha}}\,dy = I + II\,.$$

For the integral I we get,

$$|I| \leq \frac{c}{(1-x)^{s+1+\beta}} \left(\frac{-x}{2}\right)^{\alpha} \leq \frac{c}{(-x)^{s+1+\beta-\alpha}} \leq \frac{c}{(-x)^{s+1+\beta'-\alpha}} .$$

In order to deal with II, we define

$$h_s(x) = \int_{-\infty}^{x} D^{s-1}\sigma(z)\, dz .$$

Since $|D^{s-1}\sigma(z)| \leq c/(1-z)^{s+\beta}$, then $|D^{s-1}\sigma(z)| \leq c/(1-z)^{s+\beta'}$ and using this estimate for $D^{s-1}\sigma$ it follows that

$$|h_s(x)| \leq \frac{c}{(1-x)^{s-1+\beta'}} \quad , \quad |Dh_s(x)| = |D^{s-1}\sigma(x)| \leq \frac{c}{(1-x)^{s+\beta'}}$$

and

$$|D^2 h_s(x)| = |D^s\sigma(x)| \leq \frac{c}{(1-x)^{1+s+\beta'}} \quad \text{for} \quad x < 0 .$$

Moreover, $Dh_s(0) = D^{s-1}\sigma(0) = 0$ and $h_s(0) = \int_{-\infty}^{0} D^{s-1}\sigma(z)\, dz = 0$. Then, integrating II by parts, we get

$$\begin{aligned} II \;&=\; -Dh_s\left(\frac{x}{2}\right)\left(\frac{2}{-x}\right)^{1-\alpha} - (1-\alpha)\, h_s\left(\frac{x}{2}\right)\left(\frac{2}{-x}\right)^{2-\alpha} \\ &\quad + (1-\alpha)(2-\alpha) \int_{-x/2}^{-x} \frac{h_s(x+y)}{y^{3-\alpha}}\, dy . \end{aligned}$$

The first two terms are bounded by $c/(-x)^{1+s+\beta'-\alpha}$. The third term is bounded by

$$c \int_{-x/2}^{-x} \frac{1}{y^{3-\alpha}} \frac{1}{(-x-y)^{s-1+\beta'}}\, dy = \frac{c}{(-x)^{1+s+\beta'-\alpha}} \int_{1/2}^{1} \frac{1}{y^{3-\alpha}} \frac{1}{(1-y)^{\beta'}}\, dy .$$

Therefore, we have shown that

$$|I_\alpha^+(D^s\sigma)(x)| \leq \frac{c}{(-x)^{1+s+\beta'-\alpha}}, \quad 1 \leq s \leq \gamma , \tag{2.1}$$

for any β', $\alpha < \beta' < \beta$.

Now, we shall show that

$$\int_{-\infty}^{x} I_\alpha^+(D^s\sigma)(z)\, dz = D^{s-1} I_\alpha^+(\sigma)(x), \quad 1 \leq s \leq \gamma . \tag{2.2}$$

By (2.1) we know that $I_\alpha^+(D^s\sigma)$ is integrable and by the observation at the beginning of the proof $I_\alpha^+(D^s\sigma)$ is also continuous. Then, if (2.2) holds, it follows that $I_\alpha^+(D^s\sigma) = D^s I_\alpha^+(D\sigma)$, $1 \leq s \leq \gamma$, and we obtain parts (i) and (ii) of this lemma. Let us prove (2.2). For $x < 0$, we have

$$\begin{aligned} \int_{-N}^{x} I_\alpha^+(D^s\sigma)(z)\, dz &= \int_{-N}^{x} \left(\int_0^{-z} \frac{D^s\sigma(z+y)}{y^{1-\alpha}}\, dy\right) dz \\ = \int_0^{-x} \frac{1}{y^{1-\alpha}} \left(\int_{-N}^{x} D^s\sigma(z+y)\, dz\right) dy &+ \int_{-x}^{N} \frac{1}{y^{1-\alpha}} \left(\int_{-N}^{-y} D^s\sigma(z+y)\, dz\right) dy \\ &= I_\alpha^+(D^{s-1}\sigma)(x) - I_\alpha^+(D^{s-1}\sigma)(-N) . \end{aligned}$$

Then taking the limit for N tending to infinity, we get that (2.2) holds. Let us prove (iii). From (2.1) and (2.2) we obtain

$$|I_\alpha^+(\sigma)(x)| \leq \frac{c}{(-x)^{1+\beta'-\alpha}} \quad \text{if} \quad x \leq 0 .$$

By Fubini's Theorem, a change of variables, and (c)

$$\int_{-N}^{0} I_\alpha^+(\sigma)(x)\,dx = \int_{-N}^{0} \left(\int_0^{-x} \frac{\sigma(x+y)}{y^{1-\alpha}}\,dy \right) dx$$

$$= \int_0^N \frac{1}{y^{1-\alpha}} \left(\int_{-N+y}^{0} \sigma(z)\,dz \right) dy = -\int_0^N \frac{1}{y^{1-\alpha}} \left(\int_{-\infty}^{-N+y} \sigma(z)\,dz \right) dy .$$

The absolute value of the last integral is bounded by a constant times

$$\int_0^N \frac{1}{y^{1-\alpha}} \frac{1}{(1+N-y)^{\beta'}}\,dy = \frac{1}{N^{\beta'-\alpha}} \int_0^1 \frac{1}{y^{1-\alpha}} \frac{1}{(1-y)^{\beta'}}\,dy .$$

Then taking the limit for N tending to infinity it follows that

$$\int_{-\infty}^{0} I_\alpha^+(\sigma)(x)\,dx = 0 .$$

ending the proof of the lemma. □

Lemma 2.

Let $\varphi \in \mathcal{S}(\mathbb{R})$, m a positive integer and $\rho(x) = x^m \varphi(x)$. Then, for every $t > 0$, holds

$$\partial_t^m \varphi_t(x) = \partial_t^m \left[\frac{1}{t} \varphi\left(\frac{x}{t}\right) \right] = \frac{(-1)^m}{t^{m+1}} \left[\left(\frac{d}{dx}\right)^m \rho \right] \left(\frac{x}{t}\right) .$$

Proof. The proof is simple and shall be omitted. □

Lemma 3.

Let $w \in A_q^+$, $1 < q < \infty$ and $0 < t \leq 1$. Then for every non-negative $u \in L^{q'/t}(w)$ there exists a non-negative $v \in L^{q'/t}(w)$ such that

(a) $u(x) \leq v(x)$ a. e.,

(b) $\|v\|_{L^{q'/t}(w)} \leq 2\|u\|_{L^{q'/t}(w)}$ and

(c) $vw \in A_p^+$ if $p = (1-t)q + t$.

Proof. This is the one–sided version of Lemma 5.17 of [5, page 447]. The sublinear function used in the proof of our lemma is $S(u) = (M^-(|u|^{1/t}w)w^{-1})^t$. □

Lemma 4.

Let B be a Banach space and $1 < r < \infty$. Let U_j be a sequence of linear operators such that

$$\left(\int \|U_j f(x)\|_B^r\, \rho(x)\,dx \right)^{1/r} \leq c_r(\rho) \left(\int |f(x)|^r \rho(x)\,dx \right)^{1/r} \tag{2.3}$$

holds for every $\rho \in A_r^+$ with a constant $c_r(\rho)$ not depending on j. Then, for every $1 < p < \infty$ and $1 < q < \infty$, we have that

$$\left(\int \left(\sum_j \|U_j f_j(x)\|_B^p \right)^{q/p} w(x)\,dx \right)^{1/q} \tag{2.4}$$

$$\leq c_{p,q}(w)\left(\int\left(\sum_j |f_j(x)|^p\right)^{q/p} w(x)\,dx\right)^{1/q}$$

holds for every $w \in A_q^+$.

Proof. We observe that by extrapolation, see [9], if (2.3) holds for a given r, $1 < r < \infty$, then it holds for every r, $1 < r < \infty$. If $p = q$, the proof of the theorem is trivial. Let $p < q$, by Lemma 3 with $t = (q-p)/(q-1)$ given $0 \leq u \in L^{(q/p)'}(w)$ there exists v such that

$$\|v\|_{L^{(q/p)'}(w)} \leq 2\,\|u\|_{L^{(q/p)'}(w)},$$

and

$$\left(\int \|U_j f\|_B^p\, u(x)w(x)\,dx\right)^{1/p} \leq \left(\int |f(x)|^p v(x)w(x)\,dx\right)^{1/p},$$

where c does not depend on j. Then proceeding as in Theorem 6.1 of [5, page 519] we get (2.4) for $1 < p < q < \infty$. As for the case $p > q$ we have

$$\left(\int\left(\sum_j \|U_j f_j(x)\|_B^p\right)^{q/p} w(x)\,dx\right)^{1/q} = \int \sum_j \|U_j f\|_B\, g_j(x)\,dx,$$

where $\{g_j(x)\} \in L^{q'}_{\ell^{p'}}(w^{-q'/q})$. Now proceeding as in the proof of Theorem 6.4 of [5, page 519] we obtain that (2.4) holds for $p > q$. □

3. One-Sided Littlewood-Paley and Lusin Area Functions

In this section we shall develop the theory of the one-sided versions of the Littlewood-Paley and Lusin area functions using vector-valued methods.

Theorem 1.

Let H_1 and H_2 be Hilbert spaces and k a strongly measurable $\mathcal{L}(H_1, H_2)$-valued function defined for $x \neq 0$, and strongly measurable. Assume that k satisfies the conditions

(i) $\|k(x)\|_{\mathcal{L}(H_1,H_2)} \leq B_1 \frac{1}{|x|}$,

(ii) for $x \neq 0$, $Dk(x)$ exists in the $\mathcal{L}(H_1, H_2)$-norm and

$$\|Dk(x)\|_{\mathcal{L}(H_1,H_2)} \leq B_2 \frac{1}{|x|^2},$$

(iii) for any pair (ϵ, N), $0 < \epsilon < N$,

$$\left\|\int_{\epsilon<|x|<N} k(x)\,dx\right\|_{\mathcal{L}(H_1,H_2)} \leq B_3, \text{ and}$$

(iv) if $x > 0$, then $k(x) = 0$.

