Theory usage and theoretical trends in Europe: A survey and preliminary analysis of CERME4 research reports

Bharath Sriraman (USA)
Gabriele Kaiser (Germany)

Abstract: The 29th PME research forum on theories included only one European perspective on mathematics education. In order to convey trends in theory usage in Europe we compile, survey and analyze a large subset of the research papers from the 4th European Congress on Mathematics Education (CERME4). That is, this paper includes a discussion of trends seen within CERME4 reports[1] on theory usage by European researchers in *seven of the fourteen working groups* and (a) Outlines similarities and differences in theory usage and (b) takes a futuristic stance on ways in which researchers from different traditions could understand each other. Such an enterprise would further Hans-Georg Steiner's vision of bridging theoretical traditions which are independently formulated in different regions of the world.

ZDM Classification: D20

Introduction

In the first part of the special issue on theories, Sriraman & English (2005) posed several questions to the community related to researchers' preferences of theories depending on their philosophical orientations and locations. Other open questions were how theories of learning were being used in current research, and what was required to propel the field forward? Another question was whether researcher's country specific location influenced the choice of frameworks? In this paper we embark on constructing preliminary observations which could lay the foundation of answering these questions for the mathematics education research scene in Europe, based on the reported research at the recently concluded CERME4 in Spain, 2005.

Mathematics education research frameworks in Europe are much more heterogeneous compared to other regions of world, in spite of geographic proximity and shared borders between countries. The work initiated by Hans-Georg Steiner within the TME group started a dialogue between researchers situated within varying institutional, linguistic, political and historical structures. Given the political backdrop of Europe coming together under a shared economic structure, and the global nature of mathematics education research, we were interested in whether theoretical traditions within which mathematics education research in Europe is situated is converging towards uniformity or whether the view remains plural?

On a much larger geographic scale, based on his empirical analysis of PME reports from 1985-2005, Lerman (2006, this issue) contends that the multiplicity and divergence are neither surprising nor necessarily damaging to the field. Lerman observes that the plurality of theories can be traced to the intellectual communities and the creative products produced by the communities within which researchers are situated in. Within PME, these intellectual communities in the last 20 years or so have increasingly been sociology, philosophy, semiotics, anthropology. Similarly, based on his overview of reported research within the American context, Lester (2005) posited that researchers typically situated their research within a combination of theoretical and practical frameworks, which he termed a conceptual framework. However Lester pointed out to the problem of misalignment between researchers' philosophical orientation and the research conducted as a source of conflict when collecting and analyzing data. Both Lerman (2006, this issue) and Lester (2005) suggest we pay careful attention to the underlying inquiry systems.

Given this context, in this paper we discuss of trends seen within CERME4 reports on theory usage by European researchers in seven of the fourteen working groups. The working groups that we purposefully chose were (WG1) Metaphors and embodied cognition; (WG2) Affect and Beliefs; (WG3) Structures; (WG4) Argumentation and Proof; (WG6) Algebraic Thinking; (WG7) Geometric Thinking and (WG13) Modelling and Applications. The reasoning between choosing these particular groups was the specific focus on a particular aspect of mathematical thinking and learning and our familiarity with the established body of existing research. We summarize the theoretical frameworks used in research reports within these groups; discuss similarities and differences within frameworks used, and examine the issue of tendencies. We also try to take a futuristic stance on ways in which researchers from different traditions could understand each other and avenues of possible interaction.

[1] In our tables we have excluded the very few papers from the US and Canada because of our specific focus on European theory usage. We also assume the reader has some familiarity with the canonical literature in the various research domains of inquiry of the working groups.

Working Group 1: Metaphors and Embodied Cognition

Working Group 1 Metaphors and Embodied Cognition	Primary theoretical Framework[2](s)	Data sources/ Type of Study	Motivating research questions
Paper1 (Spain) Teacher-student interaction during discourse	Lakoff & Nunez Sfard (reification) Presmeg (Prototypes, metonymies, Metaphors)	Classroom observation Video recording Transcripts from videos Written artifacts (Qualitative)	Types of metaphors used by teachers to explain graphic representation of functions
Paper 2 (Sweden) Impact of concrete manipulatives in 3-D visualization	Hall (concrete embodiment) Hall (procedural analogy)	Videotapes of student work in groups Transcripts from videos (Qualitative)	Student actions/interactions with the concrete model and its influence on solution processes
Paper 3 (France) Shifts in students' understanding of geometry	Lakoff & Nunez Houdement & Kuzniak (3 mathematical paradigms) Presmeg	Problem based instrument/Questionnaire Data gathered from 700 pre-service elementary students (Qualitative analysis of solutions from questionnaire)	Factors influencing shifts in understanding from visual representations to purely symbolic representations
Paper 4 (Germany) Shifts in student interpretations/understanding of the meaning of visual representations	Goldin & Shteingold Steinbring Bruner (secondary reference)	Clinical Interviews with 15 children (6-10) (Qualitative)	Factors enabling shift from concrete (empirical) to abstract (underlying structures)
Paper 5 (Cyprus) Survey/analysis of extant literature on representations in Greece/Cyrpus	Duval Skemp Kaput Hiebert	2 task based instruments on "translating" representations from one form to another Statistical Implicative analysis (Quantitative)	Contextual effects of representations in mathematical learning
Paper 7 (Cyprus) Longitudinal study of a dyslexic child's visual and enactive counting strategies in arithmetic	Dubinsky Kaput	Diagnostic Batteries Clinical Interviews Quantitative/Qualitative	Development of counting procedures from concrete to abstract
Paper 8 (Italy) Metaphorical and artistic intertwining of mathematics and real life experiences	Lakoff & Nunez Gestures/embodiment	Discourse Group activities (Qualitative)	Reflection on personal professional experiences, communication and cognitive styles for adaptation in the classroom
Paper 9 (Germany) The role of mental models (carrying meaning) (GV's) in the process of mathematical modelling	Blum (Grundvorstellung GV)	Quantitative batteries Normed instruments to measure achievement, competence etc	

[2] The theoretical frameworks are further elaborated in a separate table for six out of the seven working groups following the overall summaries of each working group.

WG1 Metaphors and Embodied Cognition	Primary Theoretical Frameworks		
Paper1 (Spain)	Lakoff & Nunez (2000) (Embodied Cognition)	Sfard (1994) (reification)	Presmeg (1992,1997) (Prototypes, metonymies, Metaphors)
Paper2 (Sweden)	Hall (1991) (concrete embodiment)	Hall (1998) (Procedural analogy/concrete representations)	Resnick & Omansson (1987)
Paper3 (France)	Lakoff & Nunez (2000) (Embodied Cognition)	Houdement & Kuzniak (1998) Three mathematical paradigms	Presmeg (1992,1997) (Prototypes, metonymies, Metaphors
Paper4 (Germany)	Goldin & Shteingold (2001) (representations)	Steinbring (2005) (Epistemological perspective on the construction of new mathematical knowledge)	
Paper5 (Cyprus)	Duval (2002) (cognitive analysis)	Skemp (1986) (relational vs instrumental)	Kaput (1987) (representations)
Paper7 (Cypus)	Dubinsky (1991) (notion of function)		Kaput (1987) (representations)
Paper8 (Italy)	Lakoff & Nunez (2000) (Embodied Cognition)		
Paper 9 (Germany)	Blum (1998) (Grundvorstellung)	Freudenthal (1983) (Didactical Phenomology)	Fischbein (1987) (Intuition in science and mathematics)

Discussion of WG1

As the tables for working group 1 indicate, although there were small geographic variations in the theoretical frameworks (depending on the particular mathematics content being researched) WG1 shows remarkable consistencies in the theoretical frameworks used. The primary sources typically included the work of Lakoff & Nunez and closely related literature on embodied cognition, the role of gestures and metaphors (Presmeg), and the role of representations in the learning of mathematics. Table 1, which contains our summary of the different topics of research within this domain, the questions motivating this particular line of inquiry, the theoretical frameworks used, the research methodology and data collection techniques, indicates the heterogeneity in research methods employed as well as the wide scope of this area of investigation. The predominantly qualitative nature of the research in these reports indicates the difficulty of quantifying or measuring the use of metaphors, gestures, and semiotic notions in general. The next-generational aspect as well as the importance of research in this domain of mathematics education is evident in established journals such as *Educational Studies in Mathematics* (e.g., ESM,2004, vol57,no3) deviating from the norm of researchers passively reporting in written form on the use of gestures and metaphors in mathematical thinking and learning, to the use of video-issues with videos within the body of research papers, which transport the reader into the classroom , convey a first-person perspective on the data, as well as allow the reader to re-construct (or as Sierpinska (2004) puts it „co-construct“) the interpretations of the researchers. The general consensus on the canonical literature in this domain of inquiry is particularly heartening for researchers interested in conducting research in this area of mathematics education. One of the philosophical challenges confronted by researchers working within this domain and which has been subject of much debate after the release of Lakoff & Nunez's (2000) book is to view mathematics as a socio-cultural artifact part of a larger linguistic system as opposed to the Platonist objective view. Recently Burton (2004) appealed to work by Lakoff and Núñez (2000) and Rotman (2000) among others to illuminate this viewpoint in her empirical enquiry on how research mathematicians transition from learners to creators of original mathematics. Lakoff and Núñez (2000) suggested that mathematics is a collective socio-cultural artifact, a product shaped by human brains, societies and culture. Their concept of "the embodied mind" maintains that the body and brain together with everyday experiences structure our conceptual systems. This concept in conjunction with the notions of "cognitive unconscious" and "conceptual metaphor were used by Lakoff & Nunez to re-examine mathematical ideas such as transfinite numbers, use of infinitesimals, Dedekind cuts etc. Numerous reviews (Auslander, 2001; Gold, 2001; Goldin, 2001) have pointed out the mathematical errors in the book, and questioned the applicability of the conceptual metaphors to concepts in mathematics. Lakoff and Núñez have written rejoinders to these criticisms (such as the following reply to Gold)

> …[T]here is a whole group of concepts and analyses we present that are incorrectly called "mathematical errors" because they are taken as mathematical analyses and not as cognitive analyses… Our job is not to improve mathematics (how could we possibly do that?)…And it is certainly not our job to tell mathematicians how to do mathematics (just as it not the job of a zoologist to tell a bird how to fly!). As cognitive scientists our job is to give cognitive accounts of largely unconscious mechanisms of mind used to characterize ideas - ideas of all sorts, both mathematical and otherwise. (Lakoff and Núñez, MAA online)

It therefore serves the community of mathematics education researchers well, if this area of inquiry were further developed with more warrants to evidence in the classroom and other mathematical settings where semiotic systems play an important role in both learning and teaching. Another feature of this group was the balanced use of original theoretical frameworks and practical frameworks based on prior empirical studies. The frameworks employed by researchers in this working group also pointed to a synthesis between theories arising from the domain of cognition and learning theories that are social constructivist in their orientation.

