

# Simple Analysis and Working Equations for the Solidification of Cylinders and Spheres

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The analysis of solidification processes is complicated by a nonlinear boundary condition at the moving solid-liquid interface, and exact solutions are rare. Various attempts to predict the rate of solidification are available in the literature but most of the results seem to be of limited use for operation and design studies on metallurgical processes. In this article we present a physical model which can be solved analytically for the most commonly encountered boundary conditions; that is constant temperature at the cooling wall or finite heat transfer to the cooling fluid. The model is based on the assumption of a linear temperature profile in the solidified shell and a corresponding differential removal of internal energy. As a result one obtains a very simple expression for the solidification time as a function of the space variable and the pertinent system parameters. By comparison with numerical results the prediction error is shown to be less than 10 pct over a wide range of parameter combinations. In extreme situations, where a larger error may occur, equally accurate working equations can be generated by slightly modifying the basic results.

THE prime objective of solidification studies is to predict the position of the solid-liquid interface at any given time subject to certain boundary conditions and values of the system parameters. For a simple geometric configuration, a sharp solid-liquid interface and a well defined phase change temperature various theoretical results have been published, some of which agree well with the few experimental data available.

In metallurgical applications only numerical or seminumerical techniques have been successful until now and the reason for this is well known—with metals the removal of sensible heat from the material already solidified is usually as significant as the removal of latent heat from the interface. Therefore, if one neglects the change in internal energy of the solid, one obtains analytic results<sup>1,2</sup> which heavily overpredict the rate of solidification. On the other hand, the complete set of governing equations cannot be solved analytically if we exclude the classical case known as Stefan's problem.<sup>3</sup>

Under unidirectional heat-flow conditions, that is for the solidification of slab shaped bodies, the approximate analyses<sup>4-7</sup> have been quite successful in predicting metallurgical processes; the results agree well with numerically exact data<sup>8-10</sup> and experiments.<sup>11,12</sup> However, for cylindrical and spherical systems the analytic approximations so far available<sup>13-17</sup> either do not compare favorably with numerical results<sup>9,18</sup> or they are so involved that a digital computer is required to calculate actual freezing times.

These remarks may indicate that a considerable effort has been made in this field, particularly in recent years, but that no "engineering" solutions of simple analytic structure and sufficient accuracy have been developed. It is the aim of this paper to overcome the present difficulties; on the basis of a very simple physical concept, originally developed for

unidirectional heat flow,<sup>7</sup> analytic results for the solidification times of cylinders and spheres are presented which are easy to handle and much more accurate than previous results of equally simple structure.

## 1. MODEL AND MATHEMATICAL FORMULATION

The model to be discussed and evaluated here is formally the same as the one for slab-shaped bodies.<sup>7</sup> However, with cylindrical or spherical systems the physical significance of the model as well as the quality of the results are expected to be quite different and we have to draw on previous conclusions in order to justify the present procedure.

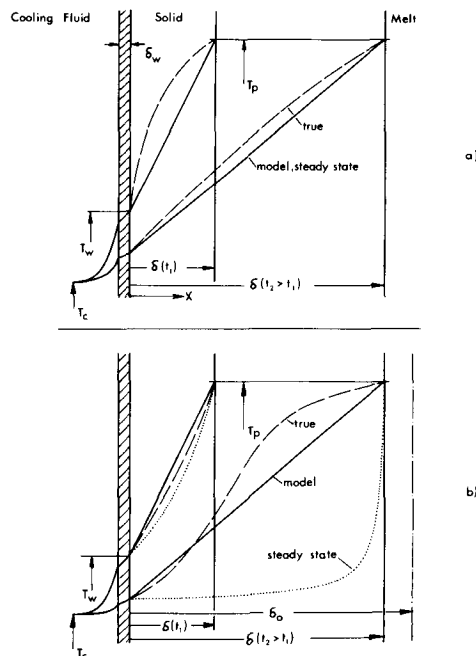


Fig. 1—(a) Assumed temperature distribution during solidification—slab analysis. (b) Assumed temperature distribution—cylindrical and spherical solidification.

