The Journal of Fourier Analysis and Applications

Volume 3, Special Issue, 1997

Sharp Inequalities and Geometric Manifolds

William Beckner

ABSTRACT. Sharp constants for function-space inequalities over a manifold encode information about the geometric structure of the manifold. An important example is the Moser–Trudinger inequality where limiting Sobolev behavior for critical exponents provides significant understanding of geometric analysis for conformal deformation on a Riemannian manifold [5, 6]. From the overall perspective of the conformal group acting on the classical spaces, it is natural to consider the extension of these methods and questions in the context of $SL(2, R)$, the Heisenberg group, and other Lie groups. Among the principal tools used in this analysis are the linear and multilinear operators mapping $L^p(M)$ to $L^q(M)$ defined by the Stein–Weiss integral kernels which extend the Hardy–Littlewood–Sobolev fractional integrals

$$K(x, y) = |x|^{-\alpha} |x - y|^{-\lambda} |y|^{-\beta} , \tag{1}$$

conformal geometry, and the notion of equimeasurable geodesic radial decreasing rearrangement. To illustrate these ideas, four model problems will be examined here: (1) logarithmic Sobolev inequality and the uncertainty principle, (2) $SL(2, R)$ and axial symmetry in fluid dynamics, (3) Stein–Weiss integrals on the Heisenberg group, and (4) Morpurgo's work on zeta functions and trace inequalities of conformally invariant operators.

1. Logarithmic Sobolev Inequality and the Uncertainty Principle

Geometric information about a manifold is determined by the classical inequalities. The uncertainty principle is a quantitative statement about both the dilation structure and the product structure on a manifold. A new formulation of the uncertainty principle can be given as a logarithmic indeterminate expression which is additive rather than multiplicative [7]. Let P, Q designate canonical momentum and position variables with mean values denoted by $\widehat{P}$, $\widehat{Q}$. Then logarithmic uncertainty is given by

$$\langle \ln |Q - \widehat{Q}| \rangle + \langle \ln |P - \widehat{P}| \rangle \geq C \, . \tag{1.1}$$

More precisely, for functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and the Fourier transform defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} f(x)\, dx \, ,$$

Math Subject Classifications.
Keywords and Phrases.
Acknowledgements and Notes. This work was partially supported by the National Science Foundation and the Texas Advanced Research Program.

© 1997 CRC Press LLC
ISSN 1069-5869

then

$$\int_{\mathbb{R}^n} \ln|x|\, |f(x)|^2\, dx + \int_{\mathbb{R}^n} \ln|\xi|\, \left|\hat{f}(\xi)\right|^2 d\xi \geq D_n \int_{\mathbb{R}^n} |f(x)|^2\, dx$$

$$D_n = \psi(n/4) - \ln \pi \qquad \psi(t) = \frac{d}{dt} \ln \Gamma(t)\,. \tag{1.2}$$

Since the individual terms on the left-hand side of this expression may be indeterminate on $L^2(\mathbb{R})$, this inequality is realized as an *a priori* limit. Consider product functions of the form $\prod f(x_k)$ where each $x_k \in \mathbb{R}^m$ and k runs from 1 to ℓ. Further, adjust (1.2) to include the variation from the mean values $\hat{x}, \hat{\xi}$ of the respective distributions $|f(x)|^2$ and $|\hat{f}(\xi)|^2$. Then the concavity of the logarithm function implies that for $\|f\|_{L^2(\mathbb{R}^m)} = 1$,

$$\ln\left[\ell^2 \int_{\mathbb{R}^m} |x - \hat{x}|^2 |f(x)|^2\, dx \int_{\mathbb{R}^m} \left|\xi - \hat{\xi}\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi\right]^{1/2}$$

$$\geq \int_{\mathbb{R}^{\ell m}} \ln|x - \hat{x}| \prod_{k=1}^{\ell} |f(x_k)|^2\, dx + \int_{\mathbb{R}^{\ell m}} \ln|\xi - \hat{\xi}| \prod_{k=1}^{\ell} \left|\hat{f}(\xi_k)\right|^2 d\xi$$

$$\geq \left[\psi\left(\frac{m\ell}{4}\right) - \ln\left(\frac{m\ell}{4}\right)\right] + \ln\left(\frac{m\ell}{4\pi}\right)\,.$$

Since $-\frac{1}{z} \leq \psi(z) - \ln z \leq 0$, the limit $\ell \to \infty$ gives the classical Heisenberg–Weyl inequality

$$\left[\int_{\mathbb{R}^m} |x - \hat{x}|^2 |f(x)|^2\, dx \int_{\mathbb{R}^m} \left|\xi - \hat{\xi}\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi\right]^{1/2} \geq \frac{m}{4\pi} \int_{\mathbb{R}^n} |f(x)|^2\, dx\,. \tag{1.3}$$

The basic inequality (1.2) is a consequence of computing the sharp constant for Pitt's inequality on $\mathbb{R}^n$ [7]. For $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 \leq \alpha < n$

$$\int_{\mathbb{R}^n} |\xi|^{-\alpha} \left|\hat{f}(\xi)\right|^2 d\xi \leq C_\alpha \int_{\mathbb{R}^n} |x|^\alpha |f(x)|^2\, dx$$

$$C_\alpha = \pi^\alpha \left[\Gamma\left(\frac{n-\alpha}{4}\right) / \Gamma\left(\frac{n+\alpha}{4}\right)\right]^2\,. \tag{1.4}$$

This inequality can be realized as an equivalent Stein–Weiss integral inequality on $\mathbb{R}^n$:

$$\left|\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) \frac{1}{|x|^{\alpha/2}} \frac{1}{|x-y|^{n-\alpha}} \frac{1}{|y|^{\alpha/2}} g(y)\, dx\, dy\right|$$

$$\leq C_\alpha \left[\pi^{\frac{n}{2}-\alpha} \Gamma\left(\frac{\alpha}{2}\right) / \Gamma\left(\frac{n-\alpha}{2}\right)\right] \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}\,. \tag{1.5}$$

The sharp constant here is calculated by using radial symmetrization and reducing this inequality to Young's inequality for convolution on the multiplicative group $\mathbb{R}_+$.