Working Group 2: Affect and Beliefs			
Working Group 2 Affect and Beliefs	Primary theoretical Framework(s)	Data sources/ Type of Study	Motivating research questions
Paper1 (Finland) Pre-service teachers views about mathematics	See next table	Fennema-Sherman battery for attitude scale Task based questionnaires Interviews (Quantitative/ Qualitative)	The effectiveness of autobiographical narratives to study beliefs. The impact of prior mathematical experiences in schools
Paper 2 (Northern Ireland) Views of attitudes, competence and confidence in mathematics	Ernest Nespor Bolhius & Voehten	Likert Questionnaire Semi-structured Interviews (Quantitative/Qualitative)	Self-reflection as a means of motivating change Comparision/measuring change of beliefs/attitudes between post-graduate and undergraduate students
Paper 3 (Spain) The interaction between mathematical thinking and affect in a multicultural context	Gomez-Chacon et al Goldin Cobb & Hodge	Interviews Questionnaires/Instruments Mood-maps Field notes Ethnography (Qualitative/Quantitative)	Establish relationships between cognition and affect Trace origins of affective responses Analyze and interpret emotional responses from the social and culture perspective
Paper 4 (Italy) The relationships between drop-out rates and affect about mathematics	See next table	Survey/Questionnaire Interviews Analysis of data from surveys and from schools Quantitative/Qualitative	Identification of endogenous variables constituting mathematical discomfort which play a role in drop-out
Paper5 (Austria) The relationship between meta-affect and learning strategies employed by students with learning difficulties	DeBellis & Goldin Goldin Ciompi Hannula Schlöglmann	Interviews Qualitative	Identification of meta-affective routines/responses
Paper 6 (Italy) One year teaching experiment on changing attitudes towards mathematics	Hannula Di Martino & Zan Furinghetti & Pehkonen Phillipou & Christou	Questionnaire (with open-ended responses) Teaching experiment Quantitative/Qualitative	Effectiveness of interventions aimed at changing attitude towards mathematics in students with strong emotions/opinions.
Paper7 (Italy) The aesthetic, affective and cogntive components of creativity	DeBellis & Goldin Sinclair Maslow	Problem based case study of one student Qualitative analysis of student work	Documenting the powerful role of intuition/insight and aesthetic values as contributing to creativity in novice problem solvers
Paper 8 (Finland) Structure of elementary pre-service math teachers view about mathematics	Hannula Op ’t , De Corte, E. & Verschaffel Furinghetti & Pehkonen	Two questionnaires measuring competencies and views of mathematics Quantitative (Principal component analysis and factor analysis)	Statistically identify inter-related background variables related to confidence, like or dislike of mathematics
Paper 9 (Italy) Mutidimensional characterization of the construct of attitude and its use	Hannula Di Martino & Zan McCleod Bruner Pajares	Questionnaire designed for the specific population Quantitative Study	Enable teachers to identify specific indicators of negative attitudes

WG2 Affect and Beliefs[3]	Primary Theoretical Frameworks			
Paper1 Finland	Autobiographical lit Ricouer(1992) Bruner (1986) Linde (1993)	Mathematical-identity Bikner-Ahsbahs (2003)	(View of mathematics) Schoenfeld (1985) Pehkonen & Pietilä (2003)	Op 't Eynde, P., De Corte, E. & Verschaffel, L (2002)
Paper2 Northern Ireland	Ernest (1988,1989)	Nespor (1987)	Bolhius & Voehten (2004)	
Paper3 Spain	Gomez-Chacon (2000)	Goldin (1998)	Cobb & Hodge (in press) Interpretive schemes	
Paper4 Italy	Mcleod (1992) Prawat & Anderson (1994) Di Martino & Zan (numerous citations)	Alro & Skovsmose (2002)	Schoenfeld (1985)	Arcavi (1994) Kieran (1992) Sfard (1991)
Paper5 Austria	Meta-affect DeBellis & Goldin (1997) Goldin (2000, 2002, 2004)	Affect-logic Ciompi (1982, 1988, 1991)	Hannula (2002) Schlöglmann (2000, 2002)	
Paper6 Italy	Hannula (2002) Di Martino & Zan (numerous citations)	Furinghetti & Pehkonen (2002)	Phillipou & Christou (1998)	
Paper7 Italy	DeBellis & Goldin (1999) (internal representations)	Sfard (1991) Sinclair (2004)	Maslow (1962) (hierarchy)	
Paper8 Finland	Hannula (2002)	Furinghetti & Pehkonen (2002)	Op 't Eynde, P., De Corte, E. & Verschaffel, L (2002)	
Paper 9 Italy	Hannula (2002) Di Martino & Zan (various citations)	Mcleod (1992)	Bruner (1990)	Pajares (1992)

[3] Please note that we report on only 9 out of the 11 papers as two were unavailable at the time of our survey

Discussion of WG2

The role of affect as a central issue in mathematics education has periodically been reaffirmed in various curricular documents worldwide such as the *Principles and Standards* (NCTM, 2000) and the Australian Educational Council (AEC, 1990). Among the major goals stated in the NCTM (2000) document deal with helping students understand the value of mathematics and with developing student confidence. The term affect as defined by Mcleod (1991) refers to “a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition” (p.56). Although researchers in mathematics education initially ignored the role of affect, this has began to change largely due to the initiatives of researchers such as Erkki Pehkonen, Marku Hannula, Günter Törner and others in teacher education; Frank Lester, Alan Schoenfeld and Gerald Goldin in problem solving, and numerous others cited in the table. The recently released Kluwer book in the mathematics education library series indicates that the international community is actively investigating and consolidating research in this area. John Dewey (1903) drew educators’ attention to the importance of addressing affective aspects of learning in addition to the cognitive aspects. His writings were focussed in the area of Euclidean geometry instruction and proof. However Dewey did not elaborate on the emotions or habits that might indicate whether or not the student was prepared to study a geometry course that emphasized proof. Theorists like Goldin (2000, Kaput (1989), Lester, Garofalo & Kroll (1989), Mandler (1984, 1989), and Presmeg (2001), among others have pointed to the influence of affect on the cognitive and meta-cognitive behaviours of student’s during problem-solving and how affective factors arise out of emotional responses to the interruption of plans or planned behaviour (Mander). Mandler provided a general framework for research on affect which simply stated says plans arise from the activation of a schema, which in turn produces an action sequence. Now if the „anticipated schema“ is blocked for whatever reason then it causes the physiological response of arousal. Finally, the individual attempts to evaluate the meaning of this blockage. Mandler’s theory is similar to Piaget’s theory of equilibrium. The process of equilibration is the learner’s self-regulatory and self-correcting process that regulates the individual’s specific interactions with the environment as well as physical experience, social experience, and physical development (Piaget, 1980 as quoted in Bell-Gredler, 1986). The survey of this working group reveals that a consistent body of theoretical literature being used by researchers. In spite of the heterogeneity in the locations and specific areas of affect and beliefs in which these researchers worked, the literature was not restricted to specific European affiliations but to the international body of research which constitutes the canon of this domain of inquiry. The cross-referencing consistencies of the primary theoretical frameworks employed by this working group were also remarkable. This was tested by using several common primary references found in these papers such as Mcleod (1992) or Goldin (2000) or DeBellis & Goldin (1997) or Hannula (2002) in three different databases such as Academic Search Premier, Psychinfo and Eric. For instance if one performed a search with Goldin (2000), the list of references in this paper included Schoenfeld, Mcleod, DeBellis & Goldin, etc, which in turn were cited by other papers in this group. If one performed a search say „Goldin and affect “to be found anywhere in the texts in refereed educational journals, this yielded numerous articles which were cited by the papers in this working group.

The papers in this working group revealed the most diversity in the combination of qualitative and quantitative research methodologies. Another major theme was the necessity as well as the justification of the instrumentation used for empirically measuring beliefs. In the North American context, the work engaged in by Schoenfeld's research group at Berkeley, California has resulted in teacher's decision-making model of "teaching in context" which provides a fine-grained characterization of the teacher's decision-making, grounded in the analysis of teacher's knowledge, teacher's goals, and teacher's beliefs (see Sriraman & English, 2005 for citations of: Schoenfeld 1998, 1999a, 1999b, 2002a). In the European context, the evolution of the research in this domain of inquiry as seen in the reports of this working group suggests that although uniformity in instrumentation is still an issue, researchers in Europe are systemically building off existing models developed to understand the beliefs and attitudes of pre-service teachers, and more practically to help teachers understand the affective traits of students and its relationship to learning.

Working Group 3: Structures

WG3 Structures	Primary Theoretical Frameworks	Data sources/Type of Study	Motivating research questions
Paper 1 (Germany) The alignment/misalignment of teachers and students conceptual networks about linear functions	Novak (Concept mapping) Brinkmann (mind mapping)	Questionnaire (broadly stated questions to more specific questions), task based tests to students, interviews with teachers.	Pinpointing the structural changes occuring within existing mental conceptual maps/webs as a result of teaching and learning in the context of linear equations
Paper2 (Czech/UK) The interplay between tactile and visual perceptions on student's conceptions of geometric solids	Bell Design principles for teaching Hejny Understanding and Structure Van-Hiele Structure and Insight	Classroom experiment with 9 students in Czechia, 5 in the UK (inner-city school) , 4 in Czechia (inner-city school) Video-recording of student's manipulations, and transcipts with student responses	Understand the structure building process of students in 3-D geometry through tactile and visual sensory inputs
Paper 3 (Turkey) Fragmentation in existing knowledge structures impeding the formulating of generalizations in the context of the triangle inequality	Gray, Pitta & Tall Hejny Dahl	Participant observation Student-student Student-teacher Videotaping of student group work. Transcripts of video-tapes. Interpretations of meanings taken in the context of the specific tasks provided	Pedagogical considerations for the teaching of mathematical concepts with consideration for students existing knowledge structures
Paper 4 (Germany) The role of a metaphorical operating system (the contract metaphor) in linking new knowledge to existing structures.	Lakoff (metaphors) Davis (Modelling thinking processes) Wittman (Mathematics education as a design science)	Qualitative analysis of three student solutions on assessment items from the curriculum.	Outline a mechanism to counteract the fragmenting of student knowledge.
Paper 5 (Slovak Republic)	Gray, Pitta & Tall Hejny Hiebert Sfard	Historical case study of the development of structures in Algebra (rules-formulas-forms)	Philosophical argument to slow down the pace of instruction in Algebra in order to faciliate the development of mathematical structures in students analogous to their historical development.
Paper 6 (UK and France) Thought experiment in the Lakatosian sense between a mathematician and didactian on the nature of mathematical „research" experiences provided to students	Lakatos Popper Brousseau Borasi Hejny Piaget (genetic epistemology)	Philosophical examination of major primary writings to analyze the nature of mathematics research from different viewpoints.	
Paper 7 (Italy) Meaningful approaches to the distributive law in early algebra	Malara & Navara Tirosh et al	Detailed analysis of a teaching situation lasting 3 hours in 3 sessions. Detailed analysis of emergent representations	Present didactical pathways via which students aged 8-10 can unify multiple representations and conceptually intuit the distributive law.