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### 1.1. Conclusions From the Slab Analysis

For the purpose of summarizing the slab analysis we refer to Fig. 1(a) and assume that there is a linear temperature profile in the solidified layer at any time. Neglecting the heat capacity of the solid one obtains a simple analytic result for the solidification time as a function of the layer thickness  $\delta$ . This result, known as the quasistationary solution, clearly provides a lower bound for the solidification time because in reality not only latent heat but also some sensible heat has to be removed. This is easily seen by following the temperature-time history of some volume element at  $0 < x < \delta$ . On the other hand we know that the true temperature profile is always curved so that the temperature at any point  $x < \delta$  is larger than the one specified by a linear temperature gradient. Therefore, if we assume that after an incremental increase of  $\delta$  internal energy is removed until this steady profile is reached then less capacity is available for latent heat removal and an upper bound for the solidification time should result. Again, the result is obtained from an energy balance and consists of an extremely simple relationship between  $t$ ,  $\delta$  and the system parameters.

It is pointed out that the bounding character of the second result has not yet been proved mathematically but was verified by comparison with all the numerically and analytically exact results which are available from the literature. Indeed, for large temperature changes over the solid layer and large solidification rates (typically encountered in metallurgical applications) heavy overpredictions for the solidification time occur; the temperature profile is curved markedly and far less internal energy is removed in the true process than is anticipated in the model. This suggested the formulation of a weighted mean of the two limiting solutions and high accuracy was achieved by using two-thirds of the second and one third of the first (quasistationary) result. Since the latter is part of the former the final result is of equally simple structure.

Two conclusions can be drawn from the previous discussion. Firstly, no meaningful equivalent upper bound can be devised for cylindrical and spherical systems because the steady-state temperature profile approaches  $T_w = \text{constant}$  when the solidification front approaches the center (see Fig. 1(b)). In order to remove all the internal energy above  $T_w$  an infinite length of time is required, that is the upper bound for the solidification time simply becomes infinity. Secondly, the true temperature profile may be either convex or concave, depending on the system parameters and on time. Typically the profile starts off being slightly concave and becomes convex towards the end of the process when the linear velocity of the solid-liquid interface increases drastically.

These physical considerations may indicate that useful results can be obtained with a linear temperature profile in the solidified shell. The differential internal energy removal corresponding to such a profile can be calculated and incorporated in an energy balance. It is noted that the model makes qualitative provision for the fact that the true solidification time always lies in between those for zero and maximum removal of internal energy. However, quantitative in-

formation can only be obtained from a comparison of the results with numerically exact data.

### 1.2. Formulation of the Problem

The assumptions inherent in the formulation of the mathematical model are as follows: (a) the heat flow is geometrically one dimensional; (b) constant physical properties are assumed throughout; (c) the solidification process occurs between the two constant temperatures of melt and cooling fluid; (d) the system is cooled subject to a constant film coefficient at the cooling wall; and (e) volume changes due to different densities of liquid and solid are neglected.

Under these conditions the overall energy balance reads (see Nomenclature for notation)

$$h'_c A_o (T_w - T_c) = \lambda' \rho A_\delta \frac{d\delta}{dt} + \frac{dU_s}{dt}. \quad [1]$$

The heat to be removed from the system comprises the latent heat of phase change, which is proportional to the rate of solidification, and a change in internal energy of the solidified layer.

The overall heat transfer coefficient  $h'$ , representing the sum of the resistances of the solidified layer, the outer wall and the cooling fluid, can be expressed as follows:

$$\frac{1}{h' A_m} = \frac{1}{h'_c A_o} + \frac{\delta}{A_m k}. \quad [2]$$

The change in internal energy of the cooling wall is neglected so that the wall resistance can be combined with the heat transfer coefficient on the cooling fluid side, *i.e.*

$$h'_c = \left( \frac{1}{h_c} + \frac{\delta w}{k_w} \right)^{-1}. \quad [3]$$

Further, any desuperheating of the melt is incorporated in the latent-heat term such that

$$\lambda' = \lambda + c_m (T_m - T_p). \quad [4]$$

This provides a good approximation as long as the superheating of the melt is small.<sup>19</sup> For a large superheat more sophisticated analyses have to be employed.<sup>20</sup>

The change in internal energy of the solidified layer is

$$\frac{dU_s}{dt} = \rho c \frac{d}{dt} \{V(T_p - \bar{T})\} \quad [5]$$

where  $\bar{T}$  is the integrated mean temperature of the layer.