Then, the operator

$$K^* f(x) = \sup_{\epsilon>0}\left\|\int_{|x-y|>\epsilon} k(x-y)f(y)\,dy\right\|_{H_2},$$

satisfies:
if $w \in A_p^+$, $1 < p < \infty$, *there exists a constant c such that*

$$\int K^* f(x)^p\ w(x)\, dx \ \leq\ c \int \|f(x)\|_{H_1}^p w(x)\, dx$$

holds, and if $w \in A_1^+$, *there exists a constant c such that*

$$\lambda\, w\left(\{x : K^*(f)(x) > \lambda\}\right) \ \leq\ c \int \|f(x)\|_{H_1} w(x)\, dx$$

holds for any $\lambda > 0$. *The constants c depend on* p, B_1, B_2, B_3 *and the constant of the condition* A_p^+ *for* w.

Proof. The proof is a straightforward generalization of the proof given in [1] for the scalar case, i.e., $H_1 = H_2 = \mathbb{C}$ the complex numbers. □

Theorem 2.

If in addition to conditions (i), (ii), (iii) and (iv) of Theorem 1 we assume that
(v) For any $u \in H_1$ *and* $v \in H_2$,

$$\lim_{\epsilon \to 0} \left\langle v, \left(\int_{\epsilon < |x| < 1} k(x)\, dx \right) u \right\rangle \tag{3.1}$$

exist, then

$$Kf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} k(x-y) f(y)\, dy = \lim_{\epsilon \to 0} K_\epsilon f(x) \tag{3.2}$$

exists weakly in H_2 *for any* $f \in C_0^1(H_1)$,

$$\int \|Kf(x)\|_{H_2}^p\ w(x)\, dx \ \leq\ c_p \int \|f(x)\|_{H_1}^p\ w(x)\, dx \tag{3.3}$$

holds for $1 < p < \infty$, $w \in A_p^+$, *and*

$$\lambda\, w\left(\left\{x : \|Kf(x)\|_{H_2} > \lambda\right\}\right) \ \leq\ c_1 \int \|f(x)\|_{H_1} w(x)\, dx \tag{3.4}$$

holds for any $\lambda > 0$ *and* $w \in A_1^+$.

Moreover, Kf *can be extended to* $L_{H_1}^p(w)$ *and so that (3.3) and (3.4) hold and the limit in (3.2) exists weakly in* H_2 *a.e. for a general* $f \in L_{H_1}^p(w)$, $w \in A_p^+$, $1 \leq p < \infty$.

Proof. Assumption (v) is equivalent to assume that there exists $l \in \mathcal{L}(H_1, H_2)$ such that the limit in (3.1) is equal to $\langle v, lu \rangle$. Then, if $f \in C_0^1(H_1)$ and $v \in H_2$

$$\lim_{\epsilon \to 0} \langle v, K_\epsilon f(x) \rangle = \left\langle v, \int_{|x-y|<1} k(x-y)[f(y) - f(x)]\, dy \right\rangle$$
$$+ \left\langle v, \int_{|x-y|>1} k(x-y) f(y)\, dy \right\rangle + \langle v, lf(x) \rangle = \langle v, Kf(x) \rangle\,.$$

Now, Theorem 1 and standard arguments (see, for instance [3, page 110]), allow us to extend K to a general f, $f \in L_{H_1}^p(w)$, remaining valid (3.2), (3.3), and (3.4). □

Given an integer $\gamma \geq 1$ and $x \in \mathbb{R}$, we shall say that a C_0^∞–function ψ belongs to the class $\Phi_\gamma(x)$ if there exists a bounded interval $I_\psi = [x, \beta]$ containing the support of ψ such that $D^\gamma \psi$ satisfies

$$|I_\psi|^{\gamma+1} \|D^\gamma \psi\|_\infty \leq 1\,.$$

Let F be a distribution on $\mathcal{D}'(r, \infty)$, $-\infty \le r < \infty$. We define the one-sided maximal function $F^*_{+,\gamma}(x)$ as

$$F^*_{+,\gamma}(x) = \sup\{| < F, \psi > | : \psi \in \Phi_\gamma(x)\}$$

for every $x > r$.

Fixed $w \in A^+_q$, $q \ge 1$, we shall consider $x_{-\infty}$ and x_∞ as in (1.1). Given $0 < p$, and $\gamma \ge 1$ satisfying

$$(\gamma + 1)p \ge q > 1 \text{ or } (\gamma + 1)p > q = 1\,, \tag{3.5}$$

we shall say that the distribution F in $\mathcal{D}'(x_{-\infty}, \infty)$, belongs to $H^p_{+,\gamma}(w)$ if

$$\|F\|_{H^p_{+,\gamma}(w)} = \left(\int_{x_{-\infty}}^{\infty} F^*_{+,\gamma}(x)^p w(x)\, dx\right)^{1/p},$$

is finite.

Remark 1.

We observe that if γ_1 and γ_2 satisfy the condition (3.5) and $\gamma_1 \le \gamma_2$, then, taking into account the definition of $F^*_{+,\gamma}$, we have the inclusion $H^p_{+,\gamma_1}(w) \subseteq H^p_{+,\gamma_2}(w)$. On the other hand, in virtue of the decomposition into atoms obtained in Theorem 2.2 of [11], it follows that $H^p_{+,\gamma_2}(w) \subseteq H^p_{+,\gamma_1}(w)$, and therefore $H^p_{+,\gamma_1}(w) = H^p_{+,\gamma_2}(w)$. Finally, we remark that the set of all bounded functions f with bounded support belonging to $H^p_{+,\gamma}(w)$ is dense in $H^p_{+,\gamma}(w)$. Also, it can be shown that the set of C^1_0-functions f belonging to $H^p_{+,\gamma}(w)$ is dense in $H^p_{+,\gamma}(w)$.

Theorem 3.

Let γ be a positive integer and $0 < p < \infty$ such that $p(\gamma + 1) > 1$. Let K be a singular integral operator as in Theorem 1 for $H_1 = \mathbb{C}$ and $H_2 = H$, a Hilbert space. Moreover, we assume that the kernel k of K satisfies

$$\|D^\ell k(x)\|_{\mathcal{L}(\mathbb{C},H)} \le B_{2,\ell} \frac{1}{|x|^{\ell+1}}\,, \tag{3.6}$$

for every ℓ, $0 \le \ell \le \gamma$.

Then, if $w \in A^+_{p(\gamma+1)}$ we have

$$\int_{x_{-\infty}}^{\infty} \|Kf(x)\|^p_H\, w(x)\, dx \le c \int_{x_{-\infty}}^{\infty} f^*_{+,\gamma}(x)^p\, w(x)\, dx \tag{3.7}$$

with a constant c not depending on f.

Proof. Let f be a bounded function with bounded support. Since f induces a distribution in $\mathcal{D}'(-\infty, \infty)$, then we consider the maximal function $f^*_{+,\gamma}(x)$ defined for every real number x. The sets

$$\Omega_i = \{x : f^*_{+,\gamma}(x) > 2^i\}\,, \quad i \in \mathbb{Z}\,,$$

are open and bounded. Then, applying Theorem 2.2 of [11], with respect to $w \equiv 1$, if $I_{i,j}$ stands for the connected components of Ω_i, there exist functions $a_{i,j}(x)$ such that

(i) $\|a_{i,j}\|_\infty \le C$,

(ii) $supp(a_{i,j}) \subseteq I_{i,j}$,

(iii) $\int a_{i,j}(x)x^s\, dx = 0$ for every s, $0 \le s \le \gamma - 1$,

and

(iv) $f(x) = \sum_i 2^i \sum_j a_{i,j}(x)$ in L^2.

Thus,

$$Kf(x) = \sum_i 2^i \sum_j Ka_{i,j}(x)$$

in the sense of L^2_H, and therefore

$$\|Kf(x)\|_H \le \sum_i 2^i \sum_j \|Ka_{i,j}(x)\|_H .$$

Given a bounded interval $I = [\alpha, \beta]$, we denote $\tilde{I} = [3\alpha - 2\beta, \beta]$. Since (3.6) holds, it can be shown, as usual, that for $x \notin \tilde{I}_{i,j}$ we have

$$\|Ka_{i,j}(x)\|_H \le c\,[M^+(\chi_{I_{i,j}})(x)]^{\gamma+1} . \tag{3.8}$$

If $x \in \tilde{I}_{i,j}$ and $\|Ka_{i,j}(x)\|_H \le 1$ we see that (3.8) holds with a constant which is a fixed multiple of the former c. Finally, if $x \in \tilde{I}_{i,j}$ and $\|Ka_{i,j}(x)\|_H > 1$, then

$$\|Ka_{i,j}(x)\|_H \le \|Ka_{i,j}(x)\|_H^{\gamma+1} .$$

Thus, we have shown that for any x

$$\|Ka_{i,j}(x)\|_H \le c\,[M^+(\chi_{I_{i,j}})(x)^{\gamma+1} + \|Ka_{i,j}(x)\|_H^{\gamma+1}]$$

holds. Then,

$$\int \|Kf(x)\|_H^p\, w(x)\,dx \le c_p \int \left(\sum_{i,j} M^+(2^{i/(\gamma+1)}\chi_{I_{i,j}})(x)^{\gamma+1}\right)^{\frac{p(\gamma+1)}{\gamma+1}} w(x)\,dx$$
$$+ c_p \int \left(\sum_{i,j} \|K(2^{i/(\gamma+1)}a_{i,j})(x)\|_H^{\gamma+1}\right)^{\frac{p(\gamma+1)}{\gamma+1}} w(x)\,dx .$$

Since $w \in A^+_{p(\gamma+1)}$, by Lemma 4 applied to the operators M^+ and K, we obtain

$$\int \|Kf(x)\|_H^p\, w(x)\,dx \le c \int \left(\sum_i 2^i \sum_j \chi_{I_{i,j}}(x)\right)^p w(x)\,dx$$
$$\le c \int \left(\sum_i 2^i \chi_{\Omega_i}(x)\right)^p w(x)\,dx \le c' \int f^*_{+,\gamma}(x)^p\, w(x)\,dx .$$

By Remark 1, (3.7) holds for every $f \in H^p_{+,\gamma}(w)$. □

Theorem 4.

Let γ be a positive integer and $0 < p < \infty$ such that $p(\gamma+1) > 1$. Let ϕ be a function satisfying

(i) $\phi(x) = 0$ if $x > 0$ and $\lim_{x\to-\infty} D^\ell\phi(x) = 0$ for every ℓ, $0 \le \ell < \gamma$,

(ii) $D^{\gamma-1}\phi$ is continuously differentiable on $(-\infty, 0)$, and for a β, $0 < \beta \le 1$

$$|D^\gamma\phi(x)| \le \frac{c_\gamma}{(1-x)^{1+\gamma+\beta}} \quad \textit{if} \quad x < 0 .$$

These conditions imply

$$|D^\ell\phi(x)| \le \frac{c_\ell}{(1-x)^{1+\ell+\beta}} \quad \textit{if} \quad x < 0 \textit{ and } 0 \le \ell \le \gamma . \tag{3.9}$$

In addition to (i) and (ii), let us assume
(iii) $\int_{-\infty}^{\infty} \phi(y)\, dy = 0$.