WG3 Structures	Primary Theoretical Frameworks			
Paper 1 (Germany)	Novak (1990, 1996) (Concept mapping)	Brinkmann (1999, 2001, 2003a,2003b) (mind mapping)		
Paper2 (Czech/UK)	Bell (1993) Design principles for teaching	Hejny (2003) Understanding and Structure	Van-Hiele (1986) Structure and Insight	
Paper 3 (Turkey)	Gray, Pitta & Tall (1999, 2000) (APOS)	Hejny (2003) Understanding and Structure	Dahl (2004) Application of APOS	
Paper 4 (Germany)	Lakoff (1980) (metaphors)	Davis (1979) Modelling thinking processes	Wittman (1995) Mathematics education as a design science	Cohors-Fresenborg & Kaune (1987, 2001, and others) Metacognitive mechanisms and its relation to teaching
Paper 5 (Slovak Republic)	Gray, Pitta & Tall (1999,2000) APOS	Hejny (2001,2002) Understanding and Structure	Hiebert (1986) Procedural versus conceptual knowledge	Sfard (1989) Transition from operational to structural (conception of function)
Paper 6 (UK and France)	Theoretical paper involving a thought experiment...numerous philosophical and didactical references [Too many to accurately list]			
Paper 7 (Italy)	Malara & Navara (various citations, see bibliography)	Tirosh et al (1991)		

Discussion of WG3

The papers in Working Group 3 addressed aspects of the cognitive structures built in different mathematical contexts (such as linear functions or algebraic operators or inequalities). There was no uniformity per se in the various themes of the papers although the APOS theory and Hejny's theory on understanding and structure were often cited. An attention to practice was a major focus of some of the studies. That is, many of the studies were relevant for teachers' interested in learning pedagogical intervention to prevent fragmentation of mathematical ideas learned in the classroom. While the notions of Skemp and Hiebert were invoked to argue for a consolidation and the linking of new ideas with prior ones, there was no evidence of attempts to inform current theories on how structural mechanisms reorganize themselves. From a Piagetian perspective, many of the studies reported in this group were at the cusp of formal operational mechanisms. That is, as opposed to the stage of concrete operations which is characterized by operations that are insufficiently formalized, the stage of generalized formal operations is characterized by the organization of operations in a structural whole. Readers will recall that one of the main characteristics of this stage is that possibility is not bound to be an extension of empirical situations. At this level, students supposedly are able to create hypothetical situations and draw out necessary consequences without ever observing these consequences. Piaget (1958) suggested that at the stage of formal operations, there is a "structural mechanism" which enables students to compare various combinations of facts and decide which facts constitute necessary and sufficient conditions to ascertain truth. This structural mechanism is functional only after students are able to transform propositions about reality, so that the relevant variable can be isolated and relations deduced. Another characteristic of the stage of generalized formal operations is the relative ease with which reversibility of thought operations occurs. This was an interesting claim substantiated by decades of research. Many of the studies in this working group contain elements of reversibility in the vignettes and other evidence which can potentially Piaget's claims. Another major part of Piaget's characterization of knowledge and cognition is that they are forms of biological adaptation. Within this characterization structures of *action*, in the form of individual sensori-motor schemes play a role. Mathematics, unlike the other sciences is special in its concern with the creation of structure. Piaget in fact said that "the whole of mathematics could be thought of in terms of creation of structures" (p.70). These constructions are of course not physical ones, but operations carried out in the conceptual and idealized world of the mathematician. The relationship between the two is as follows: the idealized constructions emerge as a result of a series of abstractions from their literal counterparts, which are the real actions and physiological movements human beings make in the world. Piaget's psycho-genetic account of mathematics retraces this descent from its roots in the sensorimotor schemes of infancy through to abstract thought, namely how sensorimotor *actions* first become *internalized*, then by achieving *reversibility* become *operations*, that in turn serve as the objects of hypothetico-deductive thought, which in turn serve as the building blocks of formal mathematical structures. This passage from actions to formal thinking, in Piaget's account is one of increasing abstraction and generalization. Beth & Piaget (1966) compared his operator structures of thinking to the structures espoused by the Bourbaki. The Bourbaki identified three fundamental structures on which mathematical knowledge rests. They are (1) algebraic structures; (2) structures of order; and (3) topological structures. Piaget claimed that there existed a correspondence between the mathematical structures of the Bourbaki and the operative structures of thought. He felt that in the teaching of mathematics a distinctive synthesis would occur between the psychologist's operative structures of thought and the mathematician's mathematical structures. Paper 5 of this working group which was a historical case study of the development of structures in Algebra from rules-formulas-forms contained a subset of ideas which were very Piagetian in nature with the potential of being studied and tested in the classroom in the context of Algebra.

In this working group, two new theories which extend Piagetian notions on structure building were used as theoretical frameworks (namely Dubinsky's APOS theory and Hejny's theory). Dubinsky (1991) used Piaget's notion of reflective abstraction, to develop a model using "schemas" for understanding the construction of advanced mathematical knowledge. Simply put, a schema is composed of cognitive objects and internal processes. A schema is composed of various actions and constructions that can form new schemas. Dubinsky (1991) defined cognitive objects as "constructions which a subject makes in order to deal with various experiences. These experiences, typically, "involved sensory perceptions, motor activities, or thought" (p.166). Internal processes are defined as "the subject's ability to mentally manipulate cognitive objects" (p.166). Since we are now familiar with Piagetian terminology, we can view internal processes as *actions* performed on *objects*. This internal process is described as follows:

> An internal process…consists of an individual actively moving mental objects around, calling them into awareness, combining them, ignoring them, transforming them, etc. all in her or his mind. The subject is not only aware of the individual steps of the process but has a total picture of it and can move back and forth performing and reversing the mental activities. As a result of this awareness the individual can reflect on the process itself, combine it with other processes, reverse it, reason about it, etc. (Dubinsky, 1991, p.166).

Based on his observations of his students, Dubinsky inferred five means of construction. They are interiorization, encapsulation, coordination, reversal and generalization. Interiorization means that an object is organized into a process. Dubinsky cites the example where a student perceived the domain and range of a function as sets of objects. A student can then construct a mental process, which transforms the elements of the domain into the elements of the range. Encapsulation entails viewing the process as an object. For example when one considers a set of sets. The concept of set is a process of putting objects together. Now this set can become an object in another set. Another example provided by Dubinsky is the process of mathematical induction, where the inductive step is viewed as an object by itself. Coordination is "the cognitive act of taking two or more processes to construct a new process" (Dubinsky, 1991, p.174). For example: the nested loops in an algorithm. Dubinsky (1991) noted that reversal is essential when one transforms a process into another one, to form a new process. For example: finding the inverse of a function involves reversal. It is particularly interesting that Dubinsky uses the notion of one-to-one correspondence to illustrate how reversal occurs. "To think about the idea of 1-1…one must again apply the function process to everything in the domain, then move over (in one's mind) to the range and, considering each object there, *look back to the domain*, and see if it came from more than one object in the domain " (Dubinsky, 1991, p.177). Finally, Dubinsky views generalization as the process by which "an existing schema is represented and used in a new situation different from previous one" (p.179) Thus generalization involves the combination of objects and processes. This involves a high degree of awareness of the part of the subject. On the other hand, Hejny's (2003) theory of generic models, which was used as a theoretical framework in three of the papers in this working group (two of which were from the geographic region of the researcher) also contain substantial extensions of Piagetian notions. Hejny (2003) outlines the interplay between abstraction and generalization in knowledge structures. Hejny's theory of generic models consists of two levels. The first level is that of experiences→ generalization→ generic models. This level consists of concrete experiences, stored in memory as "isolated" models, which then begin to link or refer to each other and constitute a set of models (referred to as a group). The references are either confirmatory or contradictory; the former leads to consolidation, whereas the latter leads to the formation of different models. In the second level, termed the "abstraction" level the generic model serves as a starting point for a higher order re-structuring (generic model→abstraction →abstract knowledge) characterized by linguistic and symbolic shifts in the representation of the generic model. Three of the papers in this working group provide illustrations of how this theory works in the mathematical contexts of 3-D geometry and the triangle inequality with pedagogical implications for the teacher.

There are numerous other possibilities for the research on building of structures which connect to one of the papers (Pegg & Tall, 2005) in Part A of these ZDM issue on theories and in the commentary on Pegg & Tall in this issue (see Dahl, 2006, this issue). Pegg and Tall (2005) acknowledged the presence of global frameworks within which the longitudinal consolidation of knowledge could be framed, such as the Piagetian stage theory, van-Hiele theory of the growth of geometric knowledge and Brunerian theory of knowledge development in enactive-iconic-symbolic cycles. There are also local theories of conceptual growth such as the action-process-object-schema (APOS) theory of Dubinsky. Pegg & Tall (2005) proposed a neo-Piagetian framework, called SOLO within which common cycles of conceptual development could be identified which in a sense cut-across the aforementioned theories and also attempts to fill in the deficiencies within Piaget' framework. What is in common for the theories is that they involve "a shift in focus from *actions* on already known objects to thinking of those actions as maniputable mental *objects*. This cycle of mental construction has been variously described as: *action*, *process*, *object* (Dubinsky, 1991); *interiorization*, *condensation*, *reification* (Sfard, 1991); or *procedure*, *process*, *procept*—where a procept involves a symbol such as 3+2 which can operate dually as *pro*cess or *concept*" (Pegg & Tall 2005, p. 471).

Working Group 4: Argumentation and Proof

Working Group 4 Argumentation and Proof	Primary theoretical Framework(s)	Data sources/ Type of Study	Motivating research questions
Paper1 (France) Comparative study of proof styles in French and German textbooks	Balacheff Social nature of proof Polya Mathematics and plausible reasoning Toulmin The uses of arguments	Analysis of textbooks Archival research	Designing a common framework to compare validation schemes in France and Germany
Paper 2 (Spain) The use of inductive reasoning	Healy & Hoyles Polya	Problem-based Individual clinical interviews with a pre-framed protocol Even + Even =?	Identifying categories and sub-categories of inductive reasoning
Paper 3 (France) Proof analysis	Toulmin The notion of warrants	Theoretical analysis	Outlining a framework for understanding student use of logical quantifiers for analyzing proofs
Paper 4 (Spain) Visual proofs and conjectures	Dreyfus Visualization and imagery literature Arcavi The role of visual representations Fischbein The theory of figural concepts	Theoretical analysis	Analysis of geometric constructions/conjectures to solve Heron's problem
Paper 5 (Germany) Student thinking in proof situations	Balacheff Hanna & Jahnke Fischbein(Intuition and proof) Coe & Ruthven Proof practices of advanced math students	Theoretical analysis	The use of empirical work to formulate and motivate "local" theories in geometry before constructing global theories.
Paper 6 (UK) Empirical versus structural reasoning	Balacheff Polya Coe & Ruthven Healy & Hoyles	Instrument (with multiple choice and open-ended responses) Quantitative/Qualitative analysis	Analyze the development of reasoning in high ability students
Paper 7 (Sweden) Analysis of proof in Swedish upper secondary textbooks	Lave & Wenger's Notion of situated learning	Analysis of textbooks Archival research	Alignment/misalignment of textbook content to curricular goals

WG 4 Argumentation and Proof	Country (Authors)	Theoretical Frameworks		
Paper1	France	Balacheff (1991) Social nature of proof	Polya (1954) Mathematics and plausible reasoning	Toulmin (1958) The uses of arguments
Paper2	Spain	Healy & Hoyles (1998) Justifying and proving in school mathematics	Polya(1954) Mathematics and plausible reasoning Mathematical Discovery	Lampert (1990) Classroom discourse to facilitate discovery of mathematical relationships
Paper3	France	Studies that use Toulmin's (1958) model		Toulmin (1958) The notion of warrants
Paper4	Spain	Dreyfus (1994) Visualization and imagery literature	Arcavi (2003) The role of visual representations	Fischbein (1993) The theory of figural concepts
Paper5	Germany	Balacheff (1991, 1998)	Hanna & Jahnke (1993, 1996, 2002) Fischbein (1982) (Intuition and proof)	Coe & Ruthven (1994) Proof practices of advanced math students Healy & Hoyles (to appear) Curriculum change and geomteric reasoning
Paper6	UK	Balacheff (1991)	Polya (1954) Mathematics and plausible reasoning	Coe & Ruthven (1994) Healy & Hoyles(1998)
Paper7	Sweden	Lave & Wenger's (1991) notion of situated learning (transparency of artefacts)		Hanna(1995) Challenges to the importance of proof

Discussion of WG4

Before embarking on a discussion of WG4, we think it is necessary to situate the discussion in the proper historical context. The words argumentation and proof bear a rich and varied nearly 4000 year old history which make it difficult to construct comparisons. However both these words involve elements of reasoning as well as logic systems which can serve as the backbone for the discussion on one common focus of the various papers in this group, namely epistemological issues. Garrett Birkhoff in his retiring address to the SIAM argued for the human dimension of proving. He remarked that Boole's (1854) classic treatise *Investigation into the Laws of Thought*, was instrumental in the development of the algebra of logic, commonly referred to as Boolean Algebra. According to Birkhoff (1969) Boole's (1854) formalization of his laws of thought constituted a major breakthrough in mathematical psychology as the first major step in symbolic logic, i.e., towards the mechanization of mathematical thinking dreamed of by Leibniz and Descartes. This was followed by Whitehead and Russell’s (1908) monumental treatise, *Principia Mathematica*, an attempt to show that mathematics and logic were in essence identical. Even David Hilbert in the latter stages of his career regarded classical mathematics as "a combinatorial game played with the primitive symbols, and we must determine in a finite combinatorial way to which combinations of primitive symbols the construction methods or proofs lead" (Birkhoff, 1969, p.434). Finally, Gödel's incompleteness theorems of the 1930's showed that, given a formal system containing ordinary arithmetic, it was impossible to prove all propositions that are true, nor was it possible to show that the formal system was free of contradictions. Thus, any attempt to mechanize mathematical thought via use of symbolic logic is futile.