With the further assumption of a linear temperature profile we set

$$\begin{aligned} (T_p - \bar{T}) &= \frac{1}{2} (T_p - T_w) \\ &= \frac{1}{2} (T_p - T_c) \left\{ \frac{1}{\frac{A_m k}{A_o h'_c \delta} + 1} \right\}. \end{aligned} \quad [6]$$

For convenience the following dimensionless variables and parameters are introduced

$$\tau = \frac{t\alpha}{\delta_0^2}; \quad \delta^* = \frac{\delta}{\delta_0}; \quad Bi = \frac{h'_c \delta_0}{k}; \quad Ph = \frac{\lambda'}{c(T_p - T_c)}. \quad [7]$$

The reference length  $\delta_0$  is taken to be the radius of the cylinder or sphere.

With the variables and parameters so defined Eq. [1] becomes

$$\frac{1}{\frac{A_m}{A_0 Bi \delta^*} + 1} = \frac{A \delta}{A_m} Ph \delta^* \frac{d\delta^*}{d\tau} + \frac{\delta^*}{2A_m \delta_0} \frac{d}{d\tau} \left\{ V \left( \frac{1}{\frac{A_m}{A_0 Bi \delta^*} + 1} \right) \right\}. \quad [8]$$

In order to integrate this equation, one has to specify the areas  $A_m$ ,  $A_0$  and  $A_\delta$  which depend on the geometry of the system.

## 2. CYLINDER

### 2.1. Analysis

Eq. [8] requires expressions for  $A_m$  and  $V$ . The latter is simply

$$V = \pi \{ \delta_0^2 - (\delta_0 - \delta)^2 \} \quad [9]$$

whereas the mean area is approximated by

$$A_m = (A_0 + A_\delta)/2 = \pi(2\delta_0 - \delta). \quad [10]$$

Thereby the governing equation for inward cylindrical solidification becomes

$$\frac{1}{\frac{1 - \delta^*/2}{Bi \delta^*} + 1} = 2Ph \frac{(1 - \delta^*)\delta^*}{2 - \delta^*} \frac{d\delta^*}{d\tau} + \frac{\delta^*}{2(2 - \delta^*)} \frac{d}{d\tau} \left\{ \frac{[1 - (1 - \delta^*)^2] \delta^* Bi}{1 - \delta^*(1/2 - Bi)} \right\}. \quad [11]$$

After some lengthy but elementary calculus, involving the integration between 0 and  $\delta^* \leq 1$ , an expression for the partial or total solidification time is obtained.

For total solidification,  $\delta^* = 1$ , the dimensionless time is given as

$$\tau_1 = \frac{Ph}{2} \left( \frac{1}{Bi} + 6 - 8 \ln 2 \right) + \frac{1}{4} (6 - 8 \ln 2) + \frac{1}{2(Bi - 1/2)} \times \left[ 1 - \frac{1}{Bi - 1/2} \ln(Bi + 1/2) \right]. \quad [12]$$

and is seen to be a function of the two parameters  $Bi$  and  $Ph$ .

For large  $Ph$  values Eq. [12] reduces to

$$\tau_1 (Ph \rightarrow \infty) = \frac{Ph}{2} \left\{ \frac{1}{Bi} + 6 - 8 \ln 2 \right\}. \quad [13]$$

In comparison with the exact quasistationary solution,<sup>1</sup> given by

$$\tau_1 (Ph \rightarrow \infty) = \frac{Ph}{2} \left\{ \frac{1}{Bi} + \frac{1}{2} \right\}, \quad [13a]$$

a slight underprediction of  $\tau_1$  (reaching a maximum of 9 pct for  $Bi \rightarrow \infty$ ) is registered. This discrepancy will be discussed below.

