A Framework for Identifying Student Blockages during Transitions in the Modelling Process

Peter Galbraith, The University of Queensland
Gloria Stillman, The University of Melbourne

Abstract: *In this article we present, illustrate, test and refine a framework developed by Galbraith, Stillman, Brown and Edwards (2006) for identifying student blockages whilst undertaking modelling tasks during transitions in the modelling process. The framework was developed with 14-15 year old students who were engaging in their first experiences of modelling at the secondary level.*

ZDM-Classification: C70, M10

1. Introduction

The term *mathematical modelling* as it is used in curricular discussions and implementations does not have a single meaning. One interpretation sees a role for mathematical modelling primarily for the purpose of motivating, developing, and illustrating the relevance of particular mathematical content (e.g., Zbiek & Conner, 2006). For example, *Emergent modelling,* as a conceptual framework and modus operandi (Gravemeijer, 1999), is located essentially within this purpose. We recognise and respect the work of those who adopt this approach in their efforts to improve the teaching and learning of mathematics, and now proceed to clarify the different emphasis that underlies our own work. We do not use applications and modelling primarily as a means for achieving some other mathematical learning end, although at times this may occur as an additional benefit. Rather we retain, as far as possible, an approach in which the modelling process is driven by the desire to obtain a mathematically productive outcome for a problem with genuine real-world motivation (e.g., Galbraith, 2006; Galbraith, Stillman, Brown, & Edwards, 2006; Pollak, 1997; Stillman 2002, 2006). At times this is directly feasible, while at other times some modifications make the descriptor "life-like" more accurate. The point is that the solution to such a problem must take seriously the context outside the mathematics classroom, within which the problem is located, in evaluating its appropriateness and value. This approach is consistent with Julie's (2002) term "modelling as content" as distinct from "modelling as vehicle" which describes the former approach. This view of modelling has characterised the ICTMA series of conferences, and the curricular call to arms on the part of those such as Henry Pollak (e.g., Pollak, 2006). The learning goal is to support the students' development of modelling competence; however, as researchers we are guests in teachers' classrooms and therefore do not always have influence over the approach to modelling that a particular teacher adopts within his/her classroom.

2. Modelling process

Various diagrammatic representations of the modelling process are common in the literature (e.g., Edwards & Hamson, 1996; Galbraith, Goos, Renshaw, & Geiger, 2003; Stillman, 1998), where most of these can be recognised as relatives or descendants of a diagram initially provided among the teaching materials of the Open University (UK). Such diagrams illustrate key stages in an iterative process that commences with a real world problem and ends with the report of a successful solution, or a decision to revisit the model to achieve a better outcome. The purpose is to provide a scaffolding infrastructure to help modellers through stages of what can appear as a challenging and opaque task. It is based on procedures that real world problem solvers undertake. However, as pointed out by Blum and Leiß (2006), when interests in teaching and learning are also central we need a version that is more oriented towards the *problem solving individual*, to give not only a better understanding of what students do when solving (or failing to solve) modelling problems, but also a better basis for teachers' diagnoses and interventions. The diagrammatic representation (Figure 1) contains a structure that encompasses both the task orientation of the original approach, and the need to capture what is going on in the minds of individuals as they work, often idiosyncratically on modelling problems.

The respective entries A-G (expressed as nouns) represent stages in the modelling process, and the heavy arrows signify transitions between the stages. The total problem solving process is described by following these heavy arrows clockwise around the diagram from the top left. It culminates either in the report of a successful modelling outcome, or a further cycle of modelling if evaluation indicates that the solution is unsatisfactory in some way. Some (e.g., Blum & Leib, 2006) see A-B as involving a further stage (the situation model). We do not make this distinction here.

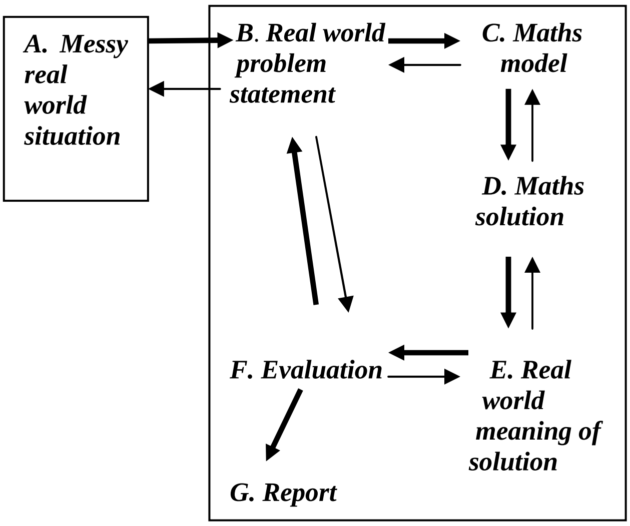


Figure 1. Modelling Process Chart

Now, adding an educational focus, we turn attention to the kinds of mental activity that individuals engage in as they move around the modelling cycle. As the term ‘activity’ suggests, these can be expressed in terms of *verbs* that describe what happens as modellers achieve a successful transition (or not) from one modelling stage to the next, (where there is special interest in identifying blockages that impede progress). At a theoretical level these may be thought of as generic activities as illustrated for the transitions below.

A → B: Understanding, structuring, simplifying, interpreting context
B → C: Assuming, formulating, mathematising
C → D: Working mathematically
D → E: Interpreting mathematical output
E → F: Comparing, critiquing, validating
F→ G: Communicating, justifying, report writing (if model is deemed satisfactory)
OR
F → B: Revisiting the modelling process (if model is deemed unsatisfactory).

These descriptors are illustrative of broad cognitive activity – finding what their detail looks like in particular cases, as a precursor to generating robust generalised structure, is one of the purposes of our research.

The remaining structure included in Figure 1 consists of light arrows that are in the reverse direction to the heavy arrows denoting the direction of the modelling cycle. These light arrows are included to emphasise that the modelling process is far from linear, or unidirectional, and to indicate the presence of metacognitive activity that permeates every part of the process (Stillman, 1998). While, for example, a model will be evaluated in terms of what it provides at stage E, processes of evaluation and attendant action will be undertaken by competent modellers throughout the process. For example in the transition **A → B,** judicious referencing back to the messy context of a problem will assist in the construction of a feasible problem statement; within the transition **B → C** reference back and forth to the real world problem statement is essential in designing a mathematical model that captures its necessary features; within the transition **C → D** checking of mathematical solutions in terms of the type of output expected from the model is important in monitoring the correctness of mathematical processing - and so on for the other transitions.

In our study we aim to learn more about critical aspects within transitions between stages in the modelling process as follows:

1. Messy real world situation → real world problem statement;
2. Real world problem statement → mathematical model;
3. Mathematical model → mathematical solution;
4. Mathematical solution → real world meaning of solution;
5. Real world meaning of mathematical solution → (evaluation) revise model or accept solution.

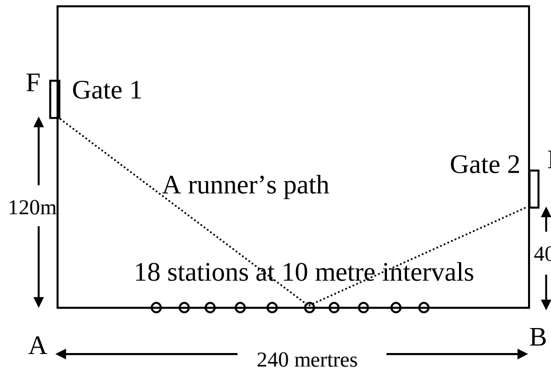
We therefore examine in detail how students approach and perform in these areas of transition, while learning modelling in an environment characterised by the interactions between modelling, mathematical content, and technology. The technological component is significant to the purpose of addressing realistic problems. Real measurements are messy, and in order to make their inclusion tractable, technology is needed. In turn this implies knowing both what alternative technologies (e.g., graphing calculators and spreadsheets) can do, and facility with them to carry out operations to achieve desired mathematical ends.