Birkhoff (1969) further argued about the shortcomings of theorem proving machines because of *the inability of computers to sense the significant*, a qualitative ability possessed by good mathematicians (Ervynck, 1991; Hadamard, 1945; Poincaré, 1948; Polya, 1954). Simply put, the ability of good mathematicians to sense the significant and to avoid undue repetition is hard to computerize; without a qualitative feel for the problem, the computer has to pursue millions of fruitless paths avoided by experienced human mathematicians.

The preceding paragraph shows the shortcomings of formal logical systems as well as the finite capabilities of machines. Given this preamble, one major premise of the papers in this group supported by the historical writings of eminent mathematicians was the qualitative dimension of mathematical inquiry and proof. In general the subjective dimension to mathematical inquiry is not readily apparent in textbooks or finished proofs found in journals. Usually the mathematician first forms a personal belief about the truth of an idea and uses this as a guide for more formal analytic methods of establishing truth. For example, a mathematician may intuitively arrive at the result of a theorem, but realize that deduction is needed to establish truth publicly. Thus, intuition serves to convince oneself about the truth of an idea while serving to organize the direction of more formal methods, namely the construction of a proof to establish the validity of the finding (Fischbein, 1980). The proof has to be sanctioned by a group of experts in order to get included in the body of knowledge. Proof is a socio-cultural process through which the mathematical community validates the mathematician's creative work. The Russian logician Manin (1977) said “a proof becomes a proof after the social act of accepting it as a proof. This is true of mathematics as it is of physics, linguistics, and biology" (p.48).

In this working group, two of the seven papers were archival in nature and involved the analysis of proof in textbooks with implications for curricular goals. Five out of the seven papers were theoretical in nature with expositions on the epistemological nature of proof, on frameworks for analyzing proof schemes of students, among others. The authors that were most frequently cited in theoretical frameworks (in alphabetical ordwer) were Balacheff, Coe& Ruthven, Fischbein, Healy & Hoyles, Polya and Toulmin. The papers from France were grounded in theoretical frameworks developed by French researchers. However, the papers from Germany and the UK also cited these sources in their reports which suggests signs of a healthy cross-fertilization of ideas.

There are several issues which can be examined from the point of view of knowledge and theory development in the domain of proof. It seems to us that research in this domain can be divided into non-mutually exclusive categories such as

(a) the use of Polya style heuristics/conjectures
How does one arrive at a mathematical result in the first place before embarking on proving its truth?

(b) uses of evidence
What types of evidence are used in order to consolidate the truth of a result. For instance, the use of empirical data, theoretical examples, verification on computer via manipulation or checking cases.

(c) reasoning/argumentation/generalization (inductive, inferential, logico-deductive (axiomatic), informal, visual etc)

(d) epistemology (questioning the very nature of proof).

The types of proof accepted by the particular community one is working within, which would take into consideration (b) and (c). This aspect involves examining the place and use of definitions, levels of evidence etc.

(e) pedagogical and curricular implications of (a)-(d).

In Moreno & Sriraman (2005) the impact of new computational technologies particularly in the domain of dynamic geometry was addressed as well as the epistemological and pedagogical implications for what they considered a „ situated proof“ in the classroom. In a sense, *every proof is situated* but emphasizing the situated-ness while working within a computational environment pays an extra bonus. Based on the premise that explorations within a computational environment eventually allow students to generate and articulate relationships that are general in the environment in which they are working, Moreno & Sriraman (2005) proposed the notion of situated proofs as a possible way to take into account the role of technology in facilitating student’s notions of proof. Noss & Hoyles (1996) have called relationships which encapsulate general statements *situated abstractions*, precisely because they are bound into the medium in which they are expressed.

Some of the papers in this working group examined the misalignment of textbook proof content with curricular goals. Another paper examined the differences in validation schemes employed by textbooks in Germany and France and proposed a framework via which the two could be compared. One overwhelming theme in all the papers was the social/institutional nature of proof. In terms of available theoretical frameworks which can inform research in this domain, particularly classroom studies investigating students’ understanding of proof, one possibility is Chevellard’s extension of Brousseau’s theory of didactical situations into the Anthropological Theory of Didactics (ATD). Chevellard’s motivation for proposing a theory much larger in scope than TDS was to move beyond the cognitive program of mathematics education research, namely classical concerns (Gascon, 2003) such as the cognitive activity of an individual explained independently of the larger institutional mechanisms at work which affect the individuals learning. Chevellard’s (1985,1992,1999a) writings essentially contend that a paradigm shift is necessary within mathematics education, one that begins within the assumptions of Brousseau’s work, but shifts its focus on the very origins of mathematical activity occurring in schools, namely the institutions which produce the knowledge (K) in the first place. The notion of didactical transposition (Chevellard, 1985) is developed to study the changes that K goes through in its passage from scholars/mathematicians →curriculum/ policymakers→teachers→students. This framework would be particularly useful to compare studies conducted across classrooms in different countries on the social dynamics of proof, particularly the transposition of meaning in its passage through the stages mentioned above.

Working Group 6: Algebraic Thinking

Working Group 6 Algebraic Thinking	Primary theoretical Framework(s)	Data sources/ Type of Study	Motivating research questions
France Case Study (or Application) of Chevellard's ATD	Chevellard Brousseau Schubring	Historical Analysis of the development of number concepts. Transcripts of video-taped classroom observations. Interviews with students of varying abilities. Analysis of artifacts.	Determine the causes of student's difficulties with subtraction of whole numbers
Italy Historical Case Study of Inequalities and Equations	Haeckel Brousseau Piaget& Garcia Vinner & Tall	Historical mathematical artifacts Theoretical Analysis	Pedagogical implications of the historical difficulties of the analogies between equations and equalities
France Diagnostic framework for autonomous learning	Bachelard Chevallard	Analysis of textbook and related curricular materials of 10th grade course. Theoretical analysis of epistemological obstacles/objects	Construct a diagnosis of student's difficulties with algebraic expressions in terms of the field of epistemological objects and relations that emerge during the study process.
Germany Early algebraic thinking	Booth Carpenter & Levi Malara & Navarra Warren	Videotapes of pairs of students working on a task Semi-structured interviews with each pair Qualitative Analysis of transcripts and artifacts	What are primary students' thinking and solution processes on algebraic problems in a fourth grade classroom ?
Canada/Italy Semiosis and microgenesis of algebraic generalizations	Radford & Radford et al., extensive work on semiosis	Micro-genetic analysis of video-taped classroom activities (teacher-students; student-student and the milieu) over a 5 year period. Transcripts of video tapes Qualitative Analysis	What is the role played by semiotic systems in students' changing understandings of patterns and algebraic generalizations.

WG6 Algebraic thinking	Primary Theoretical Frameworks			
France	Chevellard(1991, 1999) Anthropological theory of didactics (ATD)	Brousseau (1998) Theory of didactical situations	Schubring (1986, 1988) Historical analysis of German, French and English textbooks	
Italy	Haeckel's law of recapitulation Duval (1995) The role of semiosis in human thinking Lakoff & Nunez (2000)	Brousseau (1983, 1989) Epistemological obstacles and its relationships to mathematics didactics	Piaget & Garcia (1989) Psychogenesis and the History of Science	Vinner and Tall (1981) Concept Image Concept Definition
France	Bachelard (1949) Applications of rationalism The rational pupil	Chevellard (1995) Problematizing Studying as the link between teaching and learning		
Germany	Booth (1988) Children's difficulties in beginning Algebra	Carpenter & Levi (2000) Malara & Navarra (2003) Concepts/pathways into early algebraic reasoning/thinking	Warren (2003) The role of arithmetic structures in the transition from arithmetic to algebra	
Canada/Italy	Kaput & Sims-Knight (1983) Errors in translations to algebraic equations	Radford (2000, 2002, 2003) & Radford et al's (2003) work on semiosis		

Discussion of WG 6

The papers in the working group on algebraic thinking were very diverse. The papers from France predominantly used the frameworks of Brousseau, Chevellard and Bachelard. The paper from Italy was a classical historical study in the Italian tradition of integrating the history of mathematics in didactics. In this particular case, the authors were tracing the phylogenesis of the development of notions within equations and equalities, and how analogies from one domain were not always transferable to the other with historical evidence to support the difficulties encountered in the classroom by students. The second paper from Italy was a longitudinal case study with a collaborator from Canada, and involved a micro-genetic analysis of the classroom milieu over a 5 year period to isolate how semiotic systems mediated student's understandings of patterns and the formulation of algebraic generalizations. This paper made use of the extensive work of Radford on semiotic systems. The paper from Germany utilized frameworks from the Anglo-American literature on early algebraic thinking. Three of the five papers included historical analysis whereas two of the papers were qualitative studies. The theoretical frameworks used by the papers from Italy and France tended to be steeped in their particular didactic traditions. The analyses of epistemological obstacles in reasoning in several papers, understandably, indicated some overlap with papers in WG4 on reasoning patterns in proof.