We note that by integrating Eq. [11] over a shell of thickness  $\delta^* < 1$  expressions are obtained which have the same structure as Eq. [12]. For example, the

time to solidify up to the half radius  $\delta^* = 1/2$  is given by

$$\tau_{1/2} = \frac{Ph}{2} \left( \frac{3}{4Bi} + \frac{5}{2} - 8 \ln(4/3) \right) + \frac{1}{4} \left( \frac{5}{2} - 8 \ln(4/3) \right) + \frac{1}{2(Bi - 1/2)} \left[ \frac{1}{2} - \frac{1}{Bi - 1/2} \ln \left( 1 + \frac{Bi - 1/2}{2} \right) \right]. \quad [14]$$

### 2.2. Comparison With Previous Results and Working Equation

A quantitative estimate of the reliability of our model can only be made by comparison with exact results. We shall therefore not consider previous approximate analyses but rather the numerical results of Stephan and Holzknecht;<sup>9</sup> they employed a Crank-Nicolson scheme and showed that the maximum relative error in their data is only  $10^{-4}$ .

In Figs. 2 and 3 we have plotted Eqs. [12] and [14] respectively for various values of the phase change parameter  $Ph$  and as a function of the Biot modulus  $Bi$ . The solutions for constant wall temperature are included because in that case  $Bi \rightarrow \infty$ , that is  $1/Bi \rightarrow 0$ . The numerical points indicate clearly that the model over predicts solidification times for  $Ph < 1$  and under predicts, to a lesser degree, for  $Ph > 1$ . This is, in fact, expected on purely physical grounds and can be explained as follows (see Fig. 1(b)).

A large value of  $Ph$  results in a slow solidification rate such that the true temperature profile is closer to the steady-state one than the assumed linear profile. Consequently, too little sensible heat is removed in the model process. On the other hand, small  $Ph$  values provide a rapid growth of the solidified shell in which case the true temperature profile is convex throughout the process. Thus with a linear temperature profile too much internal energy is removed and the calculated solidification time becomes too long. This argument also explains why the prediction error increases with increasing Biot number, *i.e.* cooling intensity, especially in the case of half-radius solidification (Fig. 3).

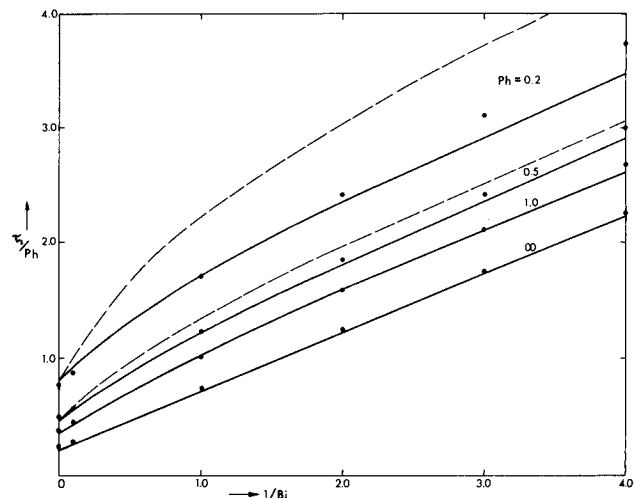


Fig. 2—Time for complete inward cylindrical solidification. Comparison between Eq. [12] (---), Eq. [16] (—) and numerically exact results (•).

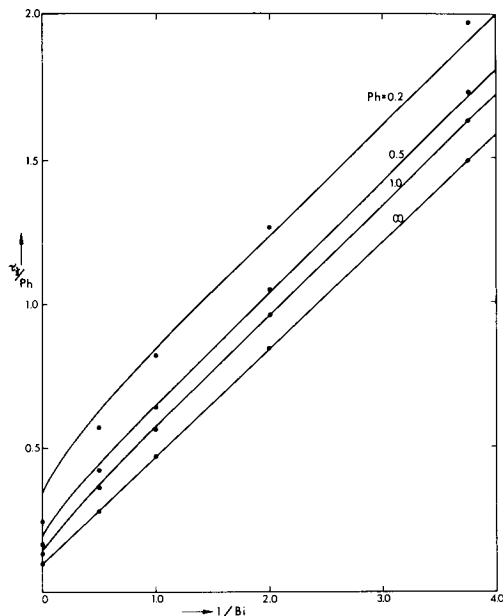


Fig. 3—Time for inward cylindrical solidification to  $\delta^* = 0.5$ . Comparison between corrected analysis (—) and numerical results ( $\bullet$ ).