3. Purpose of the Research

Data for this paper have been generated within RITEMATHS, an Australian Research Council funded project of the University of Melbourne and the University of Ballarat with six schools and Texas Instruments as industry partners (see http://extranet.edfac.unimelb.edu.au/DSME/RITEMATHS/). The research being undertaken is part of a design experiment (Barab & Squire, 2004) which was in its second cycle at the time of data collection. The research focuses at the level of individual student learning and applying modelling skills in classrooms where electronic technologies are available and expected to be used. The purpose of the research is to develop, test and refine a framework, to identify potential blockages for students in the transitions between stages in the modelling process. As pointed out by Gellert, Jablonka, and Keitel (2001), the mathematical knowledge as well as the technological tools at their disposal, will influence the models that are produced by the students and this is particularly true of students at the intermediate secondary level.

One school in RITEMATHS has been developing a lower secondary mathematics curriculum (Years 8–10) which provides opportunities for engagement in extended investigation and problem solving tasks set in real-world contexts considered meaningful for the students by their teachers. A major focus to date has been in Year 9 when students (14-15 year olds) are required for the first time to have laptop computers and graphing calculators. In keeping with local curriculum requirements (VCAA, 2005, p. 36), students are introduced in Year 9 to a mathematical model being able to be used to describe the relationship between variables in a real situation and this model being used to predict an outcome in terms of a response variable when a control variable is altered. A series of extended real-world tasks (e.g., *Cunning Running* in Figure 2) have been designed, implemented and refined by teachers at this school.

Cunning Running: In the Annual “KING OF THE COLLEGE” Orienteering event, competitors are asked to choose a course that will allow them to *run the shortest possible distance*, while *visiting a prescribed number of checkpoint stations.* In one stage of the race, the runners enter the top gate of a field, and leave by the bottom gate. During the race across the field**,** *they must go to one of the stations* on the bottom fence. Runners claim a station by reaching there first. They remove the ribbon on the station to say it has been used, and other runners need to go elsewhere. There are 18 stations along the fence line at 10 metre intervals, the station closest to Corner A is 50 metres from Corner A, and the distances of the gates from the fence with the stations are marked on the map.



THE TASK

Investigate the changes in the total path length travelled as a runner goes from gate 1 to gate 2 after visiting one of the checkpoint stations. To which station would the runner travel, if they wished to travel the shortest path length?

For the station on the base line closest to Corner A, calculate the total path length for the runner going Gate 1 – Station 1 – Gate 2. Use Lists in your calculator to find the total distance across the field as 18 runners in the event go to one of the stations, and draw a graph that shows how the total distance run changes as you travel to the different stations.

Observe the graph, then answer these questions. Where is the station that has the shortest run total distance? Could a 19th station be entered into the base line to achieve a smaller total run distance? Where would the position of the 19th station be?

If you were the sixth runner to reach Gate 1, to which station would you probably need to travel? What is the algebraic equation that represents the graph pattern? Draw the graph of this equation on your plot of the points. If you could put in a 19th station where would you put it, and why?

(Additional suggestions were provided as to how the work might be set out, and for intermediate calculations that provide some task scaffolding.)

Figure 2. Cunning Running Task

We seek to identify and document (a) key activities involved in moving successfully through the modelling cycle; and (b) the occurrence or removal of blockages that emerge at such critical points, when the process includes respective interactions between modelling, mathematical content, and technology. The purpose is to enable an in-depth analysis of issues and activities impacting on transition phases 1 - 5 above, with implications for both the learning and teaching of mathematical modelling at an intermediate stage of secondary education.

4. Emergent Framework for Identifying Student Blockages in Transitions

The structural framework as in Figure 3 consisted initially of the transitions arising from our theoretical framework (Figure 1) of which five have been included for present purposes. Initially the contents of the respective transition sections were empty. Each element in a transition section in Figure 3 has two parts which have been generated from intensive analysis of data from the implementation of two modelling tasks at this first school which resulted in the development of the emergent framework (Galbraith, Stillman, Brown, & Edwards, 2006) shown. We refer to it as an "emergent framework" as it is our first result from empirical study and was at that stage yet to be refined and tested with other tasks and implementations with other teachers with different motivations for using such tasks. The empirics gave rise to case specific categories and generalisations of these that form the various elements in each transition section in Figure 3. Key (generic) categories in the transitions between phases of the modelling cycle are indicated (in normal type), and illustrated (in capitals) with reference to one of the tasks, *Cunning Running*. Our research indicates there is potential for blockages to occur when any of these component activities have to be undertaken.

Intensive data were generated from implementations of the two tasks in order to develop this framework. These data were collected by means of student scripts (24 and 28 respectively), videotaping of teacher and selected students, video and audiotaped records of small group collaborative activity, and selected post-task interviews (8 and 4 respectively). In order to identify and document characteristic levels of performance; occurrence or removal of blockages; use of numerical, graphical, and algebraic approaches; quality of argumentation; and the respective interactions between modelling, mathematical content, and technology, these data were entered into a NUD.IST database (QSR, 1997) and analysed through intensive scrutiny of the data to develop and refine categories related to these themes. The point of the approach was to closely observe students at work on a task in a normal classroom setting, identify what is happening in terms of task specific activities, seek to describe these in terms of generic attributes that can be confirmed or supplemented by observing the implementation of a second task with the same class and teacher. The resulting emergent framework was then to be refined and tested by looking at performance on other tasks in a different setting. Thus, our methodological approach is basically a version of a Grounded Theory approach. The conceptual categories in the framework arise "through our interpretations *of* data rather than emanating *from* them or our methodological practices". Thus the framework is the result of our "interpretive renderings of a reality" (Charmaz, 2005, pp. 509-510).

We note that technology features strongly in transitions 2 and 3. In the former the knowledge of the capabilities of different calculator or computer operations impact directly on how a model is formulated. Obtaining results from the model then depends on the ability to actually perform the respective operations. Hence the regular occurrence of the terms *choosing* in section 2, and *using* in section 3 of the framework.