Working Group 7: Geometric Thinking

Working Group 7 Geometric Thinking	Primary theoretical Framework(s)	Data sources/ Type of Study	Motivating research questions
UK Cross national comparison of textbooks and of geometry lessons designed by expert teachers	Situated within the genre of TIMMS related studies	Teacher's lesson plans and textbooks from China and Japan Valverde et al's framework for cross-national analysis of textbook lessons	What are the similarities and differences of Chinese and Japanese expert teachers' lesson suggestions in the context of enhancing geometric reasoning?
France Geometric thinking of pre-service school teachers	Duval's geometry from a cognitive viewpoint Fischbein's Intuition and Science Houdement & Kuzniak (various studies) Hershkowitz and Vinner Steinbring's classification of epistemological knowledge	Student written artifacts on specifically chosen items Qualitative analysis of student's argumentation within various paradigms	What is retained in the long-term from geometric concepts learned by pre-service primary teachers?
Italy Student's naive strategies in a measurement module.	Fischbein's theory of figural concepts Van Hiele's theory Tall and Vinner; concept image versus concept definition Sfard's duality in mathematics conceptions Vergnaud's theory of conceptual fields	Teaching experiment Video taped Classroom observations Statistical analysis of task based instrument administered to students Analysis of student's written reflections on problem difficulty Qualitative/Quantitative	What are the implications of student's naïve problem strategies in the context of measure for geometry teaching?
Greece Geometry and spatial reasoning	Battista & Clemens Students' understanding of 3-d properties (rectangular arrays in cubes) Van Hiele's theory John Pegg's extension of descriptors within Van Hiele levels	N=20 6th grade children Clinical Interviews Coding by operationalizing theories Qualitative	How do student's understanding of geometric properties change when exposed to dynamic transformations
Italy Cognitive and epistemological obstacles in student understanding of geometric concepts	Vergnaud's theory of conceptual fields Bonotto: Use of cultural artifacts to research mathematical thinking	Worksheet analysis Clinical Interviews Qualitative/Quantitative	What are children's pre-conceptions and spontaneous procedures on perimeter-area concepts? What is the nature of this epistemological obstacle on student learning?

WG7 Geometric Thinking	Primary Theoretical Frameworks			
UK	Studies within the TIMMS genre			
France	Duval's (1998) theory of geometry from a cognitive viewpoint	Fischbein's (1987) Intution and Science	Steinbring's (1998) classification of epistemological knowledge	Houdement, C. and Kuzniak, A. (2003)
Italy	Sfard's(1991) duality theory	Fischbein's (1993) theory of figural concepts	Vergnaud's (1990) theory of conceptual fields	Tall & Vinner (1991) Concept image vs Concept definition Van Hiele (1986) Structure and Insight
Greece	Battista & Clemens (1996) Student's understanding of 3-D objects Clements & Battista Student's spatial reasoning	Van Hiele (1986)	John Pegg's (1997) descriptors of Van Hiele	
Italy	Bonotto (1999) Cultural artifacts in research	Tirosh & Stavy (1999) Intuitive rules: A schema for predicting and explaining student's reasoning	Vergnaud (1990)	

Discussion of WG 7

As the preceding tables indicate, the papers in this working group were also very heterogeneous in terms of their focus and frameworks employed. In our analysis there were some common threads which are important from the view of theory development. For instance two of the studies attempted to inform the van Hiele theory of geometric concept development in the context of students' understanding of elementary measures and their exposure to invariant geometric properties via the use of dynamic software. The van Hiele model (1986) of geometric thought emerged from the doctoral works of Dina van Hiele-Geldof and Pierre van Hiele in the Netherlands. The model consists of five levels of understanding, labelled as visualization, induction, induction with informal deduction, formal deduction and finally proof. These labels describe the characteristics of the thinking at each stage. The first level is characterized by student's recognizing figures by their global appearance or seeing geometric figures as a visual whole. Students at the second level (analysis) are able to list properties of geometric figures; the properties of the geometric figures become a vehicle for identification and description. The third level students begin to relate and integrate the properties into necessary and sufficient sets for geometric shapes. In the fourth level students develop sequences of statements to deduce one statement from another. Formal deductive proof appears for the first time at this level. In the fifth level students are able to analyze and compare different deductive systems. The van Hiele levels of geometric thinking are sequential and discrete rather than continuous, and the structure of geometric knowledge is unique for each level and a function of age. Van Hiele believed that instruction plays the biggest role in students moving from one level of geometric thinking to the next higher level. He also claimed that without instruction, students may remain indefinitely at a given level. The advent of dynamic geometry software presents several challenges for van Hiele's theory especially in the first three levels. As previously discussed Moreno & Sriraman (2005) proposed the notion of „situated proofs" which can serve as a transitory stage between van Hiele levels 2-3 and 3-4 because of the systematic exploration possible within a computational environment. Moreno & Sriraman (2005) contended that teachers can engineer situations in which students exploit the tools provided by the computing environment to explore mathematical relationships and to "prove" theorems (in the sense of situated proofs). This however presents the possibility of a new epistemology emerging from the lodging of these computational tools into the heart of today's mathematics, particularly what it means to "prove" a theorem in geometry. Several of the papers in this working group present the possibility of developing the van Hiele theory for computational contexts. The paper from France examined the rigor and the types of argumentation present in pre-service teachers understanding of geometry on questions administered after a period of prior exposure to these ideas in geometry. Duval's theory of cognitive development, within the specific context of geometry, used as a theoretical framework in this paper can easily be adapted to examine the findings of the other papers in this group. Another theme present in some of the papers was the nature of students' intuitions and informal understanding of geometry. In particular these papers bring to attention what constitutes evidence in geometry with considerable overlap with papers from the working group on argumentation and proof. Consolidating the epistemological value of empirical evidence presents the possibility for merging the work of several papers in this working group.

Working Group 13: Applications and Modelling

WG 13 Applications and modelling	Primary Theoretical Frameworks	Data sources/ Type of Study	Motivating research questions
Paper 1 (Germany) Independence-preserving teacher intervention in lessons with demanding modelling tasks	Reusser (thinking structures) Weinert (teaching culture)	Lab sessions and regular lessons Complex modelling task	Analysis of teacher interventions within a complex modelling task, analysis of the form of the interventions and influence on the model developed
Paper 2 (France) Experiment with students in economics	Brousseau and Margolinas (contract didactique/milieu)	Teaching experiment in university course, analysis of the students' approaches	Analysis of modelling approaches of university students in economics, possibility to promote modelling competencies
Paper 3 (Spain) Mathematical Praxeologies of Increasing Complexity	Chevallard (Anthropological theory of didactics, mathematical praxeologies	Curriculum and textbook analysis in Spanish secondary education, development of an own study	Curriculum and textbook analysis and development of own curricular approach on the basis of theoretical framework
Paper 4 (UK) Getting to grips with real world contexts	Novice-expert behaviours (Lesgold et al.)	Evaluation of modelling course with engineering students, modelling test and analysis of students' responses in this test	Analysis of the development of the students' modelling competencies via test
Paper 5 (Germany) Levels of modelling competence	PISA-consortium Niss (modelling competencies)	Development of a theoretical framework and analysis of PISA-items	Development of a competence-oriented approach of modelling
Paper 6 (Sweden) Applied or pure mathematics	Intuition (Fischbein) Hardy versus John von Neumann (applied mathematics versus pure mathematics)	Analysis of students' responses to modelling problems, mainly concerning students' problem solving techniques	Analysis of reactions of prospective teachers on different kind of word problems
Paper 7 (Netherlands) Assessment of mathematics in a laboratory-like environment	Freudenthal De Lange Treffers on curriculum Realistic mathematics education	Alternative assessment methods on nation-wide scale, usage of hands-on-tasks in lab-like environment	Analysis of the validity of alternative testing method

Discussion of WG 13

The discussions of the working group at CERME 4 are strongly influenced by different approaches towards applications and modelling, presented within the various papers. These discussions demonstrate that there does not exist a homogeneous understanding of modelling and its epistemological backgrounds within the international discussion on applications and modelling.

However, this is not a new situation at all. Nearly twenty years ago, Kaiser-Messmer (1986) showed in her analyses that within the applications and modelling discussion of that time various perspectives could be distinguished, internationally, namely a more pragmatic perspective focussing on utilitarian goals, such as the ability of learners to apply mathematics to solve practical problems. On the other hand there existed a scientific-humanistic perspective which was oriented towards mathematics as a science and humanistic ideals of education with focus on the ability of learners to create relations between mathematics and reality.
Meanwhile, the current discussion on applications and modelling has developed further and become more differentiated. New perspectives can be identified which, as it became obvious from detailed analyses, emerged from the above described traditions or partly can be regarded as their continuations.

When analysing the papers discussed during the sessions of the working group applications and modelling at CERME 4, one finds out that the apparent uniform terminology and its usage masks a great variety of approaches.
As first attempt the following approaches can be discriminated:

- Epistemological modelling
- Realistic or applied modelling
- Educational modelling
- Cognitive modelling
- Contextual modelling

If we examine in more detail these various attempts it is remarkable that now, after a longer period of time, attempts from Roman language speaking countries were brought into the discussion on applications and modelling which start out from a more theory-related background. Partly they refer to the anthropological theory of didactics and to the approach of mathematical praxeologies of Chevallard emerging from anthropological theory, or they refer to approaches like that of Brousseau concerning “contract didactique”. These approaches give less importance to the reality aspect in the examples they deal with. Both, extra-mathematical and mathematical topics may be dealt with, while the latter is then described as "intra-mathematical modelling". If the approach of praxeology becomes the main orientation, this leads to the fact that every mathematical activity is identified as modelling activity for which modelling is not limited to mathematising of non-mathematics issues. As a consequence these approaches show a strong connection to the scientific-humanistic perspective mainly shaped by the early Freudenthal. In his earlier work, Freudenthal (1973) understands mathematisation as local structuring of mathematical and non-mathematical fields by means of mathematical tools for which the direction from reality to mathematics is highly important. Freudenthal distinguishes local and global mathematisation, and for global mathematisation the process of mathematising is regarded as part of the development of mathematical theory. These approaches continue with a distinction developed by Treffers (1987): horizontal mathematising, meaning the way from reality to mathematics, and the vertical mathematising, meaning working inside mathematics. Freudenthal (like his successors) consistently uses the term mathematising. According to Freudenthal mathematical models are only found at the lowest level of mathematising when a mathematical model is constructed for an extra-mathematical situation.

Likewise, analyses show that the approaches from the pragmatic perspective were sharpened further until they became the approach of realistic modelling. For these kinds of approaches, authentic examples from industry and science play an important role. Modelling processes are carried out as a whole and not as partial processes, like applied mathematicians would do in practice. In summary, it can be stated that a characteristic of these approaches is one in which modelling is understood as activity to solve authentic problems and not the development of mathematical theory. The described empirical studies even point out that newly learned knowledge cannot be applied directly in modelling processes, only with some delay. This fact has already been pointed out in earlier reports based on anecdotal knowledge (e.g. Burghes & Huntley 1982). In general, the presented empirical studies aimed at fostering modelling competencies demonstrate well underlying complexities which makes it difficult to achieve progress.

Besides these quasi polarising approaches, the realistic modelling and the epistemological modelling, there exists a continuation of integrative approaches within the educational modelling which puts the structuring of learning processes and fostering the understanding of concepts into the centre. However, the approach of educational modelling may also be interpreted as continuation of the scientific-humanistic approaches in its version formulated by Freudenthal in his late years and the continuation done by Treffers (1987) or respectively by De Lange (1987) for whom real-world examples and their interrelations with mathematics become a central element for the structuring of teaching and learning mathematics .

Within the discussion on applications and modelling, one approach is new, the approach of cognitive modelling, which examines modelling processes under a cognitive perspective. Of course, the analysis of thinking processes by means of the approach of modelling is not new and is found in many theories of learning or cognitive psychology (see for example Skemp 1987). However, the analysis of modelling processes with a cognitive focus must be regarded as a new perspective as only recently a few studies were carried out.

The approach of solving word problems named contextual modelling, has a long tradition, but with the model eliciting perspective an explicitly theory based perspective has been established which is clearly going far beyond problem solving at school. We will not describe this approach in more detail, because it has been developed especially in the American realm.

