affine space at scale $r > 0$ is measured by the infimum of the scaled Hausdorff distances between the boundary and n-planes through Q, namely

$$\theta(r, Q) = \inf_L \left\{ \frac{1}{r} D[\partial\Omega \cap B(r, Q), L \cap B(r, Q)] \right\},$$

where the infimum is taken over all n-planes L containing Q. Our work requires uniform control of several quantities on compact sets; thus, for each compact set $K \subset \mathbb{R}^{n+1}$ we define

$$\theta_K(r) = \sup_{Q \in \partial\Omega \cap K} \theta(r, Q).$$

The quantity $\theta_K(r)$ provides a uniform measurement over K of how far $\partial\Omega$ is from being an affine plane at scale $r > 0$. It also gives an upper bound for the oscillation of the approximating affine spaces at scale r. In this sense it is a good replacement for the oscillation of the unit normal to the boundary (which measures the oscillation of the tangent planes). If $\Omega \subset \mathbb{R}^{n+1}$ is smooth, the fact that the approximation of $\partial\Omega$ by affine spaces improves as r tends to 0 translates into the following statement: for each compact set $K \subset \mathbb{R}^{n+1}$ we have

$$\lim_{r \to 0} \theta_K(r) = 0.$$

Definition 1.
Let $\Omega \subset \mathbb{R}^{n+1}$. We say that Ω is a Reifenberg flat domain if there exists $\delta \in (0, 1/8)$ so that for each compact set $K \subset \mathbb{R}^{n+1}$ there exists $R > 0$ such that

$$\sup_{0 < r \le R} \theta_K(r) \le \delta. \tag{1.1}$$

In the definition of a Reifenberg flat domain, the parameter δ could have been chosen to be any positive number. On the other hand, (1.1) only provides significant information for δ small. The choice of $1/8$ as an upper bound for δ is slightly arbitrary, but it is enough to rule out some nasty examples (see Remark 3 below). An important example of a Reifenberg flat domain is $\Omega = \{(x, t) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ t > \varphi(x)\}$ where φ is a Lipschitz function with Lipschitz constant less than $1/8$.

Definition 2.
Let $\Omega \subset \mathbb{R}^{n+1}$. We say that Ω is a Reifenberg vanishing domain if Ω is a Reifenberg flat domain and if for each compact set $K \subset \mathbb{R}^{n+1}$

$$\lim_{r \to 0} \theta_K(r) = 0.$$

Summarizing the fact that Ω is a Reifenberg flat domain guarantees that at small scales $\partial\Omega$ can be approximated by n-planes. This approximation is uniform on compact sets. The deviation of $\partial\Omega$ from being an n-dimensional affine space only depends on the parameter $\delta \in (0, 1/8)$. If Ω is a Reifenberg vanishing domain, the approximation improves as the scale diminishes. On the other hand, it is not true that if Ω is a Reifenberg vanishing domain then $\partial\Omega$ admits tangent planes. In fact, let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}}$$

The function φ can be shown to belong to λ_*, the little-o Zygmund class (see [28, pg. 47]). This implies that φ is well approximated by affine functions whose graphs are affine spaces. Using this information it is not difficult to show that $\Omega = \{(x, t) \in \mathbb{R}^2 : t > \varphi(x)\}$ is a Reifenberg vanishing domain. On the other hand, φ is a continuous function which is nowhere differentiable (in particular it is a variant of the Weierstrass function). Thus, $\partial\Omega$ is not rectifiable (for a precise definition of rectifiability, see [26]), which in particular implies that $\partial\Omega$ does not admit tangent planes almost everywhere, i.e., there is not even a weak notion of the unit normal vector to $\partial\Omega$. Furthermore, $\partial\Omega$ has locally infinite 1-dimensional Hausdorff measure (see [27]). Thus, the surface measure to $\partial\Omega$ is not well defined.