However, the fact that the model is a very accurate representation of the true process at  $Ph = 1$  (here the error is less than 5 pct for  $0.1 \leq Bi \leq 2$ ) suggests that the sensible-heat term in our result be pre-multiplied by some function of  $Ph$ . Evidently, the model leads to a solution of the form

$$\tau(\delta^* \leq 1) = f_1(Ph, Bi) + f_2(Bi) + C \quad [15]$$

where  $f_1$  represents the quasistationary solution and  $f_2$  as well as the constant  $C$  account for the removal of internal energy from the solidified material. For  $Bi \rightarrow \infty$  the function  $f_2$  vanishes and the constant  $C$  provides excellent results for complete solidification (< 10 pct error, see Fig. 2) but rather heavy over predictions for half radius solidification and  $Ph < 1$  (26 pct error at  $Ph = 0.5$ ,  $\delta^* = 0.5$ ,  $Bi = \infty$ ). The reason for this discrepancy is simple: if only a shell of thickness  $\delta^* \leq 0.5$  is to be solidified then the typical cylinder characteristics become insignificant and the unidirectional heat flow model<sup>7</sup> should be more appropriate. For the latter it was shown that, with the above parameter combination, the linear temperature profile leads to a marked over prediction. On the other hand, we believe that a constant wall temperature together with a phase change parameter  $Ph \leq 0.5$  is an unlikely or, at least, undesirable configuration in most metallurgical applications. If the temperature difference  $T_p - T_c$ , appearing in the denominator of  $Ph$ , becomes very large then buckling or even cracking of the billet may occur.

In consequence, we propose to apply the empirical correction only to  $f_2$  and note that an extreme situation such as the one discussed above may require an additional correction applied to the constant  $C$ .

In matching the analytic with the numerical results we prefer to retain the basically simple structure of Eqs. [12] and [14] rather than aim for the highest possible accuracy. Thereby we should obtain useful working equation for performance studies on continuous casting machines and other phase-change

equipment as well as for process design purposes. After various trials we found a very simple correction factor which results in possibly the best compromise between simplicity and accuracy. By pre-multiplying  $f_2$  with the square root of  $Ph$  we obtain for complete solidification of a cylindrical system

$$\tau_1 = \frac{Ph}{2} \left( \frac{1}{Bi} + 0.455 \right) + \frac{\sqrt{Ph}}{2Bi-1} \left[ 1 - \frac{1}{Bi-0.5} \ln\{Bi+0.5\} \right] + 0.114. \quad [16]$$

This result, represented by the solid curves in Fig. 2, is in excellent agreement with the numerical data. The maximum relative error  $r_{\max}$ , involved in using Eq. [16] can be specified as follows:

$$\left. \begin{aligned} 0.2 \leq Ph \leq \infty; 0.1 \leq Bi \leq 10; &\rightarrow |r_{\max}| \approx 8 \text{ pct}; \\ 0.2 \leq Ph \leq \infty; 10 \leq Bi \leq \infty; &\rightarrow |r_{\max}| \approx 11 \text{ pct}. \end{aligned} \right\} \quad [17]$$

Similarly, for half radius solidification and with the same correction factor the error is

$$\left. \begin{aligned} 0.2 \leq Ph \leq \infty; 0.1 \leq Bi \leq 1; &\rightarrow |r_{\max}| \approx 5 \text{ pct}; \\ 0.5 \leq Ph \leq \infty; 1 < Bi \leq 5; &\rightarrow |r_{\max}| \approx 9 \text{ pct}; \\ 1 \leq Ph \leq \infty; 5 < Bi \leq \infty; &\rightarrow |r_{\max}| \approx 8 \text{ pct}. \end{aligned} \right\} \quad [18]$$

The range of larger errors ( $Bi > 5$ ,  $Ph < 1$ ) has been excluded for the reasons given before.

In conclusion, it is found that Eq. [16] or the equivalent equation for solidification up to  $0.5 \leq \delta^* \leq 1$  constitutes a very accurate and simple result for most parameter combinations of practical interest.