To summarise, these analyses demonstrate on the one hand that currently significant further developments are taking place within the discussion on applications and modelling, while on the other hand it became clear that these new approaches still go along with existing traditions and that they have developed further earlier approaches or fall back on them. However, the frequent usage of concepts from the modelling discussion should not be mistaken about the fact that the underlying assumptions and positions of the various modelling approaches differ widely. A precise clarification of concepts is necessary in order to sharpen the discussion and to contribute for a better mutual understanding. Thus, this suggestion for a description of the current discussion on applications and modelling is meant to be a first step into this direction.

Concluding Notes

Our transcripts of the audio-taped end reports constructed by the fourteen working groups at the conclusion of CERME4 indicated that many of these working groups encountered difficulties in understanding the "domains of inquiry" of the others. This motivates us to construct a deeper examination of the research papers in seven out of the fourteen working groups to examine whether research methods and conceptual frameworks contributed to potential misunderstandings. Based on our present analysis, we hypothesize that these miscommunications and misconceptions were in some cases a function of language[4] but in other instances could be clearly attributed to the unfamiliarity with motivating questions, the research designs and theoretical frameworks employed in mathematics education research by researchers in other countries. For instance while researchers trained in the French school of thought[5] speak with a specific grammar which uses terms such as *institutions, praxeologies, mileus, didactical contracts* and *anthropological theories*, researchers from some other countries in Europe speak in terms of a grammar that contains terms such as *operationalized variables, research design, instruments, reliability, validity, quantitative/qualitative design, instrumentation, data analysis* etc. The latter consists of terminology evolving from the shifts within theories in psychology (from associationism/behaviorism onto cognitive science) grounded in empirical methods, whereas the former is more aligned to terminology used in socio-cultural theories with words whose meanings are not satisfactorily transferable from French to English. This suggests that it is essential for the community to try and define terms understandable by the others. That is, develop the framework of a grammar which will allow for a wider dissemination of research findings from one school of thought to another. In the preceding sections of this article, we have presented a brief overview of theory usage within Europe based on the last CERME4 relevant for this discussion. The question confronting us as this stage is whether there any patterns in the types of theoretical frameworks are used by researchers in various European countries? Are researchers entrenched in perspectives which evolved out of their countries or are they using frameworks from else where? Our summaries and analysis of research reports from seven of the fourteen working groups at CERME4 indicate that in domains such as embodied cognition; affect and beliefs, argumentation and proof, there was some uniformity in the use of theoretical frameworks and the types of research questions addressed. The theoretical frameworks of these groups were international in character. However in the domains of mathematical structures, algebraic thinking, geometric thinking and modelling there was a much wider variation in the types of theoretical frameworks used, the focus of the research and motivating questions, as well as tendencies of researchers from various European countries to use "home grown" frameworks. For instance, in the working group on proof and argumentation, authors that were most frequently cited in theoretical frameworks (in alphabetical order) were Balacheff, Coe& Ruthven, Fischbein, Healy & Hoyles, Polya and Toulmin. The papers from France were grounded in theoretical frameworks developed by French researchers. However, the papers from Germany and the UK also cited many French sources in their reports which suggested signs of a healthy cross-fertilization of ideas. Some of the papers in this working group examined the misalignment of textbook proof content with curricular goals. Another paper examined the differences in validation schemes employed by textbooks in Germany and France and proposed a framework via which the two could be compared. The papers in the working group on algebraic thinking also showed tendencies to use "home grown" frameworks. The papers from France

[4] Some of the working groups at CERME4 jokingly remarked that communication was only possible in a new language called BBEFML, which stands for BBE (Basic Broken English) and FML (Francais mais lentement).

[5] This school of thought dominates theoretical frameworks used by researchers in France, Spain and South America as noted in the recent announcement of a Congress on the Anthropological Theory of Didactics in Baez, Spain.

predominantly used the frameworks of Brousseau, Chevellard and Bachelard. The papers from Italy were classical historical studies in the Italian tradition of integrating the history of mathematics in didactics. In this particular case, the authors were tracing the phylogenesis of the development of notions within equations and equalities, and how analogies from one domain were not always transferable to the other with historical evidence to support the difficulties encountered in the classroom by students. The paper from Germany utilized frameworks from the international literature on early algebraic thinking. In the working group on modelling similar tendencies were present which were discussed in preceding section. Based on our overall analysis we have arrived at the following conclusions for patterns within research reports from Germany, France and Italy. More often than not, the theoretical frameworks used by the papers from France tended to be steeped in their particular didactic traditions (Bachelard, Brousseau, Chevellard, Duval), the Italian papers tended towards historical studies of mathematical notions with didactic recommendations based on the relationship between phylogeny and ontogeny. Although the German papers were the most heterogeneous and at first did not reveal tendencies to favour home-grown frameworks, a closer inspection revealed that a subset of these papers did include substantial citations of research conducted by focus groups within Germany (see Sriraman & Törner, forthcoming). These tendencies are not necessarily damaging to our field but do pose the community of researchers the challenge of finding meaningful modes of interaction. As Lerman (2006, this issue) has pointed out, the problem of plurality can be viewed positively provided it does not hinder communication between researchers from different traditions. Bikner Ahsbahs & Prediger (2006, this issue) present the notion of "networking" based on their work in the working group on theories at CERME4, as a means of facilitating communication between researchers entrenched within their particular traditions. Lester's (2005) ideas that we declare our underlying inquiry systems so that other researchers reading papers become aware of the types of evidence used to put forth claims and theories is another promising way of comparing future research. The present paper is a valiant attempt to present an overall picture of the research in various domains of inquiry and possible avenues of theory development for the European community to expand on this initiative and continue this line of inquiry.

References

Alro., H. & Skovsmose, O.: 2002, *Dialogue and learning in Mathematics education*, Kluwer Academics Publishers, Dordrecht-Boston-London.

Arcavi, A. (2003): The role of visual representations in the learning of mathematics. Educational Studies in Mathematics, 52, 215–241.

Arcavi, A.:1994, 'Symbol Sense: Informal Sense Making in Formal Mathematics', *For the learning of Mathematics*, vol 14(3), 24-35.

Australian Education Council (1990). *A national statement on mathematics for Australian schools*. Melbourne, VC: Australian Educational Council.

Auslander, J. (2001). Review of G. Lakoff & R. Núñez, Where Mathematics Comes From. *The American Scientist*.

Bachelard, G. (1949). Le rationalisme appliqué, Collection PUF, Paris, 1986.

Balacheff, N. (1988). Aspects of Proof in Pupils' Practice of School Mathematics. In: D. Pimm(ed.), Mathematics, Teachers and Children.. London, Hodder & Stoughton: 216-238.

Balacheff N (1991). The benefits and limits of social interaction : the case of mathematical proof In A. J. Bishop et al (Eds). Mathematical knowledge: its growth through teaching, pp.175-192. Kluwer Academic Publishers.

Balacheff, N. (1991b). Construction et anlyse d'une situation didactique. Le cas de 'la somme des angles d'un triangle'. Journal für Mathematik-Didaktik 12(2/3): 199-264.

Battista, M. T. and Clements, D. H.: 1996, 'Students' understanding of three-dimensional rectangular arrays of cubes', *Journal for Research in Mathematics Education* 27 (3), 258-292.

Bell, A. (1993). Principles for the design of teaching. *Educational Studies in Mathematics*, 24, 5-34.

Bell- Gredler, M. E. (1986). *Learning and instruction* (pp.191-224).New York, NY: Macmillan.

Beth, E. W and Piaget, J. (1966). *Mathematical Epistemology and Psychology*. D. Reidel, Dordrecht, The Netherlands.

Birkhoff, G. (1969). Mathematics and psychology. *SIAM Review, 11*, 429-469.

Bikner-Ahsbahs, A. (2003). A social extension of a psychological interest theory. In: *Proceedings PME 27*. http://onlinedb.terc.edu/PME2003/PDF/RR_bikner-ashbahs.pdf.

Bonotto, C. (1999). About the use of cultural artifacts in the teaching/learning of mathematics', *L'Educazione matematica*, Anno XX, Vol. 1, (2), 62-95.

Bolhuis, S. & Voeten, J.M. (2004). Teachers' Conceptions of Student Learning and Own Learning', *Teachers and Teaching: theory and practice*, 10 (1), 77-98.

Boole, G. (1854). *An Investigation of The Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*. Macmillan, London.

Booth, L. (1988). Children's difficulties in beginning algebra', in A.F. Coxford and A.P. Schulte (eds.), The ideas of algebra, K-12. Yearbook. National Council of Teachers of Mathematics, Reston, VA.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes in mathématiques, Recherches en Didactique des Mathématiques, 4, 2, 165-198.

Brousseau, G. (1989) Les obstacles épistémologiques et la didactique des mathématiques, Bednarz, N. & Garnier, C. (Eds.), Constructions des savoirs, obstacles et conflits, Agence d'Arc, Montreal, 41-64.

Brousseau G. (1998). Théorie des situations didactique, Paris : La pensée sauvage.

Burton,L. (2004). *Mathematicians as Enquirers*. Kluwer Academic Publishers.

Blum, W. (1998). On the Role of 'Grundvorstellungen' for reality-related Proofs – Examples and Reflections', in P. Galbraith, W. Blum, G. Booker and I. D. Huntley (eds.), *Mathematical Modelling – Teaching and Assessment in a Technology-Rich World*, Horwood, Chichester, pp. 63 – 74.

Brinkmann, A. (1999) 'Mathematical nets in curricular frames', in: G. Philippou (ed.),*MAVI-8 Proceedings, Eighth European Workshop in Cyprus 1999, Research on Mathematical Beliefs*, University of Cyprus, Nicosia, pp. 18-27.

Brinkmann, A. (2001). Mind Mapping – Eine Methode zur

Förderung der Kreativität und Lerneffektivität im Mathematikunterricht, *Lernwelten* 2/2001, Pädagogischer Zeitschriftenverlag, Berlin, pp. 101-104.

Brinkmann, A. (2003a). Mind Mapping as a Tool in Mathematics Education', *Mathematics Teacher,* 96 (2) ,96-101.

Brinkmann, A. (2003b). Graphical Knowledge Display – Mind Mapping and Concept Mapping as Efficient Tools in Mathematics Education', in: P. Perks and S. Prestage (eds.), *Mathematics Education Review, The Journal of Association of Mathematics Education Teachers,* Number 16, pp. 39-48.

Bruner, J. (1986). *Actual Minds, Possible Worlds*. Cambridge: Harvard University Press.

Bruner J. (1990). *Acts of Meaning.* Harvard University Press, Cambridge.

Carpenter, T. P., Levi, L. (2000), Developing conceptions of algebraic reasoning in the primary grades. University of Wisconsin-Madison, Report No. 2.

Chevallard Y. (1991). La transposición Didáctica, Paris : La pensée sauvage.

Chevallard, Y. (1995). Familière et Problématique, la figure du professeur, VIIIe Ecole d'été de didactique des mathématiques, Saint - Sauves.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique dudidactique, Recherches en didactique des mathématiques, 19 (2), 221-266.

Ciompi, L. (1982). *Affektlogik*, Stuttgart: Klett - Cotta.

Ciompi, L. 1988. *Innenwelt - Aussenwelt*, Göttingen: Vandenhoeck & Ruprecht.

Ciompi, L. (1991). Affects as Central Organising and Integrating Factors. A New Psychological/Biological Model of the Psyche. *British Journal of Psychiatry* 97-105.

Ciompi, L. 1999. *Die emotionalen Grundlagen des Denkens*. Göttingen: Vandenhoeck& Ruprecht.