Then, if we define

$$g^+(f)(x) = \left(\int_0^\infty |(\phi_t * f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

there exists a constant c such that

$$\|g^+(f)\|_{L^p(w)} \le c\, \|f^*_{+,\gamma}\|_{L^p(w)} \tag{3.10}$$

holds if $w \in A^+_{p(\gamma+1)}$. *In consequence, if* $p > 1$ *and* $w \in A^+_p$, *there exists a constant* c' *such that*

$$\|g^+(f)\|_{L^p(w)} \le c'\, \|f\|_{L^p(w)}\,. \tag{3.11}$$

Moreover, for $\lambda > 0$

$$\lambda\, w\left(\left\{x : g^+(f)(x) > \lambda\right\}\right) \le c\, \|f\|_{L^1(w)} \tag{3.12}$$

holds if $w \in A^+_1$.

If for $a > 0$, *we define*

$$S^+_a(f)(x) = \left(\int\int_{0 \le z < at} |(\phi_t * f)(x+z)|^2 \frac{dz dt}{t^2}\right)^{1/2}$$

we obtain that (3.10), (3.11), and (3.12) hold substituting $S^+_a(f)$ *for* $g^+(f)$.

Proof. We give the proof for $g^+(f)$. The proof for $S^+_a(f)$ is similar. Let $H_1 = \mathbb{C}$, the complex numbers, and $H_2 = H = L^2\left(\mathbb{R}^+, \frac{dt}{t}\right)$. We shall show that the kernel $k(x) = \phi_t(x) = t^{-1}\phi(x/t)$ satisfies the hypotheses of Theorem 2 and therefore the conclusions of that theorem. Moreover, in this case the operator K can be given explicitly as

$$Kf(x) = (\phi_t * f)(x)\,, \tag{3.13}$$

almost everywhere on the halfline $x_{-\infty} < x$.

We observe that an operator $M \in \mathcal{L}(\mathbb{C}, H)$ coincides with a function $m(t)$ in the sense $Mu = m(t).u$ for any complex number u, and $\|M\|_{\mathcal{L}(\mathbb{C},H)} = \|m\|_H$.

Let us prove that condition (i) of Theorem 1 holds for $k(x) = \phi_t(x)$. If $x \neq 0$, then

$$\begin{aligned}\|k(x)\|_{\mathcal{L}(\mathbb{C},H)} &= \left(\int_0^\infty |\phi_t(x)|^2 \frac{dt}{t}\right)^{1/2} \le c\left(\int_0^\infty \left(\frac{t}{t+|x|}\right)^{2+2\beta} \frac{dt}{t^3}\right)^{1/2} \\ &= \frac{c}{|x|}\left(\int_0^\infty \left(\frac{t}{t+1}\right)^{2+2\beta} \frac{dt}{t^3}\right)^{1/2} < \infty\,.\end{aligned}$$

Next, we show inductively that condition (3.6) of Theorem 3 holds. If $x \neq 0$, $|h| < \frac{|x|}{2}$, and $0 \le s \le 1$, we have $|x+sh| \ge \frac{|x|}{2}$. Then, for $0 < \ell \le \gamma$, and applying (3.9),

$$\begin{aligned}\left|\frac{1}{h}[D^{\ell-1}\phi_t(x+h) - D^{\ell-1}\phi_t(x)]\right| &= \left|\frac{1}{h}\frac{1}{t^{\ell-1}}\left[D^{\ell-1}\phi\left(\frac{x+h}{t}\right) - D^{\ell-1}\phi\left(\frac{x}{t}\right)\right]\right| \\ &\le \frac{1}{t^{\ell+1}}\int_0^1 \left|D^\ell\phi\left(\frac{x+sh}{t}\right)\right| ds \\ &\le c'_\ell\, t^{-\ell-1}\left(\frac{t}{t+|x|}\right)^{1+\ell+\beta},\end{aligned}$$

and thus,

$$\left|\frac{1}{h}\left[D^{\ell-1}\phi_t(x+h)-D^{\ell-1}\phi_t(x)\right]-\frac{1}{t^{\ell+1}}D^{\ell}\phi\left(\frac{x}{t}\right)\right|\le c_\ell'' t^{-\ell-1}\left(\frac{t}{t+|x|}\right)^{1+\ell+\beta}.$$

Squaring and integrating with respect to the measure dt/t, and applying Lebesgue's Dominated Convergence Theorem, our claim follows.

In order to prove condition (iii) of Theorem 1 we observe that hypothesis (iii) implies

$$\left|\int_{|x|<r}\phi(x)\,dx\right|\le c\frac{r}{(1+r)^{1+\beta}}, \tag{3.14}$$

see [2, page 363]. We have

$$\left\|\int_{\epsilon<|x|<N}k(x)\,dx\right\|_H\le\left\|\int_{|x|<\epsilon}\phi_t(x)\,dx\right\|_H+\left\|\int_{|x|<N}\phi_t(x)\,dx\right\|_H. \tag{3.15}$$

Since by (3.14) and a change of variables

$$\left|\int_{|x|<r}\phi_t(x)\,dx\right|\le c\,\frac{r/t}{(1+r/t)^{1+\beta}},$$

we get that $\left\|\int_{|x|<r}\phi_t(x)\,dx\right\|_H\le C$. Then, the right-hand side of (3.15) is bounded by $2C$.

Condition (iv) of Theorem 1 is obvious for $k(x)=\frac{1}{t}\phi\left(\frac{x}{t}\right)$.

Finally, we shall prove the condition (v) of Theorem 2 for $k(x)=\phi_t(x)$. We want to show that for $\epsilon_1<\epsilon_2$

$$\int_0^\infty h(t)\left(\int_{\epsilon_1<|x|<\epsilon_2}\phi_t(x)dx\right)\frac{dt}{t}\le\max_{i=1,2}\int_0^\infty|h(t)|\left|\int_{|x|<\epsilon_i}\phi_t(x)dx\right|\frac{dt}{t} \tag{3.16}$$

tends to zero with ϵ_2 provided that $h(t)\in L^2\left(\mathbb{R}^+,\frac{dt}{t}\right)$. Let $\eta>0$ to be chosen. Then, splitting the domain of integration into $t<\eta$ and $t\ge\eta$, and applying Schwarz inequality, we have

$$\begin{aligned}\int_0^\infty|h(t)|\left|\int_{|x|<\epsilon}\phi_t(x)\,dx\right|\frac{dt}{t}\le\\ \left(\int_0^\eta|h(t)|^2\frac{dt}{t}\right)^{1/2}\left(\int_0^\infty\left|\int_{|x|<\epsilon}\phi_t(x)\,dx\right|^2\frac{dt}{t}\right)^{1/2}\\ +\left(\int_0^\infty|h(t)|^2\frac{dt}{t}\right)^{1/2}\left(\int_\eta^\infty\left|\int_{|x|<\epsilon}\phi_t(x)\,dx\right|^2\frac{dt}{t}\right)^{1/2}.\end{aligned} \tag{3.17}$$

By (3.14), we have

$$\left|\int_{|x|<\epsilon}\phi_t(x)\,dx\right|\le c\,\frac{\epsilon/t}{(1+\epsilon/t)^{1+\beta}}.$$

Thus,

$$\left(\int_0^\infty\left|\int_{|x|<\epsilon}\phi_t(x)\,dx\right|^2\frac{dt}{t}\right)^{1/2}\le c\left(\int_0^\infty\frac{t^{2\beta-1}}{(t+1)^{2+2\beta}}\,dt\right)^{1/2}<\infty$$

and

$$\left(\int_\eta^\infty\left|\int_{|x|<\epsilon}\phi_t(x)\,dx\right|^2\frac{dt}{t}\right)^{1/2}\le c\epsilon\left(\int_\eta^\infty\frac{t^{2\beta-1}}{(t+\epsilon)^{2+2\beta}}dt\right)^{1/2}\le c\frac{\epsilon}{\eta}.$$

Now if we choose η small enough, and then ϵ small enough, we get that the left-hand side of (3.17) and therefore that of (3.16) is as small as we please.

Let us prove that (3.13) holds. Let f belong to C_0^1, by Theorem 2,

$$Kf(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \phi_t(x-y)f(y)\,dy = \lim_{\epsilon\to 0} K_\epsilon f(x)$$

exists weakly in $H = L^2\left(\mathbb{R}^+, \frac{dt}{t}\right)$. Then, by a theorem of Banach and Sacks, [8, page 80] there exists a sequence $\epsilon_k \to 0$ such that the means

$$S_n(x) = (K_{\epsilon_1} f(x) + \cdots + K_{\epsilon_n} f(x))/n$$

converge strongly to $Kf(x)$ in H. Since f is bounded and ϕ is integrable, by applying Lebesgue's Dominated Convergence Theorem, for every fixed x we have

$$\lim_{\epsilon\to 0} K_\epsilon f(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \phi_t(x-y)f(y)\,dy = (\phi_t * f)(x)\,,$$

for every $t > 0$. Then $Kf(x) = (\phi_t * f)(x)$ as an element of H for every x.

Given f a distribution in $\mathcal{D}'(x_{-\infty}, \infty)$, let φ belong to $\mathcal{D}(x_{-\infty}, \infty)$ such that the support of φ is contained in an interval $I = [a, b]$. Since $x_{-\infty} < a$, there exists a', $x_{-\infty} < a' < a$, such that $a - a' \le |I| = b - a$. Therefore, if $\tilde{I} = [a', a]$, by the definition of $f^*_{+,\gamma}$, we have that

$$|\langle f, \varphi\rangle| \;\le\; 2^{\gamma+1}\,|I|^{\gamma+1}\,\|D^\gamma\varphi\|_\infty\, f^*_{+,\gamma}(x)\,,$$

holds for every $x \in \tilde{I}$. Taking the p–power and averaging on $\tilde{I}$, we obtain

$$|\langle f, \varphi\rangle| \;\le\; 2^{\gamma+1}\,\frac{|I|^{\gamma+1}}{w(\tilde{I})^{1/p}}\,\|D^\gamma\varphi\|_\infty\,\|f\|_{H^p_{+,\gamma}(w)}\,.$$

Now, let φ be a function with continuous derivatives up to the order γ, with support contained in the halfline $[a, \infty)$, where $a > x_{-\infty}$ and such that

$$|||\varphi|||_{\gamma,\gamma+1+\beta} = \sup_x |D^\gamma\varphi(x)|(1+|x|)^{\gamma+1+\beta} < \infty\,.$$

For simplicity and without loss of generality we assume that $a = 0 > x_{-\infty}$.