1. MESSY REAL WORLD SITUATION → REAL WORLD PROBLEM STATEMENT:
1.1 Clarifying context of problem [ACTING OUT, SIMULATING, REPRESENTING, DISCUSSING PROBLEM SITUATION]
1.2 Making simplifying assumptions [RUNNERS WILL MOVE IN STRAIGHT LINES]
1.3 Identifying strategic entit(ies) [RECOGNISING LENGTH OF LINE SEGMENT AS THE KEY ENTITY]
1.4 Specifying the correct elements of strategic entit(ies) [IDENTIFYING *SUM* OF THE *TWO CORRECT LINE SEGMENTS*]
2. REAL WORLD PROBLEM STATEMENT → MATHEMATICAL MODEL:
2.1 Identifying dependent and independent variables for inclusion in algebraic model [TOTAL RUN LENGTH AND DISTANCE OF STATION FROM CORNER]
2.2 Realising independent variable must be uniquely defined [CANNOT USE *X* FOR DISTANCE FROM BOTH A & B]
2.3 Representing elements mathematically so formulae can be applied [TOTAL LENGTH EXPRESSED IN TERMS OF EDGE DISTANCES ALONG THE FIELD]
2.4 Making relevant assumptions [PYTHAG. TH. IS APPROPRIATE GIVEN STRAIGHT LINE APPROXIMATIONS TO PATHS]
2.5 Choosing technology/mathematical tables to enable calculation [RECOGNISING HAND METHODS IMPRACTICAL]
2.6 Choosing technology to automate application of formulae to multiple cases [*LISTS* HANDLE *MULTIPLE X-VALUES*]
2.7 Choosing technology to produce graphical representation of model [SPREADSHEET OR GRAPHING CALCULATOR WILL GENERATE PLOT OF *L* FOR *DIFFERENT X-VALUES*]
2.8 Choosing to use technology to verify algebraic equation [RECOGNISING GRAPHING CALCULATOR FACILITY TO GRAPH *L* VERSUS *X*]
2.9 Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation [GRAPHING CALCULATOR CAN PRODUCE GRAPH OF FUNCTION TO FIT POINTS – SPREADSHEET CANNOT]
3. MATHEMATICAL MODEL → MATHEMATICAL SOLUTION:
3.1 Applying appropriate formulae [$L = \sqrt{}(14400+X^2) + \sqrt{}(1600+(240-X)^2)$, WITH RELEVANT *X*-VALUES SELECTED]
3.2 Applying algebraic simplification processes to symbolic formulae to produce more sophisticated functions [PRODUCING ALGEBRAIC EQUATION FROM MANIPULATION OF LIST FORMULAE]
3.3 Using technology/mathematical tables to perform calculation [SUCCESSFUL CALCULATION OF *L*-VALUE]
3.4 Using technology to automate extension of formulae application to multiple cases [EFFECTIVE USE OF LISTS]
3.5 Using technology to produce graphical representations [USE OF SPREADSHEET CHART/GRAPHING CALCULATOR]
3.6 Using correctly the rules of notational syntax (whether mathematical or technological) [CORRECT USE OF ALGEBRAIC NOTATION IN EQUATIONS]
3.7 Verifying of algebraic model using technology [GRAPHING A FUNCTION TO MATCH A DATA PLOT]
3.8 Obtaining additional results to enable interpretation of solutions [CALCULATING & PLOTTING EXTRA VALUES TO TEST HUNCHES OR SUSPICIONS]
4. MATHEMATICAL SOLUTION → REAL WORLD MEANING OF SOLUTION:
4.1 Identifying mathematical results with their real world counterparts [INTERPRETING *L*-VALUES IN TERMS OF RESPECTIVE CHECKPOINT STATIONS].
4.2 Contextualising interim and final mathematical results in terms of RW situation (routine → complex versions) [INDIVIDUAL *L*-VALUES GIVE STATION SPECIFIC RESULTS: STRATEGY DECISIONS (E.G. OPTIMUM POSITIONING OF STATION – REQUIRE EXTENDED COMPARATIVE DATA]
4.3 Integrating arguments to justify interpretations [PRESENTING REASONED CHOICE FOR OPTIMUM PLACEMENT OF STATION IN TERMS OF GRAPHICAL BEHAVIOUR]
4.4 Relaxing of prior constraints to produce results needed to support a new interpretation [PLACING 19TH STATION – NEED TO BREAK PATTERN OF SUCCESSIVE STATIONS AT 10M INTERVALS]
4.5 Realising the need to involve mathematics before addressing an interpretive question [BE UNWILLING TO SUGGEST POSITION OF OPTIMALLY LOCATED STATION WITHOUT SOME MATHEMATICAL SUPPORT]
5. REAL WORLD MEANING OF SOLUTION→ REVISE MODEL OR ACCEPT SOLUTION:
5.1 Reconciling unexpected interim results with real situation [RECALCULATING *L*-VALUE AS CONSEQUENCE OF OBVIOUS ERROR WHEN COMPARED TO NEIGHBOURING VALUES]
5.2 Considering Real World implications of mathematical results [LOCAL – DO INDIVIDUAL CALCULATIONS/GRAPHS ETC MAKE SENSE WHEN TRANSLATED TO REAL WORLD MEANINGS?]
5.3 Reconciling mathematical and Real World aspects of the problem [*L*-VALUES SHOULD VARY STEADILY WITH CHANGING STATION LOCATIONS]
5.4 Realising there is a limit to the relaxation of constraints that is acceptable for a valid solution [19TH STATION MUST BE ON AB – NOT ON STRAIGHT LINE JOINING GATES]
5.5 Considering real world adequacy of model output globally [MODEL PROVIDES ALL ANSWERS TO RW PROBLEM?]

Figure 3. Emergent framework for identifying student blockages in transitions

The following outline of key steps in the solution of *Cunning Running* is now provided, to assist the reader in the discussion that follows as we use this task and data used to generate the emergent framework to elaborate its various entries.

Following the calculation of individual distances for the respective checkpoints the solution involves the calculation of the total path as the sum of two segments, followed by graphing, construction of an algebraic model, verification, interpretation, and the search for a nineteenth station optimally located.

Figure 4 shows a typical graph produced by students who used a spreadsheet to graph the different values of path length calculated for the separate checkpoint stations obtained, for example, using the LIST facility on a TI-83 Plus graphing calculator.

Total path length is given by $L = \sqrt{(14400 + X^2)} + \sqrt{(1600 + (240 - X)^2)}$ where X is the distance to a station from corner A.

The equation can be checked, using the function plotting facility of a graphing calculator to generate the graph for L in terms of X.

Deciding which checkpoint station to use (if the sixth runner), and selecting a site for the 19th station are inferred from an understanding of the behaviour of L in terms of X, as displayed, for example, in the graphical output.

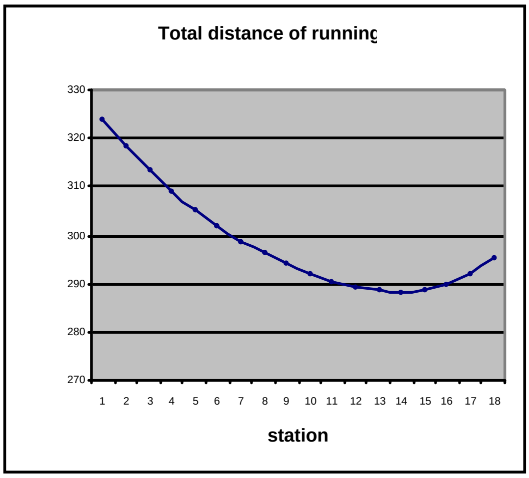


Figure 4. Spreadsheet graph for length of path

We now look at the transitions in the solution phases and possible blockages that could result for students with this level of expertise in modelling and mathematical background. Links to the framework will be shown as numerical codes such as 1.1 meaning 'clarifying context of problem' as per Figure 3.

Messy real world situation → real world problem statement

The first challenge is to identify the key elements that will be the basis of model building, namely, deciding the nature of the 'strategic entity' and then specifying the correct element of this identified strategic entity. Here it is assumed that the runner's path from gate to checkpoint station is a straight line (1.2). The key element is a distance (1.3), which is a compound entity to be constructed from other components present in the real situation being modelled, namely line segments (1.4). To avoid possible blockages in the early specification and formulation stages, the teacher provided a supporting dynamic geometry animation of the task which students watched. All students interviewed said this clarified or confirmed their thinking in some way (1.1).