Not only does the logarithmic uncertainty principle imply the classical uncertainty inequality, but it also determines the logarithmic Sobolev inequality for Gaussian measure. Observe that using equimeasurable radial decreasing rearrangement of either the function or its Fourier transform will improve the estimate (1.2). Suppose that f is radial decreasing; then for $\|f\|_2 = 1$, $|f(x)| \leq A|x|^{-n/2}$ or

$$\ln|f(x)| \leq -\frac{n}{2} \ln|x| + C_n$$

where $C_n = \frac{1}{2}\ln\Gamma(\frac{n}{2}+1) - \frac{n}{4}\ln\pi$. Now substituting in expression (1.2), one obtains

$$\frac{n}{4}\int_{\mathbb{R}^n} \ln|\xi|^2 \left|\hat{f}(\xi)\right|^2 d\xi \geq \int_{\mathbb{R}^n} |f(x)|^2 \ln|f(x)|\, dx + A_n$$
$$A_n = \frac{n}{2}\psi\left(\frac{n}{4}\right) - \frac{n}{4}\ln\pi - \frac{1}{2}\ln\Gamma\left(\frac{n}{2}+1\right) . \tag{1.6}$$

This expression is already a logarithmic Sobolev inequality because the left-hand side is a smoothness estimate while the right-hand side is a measure of entropy. A_n is not the sharp constant for this inequality, but it will suffice to derive the Gaussian logarithmic Sobolev inequality using the product structure of Gaussian measure. Again, using the fact that the logarithm is concave and inverting the resulting expression for the Fourier transform, one finds

$$\frac{n}{4}\ln\int_{\mathbb{R}^n} |\nabla f|^2\, dx \geq \int_{\mathbb{R}^n} |f|^2 \ln|f|\, dx + A_n + \frac{n}{2}\ln(2\pi) . \tag{1.7}$$

Theorem 1.

The logarithmic Sobolev inequality (1.6) obtained from the uncertainty principle (1.2) implies the logarithmic Sobolev inequality for Gaussian measure

$$\int_{\mathbb{R}^n} |g|^2 \ln|g|\, d\mu \leq \int_{\mathbb{R}^n} |\nabla g|^2\, d\mu \tag{1.8}$$

where $d\mu = (2\pi)^{-n/2}\exp(-x^2/2)\, dx$ *and* $\|g\|_{L^2(d\mu)} = 1$.

Proof. Using the interplay between Gaussian measure and Lebesgue measure, observe that the weaker inequality with a non-positive constant V_n

$$V_n + \int_{\mathbb{R}^n} |g|^2 \ln|g|\, d\mu \leq \int_{\mathbb{R}^n} |\nabla g|^2\, d\mu$$

is equivalent to the inequality

$$V_n + \frac{n}{4}\ln\left(\frac{n\pi e}{2}\right) + \int_{\mathbb{R}^n} |f|^2 \ln|f|\, dx \leq \frac{n}{4}\ln\int_{\mathbb{R}^n} |\nabla f|^2\, dx$$

by setting $g(x) = (2\pi)^{n/4}e^{x^2/4}f(x)$ in the first inequality and then making the resulting expression dilation invariant where

$$\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(d\mu)} = 1 .$$

Now inequality (1.7) implies that for $V_n = A_n + \frac{n}{4}\ln(8\pi/ne)$

$$V_n + \int_{\mathbb{R}^n} |g|^2 \ln|g|\, d\mu \leq \int_{\mathbb{R}^n} |\nabla g|^2\, d\mu .$$

Using an expression for ψ taken from Whitaker and Watson ([41, p. 251])

$$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)\left(e^{2\pi t}-1\right)}\, dt ,$$

the constant $V_n \simeq -1 - \frac{1}{4}\ln\pi - \frac{1}{4}\ln n$ as $n \to \infty$. Set $n = m\ell$ and consider the product function $\prod g(x_k)$ where each $x_k \in \mathbb{R}^m$ and k runs from 1 to ℓ and $\|g\|_2 = 1$. Then

$$\frac{1}{\ell}V_n + \int_{\mathbb{R}^m} |g|^2 \ln|g|\, d\mu \leq \int_{\mathbb{R}^m} |\nabla g|^2\, d\mu .$$

Then for fixed m and $\ell \to \infty$, $\frac{1}{\ell} V_n \to 0$ and the logarithmic Sobolev inequality (1.8) is obtained. □

In part, this result is interesting because the logarithmic Sobolev inequality has been viewed as sharpening the classical uncertainty principle [4]. Now it would seem that these two concepts are more closely intertwined. Of course, the original interest in the logarithmic Sobolev inequality was Gross' proof that it was equivalent to Nelson's hypercontractive estimates for the Hermite semigroup [24]. Subsequently it was recognized that such hypercontractive estimates followed from the sharp Young's inequality [2] or by using a symmetrization argument and the interplay between Gaussian measure and Lebesgue measure [3].