Let $(\psi_k)_{k\ge 0}$ be a sequence of non-negative C_0^∞–functions, satisfying:

For $k \ge 1$, support$(\psi_k) \subset [2^{k-1}, 2^{k+1}]$, and support$(\psi_0) \subset [-1, 2]$,

$\sum_{k\ge 0}\psi_k(x) = 1$ if $x \ge 0$, and

$\|D^s\psi_k\|_\infty \le C\,2^{-ks}$, $1 \le s \le \gamma$.

Thus,

$$\varphi(x) \;=\; \sum_{k\ge 0}\psi_k(x)\varphi(x)\,.$$

We choose an interval $J = (\max(x_{-\infty}, -1), 0]$ and for each $x \in J$ and $k \ge 0$, we denote J_k to the interval $[x, 2^{k+1}]$. Therefore, for every $k \ge 0$ we get

$$|J_k|^{\gamma+1}\,\|D^\gamma(\varphi\psi_k)\|_\infty \;\le\; C_\gamma\,|||\varphi|||_{\gamma,\gamma+1+\beta}\sum_{k\ge 0} 2^{-k\beta}\,.$$

Then, we can extend the distribution f to these functions φ as

$$\langle f, \varphi\rangle = \sum_{k\geq 0}\langle f, \varphi\psi_k\rangle ,$$

and we have that

$$|\langle f, \varphi\rangle| \leq \sum_{k\geq 0}|\langle f, \varphi\psi_k\rangle| \leq C_\gamma\, f^*_{+,\gamma}(x)\, |||\varphi|||_{\gamma,\gamma+1+\beta} \sum_{k\geq 0} 2^{-k\beta}$$
$$\leq C_{\gamma,\beta}\, f^*_{+,\gamma}(x)\, |||\varphi|||_{\gamma,\gamma+1+\beta} , \tag{3.18}$$

holds for every $x \in J$. Taking the p–power and the average on J, we obtain

$$|\langle f, \varphi\rangle| \leq C_{\gamma,\beta}\, |||\varphi|||_{\gamma,\gamma+1+\beta} \left(\frac{1}{w(J)}\int_{x_{-\infty}}^{\infty} f^*_{+,\gamma}(x)^p\, w(x)\, dx\right)^{1/p} . \tag{3.19}$$

By Remark 1, if f belongs to $H^p_{+,\gamma}(w)$ there exists a sequence $(f_n)_{n\geq 1}$ of C^1_0–functions such that f_n tends to f in $H^p_{+,\gamma}(w)$. Taking into account Theorem 3

$$\int_{x_{-\infty}}^{\infty} \|Kf_n(x) - Kf_m(x)\|^p_H\, w(x)\, dx \leq c \int_{x_{-\infty}}^{\infty} (f_n - f_m)^*_{+,\gamma}(x)^p\, w(x)\, dx$$

which implies that there exists $Kf = \lim_{n\to\infty} Kf_n$ in $L^p_H(w)$.

Since for every $\lambda > 0$, we have

$$w\left(\{x > x_{-\infty} : \|Kf(x) - Kf_m(x)\|_H > \lambda\}\right) \to 0 ,$$

then there exists a subsequence of $(Kf_n)_{n\geq 1}$ that converges in H for almost every x. Let x_0 be a point for which the subsequence converges in $H = L^2\left(\mathbb{R}^+, \frac{dt}{t}\right)$. Then, there exists a new subsequence, depending on x_0, such that we shall denote $(Kf_m(x_0))_{m\geq 1}$, satisfying

$$Kf(x_0) = \lim_{m\to\infty} Kf_m(x_0) \quad a.e. \text{ in } t > 0 .$$

On the other hand, since $f_m \in C^1_0$ we know that $Kf_m(x_0) = \phi_t * f_m(x_0)$. Taking into account (3.19)

$$|\phi_t * f_m(x_0) - \phi_t * f(x_0)|$$
$$\leq C_{\gamma,\beta}\, |||\phi_t|||_{\gamma,\gamma+1+\beta} \left(\frac{1}{w(J)}\int_{x_{-\infty}}^{\infty} (f_m - f)^*_{+,\gamma}(x)^p\, w(x)\, dx\right)^{1/p} .$$

Then, $\phi_t * f_m(x_0)$ tends to $\phi_t * f(x_0)$, and in consequence $Kf(x) = \phi_t * f(x)$ for almost every x and almost every $t > 0$. □

Let φ belong to $\mathcal{S}(\mathbb{R})$ supported on $(-\infty, 0]$. Let m and n non–negative integers such that $m + n \geq 1$, and $0 \leq \alpha < 1$. We define

$$g^+(f)(x) = \left(\int_0^\infty |t^{n+m-\alpha}\, \partial^n_x\, \partial^m_t\, I^+_\alpha(\varphi_t * f)(x)|^2\, \frac{dt}{t}\right)^{1/2}$$

and

$$S^+_a(f)(x) = \left(\int\int_{0\leq z<at} |t^{n+m-\alpha}\, \partial^n_x\, \partial^m_t\, I^+_\alpha(\varphi_t * f)(x+z)|^2\, \frac{dzdt}{t^2}\right)^{1/2} ,$$

where $0 < a < \infty$.

In order to apply Theorem 4, we observe that

$$\partial_x^n \, \partial_t^m \, I_\alpha^+(\varphi_t)(x) \;=\; I_\alpha^+(\partial_x^n \, \partial_t^m \, \varphi_t)(x) \tag{3.20}$$

and, by Lemma 2,

$$\begin{aligned} \partial_x^n \, \partial_t^m \, \varphi_t(x) &= \frac{(-1)^m}{t^{m+1}} \, \partial_x^n \left[\left(\frac{d}{dx}\right)^m \rho\left(\frac{x}{t}\right) \right] \\ &= \frac{(-1)^m}{t^{m+n+1}} \left[\left(\frac{d}{dx}\right)^{n+m} \rho \right] \left(\frac{x}{t}\right) \end{aligned}$$

where $\rho(x) = x^m \varphi(x)$. Then, (3.20) is equal to

$$\frac{(-1)^m}{t^{n+m-\alpha+1}} \, I_\alpha^+ \left[\left(\frac{d}{dx}\right)^{n+m} \rho \right] \left(\frac{x}{t}\right) .$$

Therefore, $g^+(f)$ and $S_a^+(f)$ are defined as in Theorem 4 for

$$\phi(x) = (-1)^m \, I_\alpha^+ \left[\left(\frac{d}{dx}\right)^{n+m} x^m \varphi \right] . \tag{3.21}$$

Applying Lemma 1 to $\sigma = (-1)^m D^{n+m}(x^m \varphi)$ and by Theorem 4, we have the following theorem:

Theorem 5.

Let φ belong to $\mathcal{S}(\mathbb{R})$ supported on $(-\infty, 0]$. Let m and n non–negative integers such that $n + m \geq 1$, and $0 \leq \alpha < 1$. If

$$g^+(f)(x) = \left(\int_0^\infty |t^{n+m-\alpha} \, \partial_x^n \, \partial_t^m \, I_\alpha^+(\varphi_t * f)(x)|^2 \, \frac{dt}{t} \right)^{1/2} ,$$

there exists a constant c such that

$$\|g^+(f)\|_{L^p(w)} \;\leq\; c \, \|f^*_{+,\gamma}\|_{L^p(w)} \tag{3.22}$$

holds if $p(\gamma + 1) > 1$ and $w \in A^+_{p(\gamma+1)}$. In consequence, if $p > 1$ and $w \in A_p^+$, there exists a constant c' such that

$$\|g^+(f)\|_{L^p(w)} \;\leq\; c \, \|f\|_{L^p(w)} \, . \tag{3.23}$$

Moreover, for $\lambda > 0$

$$\lambda \, w\left(\{x : g^+(f)(x) > \lambda\}\right) \;\leq\; c \, \|f\|_{L^1(w)} \tag{3.24}$$

holds if $w \in A_1^+$.

Besides, if for $a > 0$

$$S_a^+(f)(x) = \left(\int\int_{0 \leq z < at} |t^{n+m-\alpha} \, \partial_x^n \, \partial_t^m \, I_\alpha^+(\varphi_t * f)(x + z)|^2 \, \frac{dz dt}{t^2} \right)^{1/2} ,$$

we obtain that (3.22), (3.23), and (3.24) hold substituting $S_a^+(f)$ for $g^+(f)$.

4. One-Sided g_λ Function

We begin this section stating a known result of the auxiliary T_λ^+ function, see [7, page 97]. The T_λ^+ function generalizes the T_λ function introduced by Fefferman and Stein in [4, page 178].

Let Φ be an integrable function and $\lambda > 1$. We define

$$T_\lambda^+(f)(x) = \sup_{h>0}\left(\frac{1}{h^\lambda}\int_0^h\int_x^{x+h} t^{\lambda-2}|(\Phi_t * f)(y)|^2 dy dt\right)^{1/2}.$$

The function T_λ^+ has been studied in [10]. In that paper, we proved the following theorem:

Theorem 6.

Let $p > \frac{2}{\lambda}$, γ be a positive integer such that $\gamma + 1 > \frac{\lambda}{2}$ and $w \in A^+_{\frac{p\lambda}{2}}$. If $\Phi \in \mathcal{S}(\mathbb{R})$ with support contained in $(-\infty, 0]$, then

$$\|T_\lambda^+(f)\|_{L^p(w)} \le c\,\|f^*_{+,\gamma}\|_{L^p(w)}$$

holds with a finite constant c depending on λ, Φ, w, γ, and p.

The following technical lemma shall be needed in the proof of Theorem 7.

Lemma 5.