Clements, D. H. and Battista, M. T.: 1992, 'Geometry and Spatial Reasoning', in D. A. Grouws (Ed.), *Handbook on Research in Mathematics Teaching and Learning*, New York: Macmillan, pp. 420-464.

Cobb, P. & Hodge, L. L. (press) An Interpretive Scheme for Analyzing the identities that Students Develop in Mathematics Classrooms, *Journal of the Learning Sciences*.

Coe, R. & K. Ruthven (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal* 2(1): 41-53.

Cohors-Fresenborg, E.(1987).Action and Language – Two Modes of Representing Mathematical Concepts. In I. Wirszup and R. Streit (eds.), *Developments in School Mathematics Education around the World, Proceedings of the UCSMP International Conference on Mathematics Education*, NCTM, Reston, pp. 250-275.

Cohors-Fresenborg, E. and Kaune, C. (2001) Mechanisms of the Taking Effect of Metacognition in Understanding Processes in Mathematics Teaching', in *Developments in Mathematics Education in German-speaking Countries, Selected Papers from the Annual Conference on Didatics of Mathematics, Ludwigsburg 2001,* pp. 29-38.

Dahl, B.(2004). Analysing Cognitive Learning Processes Through Group Interviews of Successful High School Pupils: Development and Use of a Model, Educational Studies in Mathematics 56, pp.129-155.

Davis, R.B. and McKnight, C. (1979) Modeling the processes of mathematical thinking', *Journal of Children's Mathematical Behaviour* 2, 91-113.

Davis, P.J. & Hersh, R. (1981). *The mathematical experience*. New York: Houghton Mifflin.

DeBellis, V.A. & Goldin, G.A. 1997. The Affective Domain in Mathematical Problem Solving. In: Pekhonen, E. (Ed.) *Proceedings of the PME 21.* Vol. 2 (pp.209-216)

DeBellis V. & Goldin G.A. (1999). Aspects of affect: mathematical intimacy, mathematicalintegrity. *Proceedings of PME 25* (Haifa, Israel), vol.2, 249-256.

Dewey, J. (1903). The psychological and the logical in teaching geometry. *Educational Review, August 1903, 25,* 387-399.

Dewey, J. (1968). *Philosophy and Civilization*. Gloucester, MA: Peter Smith .

Di Martino P., Zan R. (2001a) *'The problematic relationship between beliefs and attitudes'*, in Soro R. (Ed.) Proceedings of the Mavi X, Kristianstad, p. 17 – 24.

Di Martino P., Zan R. (2001b): *'Attitude toward mathematics: some theoretical issues'*, in van den Heuvel M. (Ed.) Proceedings of the 25th PME, vol.3, Utrecht p. 351 – 358.

Di Martino P. & Zan R. (2002). An attempt to describe a 'negative' attitude toward mathematics. In P. Di Martino (Ed.) *Proceedings of the MAVI-XI European Workshop*, 22-29.

Di Martino P., Zan R. (2003) *'What does 'positive' attitude really mean?'*, in Pateman N., Dougherty B. & Zilliox J. (Eds.) Proceedings of the 2003 Joint Meeting of PME and PME-NA, vol.4, Honolulu, p. 451 – 458.

Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In D. Tall (Ed.), Advanced Mathematical Thinking (pp. 95–123). Kluwer: Dordrecht.

Dreyfus, T. (1994): Imagery and reasoning in mathematics and mathematics education. In D. Robitaille, D. Wheeler, & C. Kieran (Eds). *Selected lectures from the 7th ICME*. Université laval, Quebec.

Duval, R.: (2002). The cognitive analysis of problems of comprehension in the learning of mathematics', *Mediterranean Journal for Research in Mathematics Education* 1(2), 1-16.

Duval R. (1998).Geometry from a cognitive point of view' *Perspectives in the teaching of Geometry for the 21st Century Icmi Study* Kluwer pp 37-51.

Duval, R. (1995). Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels, Peter Lang, Paris.

Ernest, P. (1988). The Attitudes and Practices of Student Teachers of Primary School Mathematics', *Proceedings of 12th International Conference on the Psychology of Mathematics Education, Hungary,* 1988, Vol. 1, A. Borbas (ed).

Ernest, P. (1989). The Impact of Beliefs on The Teaching of Mathematics', in P. Ernest (ed), *Mathematics Teaching: The State of the Art,* London, Falmer Press.

Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.). *Advanced Mathematical Thinking. (pp. 42-53),*The Netherlands, Kluwer Academic Publishers.

Fischbein, E. (1980, August). Intuition and proof. Paper presented at the 4th conference of the International Group for the Psychology of Mathematics Education, Berkeley, CA.

Fishbein, E. (1982). Intuition and Proof. For the Learning of Mathematics 3(2): 9-24.

Fischbein E. (1987). *Intuition in Science and Mathematics*. *An educational Approach*. D. Reidel Publishing Company. Dordrecht.

Fischbein, E. (1989). Tacit Models and Mathematical Reasoning. *For the Learning of Mathematics,* 9, 9 – 14.

Fischbein, E. (1993): The theory of figural concepts. Educational Studies in Mathematics, 24, 139–162.

Freudenthal, H. (1983) *Didactical Phenomenology of Mathematical Structures*, Reidel, Dordrecht.

Furinghetti F., Pehkonen E. (2002): *'Rethinking characterizations of beliefs'*, in Leder G., Pehkonen E., Toerner G. (Eds.) Beliefs: a hidden variable in mathematics education?, Kluwer Academic Publishers, p. 39 – 58.

Gold, B. (2001). Book Review of G. Lakoff & R. Núñez,

Where does Mathematics Come From. *MAA Book Reviews Online.*
Goldin, G. (2001). Counting on the metaphorical. Book Review of G. Lakoff & R. Núñez, Where does Mathematics Come From. *Nature* , 413,18 – 19.
Goldin, G. and Shteingold, N.: 2001, ‚Systems of Representations and the Development of Mathematical Concepts', in Cuoco, A.. A. and Curcio, F. R. (ed.), *The Roles of Representation in School Mathematics*, NCTM Yearbook. Boston, Vir-ginia, 1-23.
Goldin, G.A. (1998). Representational systems, learning and Problem Solving. *Journal of Mathematical Behavior, 17* (2), 137-165.
Goldin, G.A. 2000. Affective Pathways and Representations in Mathematical Problem Solving. *Mathematical Thinking and Learning*, 17(2), (pp.209-219).
Goldin, G.A. (2002). Affect, Meta - Affect, and Mathematical Belief Structures. In: Gilah C. Leder/ Erkki Pehkonen/ Günter Törner 2002: *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht/ Boston/ London. Kluwer Academic Publishers. 59 – 72.
Goldin, G. A. 2004. Characteristics of Affect as a System of Representation. In: M.Johnsen Hoines & A. B. Fuglestad (Eds.): *Proceedings of the PME 28*, Vol.*1* (pp.109 – 114) Bergen: Bergen University College.
Gómez-Chacón, I. M.: 2000, Affective influences in the knowledge of mathematics, *Educational Studies in Mathematics,43, 149-168.*
Gray, E., Pinto, M., Pitta, D. & Tall, D. (1999). Knowledge Construction and Diverging Thinking in Elementary & Advance Mathematics', *Educational Studies in Mathematics 38*, pp.11-133.
Gray, E., Pitta, D. & Tall, D.: 2000, 'Objects, Actions and Images: A Perspective on Early Number Development', *Journal of Mathematical Behavior*, 18, 4, pp.1– 13
Hall, N.: 1991, 'A procedural analogy theory: The role of concrete embodiments in teaching mathematics', in F. Furinghetti (ed.), *Proceedings of PME 15*, Vol. II, Assisi, Italy, pp. 117-124.
Hall, N.: 1998, 'Concrete repesentations and procedural analogy theory', *Journal of Mathematical Behavior* 17, 33-51.
Hadamard, J.W. (1945). *Essay on the psychology of invention in the mathematical field*. Princeton University Press.
Hanna, G. (1995). Challenges to the Importance of Proof. *For the Learning of Mathematics*, 15(3), 42-50.
Hanna, G. & H. N. Jahnke (1993). Proof and Application. Educational Studies in Mathematics 24(4): 421 - 438.
Hanna, G. & H. N. Jahnke (1996). Proof and proving. In: A. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (eds.), International handbook of mathematics education. Dordrecht, Kluwer: 877 - 908.
Hanna, G. and H. N. Jahnke (2002). Arguments from physics in mathematical proofs: An educational perspective. For the learning of mathematics 22(3): 38-45.
Hannula, M. (1998). Changes of Beliefs and Attitudes. In: Pekhonen, E. & Törner, G. (Eds.) The State-of-Art in Mathematics- Related Belief Research: Results of the MAVI Activities. *Research Report 184*,(pp. 198 – 222) University of Helsinki,.
Hannula, M./Evans, J./Philippou, G./Zan, R. (2004). RF01: Affect in Mathematics Education – Exploring Theoretical Frameworks. In M. Johnsen Hoines & A. B. Fuglestad (Eds.): *Proceedings of the PME 28*, Vol.1(pp. 107 – 136) Bergen: Bergen University College.
Hannula M. (2002). Attitude toward mathematics: emotions, expectations and values. *Educational Studies in Mathematics*, 49, 25-46.
Healy, L. & Hoyles, C. (1998). Justifying and Proving in School Mathematics. (Technical Report on The Nationwide Survey. Institute of Education, University of London, February, 1998.
Healy, L. & C. Hoyles (to appear). Curriculum Change and Geometrical Reasoning. In: P. Boero (ed.), Theorems in School. Dordrecht, Kluwer
Hejný, M. (1999): Procept. In: *Zborník bratislavského seminára z teórie vyuèovania matematiky*, Comenius University, Bratislava, pp. 40-61.
Hejný, M. (2002): Creating Mathematical Structure. In: Novotná, J. (ed.): *European Research in Mathematics Education II*, Charles University Prague 2002, pp. 14-24.
Hejný, M.(2003). Understanding and Structure. In *CERME3 (Conference on European Research in Mathematics Education 3, Group 11)*, Feb. 28 – March 3, Bellaria, Italy, http://www.dm.unipi.it/~didattica/CERME3.
Hiebert, L. (ed. 1986): Conceptual and procedural knowledge: the case for mathematics, Lawrence Erlbaum, Hillsdale NJ.
Houdement C. & Kuzniak A (1998). Trois paradigmes mathématiques, in Petit x 51.
Houdement, C. and Kuzniak, A. (2003). Elementary Geometry Split into Different Geometrical Paradigms' *Proceedings of CERME 3*. Belaria, Italy.
Kaput, J. and J. Sims-Knight (1983). Errors In Translations To Algebraic Equations: Roots and Implications. Focus on Learning Problems in Mathematics 5: 63-78.
Kaput, J. J.: 1987, 'Representation systems and mathematics', in C. Janvier (ed.), Problems of Representation in the Teaching and Learning of Mathematics, Lawrence Erlbaum, Hillsdale, NJ, pp. 19-26.
Kaput, J.J. (1989). Information technologies and affect in mathematical experience. . In D.B. Mcleod & V.M. Adams (Eds) Affect & mathematical problem solving (pp.87-103). Springer-Verlag.
Kieran, C..(1990). Cognitive Processes involved in Learning School Algebra', in Nesher P. & Kilpatrik, J. (Ed.):*Mathematics and Cognition*, ICMI Study Series, Cambridge University Press, 96-112.
Kieran, C.: 1992, 'The Learning and Theaching of School Algebra', in Grouws D.A., (Ed.) *Handbook of Research of Mathematics Teaching and Learning*, Macmillan, NY, 390-419.
Lakoff, G. (1980) *Metaphors we live by*. University of Chicago Press, Chicago.
Lakoff .G. & Núñez, R. (2000). *Where Does Mathematics Come From*. New York: Basic Books.
Lakoff .G. & Núñez, R. Reply to B. Gold's review of Where Mathematics Comes From. *MAA Book Reviews Onlin.*
Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. American Educational Research Journal, 27, 29-36.
Lave, J. and Wenger, E. (1991). Situated Learning: Legitimate Peripheral Participation, Cambridge University Press, Cambridge.
Lester, F. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *Zentralblatt für Didaktik der Mathematik*, vol.37, no.6, pp. 457-467.
Lester, F., Garofalo,J., & Kroll, D.L. (1989). Self-confidence, interest, beliefs and metacognition. In D.B. Mcleod & V.M. Adams (Eds) Affect & mathematical problem solving, (pp. 75-88). Springer- Verlag.
Linde, C. (1993) *The Creation of Coherence*. Oxford University Press. New York.
Malara, N.A. (2004). From teachers education to classroom culture, *regular lecture at ICME 10*, (to appear)
Malara N.A., Navarra G. (2001). "Brioshi" and other mediation tools employed in a teaching of arithmetic with

the aim of approaching algebra as a language, in Chick, E. & al. (eds), *proc. 12th ICMI Study on Algebra*, vol. 2, 412-419