Resuming the analysis of the initial Sobolev inequality (1.6), one can observe that this inequality is conformally invariant and so the sharp constant can be calculated and up to conformal automorphism the extermal functions are determined as $A(1+|x|^2)^{-n/2}$:

$$\frac{n}{2}\int_{\mathbb{R}^n} \ln|\xi| \left|\hat{f}(\xi)\right|^2 d\xi \geq \int_{\mathbb{R}^n} |f(x)|^2 \ln|f(x)|\,dx + B_n$$

$$B_n = \frac{n}{2}\psi\left(\frac{n}{2}\right) - \frac{n}{2}\ln\pi - \frac{1}{2}\ln\left[\Gamma(n)/\Gamma(n/2)\right] . \tag{1.9}$$

This estimate follows from the limit $p \to 2$ for the norm of the Hardy–Littlewood–Sobolev mapping

$$f \in L^p\left(\mathbb{R}^n\right) \to |x|^{-\lambda} * f \in L^{p'}\left(\mathbb{R}^n\right)$$

where $\lambda = 2n/p'$ and $1 < p < 2$ (see [7]). Since this inequality is conformally invariant, there is an equivalent realization on the sphere S^n. The most effective representation in the compact setting is by using a Dini integral (see [3] and [8]):

$$\int_{S^n} |F|^2 \ln|F|\,d\xi \leq A_n \int_{S^n\times S^n} \frac{|F(\xi)-F(\eta)|^2}{|\xi-\eta|^n}\,d\xi\,d\eta$$

$$A_n = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})} \tag{1.10}$$

where $\|F\|_2 = 1$ and $d\xi$ denotes normalized surface measure. Extremal functions are of the form $A|1-\zeta\cdot\xi|^{-n/2}$ for $|\zeta| < 1$. The right-hand side is a quadratic form so there is also a representation in terms of spherical harmonics. For $F = \sum Y_k$, this form is given by $\sum \Delta_k(n) \int |Y_k|^2\,d\xi$ with

$$\Delta_k(n) = \frac{n}{2}\sum_{m=1}^{k-1}\left(m+\frac{n}{2}\right)^{-1} .$$

This result was first obtained in 1983 by the author to show that the Poisson semigroup on the n-dimensional sphere is hypercontractive [3]. As a corollary, it followed that the heat semigroup is hypercontractive on S^n. But the inequality (1.10) is also related to work of Calderón et al. on the existence of singular integrals for operators that commute with dilations on $\mathbb{R}^{n+1}$ where the kernel takes the form

$$K(x) = \frac{\Omega(x)}{|x|^{n+1}}$$

with Ω being homogeneous of degree zero and so a function on the unit sphere S^n. In their work, an estimate was needed to show that if Ω was integrable and satisfied an L^1 Dini condition, then Ω belonged to the Zygmund class $L\ln^+ L$. Calderón quantified an old argument of Riesz which concerned the conjugate function and used complex variables to prove the required estimate in one dimension, and then the n-dimensional form was obtained by approximation:

$$\int_{S^n} |\Omega|\ln|\Omega|\,d\xi \leq B\int_{S^n\times S^n} \frac{|\Omega(\xi)-\Omega(\eta)|}{|\xi-\eta|^n}\,d\xi\,d\eta + C$$

for $\int |\Omega|\, d\xi = 1$. This result seems to have been the first appearance of a logarithmic Sobolev inequality where "smoothness controls entropy" (see inequality 2.3 in [17]). Later the one-dimensional estimate was also used by Pichorides in his thesis (see Theorem 2.4 in [33]). A different argument was given in Fefferman's thesis to obtain this same result without sharp constants. But the constant given by Calderón using the Riesz argument was sharp in one variable. Calderón et al. remark that "it would be useful and interesting to prove this inequality directly, without using complex variables". This question was answered by the author in [8] by two different arguments.

Theorem 2.
Let $\Omega \in L^1(S^n)$ with $\int |\Omega|\, d\xi = 1$. Then

$$\int_{S^n} |\Omega| \ln |\Omega|\, d\xi \le \sqrt{\pi}\, \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})} \int_{S^n \times S^n} \frac{|\Omega(\xi) - \Omega(\eta)|}{|\xi - \eta|^n}\, d\xi\, d\eta\,. \tag{1.11}$$

Proof. This result follows from the logarithmic Sobolev inequality (1.10) by replacing F^2 by Ω, and noting first that $|F(\xi) - F(\eta)|^2 \le |F^2(\xi) - F^2(\eta)|$ for $F \ge 0$ which gives (1.11) for $\Omega \ge 0$ and in turn implies the general case. □

It is surprising that this inequality is sharp, but in part that is explained by the fact that there are no extremals for this inequality. By using symmetrization arguments, one can give a more elementary proof of (1.11) and for a wide range of singular kernels, including an improvement of the Riesz result by cutting down the range of integration in the right-hand integral.