> *Interviewer*: The Friday before you did the task you saw a GSP animation of the task. He showed you a diagram that had a yellow triangle and a green triangle.
>
> *Gary:* [showing the movement with his fingers using the task diagram] Awh, and he moved the bar. It just, it showed um, um, I understood it. It was just showing you the length and how you actually got it, the area. And it was just basically the formula you needed to know to get the answer. It showed everything that you needed really. …it was just good to see it in front of you and it doing its own little business [indicating the movement of the station along the base line with his fingers again].

In addition students were required to make a scaled plan of the field showing the checkpoint stations and the first four paths. Some students such as Mei saw this as a further source of clarification (1.1).

> *Mei:* Because you could actually see like how many metres it was away from each other and, yeah, it was easier to work out by seeing the image.

Others only did it because they had to, saying they already had a clear idea of the task situation and what specifically they were focussing on.

Real world problem statement → mathematical model

Key decisions with the potential to generate blockages in this transition include (a) deciding how to represent the sum of the lengths mathematically so formulae can be applied (2.3); (b) making assumptions necessary to support model development (2.4); (c) choosing to use technology to enable calculations such as square roots with different technologies presenting different challenges (e.g., Excel versus graphing calculator) (2.5); (d) choosing to use technology to automate extension of the application of distance formulae to multiple cases (graphing calculator LISTs or a spreadsheet) (2.6); (e) choosing to use technology (spreadsheet or graphing calculator) to produce a graphical representation of the model (2.7); (f) identifying the dependent and independent variables for inclusion in an algebraic model, respectively the total travel distance and the distance of the station from the corner of the field (2.1); (g) realising that an independent variable must be uniquely defined [e.g., Mei used a one variable expression for the total distance $\sqrt{(40^2 + x)} + \sqrt{(120^2 + x)}$ as only one station was involved at any time when her expression was evaluated. She did not see this conflicting with the *x* being the distance from the station to corner A in one part of the expression and from corner B in the other part] (2.2); and (h), choosing to use technology to verify the algebraic model (2.8).

The last of these includes an affordance of the graphing calculator (along with other function graphers but not data plotters) which is critical for students at this level who do not have the mathematical experience to fully cope with unusual functions as arise in some of these tasks. Several students did not perceive the affordance of the calculator to draw the graph of an algebraic function on the screen to match the scatter plot of the points if the equation for the function is correct (2.9). They thought the question was superfluous, stating, “the graph will be the same”, as they had used a joined scatter plot on a spreadsheet (a data plotter) or joined the points already in a by-hand graph. However, Ben perceived an alternative method of verification that could overcome this blockage was to use substitution of the station number into his model on the calculator home screen. Whether the answer was the same as that from his calculator LIST would tell him if his model was incorrect or needed revision (2.8).

Mathematical model → mathematical solution

Knowledge of mathematical procedures, technological knowledge for their automation and declarative knowledge about the rules of notational syntax associated with both mathematics and technology feature in the potential sources of blockages in this transition. It would be expected that some blockages here follow from difficulties in the formulation process (e.g., non-uniqueness of a definition of a variable). Others follow appropriate decisions made in formulation (e.g., choosing to use technology for some correct purpose), but occur for example, due to technical failures when attempting to use technology to automate extensions of formulae to multiple cases (3.4) or to produce graphical representations (3.5).

All students but one used graphing calculator LISTs to automate extension of their by-hand calculations using Pythagoras’ Theorem, the method promoted by the teacher (3.1, 3.3, 3.4). The remaining student did all 18 calculations by hand supported by home screen calculations (3.1, 3.3), and then checked her work against that of other group members who were using LISTs. This was at costs of both time, and experience in using technology to automate calculations already mastered by hand. She thus denied herself the time to develop the “reflective ness" needed in modelling to examine the appropriateness and reasonableness of the models that are constructed within the real world aspects of the situation being modelled (Gellert, Jablonka & Keitel, 2001), potentially laying the foundation for future blockages at other transitions as the time remaining was reduced substantially.

Students had the greatest difficulty in formulating a two variable algebraic model with only 2 of the 27 students who attempted the task doing this successfully but these were early days in their study of algebraic functions. The process of concatenating LIST formulae to produce an equation with just two variables was described by one student as “putting them together and then kind of simplifying it so they worked because it is pretty well what it is, just all the LISTs.” Such simplification of symbolic formulae to produce more sophisticated functions (3.2) often presents a blockage for students at this level, as does the understanding of rules of notational syntax (3.6) needed to apply them.

The verification of an algebraic model using technology requires both technological knowledge and mathematical knowledge of a high level, and the ability to integrate these. Only Sandra was able to verify that her algebraic equation for *Cunning Running* was correct by using the graphical representation of the graphing calculator (3.7).

Other blockages occur in this transition when students do not know what is necessary mathematically to obtain the additional results needed to enable an interpretation (3.8); for example, when asked interpretive questions such as: "Could a 19th station be entered into the base line to achieve a smaller total run distance? Where would the position of the 19th station be?"

Mathematical solution → real world meaning of solution

Blockages occur here as students fail to identify mathematical results with their real world counterparts (4.1). This is the most basic of interpretative acts where a mathematical result has been obtained and students need to identify what it tells them about the real world situation it was intended to model. In this task it involves the interpretation of an outcome distance in terms of implications of using the corresponding checkpoint station, or the meaning of the minimum distance in terms of a particular strategy for station selection.

Students' attempts at routine interpretations lead to blockages when they have difficulty contextualising interim or final mathematical results in terms of the real world situation (4.2). For example, when asked, "Does running via station 1, or station 2, or station 3 make any difference to the overall length of the run?" responses ranged from bald assertions such as, "It makes a difference", to justified arguments such as, "Yes, it does the closer you are to corner A, the further the distance you have to run." In some cases an adequate response requires arguments that integrate mathematical knowledge with the impact of this knowledge in the real situations to justify interpretation (4.3). There is a dilemma that students must face as to whether people really take mathematical knowledge into account or are driven by more pragmatic concerns; for example whether runners would use mathematical reasoning to select the best available station, or just go to any convenient one.

Failing to perceive that constraints can be relaxed in order to answer interpretative questions central to the real world meaning of a mathematical solution, is another source of potential blockage (4.4). Students had the greatest interpretative difficulty determining where to place a 19th station, as this entailed relaxing the previous constraint of continuing the ordered pattern (i.e., 19th must follow 18th at a distance of 10 metres). Many students simply placed the extra station 10 m away from either the first or last stations, rather than applying a minimum distance criterion.

Failing to realise the need to involve mathematics before addressing an interpretative question, can also lead to a blockage (4.5). As the aim here is to find the shortest path through the field, the essential question to be answered is: Where is the station that has the shortest total run distance? Most students used their numerical lists, the graph or a combination of the two to identify the station. Con, however, saw no need for any mathematical involvement in arriving at his answer, just writing "towards the end".

Real world meaning of solution → revise or accept model

This transition produced some potential sites for blockages, noting that interpretive aspects of tasks also cause difficulties when students have to ultimately reconcile mathematical and real-world aspects of a problem (5.3). One sub-task required that mathematical results from the model be used to indicate to which station the sixth runner would probably travel. This involved assuming the other runners would take the shortest paths available to them (1.2), and several students perceived this. Meg's response was typical, "If you were the sixth runner to reach gate 2 you would probably of [sic] travelled from Station 11. (That's if the people in front know the fastest stations.)" However, others such as Kim were more pragmatic stating, "Most likely the sixth because you wouldn't really think about the distance."