A way to understand the pathologies presented by Reifenberg vanishing domains is by thinking about them as domains which admit $C^{0,\alpha}$ parametrizations for every $\alpha \in (0, 1)$ but which might not admit $C^{0,1}$ parametrizations. As a matter of fact, Reifenberg's theorem guarantees that the boundary of a Reifenberg vanishing domain is locally representable as the image (not the graph!) via a bi-Hölder continuous map of an open subset of $\mathbb{R}^n$, with Hölder exponent as close to 1 as one wishes (see [19, 22]). On the other hand, the example above shows that the boundary of a Reifenberg vanishing domain might not contain any Lipschitz piece. As we shall see, there is a correspondence between the flatness of a domain and the doubling properties of harmonic measure.

Remark 1.

Reifenberg introduced this notion of flatness in 1960. He was interested in the existence and regularity of solutions for the Plateau problem in higher dimensions. The result mentioned above allowed him to show that an n-dimensional minimal surface with prescribed boundary is a topological manifold except for a set of n-dimensional Hausdorff measure zero.

2. Doubling

A measure ω supported in a subset Σ of $\mathbb{R}^{n+1}$ is doubling if the ω-measure of the ball of radius $2r$ and center $Q \in \Sigma$ can be controlled by the ω-measure of the ball of radius r and center Q for $r > 0$ small enough. If Ω is a smooth domain, the surface measure of its boundary is a doubling measure. On the other hand, since the surface measure of the boundary of a general Reifenberg flat domain is not well defined, we use a different measure, namely harmonic measure, which makes sense in a very general context. Let $\Omega \subset \mathbb{R}^{n+1}$ be a smooth (unbounded) domain, then there exists a harmonic function v defined in Ω satisfying

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}, \qquad \Delta = \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}, \quad \text{the Laplacian}\,.$$

The function v is uniquely determined up to multiplication by a positive constant and is called the Green's function of Ω with pole at infinity. In general, the Green's function of a domain Ω with pole at $X_0 \in \Omega$ is a positive harmonic function in $\Omega \backslash \{X_0\}$ which vanishes on $\partial\Omega$. The function v can be constructed as the limit of scaled Green's functions whose poles converge to infinity. If $\phi \in C_c^\infty(\mathbb{R}^{n+1})$, integration by parts yields

$$\int_\Omega (v\Delta\phi - \phi\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial v}{\partial\nu} \right) d\sigma\,,$$

where σ denotes the surface measure and $\partial/\partial\nu$ denotes the normal derivative at the boundary (i.e., $\partial/\partial\nu = \nu\cdot\nabla$ where ν denotes the inward unit normal and ∇ denotes the gradient). Since v is harmonic in Ω and vanishes on the boundary, the integration by parts formula above becomes

$$\int_\Omega v\Delta\phi dx = -\int_{\partial\Omega} \phi\frac{\partial v}{\partial\nu} d\sigma\,. \tag{2.1}$$

By analogy with the conventional Poisson kernel, $\frac{\partial v}{\partial\nu}$ is called the Poisson kernel of Ω with pole at infinity. The measure ω which is supported in $\partial\Omega$ and defined by

$$\omega(A) = \int_{\partial\Omega\cap A} \frac{\partial v}{\partial\nu} d\sigma, \quad \text{for any Borel set } A \subset \mathbb{R}^{n+1}\,,$$

is called the harmonic measure of Ω with pole at infinity. Both the Poisson kernel and the harmonic measure are determined up to multiplication by a positive constant. Note that the measure ω might exist, without the existence of either $\partial v/\partial \nu$ or $d\sigma$, if our domain Ω is no longer smooth.