2. $SL(2, R)$ and Axial Symmetry

The mathematical description of fluid motion is a central problem in the study of critical phenomena. Ideas concerning symmetry have been useful in the analysis of differential equations that arise in vortex dynamics. In examining axisymmetric steady flow for vortex rings, the existence of $SL(2, R)$ symmetry is apparent from the representation of the Stokes stream function ψ in terms of the vorticity ω ([35, chapter 10]):

$$\begin{aligned} \psi(x, y) &= \frac{1}{4\pi} \int\int yy'\, \omega\left(x', y'\right) \\ &\quad \left[\int_0^{2\pi} \frac{\cos\theta\, d\theta}{\left[(x - x')^2 + y^2 + y'^2 - 2yy' \cos\theta\right]^{1/2}} \right] dx'\, dy'\,. \end{aligned} \tag{2.1}$$

Let $z = (x, y)$ denote a point in the upper half-plane $\mathbb{R}^2_+ \simeq M \simeq SL(2, R)/SO(2)$. Here the invariant distance is given by the Poincaré metric

$$d\left(z, z'\right) = \frac{\left|z - z'\right|}{2\sqrt{yy'}}$$

and left-invariant Haar measure $d\nu = y^{-2}\, dy\, dx$. Then

$$\tilde{\psi}(z) = \frac{1}{4\pi} \int \tilde{\omega}\left(z'\right) \varphi\left[d\left(z, z\right)\right] d\nu \tag{2.2}$$

where $\tilde{\psi}(z) = y^{-1/2}\psi(z)$, $\tilde{\omega}(z) = y^{5/2}\omega(z)$ and

$$\varphi(t) = \int_0^{\pi} \left[t^2 + \sin^2(\theta/2)\right]^{-1/2} \cos\theta\, d\theta\,.$$

In Fraenkel's work on vortex rings with swirl [23], a non-classical weighted Sobolev inequality was used to develop existence theory. It is interesting that this problem exhibited naturally an $SL(2, R)$ symmetry and the inequality in question could be rewritten as a Sobolev embedding on the two-dimensional real hyperbolic space M. Using the representation that the gradient in this setting is given by $D = y\nabla$, then Fraenkel's inequality can be realized in the form for $p \geq 2$ [10]:

$$\left[\int_M |F|^p \, dv\right]^{2/p} \leq A_p \left[\int_M |DF|^2 \, dv + \frac{3}{4}\int_M |F|^2 \, dv\right] . \tag{2.3}$$

This connection between axial symmetry and the group $SL(2, R)$ may seem surprising at first, but it is in fact quite natural in the context of standard analysis on classical manifolds. As a paradigm to illustrate this point, $SL(2, R)$ symmetry will be used to calculate extremals for the sharp L^2 Sobolev inequalities and the Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^n$. For $n > 2$ and $1/p = 1/2 - 1/n$

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} &\leq A_p \, \|\nabla f\|_{L^2(\mathbb{R}^n)} \\ A_p &= [\pi n(n-2)]^{-1/2} \, [\Gamma(n)/\Gamma(n/2)]^{1/n} \end{aligned} \tag{2.4}$$

and up to the action of the conformal group, the sharp constant is only attained for functions of the form $A(1 + |x|^2)^{-n/p}$.

The technique of symmetrization (equimeasurable radial decreasing rearrangement) provides the reduction of this inequality to radial decreasing functions (see [3, 6]). Using duality and the Green's function for $-\Delta$

$$G(x, y) = \frac{\pi^{-n/2}\Gamma(n/2)}{2(n-2)} \, |x - y|^{-(n-2)} ,$$

the inequality (2.4) is equivalent to the Hardy–Littlewood–Sobolev inequality

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}^n} u(x)G(x, y)v(y) \, dx \, dy \right| \leq (A_p)^2 \, \|u\|_{L^{p'}(\mathbb{R}^n)} \, \|v\|_{L^{p'}(\mathbb{R}^n)} \tag{2.5}$$

where the dual exponent $p' = p/(p - 1)$. Using the argument given in ([6, p. 40]), one sees that an extremal must exist for the inequality (2.5) and it must be radial decreasing up to conformal action. Hence, by duality an extremal exists for (2.4) and up to conformal action it must be radial.

Now suppose that f is a radial extremal for the Sobolev inequality (2.4). Use the product structure for Euclidean space $\mathbb{R}^n \simeq \mathbb{R} \times \mathbb{R}^{n-1}$ with $x = (t, x')$ and set $y = |x'|$. Then the function f being radial in x is also radial in x'. Now let $g(t, y) = y^{n/p} f(t, x')$ and inequality (2.4) becomes

$$\left[\int_M |g|^p \, dv\right]^{2/p} \leq B_p \left[\int_M |Dg|^2 \, dv + \frac{n}{p}\left(\frac{n}{p} - 1\right)\int_M |g|^2 \, dv\right] \tag{2.6}$$

where

$$B_p = \frac{4}{n(n-2)} \left[\frac{n-1}{2\pi}\right]^{2/n} .$$

Apply equimeasurable geodesic radial decreasing symmetrization on M.

Since f was extremal, the inequality (2.6) cannot be improved. But now there is an extremal which is a function only of the invariant distance from the origin. On M the invariant distance is given by the Poincaré metric for $w = (t, y)$. Hence, the rearranged extremal on M is a function of

$$1 + 2d^2(w, \hat{0}) = \frac{y^2 + t^2 + 1}{2y}$$

where $\hat{0} = (0, 1)$ denotes the origin on M. Now tracing the steps back to the inequality (2.4), there must be extremal $f_\#$ of the form:

$$f_\#(t, y) = y^{-n/p} g^* \left(\frac{y^2 + t^2 + 1}{2y} \right) .$$

Up to conformal action, the only possible form for $f_\#$ is to be radial. Hence,

$$f_\# \left(y^2 + t^2 \right) = y^{-n/p} g^* \left(\frac{y^2 + t^2 + 1}{2y} \right)$$

and

$$f_\#(t, y) = A \left(1 + t^2 + y^2 \right)^{-n/p} = A \left(1 + |x|^2 \right)^{-n/p} .$$