Blockages can still occur at this transition if students fail to consider the real world implications of mathematical results (5.2), such as when answering the more open final question, “If you could put in a 19th station where would you put it and why?” Some students such as Gary searched for a shorter distance than that previously determined, introducing real world (but extraneous to the problem) implications, by choosing to make it a close race.

Gary [reading from his report during interview]: “A 19th station could be placed into the base line and achieve approximately the same running distance of 288.44 m by placing the station 178.75 m from corner A.” Because my theory on trying to get the 19th gate [sic] … roughly the same distance as [station] 14. I thought a good race is a close race so I was thinking if you have the first two runners going to the same station [distance] round about it is going to be a closer race.

Other students in giving free reign to their imaginations when responding to this question did not realise there is a limit to the relaxation of constraints that is acceptable for a valid solution (5.4). Arguably, Pat could be seen as exceeding this as he relaxed all constraints of the problem and placed his 19th station on the straight line joining the gates.

5. Refining Framework against Data from another Site and Different Tasks

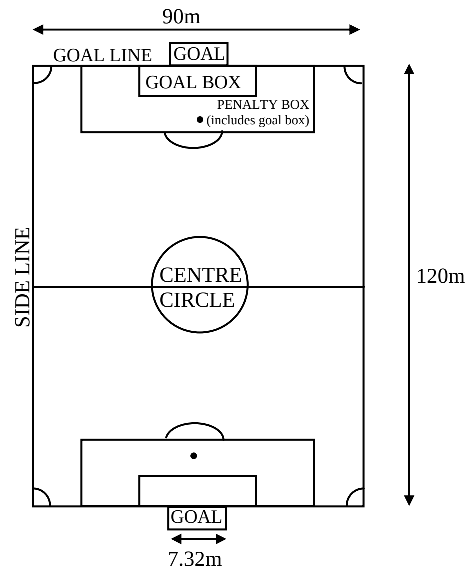
One problem of design-based research, particularly researching technology use, is adoption by other classroom teachers (Fishman, Marx, Blumenfeld, & Krajcik, 2004) with different motivations for the use of such things as modelling, real-world tasks and/or electronic technologies. This is especially true in the intermediate secondary years. Some of the tasks from the first school have been modified by members of the research team and teachers at another project school where they have been implemented to fit the conditions existing at that school. This is a girls school, in which the teachers are prepared to spend less time, no more than 2-3 class lessons (45 minutes each), on such tasks. They also make much less use of electronic technologies in their teaching at this level and students are not required to own a graphing calculator or laptop computer. Small pods of computers are available and so are class sets of graphing and CAS calculators although these are used infrequently at Year 9 level. Data from the implementation of the two tasks, *The Bungee Experience* (Figure 5) and *Shot on Goal* (Figure 6), at the second school, also during the second design experiment cycle, will be used to test the framework.

5.1. The New Tasks

The Bungee Experience: Barbie has turned 40. Her friend Ken has given her an extreme sports experience, part of which is an afternoon’s bungee jumping. Your task as the operator is to CALCULATE the length of Bungee Cord Barbie will need to jump from the given height, of the Bungee tower. Remember there is concrete below and we don’t want to mess up Barbie’s hair.
During the next two maths periods your team will:
1. Conduct measurements in the classroom to determine a model that links the fall distance to the number of rubber bands used for a shock cord.
2. Record your data, the graph for the data, and your linear equation.
3. Test your model by predicting the requirements for a fall from an unknown height.
(Additional suggestions were provided as to how to collect data, display results, and use a graphing calculator to find the equation. Students used a doll, usually Barbie or a soft toy such as a bear or Sesame Street Bert for the Bungee Jumper.)

Figure 5. *The Bungee Experience* Task

Shot on Goal: You have become a strategy advisor to the new football recruits. Their field of dreams will be the FOOTBALL FIELD. Your task is to educate them about the positions on the field that maximise their chance of scoring. This means—when they are taking the ball down the field, running parallel to the SIDE LINE, where is the position that allows them to have the maximum amount of the goal exposed for their shot on the goal?
Initially you will assume the player is running on the wing (that is, close to the side line) and is not running in the GOAL-to-GOAL corridor (that is, running from one goal mouth to the other). Find the position for the maximum goal opening if the run line is a given distance from the near post.



(Additional suggestions were provided as to how the work was to be set out, and for intermediate calculations especially in the area of graphing calculator use providing extra task scaffolding.,)

Figure 6. *Shot on Goal* Task

Methodologically the approach contains a continuous aspect, in that two versions of *Shot on Goal* have featured. The original version in the first school was the second task used to generate elements for the framework. The version used by the second school (see above) has provided some of the data for testing the framework—and one would expect consistency if the classification is to be robust.

The following outlines typical solution elements for *The Bungee Experience* using data collected for a stuffed toy bear (Table 1).

Number of Bands	Av. Drop Distance (cm)
3	70.6
4	89.0
5	105.3
6	121.0
7	144.0
8	157.0

Table 1. Sample data for a typical solution to *The Bungee Experience*

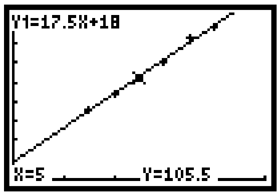


Figure 7. Plot of sample data with line of best fit by eye

Estimated line of best fit is: $y = 17.5x + 18$
Set jump height for test: 580 cm
Number of bands required = 32 (rounded down).
Note: This gives a drop distance of 578 cm.
(A more cautious modeller might under estimate and choose 31 to leave some more room for error.)

The outline of essential steps in the solution that follows is for the version of *Shot on Goal* given in Figure 6. Table 2 shows calculations obtained using the LIST facility of a TI-83 Plus graphing calculator. Calculations are shown for positions of the goal shooter at (typical) distances from the goal line of between 2 and 24 metres; along a run line that is 10 metres from the near goalpost (see Figures 6 and 8). Width of goalmouth is 7.32 metres. (The students encompassed a wider range of calculations than these, increasing the distance along the run line beyond 24 metres, and varying the lateral position of the run line.) The maximum angle and its reference points are highlighted in the table, which was generated by the LIST facility of the calculator, following hand calculations to establish a method).

Distance (m)	Angle1 (degrees)	Angle2 (degrees)	Difference (α degrees)
L1	L2	L3	L4
2.00	78.69	83.41	4.72
4.00	68.20	77.00	8.80
6.00	59.04	70.89	11.85
8.00	51.34	65.21	13.87
10.00	45.00	60.00	15.00
12.00	39.81	55.28	15.47
14.00	***35.54***	***51.05***	***15.51***
16.00	32.01	47.27	15.26
18.00	29.06	43.90	14.84
20.00	26.57	40.89	14.32
22.00	24.44	38.21	13.77
24.00	22.62	35.82	13.20

Table 2. Sample calculations from a typical solution to *Shot at Goal*

Note. Calculator LIST formulae used were L2 = "$\tan^{-1}$(10/L1)", L3 = "$\tan^{-1}$((10+7.32)/L1)", L4 = "L3 – L2".

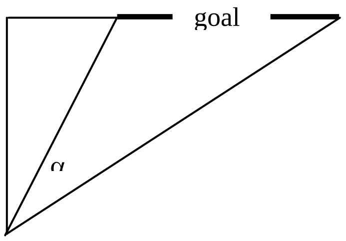


Figure 8. Angle (α) to be maximised

A graph (Figure 9), showing angle against distance along the run line is drawn, using the graph plotting facility of the calculator. Additional points near the maximum can then be calculated, to provide a numerical approach to the optimum position (15.54 degrees at 13.16 metres from the goal line—a suitable approach for early or middle secondary students—or an algebraic model can be constructed and the maximum found using the graphing calculator operations.