In particular if $\Omega = \mathbb{R}^{n+1}_+$, $v(x_1, \dots, x_{n+1}) = x_{n+1}$, the Poisson kernel of $\mathbb{R}^{n+1}_+$ with pole at infinity is identically 1 and the harmonic measure of $\mathbb{R}^{n+1}_+$ with pole at infinity is the Lebesgue measure of $\mathbb{R}^n$. Thus, (2.1) becomes

$$\int_{\mathbb{R}^{n+1}_+} x_{n+1} \Delta\phi \, dx_1 \dots dx_{n+1} = -\int_{\mathbb{R}^n} \phi \, dx_1 \dots dx_n \, .$$

The Hopf boundary lemma combined with classical boundary regularity results for the solution of the Laplace equation on a smooth domain guarantee that the harmonic measure with pole at infinity ω is *asymptotically optimally doubling*. This means that ω is a Radon measure, i.e., the ω measure of compact sets is finite, and for each compact set $K \subset \mathbb{R}^{n+1}$ such that $K \cap \partial\Omega \neq \emptyset$, and for each $\tau > 0$

$$\lim_{r\to 0} \inf_{Q\in K\cap\partial\Omega} \frac{\omega(B(r\tau, Q))}{\omega(B(r, Q))} = \lim_{r\to 0} \sup_{Q\in K\cap\partial\Omega} \frac{\omega(B(r\tau, Q))}{\omega(B(r, .Q))} = \tau^n \, . \tag{2.2}$$

On the one hand, (2.2) states that ω is a doubling measure. On the other hand, it claims that as $r \to 0$ the ratio $\omega(B(2r, Q))/\omega(B(r, Q))$ behaves more and more like the corresponding ratio for Lebesgue measure (resp. Hausdorff measure) in n-dimensional Euclidean space (resp. n-dimensional smooth hypersurface). Nevertheless, (2.2) does not imply anything about the behavior of the ratio $\omega(B(r, Q))/r^n$ for $Q \in \partial\Omega$ as r tends to 0. In fact if

$$\Omega = \{(x, t) \in \mathbb{R}^2 : t > \varphi(x)\} \text{ for } \varphi(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}} \, ,$$

then ω is asymptotically optimally doubling and for each compact set $K \subset \mathbb{R}^{n+1}$, $\sup_{K\cap\partial} \omega(B(r, Q))/r^n$ tends to infinity as r tends to 0.

Thus, a smooth domain is Reifenberg vanishing and its harmonic measure with pole at infinity is asymptotically optimally doubling. We now show that these notions of flatness and doubling are deeply intertwined, and are independent of the smoothness assumption. They provide some weak notions of regularity which suffice to answer several questions in potential theory and geometric measure theory. Before stating any results, we need to guarantee that it makes sense to talk about the harmonic measure with pole at infinity for a general Reifenberg flat domain. This is the content of the next proposition.

Proposition 1.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a Reifenberg flat domain, or a Lipschitz domain (i.e., a domain whose boundary is locally representable as the graph of a Lipschitz function), then there exists a unique (up to normalization) doubling Radon measure ω such that

$$\int_\Omega v \Delta\phi dx = -\int_{\partial\Omega} \phi d\omega \quad \forall \phi \in C_c^\infty(\mathbb{R}^{n+1})$$

where

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Here ω denotes the harmonic measure of Ω with pole at infinity, and v denotes the Green's function of Ω with pole at infinity.

3. Regularity of Harmonic Measure

In this section we discuss ways in which the regularity of the boundary of $\Omega \subset \mathbb{R}^{n+1}$ (a domain as in the proposition above) determines the regularity of its harmonic measure ω. If $\partial\Omega$ is uniformly rectifiable (i.e., rectifiable with uniform estimates, see [7] for a definition), then the surface measure σ given by the restriction of the n-dimensional Hausdorff measure to $\partial\Omega$, i.e., $\sigma = \mathcal{H}^n \lfloor \partial\Omega$, is a Radon measure. Under the appropriate geometric conditions for Ω, results by [5, 6] and [25] guarantee that ω is a doubling measure and that σ and ω are mutually absolutely continuous. In this case the Radon-Nikodyn theorem insures that the Poisson kernel $h = d\omega/d\sigma$ (the Radon-Nikodyn derivative of ω with respect to σ) exists. The Poisson kernel is the density of ω with respect to σ. In this setting we describe the regularity of ω in terms of the behavior of h. If $\partial\Omega$ is not uniformly rectifiable we concentrate on the doubling properties of ω, rather than in those of the Poisson kernel which might not even exist. The proofs of all the results discussed in this section use techniques from harmonic analysis, potential theory, and partial differential equations.