### 3. SPHERE

#### 3.1. Analysis

For inward spherical solidification the corresponding expressions for the solidified volume and the mean area become

$$V = \frac{4}{3} \pi \delta_0^3 [1 - (1 - \delta^*)^3] \quad [19]$$

and

$$A_m = \frac{A_0 + A_\delta}{2} = 2\pi \delta_0^2 [1 + (1 - \delta^*)^2]. \quad [20]$$

Upon substituting these into Eq. [8] we obtain the following differential equation:

$$\frac{Bi\delta^*}{\delta^{*2} + 2(Bi-1)\delta^* + 2} = \frac{Ph(1-\delta^*)^2 \delta^*}{1 + (1-\delta^*)^2} \frac{d\delta^*}{d\tau} + \frac{Bi}{3[1 + (1-\delta^*)^2]} \frac{d}{d\tau} \times \left\{ \frac{\delta^*[1 - (1-\delta^*)^3]}{\delta^{*2} + 2(Bi-1)\delta^* + 2} \right\}. \quad [21]$$

Integration of Eq. [21] results in the solidification time as a function of the two parameters  $Bi$  and  $Ph$ . Again, partial or total solidification can be solved for and the structure of the result is as specified by Eq. [15]. For  $\delta^* = 1$ , that is complete solidification, we find after

elementary integration that

$$f_1(Ph, Bi) = Ph \left( \frac{1}{3Bi} + 1 + \ln 2 - 2 \tan^{-1} 1 \right) \\ = Ph \left( \frac{1}{3Bi} + 0.122 \right), \quad [22a]$$

$$f_2(Bi < 1 + \sqrt{2}) = \frac{2}{3} Bi + \frac{\frac{1}{2} \ln 2 - \frac{\pi}{4}}{3Bi} \\ - \frac{4Bi^3 - 6Bi^2 - 3Bi - 1}{6Bi} \ln \left( Bi + \frac{1}{2} \right) \\ + \frac{4Bi^4 - 10Bi^3 - Bi^2 + 4Bi + 1}{3Bi\sqrt{2} - (Bi - 1)^2} \\ \times \cos^{-1} \left\{ \frac{Bi + 1}{2\sqrt{Bi + 1/2}} \right\}, \quad [22b]$$

$$f_2(Bi > 1 + \sqrt{2}) = \frac{2}{3} Bi + \frac{\frac{1}{2} \ln 2 - \frac{\pi}{4}}{3Bi} \\ - \frac{4Bi^3 - 6Bi^2 - 3Bi - 1}{6Bi} \ln \left( Bi + \frac{1}{2} \right) \\ - \frac{4Bi^4 - 10Bi^3 - Bi^2 + 4Bi + 1}{6Bi\sqrt{(Bi - 1)^2 - 2}} \\ \times \ln \left\{ 1 - \frac{2}{\frac{Bi + 1}{\sqrt{(Bi - 1)^2 - 2}} + 1} \right\}, \quad [22c]$$

$$C = \frac{1}{2} \ln 2 - \frac{\pi}{4} = -0.439. \quad [22d]$$

The two expressions for  $f_2$  result from the different integration rules to be applied at certain values of  $Bi$ . Although the final result appears to be rather complicated it is, in fact, easily evaluated for any given values of the two parameters. In the particular case of constant wall temperature Eq. [22c] reduces to

$$f_2 = 0.5$$

so that the solidification time becomes

$$\tau_1(Bi \rightarrow \infty) = 0.122 Ph + 0.061. \quad [23]$$

For the solidification of a shell of thickness  $\delta^* < 1$  expressions similar to those in Eqs. [22] and [23] are obtained; only the constants are different.

### 3.2. Comparison of Results and Working Equations

As in Section 2 we shall compare our results with exact numerical data; it will be shown that some of the previous approximate analyses are not very reliable. In doing so we observe the same trends as with the cylindrical geometry, that is over prediction for  $Ph < 1$  and under prediction for  $Ph > 1$ . In the latter case the error increases with increasing Biot number and, for practical purposes, becomes unacceptable (>20 pct) when  $Bi > 2$ . This is so because our model result does not converge towards the correct answer when the phase change parameter  $Ph$  tends to infinity. In that case the exact result is given by the quasistationary solution

$$\tau_1(Ph \rightarrow \infty) = Ph \left( \frac{1}{3Bi} + \frac{1}{6} \right). \quad [24]$$

By comparison with Eq. [22a] a 27 pct under prediction at  $Bi \rightarrow \infty$  and  $Ph \rightarrow \infty$  is registered. We note that this discrepancy is far more pronounced than with cylindrical solidification (comparison of Eqs. [13] and [13a]) and is an obvious consequence of the assumptions of a linear temperature profile in the solidified shell and of an arithmetic mean area.