This completes the proof of the sharp L^2 Sobolev inequality and gives a new application of $SL(2, R)$ methods and competing symmetry structures. An analogous argument works for the Hardy–Littlewood–Sobolev inequality. That is, the inequality

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}^n} u(x) \, |x - y|^{-\lambda} \, v(y) \, dx \, dy \right| \leq C_p \, \|u\|_{L^p(\mathbb{R}^n)} \, \|v\|_{L^p(\mathbb{R}^n)} \tag{2.7}$$

where $1 < p < 2$ and $\lambda = 2n/p'$, is equivalent to the inequality

$$\left| \int_{M \times M} g(w) \varphi_\lambda \left[d \left(w, w' \right) \right] h \left(w' \right) \, dv \, dv \right| \leq D_p \, \|g\|_{L^p(M)} \, \|h\|_{L^p(M)} \tag{2.8}$$

with $g(t, y) = y^{n/p} u(t, x')$, $h(t, y) = y^{n/p} v(t, x')$ and $x = (t, x')$, $y = |x'|$ and $w = (t, y)$.

$$\varphi_\lambda(t) = 2^{-\lambda/2} a_n \int_0^\pi \left[t^2 + \sin^2(\theta/2) \right]^{-\lambda/2} (\sin\theta)^{n-3} \, d\theta$$

$$D_p = \left[2^{-1} \pi^{-(n-1)/2} \Gamma[(n-1)/2] \right]^{2/p'} ; \qquad a_n = \pi^{-1/2} \Gamma[(n-1)/2] / \Gamma[(n-2)/2]$$

As above, the competing radial and cylindrical symmetry force the extremal to be of the form $u(x) = v(x) = A(1 + |x|^2)^{-n/p}$ up to conformal automorphism.

3. Stein-Weiss Integrals on the Heisenberg Group

The natural setting for the uncertainty principle is quantum mechanics in phase space. Since this principle constitutes a quantitative statement about both the dilation structure and the product structure on a manifold and incorporates the logarithmic Sobolev inequality, it is important to examine how the Heisenberg group fits within this framework. The tools used to achieve this geometric realization are Stein–Weiss integrals, $SL(2, R)$ symmetry, and analysis on hyperbolic manifolds.

The Heisenberg group $\mathcal{H}_n$ is realized as the boundary of the Siegel upper half-space in $\mathbb{C}^{n+1}$, $D = \{z \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > |z_1|^2 + \cdots + |z_n|^2\}$. Then $\mathcal{H}_n = \{(z, t) : z \in \mathbb{C}^n, t \in \mathbb{R}\}$ with the group action

$$(z, t) \left(z', t' \right) = \left(z + z', \, t + t' + 2 \operatorname{Im} z \bar{z}' \right)$$

and Haar measure on the group is given by $dm = dz \, d\bar{z} \, dt = 4^n \, dx \, dy \, dt$ where $z = x + iy \in \mathbb{C}^n$ and $t \in \mathbb{R}$. The natural metric here is $d((z, t), (z', t')) = d((z', t')^{-1}(z, t), (0, 0))$ with

$$d \left((z, t), (0, 0) \right) = \left| \, |z|^2 + it \right|^{1/2} = \left(|z|^4 + t^2 \right)^{1/4} .$$

Based on the relation between Pitt's inequality and the uncertainty principle (see [7]), one can formulate a Stein–Weiss integral on the Heisenberg group that incorporates an $SL(2, R)$ dilation symmetry.

Theorem 3.

For $f, g \in \mathcal{S}(\mathcal{H}_n)$ and $0 < \alpha < 2n$

$$\begin{aligned}\Big|\int_{\mathcal{H}_n\times\mathcal{H}_n} & f(z,t)\,|z|^{-\alpha/2}\,\Big||z-w|^2 + i(t-s-2\,Im\,z\bar{w})\Big|^{-(n+1)+\alpha/2} \\ & |w|^{-\alpha/2}\,g(w,s)\,dm\,dm\Big| \\ & \le C_\alpha\,\|f\|_{L^2(\mathcal{H}_n)}\,\|g\|_{L^2(\mathcal{H}_n)}\,.\end{aligned} \tag{3.1}$$

The sharp constant is given by

$$C_\alpha = 2^{\alpha/2}\,(2\pi)^{n+1}\,\Gamma(\alpha/2)\left[\frac{\Gamma(\frac{2n-\alpha}{4})}{\Gamma(\frac{2n-\alpha}{4}+\frac{1}{2})\Gamma(\frac{2n+\alpha}{4})}\right]^2 .$$

Here $\mathcal{S}(\mathcal{H}_n)$ denotes the Schwartz class. Several observations can be made about this inequality: (1) it is dilation invariant with respect to the natural dilations on the Heisenberg group, $(z,t) \to (\delta z, \delta^2 t)$; (2) fractional integration is used in order to have a real-variable formulation of the problem; and (3) the usual role of the metric is not maintained which is reflected by the range of α and the powers of $|z|$, $|w|$. This last point is determined by the underlying $SL(2, R)$ invariance of the Heisenberg group. In studying weighted inequalities on Lie groups and symmetric spaces, the complexity of the symmetry structure will allow different choices for the form of variational inequalities. Examples including both Sobolev estimates on the Heisenberg group and Pitt's inequality on $\mathbb{R}^n$ [7] suggest that here the $SL(2, R)$ invariance should be primary. But it is then surprising that there exist three simple alternate proofs of the sharp estimate (3.1) using in turn the Euclidean product structure, the hyperbolic dilation structure, and the homogeneous metric structure [9]. The multiplicity of proofs undoubtedly depends on the fact that no extremals exist for this inequality and hence a concentration phenomena gives the sharp constant. The Euclidean estimate follows by applying Hölder's inequality for integration in the t, s variables. Then