A classical boundary regularity result states that if the unit normal vector to the boundary is in some Hölder space, then so is the logarithm of the Poisson kernel. More precisely if Ω is a $C^{1,\alpha}$ domain (i.e., $\partial\Omega$ is locally representable as the graph of a $C^{1,\alpha}$ function) for $\alpha \in (0, 1)$ then $\log h \in C^{0,\alpha}$ (see [12]). Jerison and Kenig proved that if the oscillation of the unit normal vector to the boundary is small in the C^0 norm then $\log h$ has small oscillation in an integral sense, specifically $\log h$ has vanishing mean oscillation, i.e., if Ω is a C^1 domain (i.e., $\partial\Omega$ is locally representable as the graph of a C^1 function) then $\log h \in VMO(d\sigma)$ (see [11]). This ensures that $\log h$ can be well approximated by uniformly continuous functions in an integral sense (namely in the mean oscillation norm), but it does not guarantee that $\log h$ is continuous. In fact it is easy to construct examples of C^1 domains for which the logarithm of the Poisson kernel is not continuous.

Along these lines we prove that if the unit normal vector ν to the boundary of Ω has small integral oscillation, then so does the logarithm of the Poisson kernel. We show that if Ω is a chord arc domain with vanishing constant (i.e., $\nu \in VMO(d\sigma)$ see [23, 24]), then $\log h \in VMO(d\sigma)$ (see [14]). (Such domains are Reifenberg vanishing [23, 24]). Thus, we extend the result in [11]. Chord arc domains with vanishing constant thus provide a good generalization of C^1 domains from the potential theory point of view.

The first results of this type for non-smooth domains were proved by Lavrentiev ($n = 1$, see [17]) and Dahlberg ($n \geq 2$). In particular, Dahlberg showed that if Ω is a Lipschitz domain, then σ and ω are mutually absolutely continuous. In this case while the Radon-Nikodyn theorem guarantees that $h \in L^1_{\text{loc}}(d\sigma)$, he proved that in fact $h \in L^2_{\text{loc}}(d\sigma)$ ([5]). (See also [20] for VMO results.)

As we saw in the example above, in a general Reifenberg vanishing domain the surface measure of the boundary is not well defined. In this setting the regularity of the harmonic measure needs to be expressed in terms of its doubling properties, since it cannot be expressed in terms of the regularity of its density with respect to surface measure. The appropriate regularity statement is given by the following result.

Theorem 1. [14]

The harmonic measure of a Reifenberg vanishing domain is asymptotically optimally doubling.

This theorem shows that if the boundary of a domain is well approximated by affine spaces in the Hausdorff distance sense, then its harmonic measure behaves like the harmonic measure of those affine spaces from a doubling point of view. The theorem implies that: *The boundary of a Reifenberg vanishing domain supports an asymptotically optimally doubling measure.* Theorem 3, discussed in the next section, establishes the converse of this statement. This provides a complete characterization of Reifenberg vanishing domains in terms of the doubling properties of the measures they support. The main ingredients in the proof of Theorem 1 are the maximum principle, the comparison principle for non-tangentially accessible (NTA) domains (of which Reifenberg flat domains are an example), and the boundary regularity for non-negative harmonic functions on NTA domains (see [10, 14]).