Malara, N.A., Navarra G. (2003a). *ArAl project: arithmetic pathways towards favouring pre-algebraic thinking*, Pitagora, Bologna

Malara, N.A., Navarra G. (2003b)Influences of a procedural vision of arithmetic in algebra learning, *proc. WG 2 - CERME 3*, (Bellaria, Italy, february 2003)

Mandler, G. (1984). *Mind and Body: Psychology of emotion and stress*. New York: Norton.

Mandler, G. (1989). Affect and learning Causes and consequences of emotional interactions. In D.B. Mcleod & V.M. Adams (Eds) Affect & mathematical problem solving. Springer-Verlag.

Manin, Y. I.(1977). *A course in mathematical logic*. Springer-Verlag, New York

Maslow, A.H.: 1962, *Toward a psychology of being*, Van Nostrand, London, New York.

Mcleod, D. B. (1991). Research on learning and instruction in mathematics: The role of affect. In E. Fennema, T. P. Carpenter., & S. J. Lamon (eds.), *Integrating research on teaching and learning mathematics* (pp. 55- 82). Albany, NY: State Univ. of New York Press.

Mc Leod D. (1992). Research on affect in mathematics education: a reconceptualization. In D.Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp.575-596).McMillan Publishing Company.

Moreno, L & Sriraman, B. (2005). The articulation of symbol and mediation in mathematics education. *Zentralblatt für Didaktik der Mathematik*,, vol.37,no.6,pp. 476-486.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*: Reston, VA: Author.

Nespor, J. (1987). The role of beliefs in the practice of teaching, *Journal of Curriculum Studies*, 19 (4), 317-328.

Novak, J.: 1990, 'Concept Mapping: A Useful Tool for Science Education', *Journal of Research in Science Teaching* 27 (10), pp. 937-949.

Novak, J. (1996). Concept Mapping: A Tool for Improving Science Teaching and Learning', in: D.F. Treagust, R. Duit, B.J. Fraser (eds.), *Improving Teaching and* Learning in Science and Mathematics, Teachers College Press, New York.

Op 't Eynde, P., De Corte, E. & Verschaffel, L. (2002). Framing students' mathematics-related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. Leder, E. Pehkonen & G. Törner (Eds)., *Beliefs: A hidden variable in mathematics education?*, 13–37. Dordrecht: Kluwer.

Pajares F. (1992). Teachers' Beliefs and Educational Research: Cleaning Up a Messy Construct. *Review of Educational Research*, 62 (3), 307-332

Pehkonen, E. & Pietilä, A. 2003. On Relationships between Beliefs and Knowledge in Mathematics Education. In: *Proceedings of the CERME-3 (Bellaria) meeting*. http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft/TG2_draft/

Pegg, J.(1997)Broadening the descriptors of van Hiele's levels 2 and 3, *Proceedings of the 20thannual conference of the Mathematics Education Research Group of Australasia*, Rotorua, MERGA, pp. 391-396.

Pegg, J., & Tall, D. (2005). The fundamental cycle of concept construction underlying various theoretical frameworks – *Zentralblatt für Didaktik der Mathematik*. Vol. 37, No.6, pp. 468-475

Philippou, G., & Christou, C. (1998): 'The effects of a preparatory mathematics program in changing prospective teachers' attitudes toward mathematics, *Educational Studies in Mathematics*, 35, 189 – 206.

Peirce C.S. (1960) *Collected papers*, Cambridge, MA: Harvard University Press.

Piaget, J. (1958). *The Growth of Logical Thinking from Childhood to Adolescence*. Basic Books, New York, NY.

Piaget, J. & Garcia, R. (1989). Psychogenesis and the History of Science, Columbia University Press, New York.

Poincare, H. (1948). *Science and Method*. Dover: New York.

Polya, G. (1954). *Mathematics and plausible reasoning: Induction and analogy in mathematics* (Vol.II).

Prawat, R.S.& Anderson, A.L.H.(1994). 'The Affective Esperiences of Students', *The Journal of Mathematical Behavior*, 13, 201-221.

Presmeg, N. C. (1992). Prototypes, metaphors, metonymies, and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23 (6), 595-610.

Presmeg, N. C. (1997a). Reasoning with metaphors and metonymies in mathematics learning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 267-279). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Presmeg, N.C (2001). Visualization and affect in non-routine problem solving. *Mathematical Thinking and Learning*, 3, 289-313.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics* 42(3): 237-268.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics* 22(2): 14-23.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning* 5(1): 37-70.

Radford, L., Demers, S., Guzmán, J. and Cerulli, M. (2003) Calculators, graphs, gestures, and the production meaning, in: N., Pateman, B. Dougherty and J. Zilliox (eds.), *Proceedings of PME27 –PMENA25*, Vol. 4, pp. 55-62.

Resnick, L. & Omansson, S. (1987). Learning to understand arithmetic', in R. Glaser (ed.), *Advances in Instructional Psychology*, Vol 3, Erlbaum, Hillsdale, NJ.

Ricoeur, P. (1992) *Oneself as Another*. Chicago: The University of Chicago Press.

Rotman, B. (2000). *Mathematics as Sign: Writing, Imagining, Counting*. Stanford: Stanford University Press

Schlöglmann, W. (2000). Affects and mathematics learning - some remarks, In: S. Johnson/ D. Coben (Eds.): *Proceedings of the sixth international conference of Adults Learning Mathematics - A Research Conference*, (pp.51 – 59) University of Nottingham.

Schlöglmann, W. (2002). Affect and mathematics learning, In A. D. Cockburn & E. Nardi (Eds.): *Proceedings of the PME 26*, Vol. 4 (pp. 185 – 192) Norwich University of East Anglia.

Schoenfeld, A. H. (1989). Explorations of students' mathematics beliefs and behavior. *Journal for Research in Mathematics Education*, 20(4), 338-355.

Schubring (1986). Rupture dans le statut mathématique des nombres négatifs. Petit X, n° 12, pp. 5 – 32.

Schubring (1988). Discussions épistémologiques sur le statut des nombres négatifs et leur représentation dans les manuels allemands et français de mathématiques entre 1795 et 1845. *Actes du premier colloque franco-allemand de didactique des mathématiques et de l'informatique*, pp. 137 – 145. Paris: La Pensée Sauvage.

Silver E. (Ed.),*Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Hillsdale, NJ, Erlbaum.

Sierpinska, A. (2004). A note from the editor. *Educational Studies in Mathematics*, 57 (3). p.301

Sfard, A. (1989): Transition from operational to structural conception: The notion of function revisited. In: G. Vergnoud, J. Rogalski, and M. Artigue (Eds.): *Proceedings of the 13th PME*, Paris, pp. 151-158.
Sfard, A. (1991). On the Dual Nature of Mathematics Conception: Reflections on Processes and Objects as Different Sides of the Same Coin, *Educational Studies in Mathematics*, 22, 1-36.
Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44-54.
Skemp, R. (1986). *The Psychology of Learning Mathematics*. 2nd Edition, Penguin Books.
Sriraman,B., & Törner, G. (2007, forthcoming). Political Union/Mathematical Education Disunion: Building Bridges in European Didactic Traditions. To appear in L. English (Ed) *Handbook of International Research in Mathematics Education (2nd edition)*.
Sriraman, B., & English, L. (2005). Theories of mathematics education: a global survey of theoretical frameworks/trends in mathematics education research. *Zentralblatt für Didaktik der Mathematik,* vol.37,no.6, pp.450-456.
Steinbring, H. (1994) Die Verwendung strukturierter Diagramme im Arithmetikunterricht der Grundschule. Zum Unterschied zwischen empirischer und theoretischer Mehrdeutigkeit mathematischer Zeichen', *MUP,* ***IV. Quartal***, 7-19.
Steinbring H. (1998). Elements of Epistemological Knowledge for Mathematics Teachers' *Journal of Mathematics Teacher Education, Vol 1/2* Kluwer pp. 157-189.
Steinbring, H. (2005). ‚The Construction of New Mathematical Knowledge in Classroom Interaction – an Epistemological Perspective, To be published in *Mathematics Education Library (MELI), No. 38*, Springer, Berlin, Heidelberg.
Tall, D. & Vinner, S. (1981). Concept image and concept definition in Mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-69.
Tirosh D., Hadass R., Movshovic-Hadar N. (1991), Overcoming overgeneralizations: the case of commutativity and associativity, *Proeedings of PME 15*, vol.3, 310-315.
Tirosh, D., Stavy, R. (1999). Intuitive Rules: A Way to Explain and Predict Students' Reasoning', *Educational Studies in Mathematics*, 38 51-66.
Toulmin S.E. (1958). *The Uses of Arguments*, Cambridge University Press.
Van Hiele, P.M. (1986). Structure and Insight. A Theory of Mathematics Education. Academic Press, Inc., London.
Vergnaud, G. (1990). La théorie des champ conceptuels', *Recherches en Didactique des mathématiques*, vol. 10, 133 - 170.
Whitehead, A.N., & Russell, B. (1908). *Principia Mathematica*. Cambridge University Press.
Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra', *Mathematics Education Research Journal* (15) 2, 122-137.
Wittgenstein, L. (1978). *Remarks on the Foundations of Mathematics* (Revised Edition), Cambridge, Massachusetts Institute of Technology Press.
Wittmann, E. Ch. (1995). Mathematics Education as a "Design Science", *Educational Studies in Mathematics* 29, 355-374.

Authors

Prof. Dr. Bharath Sriraman
Editor, *The Montana Mathematics Enthusiast*
Dept. of Mathematical Sciences
The University of Montana
Missoula, MT 59812
USA
E-mail: sriramanb@mso.umt.edu

Prof. Dr. Gabriele Kaiser
Editor, *Zentralblatt für Didaktik der Mathematik*,
University of Hamburg
Faculty of Education
Didactics of Mathematics
Von-Melle-Park 8
20146 Hamburg
Germany
E-mail: gabriele.kaiser@uni-hamburg.de