$$\begin{aligned}&\Big|\int_{\mathcal{H}_n\times\mathcal{H}_n} f(z,t)\,|z|^{-\alpha/2}\,d\,((z,t),(w,s))^{-2n-2+\alpha}\,|w|^{-\alpha/2}\,g(w,s)\,dm, dm\Big| \\ &\quad \le 4^{2n}\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \tilde{f}(x)\,|x|^{-\alpha/2}\,|x-x'|^{-2n+\alpha}\,|x'|^{-\alpha/2}\,\tilde{g}(x')\,dx\,dx' \\ &\quad\quad \int_{-\infty}^{\infty}\left(1+t^2\right)^{-(2n+2-\alpha)/4} dt\end{aligned}$$

where

$$\tilde{f}(x) = \left[\int |f(x,t)|^2\,dt\right]^{1/2} \quad , \quad \tilde{g}(x) = \left[\int |g(x,t)|^2\,dt\right]^{1/2} .$$

The sharp Stein–Weiss inequality in the Euclidean case is given by [7]

$$\begin{aligned}&\Big|\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} f(x)\,|x|^{-\alpha/2}\,|x-y|^{-2n+\alpha}\,|y|^{-\alpha/2}\,g(y)\,dx\,dy\Big| \\ &\quad \le \pi^n\left[\frac{\Gamma(\alpha/2)}{\Gamma(n-\frac{\alpha}{2})}\right]\left[\frac{\Gamma(\frac{2n-\alpha}{4})}{\Gamma(\frac{2n+\alpha}{4})}\right]^2 \|f\|_{L^2(\mathbb{R}^{2n})}\,\|g\|_{L^2(\mathbb{R}^{2n})}\,.\end{aligned} \tag{3.2}$$

To see that this value C_α is best possible, let $f(z,t) = g(z,t) = p(z)u(t)$ with $\|u\|_{L^2(\mathbb{R})} = 1$ and then set $f_\varepsilon(z,t) = \varepsilon^{1/2}p(z)u(\varepsilon t)$ and take the limit $\varepsilon \to 0$.

An alternative proof is obtained by first reducing the problem to functions radial in $|z|$ and $|w|$ since the map is from $L^2(\mathcal{H}_n)$ to $L^2(\mathcal{H}_n)$, and then seeing by a change of variables that (3.1) is equivalent to an inequality on $M \simeq SL(2, R)/SO(2)$:

$$\left|\int_{M\times M} h(w)\psi_{n+1-\alpha/2}\left[d\left(w, w'\right)\right] k\left(w'\right)\, dv\, dv\right| \leq \frac{\Gamma(n)}{(4\pi)^n} 2^{n+1-\alpha/2} C_\alpha \|h\|_{L^2(M)} \|k\|_{L^2(m)} \tag{3.3}$$

where

$$\psi_\lambda(u) = \int_{\partial B_n} \left|\sqrt{1+u^2} - \zeta_1\right|^{-\lambda} d\zeta$$

with $d\zeta$ being normalized surface measure on the boundary of the unit ball in $\mathbb{C}^n$. The estimate is completed by applying Young's inequality for a non-unimodular locally compact group (Δ denotes the modular function):

$$\|f * g\|_{L^p(G)} \leq \|f\|_{L^p(G)} \left\|\Delta^{-1/p'} g\right\|_{L^1(G)} . \tag{3.4}$$

Here $1 \leq p \leq \infty$ and the modular function $\Delta(x, y) = 1/y$.

A third proof uses homogeneous geodesic polar coordinates and reduces to a problem on the multiplicative group $\mathbb{R}_+$ in the spirit of the real-variable case. Haar measure on $\mathcal{H}_n$ is now given by $dm = \rho^{2n+1}\, d\rho\, d\chi$ where $d\chi$ indicates integration over the boundary of the geodesic unit ball which does not have a transitive group action, and $2n+2$ is the homogeneous dimension. These proofs are all different, but they are individually important because they emphasize contrasting issues for analysis on the Heisenberg group. The essential point is that the global invariance structure involves an interlacing of lower-dimensional symmetries.

4. Zeta Functions and Trace Inequalities

Conformal invariance is a critical tool for the analysis of a wide range of problems including the hydrogen atom spectrum, the Hardy–Littlewood–Sobolev inequality, the Yamabe functional, the Moser-Trudinger inequality and its free energy-entropy dual, and the multilinear Hardy–Littlewood–Sobolev inequality [5, 6]. These inequalities constitute a rich tapestry growing almost organically though recently it has been centered on the Moser-Trudinger inequality for which the most important aspect has been its connection to the Polyakov-Onofri log determinant variation formula. Morpurgo [29] has used the zeta function and corresponding trace of the heat kernel for conformally invariant operators as a frame on which to derive many interesting results about conformal deformation, and in particular as a master partition function that incorporates many of the interesting inequalities determined by conformal invariance.