Let $J = (\alpha, \beta)$ be a bounded interval and $F \subseteq J$ a closed subset. Given μ, $0 < \mu < 1$, we define

$$D = \{x \in F : |F \cap [x - t, x]| \ge \mu\, t, \quad \forall t : 0 < t \le |J|\}.$$

If $W = \bigcup_{x\in D}\Gamma_1(x)$, where $\Gamma_1(x) = \{(z,t) : 0 \le z - x < t\}$ and $R = \{(z,t) : \alpha < z < \beta + |J|,\ 0 < t < |J|\}$, then

(i) $\int_F S_2^+(f)(x)^2\,dx \ \ge\ \mu \iint_{R\cap W} |\phi_t * f(z)|^2 \frac{dzdt}{t}$

and

(ii) $|F\backslash D| \ \le\ 15\,\mu\,|J|$.

Proof. By Fubini's Theorem we have that

$$\int_F S_2^+(f)(x)^2\,dx = \iint |(\phi_t * f)(z)|^2\,|F\cap[z-2t,z]|\,\frac{dzdt}{t^2}.$$

Thus,

$$\int_F S_2^+(f)(x)^2\,dx \ge \iint_{(z,t)\in R\cap W} |(\phi_t * f)(z)|^2\,|F\cap[z-2t,z]|\,\frac{dzdt}{t^2}.$$

We observe that if $(z,t) \in R\cap W$, then the pair (z,t) belongs to $\Gamma_1(x)$ for some x in D, and this implies that $[x-t,x] \subset [z-2t,z]$. Therefore, since $0 < t \le |J|$, we have that $|F\cap[z-2t,z]| \ge |F\cap[x-t,x]| \ge \mu t$ and

$$\int_F S_2^+(f)(x)^2\,dx \ \ge\ \mu \iint_{R\cap W} |(\phi_t * f)(z)|^2\,\frac{dzdt}{t},$$

which proves (i).

Now, let us prove (ii). We choose an open set $D \subset G$ such that

$$|G \backslash D| < 3\,\mu\,|J|\,. \tag{4.1}$$

If $x \in F\backslash G$, then $x \in F\backslash D$ and this implies that there exists t_x, $0 < t_x \le |J|$ such that

$$|F \cap [x - t_x, x]| < \mu\, t_x\,.$$

Then, we can choose ϵ_x, $0 < \epsilon_x < t_x$ satisfying

$$|F \cap [x - t_x, x + \epsilon_x]| < 2\,\mu\, t_x\,.$$

The compact set $F\backslash G$ is covered by the family $\{(x - t_x, x + \epsilon_x)\}_{x \in F\backslash G}$. Then, there exists a finite subcover $\{(x_i - t_{x_i}, x_i + \epsilon_{x_i})\}_{1 \le i \le r}$ such that

$$\sum_{i=1}^{r} \chi_{[x_i - t_{x_i}, x_i + \epsilon_{x_i}]}(x) \le 2\,\chi_{[\alpha - |J|, \beta + |J|]}(x)\,,$$

which implies that

$$\sum_{i=1}^{r} t_{x_i} \le 6\,|J|\,.$$

Thus,

$$|F\backslash G| \le \sum_{i=1}^{r} |F \cap [x_i - t_{x_i}, x_i + \epsilon_{x_i}]| < 2\,\mu \sum_{i=1}^{r} t_{x_i} \le 12\,\mu\,|J|\,.$$

By these inequalities and (4.1) we obtain (ii). □

Let φ belong to $\mathcal{S}(\mathbb{R})$ supported on $(-\infty, 0]$. Let m and n non-negative integers such that $n + m \ge 1$, and $0 \le \alpha < 1$. We denote as in (3.21)

$$\phi(x) = (-1)^m\, I_\alpha^+ \left[\left(\frac{d}{dx} \right)^{n+m} x^m \varphi \right],$$

and define

$$g_\lambda^+(f)(x) = \left(\int_0^\infty \int_x^\infty \left(\frac{t}{t + z - x} \right)^\lambda |(\phi_t * f)(z)|^2 \,\frac{dz dt}{t^2} \right)^{1/2},$$

where $\lambda > 1$. Explicitly

$$g_\lambda^+(f)(x) = \left(\int_0^\infty \int_x^\infty \left(\frac{t}{t + z - x} \right)^\lambda |t^{n+m-\alpha}\, \partial_x^n \partial_t^m I_\alpha^+(\varphi_t * f)(z)|^2 \,\frac{dz dt}{t^2} \right)^{1/2}.$$

With these notations we have the following theorem:

Theorem 7.
Let $p > \frac{2}{\lambda}$, γ be a positive integer such that $\gamma + 1 > \frac{\lambda}{2}$ and $w \in A^+_{\frac{p\lambda}{2}}$. Then

$$\|g_\lambda^+(f)\|_{L^p(w)} \le c\, \|f^*_{+,\gamma}\|_{L^p(w)}\,,$$

holds with a finite constant c depending on λ, ϕ, w, γ, and p.

The proof of this theorem consists of obtaining a good λ estimate for the g_λ^+ function and follows the lines of the proof of Theorem 2 in [7].

Proof. For any given $N > 1$ we define

$$g^{+}_{\lambda,N}(f)(x) = \left(\int_0^\infty \int_x^\infty \chi_N(z,t)\left(\frac{t}{t+z-x}\right)^{\lambda} |(\phi_t * f)(z)|^2 \,\frac{dz\,dt}{t^2}\right)^{1/2}$$

where $\chi_N(z,t)$ is the characteristic function of the rectangle $\{(z,t) : |z| \le N, N^{-1} \le t \le N\}$. It is enough to prove Theorem 7 for $g^{+}_{\lambda,N}$ instead of g^{+}_{λ} with a constant c not depending on N. Then, by Fatou's Lemma we get Theorem 7.

We shall assume that f is a bounded function with bounded support, in $H^p_{+,\gamma}(w)$. Then, $g^{+}_{\lambda,N}(f)(x)$ is a continuous function and moreover if $x > x_{-\infty}$, we have that

$$g^{+}_{\lambda,N}(f)(x) \;\le\; C_{\gamma,\beta}\, N^{2\gamma+3\beta+4}\, f^{*}_{+,\gamma}(x) \tag{4.2}$$

for any β, $0 < \beta \le 1$. In fact, by part (ii) of Lemma 1, there exists β, $0 < \beta \le 1$ such that

$$|D^{\gamma}\phi(x)| \le \frac{c}{(1-x)^{\gamma+1+\beta}}, \quad \text{if } x < 0\,.$$

Given $z \ge x$ and t, $N^{-1} < t < N$, the support of $\phi_t(z-\cdot)$ is contained in the halfline $[x,\infty)$ and

$$\begin{aligned} |||\phi_t(z-\cdot)|||_{\gamma,\gamma+1+\beta} &= \sup_y |D^{\gamma}\phi_t(z-y)|(1+|y|)^{\gamma+1+\beta} \\ &\le t^{\beta}\,\sup_y\left(\tfrac{1+|y|}{t+|z-y|}\right)^{\gamma+1+\beta} \;\le\; c'\, N^{2(\gamma+1)+3\beta}\,. \end{aligned}$$

Therefore, by (3.18), for every $y \in J = (\max(x_{-\infty}, x-1), x]$, we have

$$\begin{aligned} |f*\phi_t(z)| &= |\langle f, \phi_t(z-\cdot)\rangle| \\ &\le C_{\gamma,\beta}\, f^{*}_{+,\gamma}(y)\, |||\phi_t(z-\cdot)|||_{\gamma,\gamma+1+\beta} \;\le\; C'_{\gamma,\beta}\, f^{*}_{+,\gamma}(y)\, N^{2(\gamma+1)+3\beta}\,. \end{aligned}$$

Applying this estimate for $y = x$, we obtain that

$$\begin{aligned} g^{+}_{\lambda,N}(f)(x) &\le C'_{\gamma,\beta}\, f^{*}_{+,\gamma}(x)\, N^{2(\gamma+1)+3\beta}\left(\int_{N^{-1}}^{N}\int_{z\ge x, |z|\le N}\frac{dz\,dt}{t^2}\right)^{1/2} \\ &\le C''_{\gamma,\beta}\, N^{2\gamma+3\beta+4}\, f^{*}_{+,\gamma}(x)\,. \end{aligned}$$

We observe that by (4.2), $g^{+}_{\lambda,N}(f)(x)$ belongs to $L^p(w)$.

Since the weight w satisfies the A^{+}_{∞} condition, then by Theorem 1 in [6], there exist $K \ge 1$ and $\eta > 0$ such that

$$\frac{w(E)}{w((a,c))} \;\le\; K\left(\frac{|E|}{c-b}\right)^{\eta},$$

holds for every $a < b < c$ and every measurable set $E \subset (a,b)$.

Let $M = (2^{p+2}K)^{-1/\eta}$ and $0 < \delta < 1$ to be chosen later. We shall prove that

$$\begin{aligned} &w\left(\left\{x : g^{+}_{\lambda,N}(f)(x) > 2\alpha\,,\; S^{+}_{2}(f)(x) + T^{+}_{\lambda}(f)(x) \le \delta\alpha\right\}\right) \\ &\quad\le 2^{-(p+1)}\, w\left(\left\{x : g^{+}_{\lambda,N}(f)(x) > \alpha\right\}\right), \end{aligned} \tag{4.3}$$

holds for every $0 < \alpha < \infty$.

If we denote by A_α the set $\{x : g^{+}_{\lambda,N}(f)(x) > \alpha\}$, $0 < \alpha < \infty$, since $g^{+}_{\lambda,N}(f)(x)$ is a continuous function and by the estimate (4.2), it turns out that A_α is an open bounded set. Let

$A_\alpha = \cup I_i$, where the $I_i^{\prime}s$ denote the connected components of A_α. We choose any I_i and assume that $I_i = (a, b)$. We define the sequence $(x_n)_{n\geq 0}$ as

$$x_n = b - \frac{b-a}{2^n} \qquad , \; n \geq 0 \, .$$

Then, $x_n - x_{n-1} = 2(x_{n+1} - x_n)$. Let us denote by J_n the interval $[x_{n-1}, x_n]$ and let E_α be the set $\{x : g_{\lambda,N}^{+}(f)(x) > 2\alpha \, , \; S_2^{+}(f)(x) + T_\lambda^{+}(f)(x) \leq \delta\alpha\}$.