4. A Free Boundary Regularity Problem

In this section we discuss the converse problem. Namely, we explain that either the regularity of the Poisson kernel of a domain or the doubling properties of its harmonic measure determine the regularity of its boundary. This problem should be understood as a free boundary regularity problem in the sense that we are trying to deduce the regularity of the boundary of the set $\{v > 0\}$, where v denotes the Green's function with pole at infinity from some information about the "regularity of the normal derivative" on the boundary of this set. In the case when the boundary of the domain is rectifiable, we have a unit normal vector and the normal derivative of v at the boundary is a well-defined function, h, the Poisson kernel with pole at infinity. In the case when the boundary is not rectifiable, we do not have a unit normal vector to the boundary and therefore the "normal derivative" should be understood not as a function but as a measure, the harmonic measure with pole at infinity. (See the integral equality that appears in the Proposition 1 in Section 2).

Free boundary problems of this type were studied by Alt and Caffarelli (see [1, 2, 3]), who showed that on a domain which is sufficiently flat the behavior of the logarithm of the Poisson kernel determines the regularity and the geometry of the boundary. The strategy behind their proof is the following: in a domain whose boundary is flat enough, the uniform continuity of the logarithm of the Poisson kernel ensures that as the scale decreases this flatness improves. They proved the following theorem.

Theorem 2. *[1]*
Assume that

1. $\Omega \subset \mathbb{R}^{n+1}$ *is a set of locally finite perimeter (see [8] for the precise definition) whose boundary is Ahlfors regular (i.e., for* $Q \in \partial\Omega$ *and* $r > 0$*, the ratio*

$$\frac{\mathcal{H}^n(\partial\Omega \cap B(r, Q))}{r^n}$$

is bounded above and below by uniform constants.

2. $\Omega \subset \mathbb{R}^{n+1}$ *is Reifenberg flat domain and (1.1) holds for some* $\delta > 0$ *small enough, depending on dimension,*
3. $\log h \in C^{0,\beta}$ *for some* $\beta \in (0, 1)$.

Then Ω *is a* $C^{1,\alpha}$ *domain for some* $\alpha \in (0, 1)$ *which depends on* β*. Moreover, if* h *is identically equal to* 1*, then* Ω *is a half space.*

In [2] it was shown that if $\partial\Omega$ is *a priori* assumed to be a Lipschitz graph, *(2)* is not needed to obtain the same conclusion. In [9] Jerison later showed that in this case (Lipschitz graph), we can take $\alpha = \beta$, the optimal result. Moreover, in [9] it is also shown that in the Lipschitz graph case, if instead of *(3)*, we have that $\log h$ is uniformly continuous, then the graph function has a gradient in VMO.

Note that these results combined with those mentioned above reinforce the idea that the regularity of the boundary of a domain (described in terms of the oscillation of the unit normal vector) and the regularity of its harmonic measure (described in terms of the regularity of the logarithm of its density) are "equivalent".

Along these lines we prove that this "equivalence" prevails even when the notions of smoothness involved are weaker than the ones above. We show that on a domain which is flat enough, the behavior of the logarithm of the Poisson kernel together with the doubling properties of the harmonic measure determine the regularity and the geometry of the boundary. More precisely, if Ω is a chord arc domain with small constant (i.e., the mean oscillation of the unit normal to the boundary is small), if ω is asymptotically optimally doubling, and if $\log h \in VMO(d\sigma)$, then Ω is a chord arc domain with vanishing constant (see [9, 15]). (Both assumptions are necessary conditions, as shown by the results in the previous section).

Note that all of these results convey the idea that the oscillation of the unit normal to the boundary of a domain and the oscillation of the logarithm of its Poisson kernel are "equivalent" quantities on flat domains. For chord arc domains in $\mathbb{R}^2$, this equivalence was explicitly proved by Pommerenke. He showed, using complex analysis methods, that the unit normal to the boundary belongs to $VMO(d\sigma)$ if and only if $\log h \in VMO(d\sigma)$ (see [20]). It is the subject of current investigation whether this result is also true in higher dimensions. More precisely, we would like to show that if $\Omega \subset \mathbb{R}^{n+1}$ is a chord arc domain with small constant and $\log h \in VMO(d\sigma)$, then ω is an asymptotically optimally doubling measure. This would allow us to apply the result stated above to conclude that Ω is a chord arc domain with vanishing constant, assuming only one of the two hypotheses we used in the above theorem. (See also the conjecture at the end of the article).