Consider the zeta function $Z(s)$ on the two sphere for the operator $-\Delta$. Under conformal deformation of the metric $ds^2 = e^F\, ds_0^2$, the Polyakov formula for the determinant is given by

$$\begin{aligned} Z'_F(0) - Z'(0) &= \log[\det(-\Delta)/\det(-\Delta_F)] \\ &= \frac{1}{3}\left\{\frac{1}{4}\int_{S^2} |\nabla F|^2\, d\xi + \int_{S^2} F\, d\xi - \ln \int_{S^2} e^F\, d\xi\right\} . \end{aligned} \tag{4.1}$$

The term in brackets is the Moser–Trudinger functional which is always positive and means that the determinant of the Laplacian on S^2 under conformal deformation with fixed area is maximized by the standard metric. Using conformal invariance, Onofri proved the positivity of this functional which was later extended by the author to a class of positive-definite conformally invariant operators

$P_n(-\Delta)$ on S^n. By its action on spherical harmonics, this operator called the Paneitz operator is defined

$$\begin{aligned} P_n(-\Delta)Y_k &= k(k+1)\cdots(k+n-1)\cdot Y_k \\ P_n(-\Delta) &= \prod_{\ell=0}^{(n-2)/2}[-\Delta+\ell(n-1-\ell)] \quad n \text{ even} \\ &= \left[-\Delta+\left(\frac{n-1}{2}\right)^2\right]^{1/2}\prod_{\ell=0}^{(n-3)/2}[-\Delta+\ell(n-1-\ell)] \quad n \text{ odd} \end{aligned} \tag{4.2}$$

and for G having mean-value zero

$$P_n^{-1}(-\Delta)G = -\frac{1}{\Gamma(n)}\int_{S^n}\ln|\xi-\eta|^2\,G(\eta)\,d\eta\,. \tag{4.3}$$

This led to two dual inequalities [5]:

$$\ln\int_{S^n}e^F\,d\xi \le \int_{S^n}F\,d\xi + \frac{1}{2n!}\int_{S^n}F(P_nF)\,d\xi \tag{4.4}$$

$$-n\int_{S^n}F(\xi)\ln|\xi-\eta|^2\,G(\eta)\,d\xi\,d\eta \le -n\int_{S^n}\ln|\xi-\eta|^2\,dn + \int_{S^n}F\ln F\,d\xi + \int_{S^n}G\ln G\,d\xi \tag{4.5}$$

where in (4.5) F, G are non-negative with integral one. These inequalities have provided information about the determinant of the conformal Laplacian in dimensions 3, 4, and 6 (see [5, 12, 14, 19]). In fact, inequality (4.4) provides the leading term for the analysis of the log determinant variation. The zeta function for the Paneitz operator on S^n will have a pole of order 1 at $s = 1$ since the operator is order n. Morpurgo related the functional inequality (4.5) directly to this zeta function for the conformal change of metric $g_w = w^{2/n}g$ by showing in even dimension with $\int w\,d\xi = 1$ the regularized zeta function at $s = 1$ is minimized by the standard metric; that is,

$$\tilde{Z}_w(1) - \tilde{Z}(1) = \frac{2}{n!}\int_{S^n}w\ln w\,d\xi - \int_{S^n}\tilde{w}P_n^{-1}\tilde{w}\,d\xi \tag{4.6}$$

for $\tilde{w}$ the projection of w onto the subspace orthogonal to constants. Moreover, Morpurgo extended this result to even-dimensional compact Riemannian manifolds without boundary where the corresponding Paneitz operator is self-adjoint and nonnegative. This is a remarkable result in its own right, but Morpurgo has found other interesting results about the zeta function for conformally invariant operators. For example, he has related the other end-point limit of the Hardy–Littlewood–Sobolev inequality as given by inequality (1.10) above to the zeta function for the conformal Laplacian in dimension four. More recently, he proved the trace inequality for the conformal Laplacian Y on the sphere S^n

$$\operatorname{Tr}\left[\left(w^{-1/2}Yw^{-1/2}\right)^{-s}\right]^{1/s} \le \|w\|_{n/2}\operatorname{Tr}\left[Y^{-s}\right]^{1/s} \tag{4.7}$$

for integer $s > n/2$ and $n > 2$. The classical Sobolev inequality for Y can easily be obtained from the limit $s \to \infty$. This gives for $n > 2$

$$\lambda_1(1) \le \|w\|_{L^{n/2}(S^n)}\lambda_1(w) \le \|w\|_{L^{n/2}(S^n)}\frac{\int\varphi w^{-1/2}Y\varphi w^{-1/2}\,d\xi}{\int|\varphi|^2\,d\xi}\,.$$

Set $F = \varphi w^{-1/2}$ and $\varphi = w^{n/4}$; then $w^{n/2} = F^p$ where $1/p = 1/2 - 1/n$ so that

$$\left[\int_{S^n}|F|^p\,d\xi\right]^{2/p} \le \frac{1}{\lambda_1(1)}\int_{S^n}FYF\,d\xi\,. \tag{4.8}$$

Morpurgo's work on zeta functions together with the research programs of Branson [12] and Chang [19] are clearly opening up rich new areas of research in geometric analysis. Sharp constants are providing a deeper understanding of the geometric structure of Riemannian manifolds and Lie groups.

Acknowledgements

I would like to thank Alberto Calderón for his warm support, Tony Carbery for his invitation to participate in the Sussex Workshop on Fourier Analysis, Eugenio Hernández for his hospitality at El Escorial, and Carlo Morpurgo and Fulvio Ricci for useful remarks.

References

[1] Aubin, T. (1982). *Non-Linear Analysis on Manifolds,* Monge-Ampére Equations. Springer-Verlag.

[2] Beckner, W. (1975). Inequalities in Fourier Analysis. *Ann. of Math.* **102**, 159–182.