Given a non-negative integer n such that $|E_\alpha \cap J_n| > 0$, we consider the rectangle

$$R_n = \{(z,t) : x_{n-1} \leq z \leq b \, , \; 0 \leq t \leq |J_n|\} \, .$$

Then, if x belongs to $E_\alpha \cap J_n$, we have

$$(2\alpha)^2 \; < \; g_{\lambda,N}^{+}(f)(x)^2 \; =$$

$$\left(\iint_{\substack{(z,t)\in R_n \\ z\geq x}} + \iint_{\substack{(z,t)\notin R_n \\ z\geq x}} \right) \chi_N(z,t) \left(\frac{t}{t+z-x} \right)^{\lambda} |(\phi_t * f)(z)|^2 \, \frac{dzdt}{t^2} \tag{4.4}$$

$$= A + B \, .$$

If $(z,t) \notin R_n$ and $x \leq z < b$ then, $0 \leq z - x \leq 2t$. Therefore,

$$\iint_{\substack{(z,t)\notin R_n \\ x\leq z<b}} \chi_N(z,t) \left(\frac{t}{t+z-x} \right)^{\lambda} |(\phi_t * f)(z)|^2 \, \frac{dzdt}{t^2} \leq$$

$$\iint_{0\leq z-x\leq 2t} |(\phi_t * f)(z)|^2 \, \frac{dzdt}{t^2} \; = \; S_2^{+}(f)(x)^2 \; \leq \; (\delta\alpha)^2 \, . \tag{4.5}$$

If $b \leq z$, then $t + z - x > t + z - b$ and we have

$$\iint_{\substack{(z,t)\notin R_n \\ b\leq z}} \chi_N(z,t) \left(\frac{t}{t+z-x} \right)^{\lambda} |(\phi_t * f)(z)|^2 \, \frac{dzdt}{t^2}$$

$$\leq \; g_{\lambda,N}^{+}(f)(b)^2 \; \leq \; \alpha^2 \, . \tag{4.6}$$

The estimates (4.5) and (4.6) show that $B \leq \alpha^2(1+\delta^2) < 2\alpha^2$. Then, by (4.4) we obtain that

$$2\alpha^2 \; < \; \iint_{\substack{(z,t)\in R_n \\ z\geq x}} \left(\frac{t}{t+z-x} \right)^{\lambda} |(\phi_t * f)(z)|^2 \, \frac{dzdt}{t^2} \, , \tag{4.7}$$

holds for every x belonging to $E_\alpha \cap J_n$.

Choose a closed set $F_{\alpha,n} \subset E_\alpha \cap J_n$ such that

$$|(E_\alpha \cap J_n) \backslash F_{\alpha,n}| < \frac{M}{8}(x_{n+1} - x_n) \, , \tag{4.8}$$

and we define

$$D_{\alpha,n} = \{x \in F_{\alpha,n} : |F_{\alpha,n} \cap [x-t, x]| \geq \frac{M}{48} t \, , \; \forall t : 0 < t \leq |J_n|\} \, .$$

We observe that $D_{\alpha,n}$ is closed. By Lemma 5, we have

$$|F_{\alpha,n} \backslash D_{\alpha,n}| \; \leq \; \frac{5}{8} M (x_{n+1} - x_n) \, . \tag{4.9}$$

Let us assume $D_{\alpha,n} \neq \emptyset$. Since $D_{\alpha,n} \subset E_\alpha \cap J_n$, then integrating both sides of (4.7) with respect to x over $D_{\alpha,n}$, we have

$$2\alpha^2 |D_{\alpha,n}| \leq \iint_{R_n} \left(\int_{\substack{x \in D_{\alpha,n} \\ x \leq z}} \left(\frac{t}{t+z-x} \right)^\lambda dx \right) |(\phi_t * f)(z)|^2 \frac{dz dt}{t^2} . \tag{4.10}$$

Let $W = \bigcup_{x \in D_{\alpha,n}} \Gamma_1(x)$, where $\Gamma_1(x) = \{(z,t) : 0 \leq z - x < t\}$. By (4.10), we have

$$2\,\alpha^2\,|D_{\alpha,n}| \leq$$

$$\left(\iint_{R_n \cap W} + \iint_{R_n \setminus W} \right) \left(\int_{\substack{x \in D_{\alpha,n} \\ x \leq z}} \left(\frac{t}{t+z-x} \right)^\lambda dx \right) |(\phi_t * f)(z)|^2 \frac{dz dt}{t^2} \tag{4.11}$$

$$= I + II .$$

Let us estimate I. Since

$$\int_{\substack{x \in D_{\alpha,n} \\ x \leq z}} \left(\frac{t}{t+z-x} \right)^\lambda dx \leq \int_{-\infty}^{z} \left(\frac{t}{t+z-x} \right)^\lambda dx = \frac{t}{\lambda - 1} ,$$

we obtain that

$$I \leq \frac{1}{\lambda - 1} \iint_{R_n \cap W} |(\phi_t * f)(z)|^2 \frac{dz dt}{t} .$$

Applying Lemma 5, we have

$$I \leq \frac{1}{\lambda - 1} \frac{48}{M} \int_{F_{\alpha,n}} S_2^+(f)(x)^2\, dx \leq \frac{1}{\lambda - 1} \frac{48}{M} (\delta\alpha)^2\, |F_{\alpha,n}|$$

$$\leq \frac{1}{\lambda - 1} \frac{96}{M} \delta^2\, \alpha^2\, (x_{n+1} - x_n) . \tag{4.12}$$

Let us estimate II. Since the set $D_{\alpha,n}$ is not empty then $m = \min(D_{\alpha,n})$ exists. Denote $\{C_k\}_{k \geq 1}$ the connected componentes of $(m, b) \setminus D_{\alpha,n}$. Then,

$$II = \sum_{k \geq 1} \iint_{\substack{(z,t) \in R_n \setminus W \\ z \in C_k}} \left(\int_{\substack{x \in D_{\alpha,n} \\ x \leq z}} \left(\frac{t}{t+z-x} \right)^\lambda dx \right) |(\phi_t * f)(z)|^2 \frac{dz dt}{t^2} . \tag{4.13}$$

Choose any $C_k = (c, d)$ and consider a sequence $(d_s)_{s \geq 0}$ such that $d_0 = d$ and $d_{s-1} - d_s = d_s - c$, $s \geq 1$. For any given non–negative integer s, if $d_s \leq z \leq d_{s-1}$, $x \leq z$ and $x \in D_{\alpha,n}$, then $x \leq c$. Thus,

$$\int_{\substack{x \in D_{\alpha,n} \\ x \leq z}} \left(\frac{t}{t+z-x} \right)^\lambda dx \leq \int_{z - x \geq d_{s-1} - d_s} \frac{t^\lambda}{(z-x)^\lambda}\, dx = t^\lambda \frac{(d_{s-1} - d_s)^{1-\lambda}}{\lambda - 1} .$$

In consequence, we obtain that

$$L_s = \iint_{\substack{(z,t) \in R_n \setminus W \\ d_s \leq z \leq d_{s-1}}} \left(\int_{\substack{x \in D_{\alpha,n} \\ x \leq z}} \left(\frac{t}{t+z-x} \right)^\lambda dx \right) |(\phi_t * f)(z)|^2 \frac{dz dt}{t^2}$$

$$\leq \frac{1}{\lambda - 1} (d_{s-1} - d_s) \frac{1}{(d_{s-1} - d_s)^\lambda} \iint_{\substack{(z,t) \in R_n \setminus W \\ d_s \leq z \leq d_{s-1}}} t^{\lambda - 2} |(\phi_t * f)(z)|^2\, dz dt .$$

If $(z, t) \in R_n \backslash W$ and $d_s \leq z \leq d_{s-1}$, then $0 \leq z - c \leq 2(d_{s-1} - d_s)$ and $0 \leq t \leq 2(d_{s-1} - d_s)$. Therefore, since c belongs to E_α, we get

$$L_s \leq \frac{2^\lambda}{\lambda - 1} (d_{s-1} - d_s)\, T_\lambda^+(f)(c)^2 \leq \frac{2^\lambda}{\lambda - 1} (d_{s-1} - d_s)\, (\delta\alpha)^2 .$$

Then,

$$\begin{aligned} \iint_{\substack{(z,t)\in R_n\backslash W \\ z\in C_k}} \left(\int_{\substack{x\in D_{\alpha,n} \\ x\leq z}} \left(\frac{t}{t+z-x}\right)^\lambda dx \right) |(\phi_t * f)(z)|^2 \frac{dzdt}{t^2} \\ = \sum_{s\geq 1} L_s \leq \frac{2^\lambda}{\lambda - 1} |C_k|\, (\delta\alpha)^2 . \end{aligned}$$

Summing up in k, and taking into account (4.13) we obtain that

$$\begin{aligned} II &\leq \tfrac{2^\lambda}{\lambda-1} (\delta\alpha)^2\, |(m, b)\backslash D_{\alpha,n}| = \tfrac{2^\lambda}{\lambda-1} (\delta\alpha)^2\, 2|J_n| \\ &\leq \tfrac{2^{\lambda+2}}{\lambda-1} (\delta\alpha)^2\, (x_{n+1} - x_n) . \end{aligned}$$

Taking into account these inequalities, (4.12) and (4.11) we have that

$$2\,\alpha^2\, |D_{\alpha,n}| \leq \left(\frac{96}{M} + 2^{\lambda+2}\right) \frac{(\delta\alpha)^2}{\lambda - 1} (x_{n+1} - x_n) .$$

Since $\delta < 1$, this shows that

$$|D_{\alpha,n}| \leq \left(\frac{48}{M} + 2^{\lambda+1}\right) \frac{\delta}{\lambda - 1} (x_{n+1} - x_n) \leq 2^{\lambda+2}\, \frac{48}{M}\, \frac{\delta}{\lambda - 1} (x_{n+1} - x_n) .$$

Besides, if we choose δ such that $0 < \delta < (\lambda - 1)\frac{M^2}{2^{\lambda+4}48}$, by (4.8) and (4.9) we obtain for every $n \geq 0$,

$$|E_\alpha \cap J_n| \leq M(x_{n+1} - x_n) .$$

Taking into account that w satisfies the condition A_∞^+ and since $M = (2^{p+2}K)^{-1/\eta}$ we get

$$\begin{aligned} w(E_\alpha \cap J_n) &\leq K \left(\frac{|E_\alpha \cap J_n|}{x_{n+1} - x_n}\right)^\eta w(x_{n-1}, x_{n+1}) \\ &= 2^{-(p+2)}\, w(x_{n-1}, x_{n+1}) . \end{aligned}$$

Summing these inequalities for every $n \geq 0$, we have that

$$w(E_\alpha \cap I_i) \leq 2^{-(p+1)}\, w(I_i) ,$$

holds for every connected component I_i of A_α, which implies (4.3). Now, applying Theorem 5, Theorem 6 and standard arguments (see [7, page 108]) it follows the theorem. □

5. Application to Fractional Integrals

We begin this section by showing that the reverse inequality of (3.23) holds. More precisely:

Proposition 1.