Since from an engineering viewpoint an over prediction of the solidification time is less serious than an under prediction we propose to correct the model result (given by Eqs. [22a] to [22d]) such that it converges towards the true quasistationary solution for  $Ph \rightarrow \infty$ . Thus, for inward spherical solidification we write

$$\tau_1 = F_1(Ph, Bi) + F_2(Bi) + C$$

with

$$F_1 = \text{Eq. [24]}; F_2 = \text{Eqs. [22b, c]}; C = \text{Eq. [22d]}; \quad [25]$$

equivalent expressions are obtained for  $\delta^* < 1$ .

In Fig. 4 the final result, Eq. [25] is compared with some numerical data.<sup>9</sup> The agreement is seen to be surprisingly good and the following errors can be specified:

$$0.2 \leq Ph < 0.5; 0.1 \leq Bi \leq 2 \rightarrow |r_{\max}| \approx 15 \text{ pct}, \\ 0.5 \leq Ph \leq \infty; 0.1 \leq Bi \leq 5 \rightarrow |r_{\max}| \approx 9 \text{ pct}. \quad [26]$$

For completeness we show the corresponding results for half radius solidification in Fig. 5. As with cylindrical solidification we find good agreement with the exact data for many practically important parameter combinations but poor model results for large Biot numbers ( $Bi > 10$ ). Again, the unidirection heat flow model<sup>7</sup> seems more appropriate in such cases. Alternatively, one could easily devise an empirical correction factor to cover the described parameter range with improved accuracy.

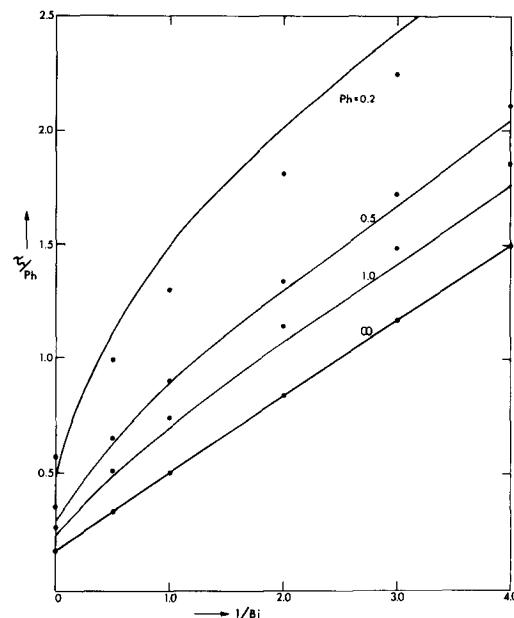


Fig. 4—Time for complete solidification of a sphere. Comparison between Eq. [25] (—) and numerical results (•).

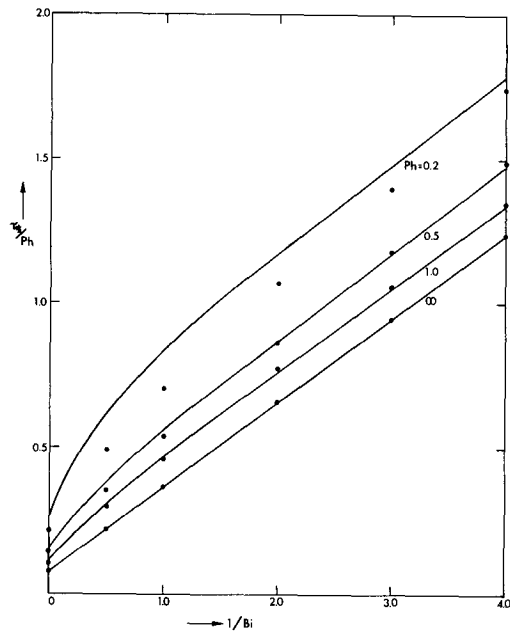


Fig. 5—Time for inward spherical solidification to  $\delta^* = 0.5$ . Comparison between corrected analysis (—) and numerical results (•).