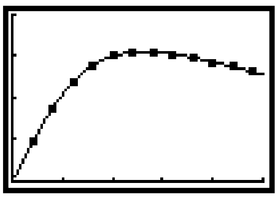


Figure 9. Function for (α) passing through scatter plot for run line 10m from near goal post

In practice a player often goes on a zigzag run and discussion can be used to infer that whatever her/his path the ultimate shot is from a position on some run line.

Intensive data were collected during the implementation of the two tasks (*The Bungee Experience* and *Shot on Goal*) by means of student scripts (21 for both tasks), videotaping of teacher and selected students, video and audiotaped records of small group collaborative activity (2 groups for both tasks), and written homework post-tests for *The Bungee Experience* (only 14 returned). These data have been subjected to the same analytic process as before using NUD.IST (QSR, 1997) except this time the generic categories within transitions in Figure 3 had already been established and it was a matter of confirming whether or not these were adequate to describe what was happening during these two task implementations.

5.2 Transitions in *The Bungee Experience*

Messy real world situation → real world problem statement

In this implementation of *The Bungee Experience* this *transition* presented no blockages to students' progress. The teacher told the students to drop (not throw) the doll off a ledge and demonstrated the attaching of bands and taping back of extraneous hair as she read through the task booklet with them before they began (1.1). At other schools some difficulties arise when students attempt their Bungee jump data collection with the doll upside down hanging from her toes or they throw the doll rather than topple it from a standing position. A major assumption (1.2) is that the bands will stretch at a constant rate and not exceed their elastic limit when the model is used to extrapolate well beyond the set of collected data (maximum drops of around 2m in a classroom). In this implementation it was also assumed that the aerodynamic characteristics of some toys would have negligible impact (1.2) and this appeared to be so in most cases, with the outstanding exception being a large stuffed toy alligator. On the test drop the alligator "went like half way down" and the group suggested they were disadvantaged because "like it is lumpy".

Real world problem statement → mathematical model

Again there were no real blockages in this implementation. This was because the task setter had already made the decisions for the students by specifying a linear model and choosing the technology to use or the task itself did not entail particular aspects in the framework.

Mathematical model → mathematical solution

Several students did not use appropriate formulae when they calculated the number of bands for their prediction (3.1). They did not use the entire linear model they had constructed but instead chose to use how far one rubber band stretched on average with their toy. Rae and Jo, for example, constructed the linear model, $y = 15.5x + 21$, using Bert from Sesame Street. They then ignored the 21 in calculating their prediction of the number of bands needed for a drop of 5.8m as is revealed in their discussion with Cate.

Rae: Well, what we did was we divided these together. ... Is that what we did?
Jo: Divided by.
Rae: That's what we did anyway divided 580 by 15.5. But I don't know if that is right.
Cate: Plus 21?
Jo: Divided by.
Rae: ... no, divided by 15.5 but I don't know if that is right.
Jo: I did the amount it is going up by. ...That's 15.5.

Mathematical solution → real world meaning of solution

Possible dilemmas for students in this transition occur when they need to contextualise interim and final mathematical results in terms of the real world situation (4.2), for example, when predicting a shock cord length for their test jump at a greater height outside the classroom, without their doll hitting the ground. The doll was to be dropped so as to stop as close to the ground as possible. Eva, for example, wondered if it was possible to have half band lengths in the shock cord.

Teacher: You will need to figure out how many rubber bands that will take.
Eva: Can it be a half or does it have to be only a whole?
Teacher: I don't quite see how we can manage a half. But I think you probably could.
Eva: Put a knot in it.
Teacher: If you put a knot halfway and held it at that spot.
Other Student: Oh, yeah!
Teacher: So yes you probably could manage halves.

The task sheet specified that the students begin finding a linear model for their data by fitting a line to a by-hand plot of the data and reading off the y-intercept. The model was then to be refined using a graphing calculator by starting with $y = 10.0x$ + '*y-intercept*' and adjusting the gradient by eye to fit the scatter plot of the points. Liz and Nancy started with $y = 10.0x + 38$ as instructed. However, Nancy had her own mathematical ideas about how she might refine the model. She calculated the differences between consecutive pairs of average drop distances.

Liz: What are you doing that for?
[Nancy has recorded the five differences.]
Nancy: Find the sum.
Liz: Awh. [Pause] How come they are not doing them?
[Nancy uses a graphing calculator to add her 5 differences and then divides the sum by 6 (sic). She records 17.2 as the new gradient.]
Nancy: On the average each rubber band makes it go up 17.2, like makes it fall that much longer.
Liz: Mmm [still looking at Nancy's work]. Yeah but doesn't it mean, yeah so we have got to put, um. Well we have just got to do the graph. So like. [Pauses and thinks.] Does it mean that one rubber band will fall 17.2?
Nancy: Yeah, but that is not what we are doing Liz.

At this stage they were refining the fit of their line to their data points. The mathematical procedure Nancy was using to do this temporarily blocked Liz's progress, and it was not until the mathematical results of the procedure made sense in the real situation that Liz was willing to accept its use (4.2 and 5.3).

When the students finally found their mathematical result for the predicted number of bands, decisions had to be made about whether they should round up, truncate their answer or over or under estimate. The real world implication (5.2), that rounding up or over estimating would mean the doll would hit the ground was not foreseen by seven students. This is not to suggest that all other students realised the implications of this, as they may have fortuitously derived a mathematical result such as 27.3, which was rounded down in keeping with expected classroom practice. In the recorded conversations no groups justified their actions either way.

Real world meaning of solution → revise model or accept solution

Some students realised the significance of the y-intercept in their mathematical model and how it could be used to partially evaluate the mathematical equation they had constructed (5.3) but most did not.

Teacher [to Eva]: So you have got your equation?
Eva: Yep, $16.9x + 18 = y$. So what is the 18?
Teacher: What do you reckon it might be?
Eva: Zero rubber bands. ... x is the rubber bands.
Teacher: If x was zero, what would you know then?
Eva: How far.

Teacher [indicates with her hands a toy falling]: How far would she fall?
Eva: 18 centimetres.
Teacher: Which is what? You have got no rubber bands and you just have?
Eva: The length of it.
Teacher: Okay, so pick it up, see if that is right. [Eva measures the length of the bear.]
Eva: Yep, he is about 18 centimetres.
Teacher: What do you reckon it means?
Eva: It proves it!

Later in a discussion with students who used the stuffed alligator for their bungee jumper, Eva explained to the group why she was able to tell their equation was incorrect - as the *y*-intercept should have been "how tall our animal is" whereas theirs clearly was not the length of the alligator.

Once the girls had tested their predictions many still faced puzzlement as to why their predictions were wrong. They had difficulty reconciling the results of their testing with the mathematics of the situation (5.1), seeking an explanation in the physics of the situation, but these possible explanations were not used to evaluate their models. Liz, for example, was puzzled by the fact that her Ken doll just hit the ground with 27 bands whereas Mel's Barbie with 34 bands performed an almost perfect jump, missing the ground by about a centimetre. Various reasons were offered by other students for the difference such as Ken being heavier, less aerodynamic and more muscular than Barbie (both the Ken and Barbie doll are 29.5cm tall). There was no discussion of the difference in their models, Liz's $y = 19.8x + 38$ and Mel's $y = 16.5x + 25$.