In general Reifenberg flat domains, the notion of surface measure of the boundary is not well defined; therefore, the regularity of the free boundary in this case depends solely on the doubling properties of the harmonic measure.

Theorem 3. *[15]*

Let $\Omega \subset \mathbb{R}^{n+1}$ be a Reifenberg flat domain whose harmonic measure is asymptotically optimally doubling, then Ω is a Reifenberg vanishing domain.

Theorem 3 is a corollary of a more general result of a geometric measure theory flavor: *A Reifenberg flat set which supports an asymptotically optimally doubling measure is Reifenberg vanishing*. Theorem 1 and 3 provide a complete characterization of Reifenberg vanishing domains in terms of the doubling properties of their harmonic measure. The proof of Theorem 3 uses tools from [18, 21]. It relies heavily on the Kowalski-Preiss classification of n-uniform measures (see [16] and Remark 2 below), as well as in the notions of tangent and pseudo-tangent measure (see [15]).

Remark 2.

Both in Alt and Caffarelli's result, as well as in Theorem 3, the assumption that Ω is a Reifenberg flat domain is crucial. In $\mathbb{R}^3$, Alt-Caffarelli [1] construct a double cone, of opening $\sim 33°$, such that, for the region outside of it, the Poisson kernel at infinity is identically 1. An example by Kowalksi and Preiss (see [16]) combined with a calculation of the harmonic measure carried out in [15] shows that in dimensions greater than 4 there exist unbounded domains whose Poisson kernel at infinity is identically 1, whose harmonic measure with pole at infinity is asymptotically optimally doubling and which are very far from being Reifenberg flat. If $n \geq 3$, let

$$\Omega = \left\{(x_1, \dots, x_{n+1}) \subset \mathbb{R}^{n+1} : |x_4| < \sqrt{x_1^2 + x_2^2 + x_3^2}\right\} .$$

Ω is an unbounded non-tangentially accessible domain whose harmonic measure ω with pole at infinity appropriately normalized satisfies

$$\omega = \sigma = \mathcal{H}^n \lfloor \partial\Omega \Rightarrow h = \frac{d\omega}{d\sigma} = 1 .$$

(See [15]). Moreover, ω is n-uniform, i.e., for $Q \in \partial\Omega$, $r > 0$, and $\tau > 0$

$$\omega(B(r, Q)) = \mathcal{H}^n(\partial\Omega \cap B(r, Q)) = r^n \Rightarrow \frac{\omega(B(r\tau, Q))}{\omega(B(r, Q))} = \tau^n .$$

(See [16]). On the other hand, it is easy to see that Ω is not Reifenberg flat. In fact $0 \in \partial\Omega$ and for $r > 0$, the Hausdorff distance between $\partial\Omega \cap B(r, 0)$ and $L \cap B(r, 0)$ for any n-dimensional plane containing the origin is at least $r/\sqrt{2}$, i.e., $\theta(r, 0) \geq 1/\sqrt{2}$.

Remark 3.

Old counter-examples of Keldysh-Laurentiev (see [13, 4]) show that in the theorem of Alt-Caffarelli [1], the hypothesis of being Ahlfors-regular is needed. These are examples of locally rectifiable domains in $\mathbb{R}^2$*, which are Reifenberg vanishing, for which* $\log h \equiv 0$*, which are not smooth. These examples, and the theorems and remarks above lead us to the following:*

Conjecture 1.

Assume that (1) and (2) in Theorem 2 above hold, and that $\log h \in VMO(d\sigma)$*. Then,* Ω *is a chord-arc domain with vanishing constant.*

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