For cylindrical solidification none of the approximate analyses (excluding the quasistationary approach) yields a closed form solution for the solidification time whereas in a spherical system the necessary integrations can usually be performed analytically. Two methods have been presented previously and it may be informative to compare those results with the present ones. However, we point out that in both cases the final result is of a much more complicated structure than Eq. [25], involving lengthy expressions in terms of both parameters,  $Bi$  and  $Ph$ . The one result<sup>21</sup> is based on the assumption of a parabolic temperature distribution in the solid and on the validity of certain rules<sup>22</sup> by which the solidification rate in a sphere is obtained directly from that in a slab. The other result<sup>17</sup> is a perturbation solution for the solidification rate with  $1/Ph$  as the perturbation parameter. Obviously, this latter method cannot yield very accurate results when the phase change parameter is small.

In Table I we list a representative set of errors involved in using either of the three results. Since in all cases good results are obtained for small  $Bi$  and large  $Ph$  values we have concentrated on the critical range of parameters discussed above. From the table we conclude that the present result provides a significant improvement over the existing solutions, particularly in view of its comparatively simple structure. It is mentioned that with a more elaborate

Table I. Approximate Relative Error in Percent of Some Analytic Solutions for Complete Solidification of a Sphere

$Bi$	$Ph$	Ref. 21	Ref. 17	Eq. [25]
1	1	-18	1	+ 5
1	0.5	-33	5	+ 0.5
1	0.2	-38	17	+15
$\infty$	0.5	-33	11	+11
$\infty$	0.2	-35	25	+27

empirical correction (as was applied to one of the other results<sup>17</sup>) a working equation of higher accuracy can be generated from our basic result. However, we feel that this would impair its usefulness for practical applications.

#### 4. CONCLUSIONS

Based on the assumption of a linear temperature profile in the solidified shell simple analytic solutions have been developed for the inward solidification of cylindrical and spherical bodies. The comparison with numerically exact data shows that the results are useful over a large range of values of the two system parameters  $Bi$  and  $Ph$ . After minor corrections to the basic results the maximum error in predicting solidification times is of the order of 10 pct for most situations of practical interest. Extreme cases, characterized by parameter values of  $Ph < 0.5$  and  $Bi > 10$ , lead to somewhat larger errors but can be represented equally well when more elaborate correction methods are employed.

With a reliable analytic result to the phase change problem at hand it is not difficult to analyze and optimize the thermal and economic efficiency of industrial solidification processes. For example, some preliminary studies on continuous casting machines have shown that the performance can be improved significantly by suitable arrangement of the various cooling zones. With the present results the necessary calculations become much simpler than with the numerical schemes devised previously; and yet, the loss in accuracy is minimal in view of other factors of uncertainty common to most of the existing theories.

#### NOMENCLATURE

##### List of Symbols

- $\alpha$  = thermal diffusivity ( $m^2/s$ ),
- $\delta$  = thickness of solidified layer (m),
- $h$  = heat transfer coefficient ( $W/m^2 K$ ),
- $\lambda$  = latent heat of solidification ( $J/kg$ ),
- $k$  = thermal conductivity ( $W/mK$ ),
- $c$  = specific heat capacity ( $J/kgK$ ),
- $\rho$  = density ( $kg/m^3$ ),
- $t$  = time (s),
- $T$  = temperature (K),
- $x$  = coordinate in the direction of the moving interface (m),
- $A$  = heat transfer area ( $m^2$ ),
- $A_m$  = mean area for heat transfer ( $m^2$ ),
- $U_s$  = internal energy ( $J/kg$ ), and
- $V$  = volume ( $m^3$ ).

##### List of Dimensionless Parameters and Variables

- $Ph$  = phase change parameter =  $\frac{\lambda'}{c(T_p - T_c)}$ ,
- $Bi$  = Biot modulus =  $\frac{h_c' \delta_0}{k}$ ,
- $\tau$  = dimensionless time =  $\frac{t\alpha}{\delta_0^2}$ , and
- $\delta^*$  = dimensionless thickness of solidified layer =  $\delta/\delta_0$ .

### Subscripts

- $c$  = relating to cooling fluid,  
 $m$  = relating to melt,  
 $p$  = at phase change conditions,  
 $w$  = relating to cooling wall,  
 $0$  = at reference time  $t_0$ ,  
 $1$  = referring to complete solidification, and  
 $1/2$  = referring to half radius solidification.

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