5.3 Transitions in the implementation of *Shot on Goal*

Messy real world situation → real world problem statement

Assumptions in *Shot on Goal* (treated as a two dimensional problem) include that the player runs on a straight line perpendicular to the goal line, that the ball travels in a straight line once kicked, and that the presence of the goal keeper (whose job it is to "close down the angle" of the attacking player) can be ignored (1.2). Here the key element is an angle (1.3), which is a compound entity, to be constructed from other angles present in the real situation (1.4). Blockages occurred despite the teacher providing a supporting physical demonstration with string lines sticky taped to a goal drawn on the board, in which class members either watched or participated (1.1). Interpretation is central here in specifying a mathematical problem in the first place. The teacher spent some time working with the girls as a class discussing the diagram of the football field in the task booklet to ensure that they understood what the terms used meant (1.1), as she did not think many played football. She deliberately pointed out the aim of the task was to try "to find out whereabouts on that [the run line] for the person shooting, is the best angle." She also had them: "Pick a spot somewhere on that line. Just so you get a picture of what is happening, then mark in the angle that the player at that spot has to kick a goal through." This language is well chosen, she was aware of the difficulties the students in the first school had in realising where the angle was (on the run line or at the goal mouth) that was to be optimised (1.1). Later when reading through the task with the class she pointed to the required angle in the diagram on the board and said, "That's here, to the near and far goal posts" (1.1). Despite this, there was still confusion between whether they should focus on the path of the ball (Lil: "Is it going to the left hand post or the right hand post?") or the shot angle. Even when they had ascertained that they were to focus on finding the shot angle, several students such as Liz and Nancy and Sui and Summer thought they should start by finding a distance using Pythagoras' Theorem but could not see how this led to finding an angle as they really had not been able to decide what were the elements of the angle that would specify it (1.4).

Later in the task when the students were asked about their initial belief about how the angle size would change as they moved along the run line, Summer instigated a dynamic concrete demonstration of how the angle might change in a discussion with Sui in an attempt to clarify her thinking about the context of the problem (1.1). On her part, Sui tried to clarify the discussion by using a diagram to show how the angle changed size (1.1).

Summer: Am I meant to say like I would have thought that the further, the closer I get the angle would get smaller?
Liz: Yeah.
Sui: I thought if you went away, like if you were here then it would get bigger.

Summer: It would get tiny, wouldn't it? [Pause] It would stay the same. [Summer uses her hands and then pens to simulate the changing angle.] Look, think about this. See moving this around, hang on it is more like that.
Sui: You are not moving it around.
Summer: Yeah, the closer you get the smaller.
Sui: It is going to be bigger!
Summer: No, look. Hang on. No you are meant to keep drawing it like.
Sui: No, no, but look at this. You have like the box [the goal box] and you're here [the shot spot] and it's going like that and here it's going really like that [draws 3 different angles from run line to goal posts.]
Summer: If you start off with the angle here and it's here like look, watch. I have no idea I just wrote smaller.
Sui: Look hang on, that is how the angle is, the closer you get the skinnier it is getting if they were long enough [referring to the pens].
Summer[laughing]: It's closer had my pens been long enough. See they get skinnier and the further out you get the fatter they get.
Sui: I don't know.

Real world problem statement → mathematical model

Blockages continued for some time as students struggled to mathematise the problem. The critical and time-consuming nature of the blockage at this stage was commented on by the teacher who was surprised by their lack of progress, "In fact most of that first period was spent on figuring out how on earth you would get that angle that you need." Even when students had determined that they should use tan and inverse tan to find the angle they were still blocked by not being able to decide how to represent the angle mathematically so formulae could be applied (2.3). The difficulty for Sui and Summer, for example, was that they were trying to apply trigonometry formulae when they did not have right-angled triangles in their representation of the required angle.

Mathematical model → mathematical solution

Even when students had identified and specified the correct component angles of the shot angle and represented these correctly geometrically in a diagram, many students still struggled in applying an appropriate method for finding the size of the shot angle (3.1). Nora and Anna, for example, took several attempts before they worked out: "To find out that one you minus it from this one" (i.e., by subtracting the angle from the run line to the near post from the angle from the run line to the far post). Kara, on the other hand, had produced such a diagram identifying the two component angles but despite knowing she had to apply tan and inverse tan, she calculated the size of only the angles to the near post for four shot distances along the run line. The other 20 students all eventually overcame this blockage.

Using technology to automate extension of their calculations of the size of the shot angle to another 26 cases (3.4) also proved an insurmountable blockage for some but 15 students were able to do this successfully. However, Jen who had successfully calculated four shot angles by hand, failed to check her calculator generated angles against these despite being advised to do so on the task sheet. Her LIST formulae to calculate component angles were correct but she divided these to find the shot angle (i.e., L4 = L2/L3). Even when asked to record between which two spot distances the maximum angle for the shot on goal occurred, she apparently did not notice she had a continually increasing table of 30 values and just wrote between "29 and 30 metres." (4.2)

Applying algebraic simplification processes to symbolic formulae in the calculator LISTs to produce a more sophisticated function using only x and y (3.2) proved challenging for those students who progressed this far in the task. Liz who worked out with Nancy how she would do this, helped Sui and Summer interpret what they were expected to do to achieve this.

Summer: Liz, have you done this bit?
Liz: Yeah.
Summer: I don't understand it.
Liz: You have to put, what one are you doing here? You have to put it all into one thing. All into one formula instead of having List 1, List 2, List 3, and List 4 you have to have just had
Sui: List 5
Liz: List 1, List 5, do you get what I mean?

To do this successfully students had to understand and apply the rules of notational syntax both mathematical and technological (3.6). This took Sui and Summer quite some time and discussion of each other's methods of finding the angle from its components and also how they had used LIST formulae to model their mathematical calculations, but finally they were able to record their new algebraic formulae correctly.

Summer: Yeah. "Now write this List formula as a function in terms of *x* and *y*". What! Ooh, it is getting tricky now, like trickier than before. What's that? What's the List formula?
Sui: This one, the L5, the L1 and L5. The formula for L5 is this.
Summer: So do we exchange the L1 and L5 for *x* and *y*?
Sui: Yes.
Summer: You agree?
Sui: I guess. It is going to make it pretty simple. Yeah.
Summer: I don't really understand.
Sui: No, just this one that you wrote here. Just instead of writing L1 you write *x* and instead of L5 you write *y*.

Liz and Alice also attempted to make an algebraic model from their symbolic LIST formulae. Liz's final equation still contained L5 in the place where *x* should be (3.2, 3.6); whereas Alice made an incorrect symbolic model multiplying the formulae for the two lists instead of subtracting them. She had subtracted them earlier to generate her list of angle values from spot distances along the run line of 1m to 30m (3.2).

Only four students verified their algebraic model using technology (3.7), as the others did not reach this stage of the solution in the 3 lessons devoted to the task. Liz was able to do this despite her formula still containing L5, for when it came to using the function menu of the graphing calculator she immediately translated L5 as *x*.

Mathematical solution → real world meaning of solution

There were several opportunities in the task for interpretative questions to present blockages. Most of these requirements were of a low interpretative level with students needing merely to identify mathematical results with their real world counterparts (4.1). For example Nancy wrote "12m out from goal" when asked to state between what two spot distances the maximum angle occurred, according to the results of her calculator generated lists of angles along her run line (4.1). Interpreting the implication from graphical output, obtained from a sketch graph of their algebraic model on the scatter plot of points was more difficult, mainly because of the novelty of the question for these students (4.1). Sui and Summer managed to do this.