Let φ belong to $\mathcal{S}(\mathbb{R})$ supported on $(-\infty, 0]$. Let m and n non-negative integers such that $n+m \geq 1$, and $0 \leq \alpha < 1$. If

$$g^{+}(f)(x) = \left(\int_0^\infty |(\phi_t * f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where ϕ is defined as in (3.21), then there exists a constant c such that

$$c\,\|f\|_{L^p(w)} \leq \|g^{+}(f)\|_{L^p(w)}, \tag{5.1}$$

holds if $1 < p < \infty$, $f \in L^p(w)$ and $w \in A_p^{+}$.

If we consider $S_a^{+}(f)$ as in Theorem 5, then (5.1) holds substituting $S_a^{+}(f)$ for $g^{+}(f)$.

Proof. Let $f \in L^1 \cap L^2$, $f(x) = f(-x)$. Since the inequality (3.11) is valid for $p = 2$ and $w = 1$, we get

$$\int\int_0^\infty |(\phi_t * f)(x)|^2 \frac{dt}{t} dx \leq c \int |f(x)|^2\,dx < \infty .$$

Then, by Plancherel's theorem

$$\int\int_0^\infty |(\phi_t * f)(x)|^2 \frac{dt}{t} dx = \int\int_0^\infty |\widehat{\phi}(tx)\widehat{f}(x)|^2 \frac{dt}{t} dx$$

$$= \int_0^\infty |\widehat{\phi}(-t)|^2 \frac{dt}{-t} \int_{-\infty}^0 |\widehat{f}(x)|^2\,dx + \int_0^\infty |\widehat{\phi}(t)|^2 \frac{dt}{t} \int_0^\infty |\widehat{f}(x)|^2\,dx .$$

Thus,

$$\int_{-\infty}^\infty \frac{|\widehat{\phi}(t)|^2}{|t|}\,dt \leq C . \tag{5.2}$$

Let $\psi(x) = \phi(-x)$. Then $\psi(x) = 0$ if $x < 0$ and we have

$$\int\int_0^\infty (\phi_t * f)(x)\overline{(\psi_t * h)(x)} \frac{dt}{t} dx = \int\int_0^\infty \widehat{\phi}(tx)\overline{\widehat{\psi}(tx)}\widehat{f}(x)\overline{\widehat{h}(x)} \frac{dt}{t} dx . \tag{5.3}$$

Since $\overline{\widehat{\psi}(x)} = \widehat{\phi}(x)$, we get that (5.3) is equal to

$$\int \left(\int_0^\infty \frac{\widehat{\phi}(tx)^2}{t} dt\right) \widehat{f}(x)\overline{\widehat{h}(x)}\,dx .$$

We have that if $x > 0$

$$\int_0^\infty \frac{\widehat{\phi}(tx)^2}{t}\,dt = \int_0^\infty \frac{\widehat{\phi}(t)^2}{t}\,dt \tag{5.4}$$

and if $x < 0$

$$\int_0^\infty \frac{\widehat{\phi}(tx)^2}{t}\,dt = -\int_{-\infty}^0 \frac{\widehat{\phi}(t)^2}{t}\,dt . \tag{5.5}$$

By (5.2) we know that $\left|\frac{\widehat{\phi}(t)^2}{t}\right|$ is integrable. On the other hand, $\widehat{\phi}$ can be extended to the upper half–plane as

$$\widehat{\phi}(z) = \int_{-\infty}^0 e^{-2\pi i z x}\phi(x)\,dx .$$

This function $\widehat{\phi}(z)$ is analytic for $\mathcal{I}m z > 0$ and

$$|\widehat{\phi}(z)| \leq \frac{C}{1+|z|} .$$

Then $\frac{\widehat{\phi}(z)^2}{z}$ is an analytic function on the upper half-plane and for $z = x + iy$,

$$\left|\frac{\widehat{\phi}(z)^2}{z}\right|^{1/2} \leq \frac{C}{1+|z|}\frac{1}{|z|^{1/2}} \leq \frac{C}{(1+|x|)|x|^{1/2}} .$$

Thus, we have that $\frac{\widehat{\phi}(z)^2}{z} \in H^{1/2}$ and since $\frac{\widehat{\phi}(t)^2}{t} \in L^1$ we get that $\frac{\widehat{\phi}(z)^2}{z} \in H^1$. Therefore,

$$\int_{-\infty}^{\infty} \frac{\widehat{\phi}(t)^2}{t}\, dt = 0 .$$

Then the integrals (5.4) and (5.5) have the same value c and we get

$$c\int f(x)\overline{h(x)}\, dx = c\int \widehat{f}(x)\overline{\widehat{h}(x)}\, dx = \int\int_0^{\infty} (\phi_t * f)(x)\overline{(\psi_t * h)(x)}\, \frac{dt}{t} dx .$$

Since $w \in A_p^+$ implies that $w^{-p'/p} \in A_{p'}^-$, by the part of this theorem that we have already proved we have

$$|c|\left|\int f(x)\overline{h(x)}\, dx\right| \leq \int g^+(f)(x) g^-(h)(x) dx$$
$$\leq \left(\int g^+(f)(x)^p w(x)\, dx\right)^{1/p} \left(\int g^-(h)(x)^{p'} w(x)^{-p'/p}\, dx\right)^{1/p'}$$
$$\leq \left(\int g^+(f)(x)^p w(x)\, dx\right)^{1/p} C\|h\|_{L^{p'}(w^{-p'/p})} .$$

We observe that $c = \int_0^{\infty} \frac{\widehat{\phi}(t)^2}{t}\, dt$ is different from zero. In fact, since $\left|\frac{\widehat{\phi}(z)^2}{z}\right| \leq \frac{C}{(1+|z|)^2|z|}$ then

$$I = \int_0^{\infty} \frac{\widehat{\phi}(iy)^2}{iy}\, diy = \int_0^{\infty} \frac{\widehat{\phi}(t)^2}{t}\, dt = c .$$

Now, if we assume that $c = 0$, we have

$$\widehat{\phi}(iy) = \int_{-\infty}^{0} e^{2\pi y\xi}\, \phi(\xi)\, d\xi$$

and

$$I = \int_0^{\infty} \frac{1}{y}\left(\int_{-\infty}^{0} e^{2\pi y\xi}\, \phi(\xi)\, d\xi\right)^2\, dy = 0 .$$

This implies that for every $y > 0$, $\widehat{\phi}(iy) = 0$. Then, since $\widehat{\phi}(z)$ is analytic for $\mathcal{I}mz > 0$ we get that $\widehat{\phi} = 0$ and thus $\phi = 0$ by the unicity of the Fourier Transform. □

Lemma 6.

Let $0 < \beta < \infty$, $-\infty \leq c < \infty$ and $f(x) \geq 0$ a function belonging to $L^1_{loc}(c, \infty)$. Assume that there exists a pair (a, b), $c < a < b < \infty$ such that

$$\int_b^{\infty} \frac{f(y)}{(y-a)^{\beta}}\, dy \quad < \quad \infty . \tag{5.6}$$

Then (5.6) holds for every pair (a, b), $c < a < b$.

Proof. The proof is easy and shall not be given. □

Proposition 2.

Let $0 < \alpha < 1$, $1 < p < \infty$ *and* $w(x) \geq 0$ *such that* $w(x)^{-p'/p} \in L^1_{loc}(c, \infty)$ *where* $-\infty \leq c < \infty$. *Then, the following statements are equivalent:*

(i) For every non-negative $f(x)$, $f \in L^p(w)$, *we have that* $I^+_\alpha(f)(x)$ *is finite a.e. on* (c, ∞).

(ii) There exists a pair (a, b), $c < a < b < \infty$, *such that*

$$\int_b^\infty \frac{w(y)^{-p'/p}}{(y-a)^{(1-\alpha)p'}}\, dy \quad < \quad \infty \,. \tag{5.7}$$

Proof. (i) implies (ii). Since

$$\int_{x+1}^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy \quad \leq \quad I^+_\alpha(f)(x) \quad < \infty \qquad \text{a.e. on } (c, \infty)\,,$$

by Lemma 6, with $\beta = 1 - \alpha$, given a pair (a, b), $c < a < b < \infty$, the integral (5.6) is finite for every $f \in L^p(w)$. Then by the Principle of Uniform Boundedness it turns out that (ii) holds.

(ii) implies (i). By Lemma 6 since $w(x)^{-p'/p} \in L^1_{loc}(c, \infty)$ it follows that for every (a, b), $c < a < b$, (5.7) holds. In particular, if $d > 0$ and $x > c$, by Hölder's inequality we obtain

$$\int_{x+d}^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy \;\leq\; \|f\|_{L^p(w)} \left(\int_{x+d}^\infty \frac{w(y)^{-p'/p}}{(y-x)^{(1-\alpha)p'}}\, dy \right)^{1/p'} \quad < \infty\,. \tag{5.8}$$

Let $c < x_1 < x_2 < \infty$. By simple changes of variables and Hölder's inequality we get

$$\begin{aligned} \int_{x_1}^{x_2} \left(\int_x^{x+d} \frac{f(y)}{(y-x)^{1-\alpha}}\, dy \right) dx \;&\leq\; \frac{d^\alpha}{\alpha} \int_{x_1}^{x_2+d} f(z)\, dz \\ &\leq\; \frac{d^\alpha}{\alpha} \|f\|_{L^p(w)} \left(\int_{x_1}^{x_2+d} \frac{w(y)^{-p'/p}}{(y-x)^{(1-\alpha)p'}}\, dy \right)^{1/p'} \quad < \infty\,. \end{aligned} \tag{5.9}$$

The estimates (5.8) and (5.9) show that (i) holds. □

Let $0 < \alpha < 1, f \in L^p(w)$, w a weight in A^+_p that satisfies (ii) of Proposition 2. By that proposition, the difference $I^+_\alpha(f)(x+y) - I^+_\alpha(f)(x)$ is well defined for almost every $y > 0$, provided $I^+_\alpha(f)(x)$ is finite. Then

$$D_\alpha(f)(x) = \left(\int_0^\infty \frac{|I^+_\alpha(f)(x+y) - I^+_\alpha(f)(x)|^2}{y^{1+2\alpha}}\, dy \right)^{1/2}$$

is well defined for almost every x, $x_{-\infty} < x$. For this $D_\alpha(f)$ we have the following theorem:

Theorem 8.