[3] Beckner, W. (1992). Sobolev inequalities, the Poisson semigroup and analysis on the sphere S^n. *Proc. Nat. Acad. Sci.* **89**, 4816–4819.

[4] Beckner, W. (1989). A generalized Poincaré inequality for Gaussian measures. *Proc. Amer. Math. Soc.* **105**, 397–400.

[5] Beckner, W. (1993). Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. Math.* **138**, 213–242.

[6] Beckner, W. (1995). Geometric inequalities in Fourier analysis. Essays on Fourier Analysis in Honor of Elias M. Stein Princeton University Press, Princeton, NJ, 36–68.

[7] Beckner, W. (1995). Pitt's inequality and the uncertainty principle. *Proc. Amer. Math. Soc.* **123**, 1897–1905.

[8] Beckner, W. (1997). Logarithmic Sobolev inequalities and the existence of singular integrals. *Forum Math.* **9**, 303–323.

[9] Beckner, W. Pitt's inequality on the Heisenberg group. (preprint).

[10] Beckner, W. Sobolev inequalities, vortex dynamics and Lie groups. (preprint).

[11] Branson. T. (1987). Group representations arising from Lorentz conformal geometry. *J. Funct. Anal.* **74**, 199–291.

[12] Branson, T. (1995). Sharp inequalities, the functional determinant and the complementary series. *Trans. Amer. Math. Soc.* **347**, 3671–3742.

[13] Branson, T. (1996). An anomaly associated with 4-dimensional quantum gravity. *Comm. Math. Phys.* **178**, 301–309.

[14] Branson, T., Chang, S.-Y.A., and Yang, P. (1992). Estimates and extremals for zeta function determinants on four-manifolds. *Comm. Math. Phys.* **149**, 241–262.

[15] Branson, T. and Ørsted, B. (1991). Explicit functional determinants in four dimensions. *Proc. Amer. Math. Soc.* **113**, 671–684.

[16] Brascamp, H.J., Lieb, E.H., and Luttinger, J.M. (1974). A general rearrangement inequality for multiple integrals. *J. Funct. Anal.* **17**, 227–237.

[17] Calderón, A. P., Weiss, M., and Zygmund, A. (1967). On the existence of singular integrals. *Proc. Symposia Pure Math.* **10**, 56–73.

[18] Chang, S.-Y.A. and Yang, P.C. (1995). Extremal metrics of zeta function determinants on 4-manifolds. *Ann. of Math.* **142**, 171–212.

[19] Chang, S.-Y.A. and Qing, J. Zeta functional determinants on manifolds with boundary. (preprint).

[20] Fefferman, R. (1978). A theory of entropy in Fourier analysis. *Adv. in Math.* **30**, 171–201.

[21] Folland, G.B. (1989). *Harmonic analysis in phase space.* Princeton University Press, Princeton, NJ.

[22] Folland, G.B. and Stein, E.M. (1982). *Hardy Spaces on Homogeneous Groups.* Princeton University Press, Princeton, NJ.

[23] Fraenkel, L.E. (1992). *On steady vortex rings with swirl and a Sobolev inequality.* Progress in Partial Differential Equations, Calculus of Variations and Applications. Longman Scientific and Technical. 13–26.

[24] Gross, L. (1975). Logarithmic Sobolev inequalities. *Amer. J. Math.* **97**, 1061–1083.

[25] Grossmann, A., Loupias, G., and Stein, E.M. (1968). An algebra of pseudodifferential operators and quantum mechanics in phase space. *Ann. Inst. Fourier* **18**, 343–368.

[26] Hardy, G.H., Littlewood, J.E., and Pólya, G. (1952). *Inequalities.* Cambridge University Press.

[27] Lieb, E.H. (1983). Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. *Ann. Math.* **118**, 349–374.

[28] Messiah, A. (1961). *Quantum Mechanics*. North-Holland.

[29] Morpurgo, C. (1996). The logarithmic Hardy-Littlewood-Sobolev inequality and extremals of zeta functions on S^n. *Geometric and Functional Analysis* **6**, 146–171.

[30] Morpurgo, C. (1997). Zeta functions on S^2. *Contemporary Math.* **201**, 213–225.

[31] Onofri, E. (1982). On the positivity of the effective action in a theory of random surfaces. *Comm. Math. Phys.* **86**, 321–326.

[32] Osgood, B., Phillips, R., and Sarnak, P. (1988). Extremals of determinants of Laplacians. *J. Funct. Anal.* **80**, 148–211.

[33] Pichorides, S.K. (1972). On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov. *Studia Math.* **44**, 165–179.

[34] Polyakov, A.M. (1987). *Gauge Fields and Strings*. Harwood Academic Publishers.

[35] Saffman, P.G. (1992). *Vortex Dynamics*. Cambridge University Press.

[36] Stein, E.M. (1970). Analytic continuation of group representations. *Adv. Math.* **4**, 172–207.

[37] Stein, E.M. (1970). Some problems in harmonic analysis suggested by symmetric spaces and semi-simple Lie groups. *Actes Congrès Intern. Math.* Nice, Tome 1, 173–189.

[38] Stein, E.M. (1970). *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, NJ.

[39] Stein, E.M. (1993). *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton University Press, Princeton, NJ.

[40] Weyl, H. (1949). *The Theory of Groups and Quantum Mechanics*. Dover.

[41] Whittaker, E.T. and Watson, G.N. (1927). *A Course of Modern Analysis*. Cambridge University Press.

[42] Zygmund, A. (1968). *Trigonometric Series*. Cambridge University Press.

Department of Mathematics
University of Texas at Austin
Austin, TX 78712