Let $0 < \alpha < 1, 1 < p < \infty$ *and* $w \in A^+_p$ *satisfying condition (5.7). Then, there exists a constant* c_1 *depending on* α, p *and* w *only, such that*

$$c_1 \|f\|_{L^p(w)} \;\leq\; \|D_\alpha(f)\|_{L^p(w)}\,. \tag{5.10}$$

On the other hand, there exists another constant c_2 *depending on* α, p *and* w *only, such that*
(a) if $\alpha > 1/2$

$$\|D_\alpha(f)\|_{L^p(w)} \;\leq\; c_2 \|f\|_{L^p(w)}\,. \tag{5.11}$$

(b) if $0 < \alpha \leq 1/2$ *and* $p > \frac{2}{1+2\alpha}$, *then (5.11) holds provided that* $w \in A^+_{p\frac{1+2\alpha}{2}} \subset A^+_p$.

Proof. Let f be a C_0^1–function with support contained in $(x_{-\infty}, x_\infty)$. We are going to show that for $0 < \mu < 1$

$$D_\alpha(f)(x) \le c\ \{g^+(f)(x) + S_1^+(f)(x) + g_\lambda^*(f)(x)\}\ , \tag{5.12}$$

holds with $\lambda = 2\alpha + \mu$. The functions g^+ and g_λ^* correspond to a kernel $\phi(x) = \partial_x I_\alpha^+[x\varphi(x)]$ and S_1 has the kernel $\phi(x) = \partial_x I_\alpha^+[\varphi(x)]$, where $\varphi(x) \in C_0^\infty$ and $\int \varphi = 1$. In fact, we have

$$\begin{aligned} I_\alpha^+(f)(x) - I_\alpha^+(f)(x+y) &= \left[I_\alpha^+(f)(x) - (\varphi_y * I_\alpha^+(f))(x)\right] + \\ \left[(\varphi_y * I_\alpha^+(f))(x) - (\varphi_y * I_\alpha^+(f))(x+y)\right] &+ \left[(\varphi_y * I_\alpha^+(f))(x+y) - I_\alpha^+(f)(x+y)\right] \\ &= I_1(x,y) + I_2(x,y) + I_3(x,y)\ , \end{aligned}$$

thus,

$$D_\alpha(f)(x) \le \sum_{j=1}^{3} \left(\int_0^\infty \frac{|I_j(x,y)|^2}{y^{1+2\alpha}}\, dy \right)^{1/2} = \sum_{j=1}^{3} A_j(x)\ .$$

Procceding as in [13, page 162], we obtain

$$A_1(x) \le c\, g^+(f)(x) \quad , \quad A_2(x) \le c\, S_1(f)(x)$$
$$\text{and } A_3(x) \le c\, g_\lambda^*(f)(x)\ ,$$

with $\lambda = 2\alpha + \mu$, $1 - 2\alpha < \mu < 1$. Therefore, (5.12) holds. Then, in virtue of Theorems 5 and 7, we get (5.11) for f in C_0^1.

If f is any function in $L^p(w)$, let $\{f_n\}$ be a sequence of C_0^1–functions with support contained in $(x_{-\infty}, x_\infty)$ converging to f in $L^p(w)$ substituting $|f - f_n|$ by f in (5.8) and (5.9) we see that $I_\alpha^+(f_n)(x)$ tends to $I_\alpha^+(f)(x)$ a.e. in $(x_{-\infty}, \infty)$. Thus, if $g(x,y) \ge 0$ satisfies

$$\left[\int_{x_{-\infty}}^{x_\infty} \left(\int_0^\infty g(x,y)^2 \frac{dy}{y^{1+2\alpha}} \right)^{p'/2} w(x)^{-p'/p}\, dx \right]^{1/p'} \le 1\ ,$$

then, by Fatou's Lemma and Hölder's inequality,

$$\begin{aligned} &\int_{x_{-\infty}}^{x_\infty} \int_0^\infty g(x,y) |I_\alpha^+(f)(x+y) - I_\alpha^+(f)(x)|\, \frac{dy}{y^{1+2\alpha}}\, dx \\ \le\ &\liminf \int_{x_{-\infty}}^{x_\infty} \int_0^\infty g(x,y) |I_\alpha^+(f_n)(x+y) - I_\alpha^+(f_n)(x)|\, \frac{dy}{y^{1+2\alpha}}\, dx \\ \le\ &\liminf \|D_\alpha(f_n)\|_{L^p(w)} \le c\ \liminf \|f_n\|_{L^p(w)} = c\ \|f\|_{L^p(w)}\ , \end{aligned}$$

which implies that (5.11) holds for any f.

As for (5.10), procceding as in [13, page 162], we get

$$g^+(f)(x) \quad \le \quad c\, D_\alpha(f)(x)$$

for $x_{-\infty} < x$. Thus, (5.10) follows by integration and Proposition 1. □

6. Application to Multipliers

Let $m(x)$ be a bounded measurable function defined on $\mathbb{R}$. The operator

$$\widehat{T_m f}(x) = m(x)\widehat{f}(x)$$

is well defined if $f \in \mathcal{S}(\mathbb{R})$. With this notation we have the following theorem:

Theorem 9.

Let $m(x)$ $(x \in \mathbb{R})$ be the boundary value of an analytic and bounded function on the upper half-plane. We assume that its derivative $Dm(x)$ exists for every $x \neq 0$ and

$$|x||Dm(x)| \leq c \quad , \quad x \neq 0 \, .$$

If $w \in A_p^+$, $1 < p < \infty$, then there exists a constant c' depending on p and w only, such that

$$\|T_m(f)\|_{L^p(w)} \quad \leq \quad c' \, \|f\|_{L^p(w)} \, .$$

Proof. Let φ be a function with the following properties:

$$\begin{cases} (i) & \varphi \in \mathcal{S}(\mathbb{R}) \text{ and } \varphi \geq 0 \, , \\ (ii) & \text{supp } (\varphi) \subset (-\infty, 0] \text{ and} \\ (iii) & \int \varphi dx > 0 \, . \end{cases} \tag{6.1}$$

We define $\phi(x) = -x\varphi(x)$ and $\psi(x) = \phi * \phi(x)/x^2$. These functions ϕ and ψ satisfy the same conditions (6.1) that φ does. Since $x^2\psi(x) = (\phi * \phi)(x)$ it follows that

$$D^2\widehat{\psi}(x) = [D\widehat{\varphi}(x)]^2 \, . \tag{6.2}$$

Let $M(x,t)$ be define by $\widehat{M}(x,t) = m(x)\widehat{\varphi}(tx)$. By (i) and (ii) we get that $\widehat{\varphi}(x)$ is the boundary value of the function $\widehat{\varphi}(z) = \int_{-\infty}^{0} e^{-2\pi izy}\,\varphi(y)\,dy$, where $\mathcal{I}m(z) > 0$ and $|\widehat{\varphi}(z)| \leq c/|z|^{k+2}$. Then, since $|m(z)| \leq c$, by the Cauchy's Theorem it follows that

$$M(x,t) = 0 \;\text{ for }\; 0 \leq x \;\text{ and }\; t > 0 \, . \tag{6.3}$$

We define

$$\widehat{U}(x,t) = m(x)\widehat{f}(x)\widehat{\psi}(tx) = \widehat{T_m f}(x)\widehat{\psi}(tx) \;\text{ and}$$
$$\widehat{u}(x,t) = \widehat{f}(x)\widehat{\varphi}(tx) \, .$$

Taking into account (6.2) it follows that

$$\partial_t^2\widehat{U}(x,t) = \partial_t\widehat{M}(x,t)\,\partial_t\widehat{u}(x,t) \, .$$

Thus,

$$\partial_t^2 U(x,t) = \int_{-\infty}^{\infty} \partial_t M(y,t)\,\partial_t u(x-y,t)\,dy \, .$$

By a change of variables and (6.3) we have

$$\partial_t^2 U(x,t) = \int_0^{\infty} \partial_t M(-y,t)\,\partial_t u(x+y,t)\,dy \, .$$

Following [13, page 96], we have that

$$g^+(T_m(f))(x) \quad \leq \quad c\, g_2^*(f)(x) \, .$$

Appealing to Proposition 1 and Theorem 7, and recalling that $\int \psi > 0$, we get

$$\|T_m(f)\|_{L^p(w)} \quad \leq \quad c' \, \|f\|_{L^p(w)} \, ,$$

whenever $w \in A_p^+$, $1 < p < \infty$. □

References

[1] Aimar, H., Forzani, L. and Martín–Reyes, F. On weighted inequalities for one–sided singular integrals, to appear in *Proccedings of the Amer. Math. Soc.*.

[2] Benedek, A., Calderón, A.P. and Panzone, R. (1962). Convolution operators on Banach–space valued functions, *Proc. Nat. Acad. Sci. U.S.A.*, **48**, 356–365.

[3] Duoandikoetxea, J. (1991). Análisis de Fourier, *Ediciones de la Universidad Autónoma de Madrid,* Madrid.

[4] Fefferman, C. and Stein, E.M. (1972). H^p spaces of several variables, *Acta Math.* **129**, 137–193.

[5] García–Cuerva, J. and Rubio de Francia, J.L. (1985). *Weighted Norm Inequalities and Related Topics,* North–Holland, Amsterdam.

[6] Martín–Reyes, F.J., Pick, L. and de la Torre, A. (1993). A_∞^+ condition, *Canad. J. Math.* **45**, 1231–1244.

[7] Muckenhoupt, R. and Wheeden, R. (1974). Norm inequalities for the Littlewood–Paley function g_λ^*, *Trans. Amer. Math. Soc.* **191**, 95–111.

[8] Riesz, F. and Sz–Nagy, B. (1966). *Functional Analysis,* Ungar, New York.

[9] Riveros, S. (1994). Extrapolación para pesos laterales, Doctoral Dissertation, Universidad Nacional del Litoral, Santa Fe, Argentina.

[10] de Rosa, L. and Segovia, C. A substitute of harmonic majorization, preprint.

[11] de Rosa, L. and Segovia, C. (1995). Weighted H^p spaces for one sided maximal functions, *Contemporary Mathematics (AMS series),* **189**, 161–183.

[12] Sawyer, E. (1986). Weighted inequalities for the one sided Hardy–Littlewood maximal functions, *Trans. Amer. Math. Soc.*, **297**, 53–61.

[13] Stein, E.M. (1970). *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, NJ.

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