Sui: Okay, what does your graph show you about your function? Well, it drew a line through it. It really did. So does it mean that it is correct?
Summer: So what does this graph show?
Sui: It showed me that my function is correct.
Summer: Is that what it meant when it drew a line through it?

Summer then wrote:
"It showed me that my graph was correct, as it drew a line through my graph. It was able to do this because my squares [marks for points on her plot] were in the correct position."

Contextualising of interim mathematical results in terms of the real world situation (4.2) did not occur to some students as necessary or even possible, until after they thought about responding to the question: "Initially, what was your belief about angle size for the *Shot on Goal* as you changed positions along the Run Line?"

Summer: What was my belief? I didn't pay attention to that bit. I just did the sums.
Sui: The answers to the angles I know but like it doesn't, it doesn't prove anything about the shot really. It just tells you what angle they're at.

When Sui derived her final mathematical results for the task again she did not consider them in terms of the real world, writing both the angle in degrees and the distance in metres to six decimal places (4.2).

An interpretative question (after the students had completed a scatter plot of the data points generated in their calculator LISTs) required them to integrate their interpretation of the scatter plot with their response to an earlier interpretative question about the location of the maximum angle for shot on goal along their run line. Alice had previously said her maximum was "between 5 and 6" metres from the goal line. She was then able to write in response to whether her plot confirmed her answer: "Yes, because 5 & 6 are the two highest ones." Liz, on the other hand, had said her maximum angle would be located "13m out from goals" (actually from the goal line). She then focussed on matching the points in her plot to the numerical values in her table when looking at the plot writing: "Yes, because they were really close at the time, all the angles.“ (4.3)

At other times when students were asked interpretative questions about the problem requiring them to integrate their beliefs and observations, several students seemed quite unperturbed by the fact that they could hold quite contradictory views at the same time, and record them in close juxtaposition. When Sui recorded her initial belief about the size of the angle for the shot on goal as she changed position along the run line she wrote: “My belief was the further away from the goals you got, the smaller the angle would become.” Her calculated angles for distances of 5m, 10m, 15m, 20m along her run line of 12m from the near goal post were 8.11°, 12.44°, 13.51°, 13.05°. She was able to interpret these results as refuting her belief explaining, “the answers did not get smaller as I had assumed; instead, the results varied.” However, when responding to a question about the effect on the angle size of the measurement that continually changed (which she identified as “the side lengths”) she wrote, “Yes, generally, the smaller the side lengths, the smaller the angle.”

Real world meaning of solution → revise model or accept solution

Several students had difficulty reconciling unexpected interim results with the real situation. At times unexpected results were the consequence of an obvious error when they were compared to other results generated (5.1). These thus did not result in a blockage although they may have for a less observant student. Nancy, for example, calculated a shot angle of 42° for a shot distance of 5m from the goal line along her run line but this was easily detected when she reconciled this with the other results for 10m, 15m, and 20m namely, 16.515°, 16.449° and 14.986°.

The students were allocated distances along the goal line from the near post for their run line. These were from 1m to 12m in whole metre values. For the students who had run lines 1m – 4m from the near post, their calculated angle sizes for 5m, 10m, 15m, 20m increased as the shot spot was further away from the goal line. For the students who had run lines 5m – 12m from the near post, their calculated angle sizes for 5m, 10m, 15m, 20m increased and decreased. Five of the seven students who calculated only increasing values were able to interpret what these results meant in terms of their initial beliefs about how the angle size would vary as they changed position along the run line. Only eight of the 14 students who had increasing and decreasing values were able to interpret these in terms of their initial belief (5.3). For example, Liz had a run line 10 m from the near post and calculated shot angle values (in degrees) of 14.328, 15.416, 14.999 and 10.462 for distances of 20m, 15m, 10m and 5m from the goal line along this run line. Her partner, Nancy, calculated 14.986, 16.449, 16.515 and 12.021 for a run line of 9m. Nancy was the student at the beginning of the task who held the string lines from the near and far posts of the board goal diagram and walked towards the back of the room away from the goal on a run line. Nancy was puzzled by her mathematical results as she expected the angle to get smaller from her point of view of the demonstration however during a discussion with the teacher, who told her she could be correct, she suddenly recalled an earlier task, *The Biggest Box*, where they drew a spreadsheet graph of volume versus height of an open box.

Nancy: Would it be like *The Biggest Box* how it turns around?
Teacher: Uhuh, similar sort of thing.
Nancy: Okay.
Teacher: Is that what you were expecting?
Nancy: No.
Teacher: What were you expecting this time?
Nancy: That it would keep getting smaller.

The girls remained puzzled by their results and were even more so after comparing with Alice who with a run line distance from the near post of 3m had angle values that continually increased. They were unable to reconcile these results with what they expected of the real world situation but after about 12 minutes discussion amongst themselves sort reassurance from the teacher that, “That can happen, it depends on the distance you are from the goal line.”

6. Testing and Refining the Framework

We review again the purpose of the research described in this paper. For reasons indicated in the introductory sections we set out to construct a framework from our theoretical model (Figure 1) to provide an analytic tool for analysing modelling processes and associated blockages that occur as individuals attempt to solve problems with real world connections. In particular we are interested in the transitions through which solvers must progress in order to produce a satisfactory total solution. We began by analysing responses to two tasks as attempted by students at a middle secondary school level in order to fill in the elements of the structural framework containing only the transitions derived from Figure 1 that were our focus. This resulted in the emergent framework shown in Figure 3. In section 5 of this paper we describe the next phase of our research, in which two further tasks (one a different version of one of the above) were undertaken by students of a comparable level, at a different school. The purpose was to obtain further data to test, and where relevant refine or perhaps reject, the categories previously identified and illustrated. An overall criterion is that to be eligible for inclusion in a classification such as Figure 3 an activity should not be idiosyncratic to a particular case; that is the activity should have a generic aspect, which will typically have different instantiations in different cases. Following the discussion in the preceding sections, based on the student responses, we make the following observations.

Within transition 1 (from messy real world situation to real world problem statement) the four activities are robust. In every case a major starting hurdle was the students' understanding of the problem context. Video, computer representation, and enactment were variously found necessary to clarify the meaning for individual students. In each case simplifying assumptions (here mainly to do with straight line approximations) were explicitly required to give structure to the original problem statement. Finally the recognition of a property of a key entity (minimise this length, maximise this angle, maximise the height of a safe drop) was necessary to formulate a tractable mathematical question.

Transition 2 (from structured real world problem to model) is notorious as one of the most challenging parts of the modelling cycle. In Figure 3 the entries fall into two groups. Activities 2.1 to 2.4 are mathematical concerning successfully setting up equations for models with an algebraic base. They were identified as significant within all the problem contexts except this particular implementation of *The Bungee Experience*. (Other kinds of models, e.g., statistical, would be expected to add parallel structure within this part of the classification). Entries 2.5 to 2.9 are technology based, and have proved critical to progress where messy real world data need to be processed especially at this level of schooling. The choice of terms such as *choosing*, and *perceiving*, emphasise the background competences that are required. Students need to know the capabilities of their technology, for example, for calculating and graphing, and of differences between such entities as data plots and function graphs. Such knowledge is essential if students are to develop models to handle situations that are beyond the practical scope of their hand methods alone.

Activities central to obtaining solutions from a formulated mathematical model are represented in transition 3. Activities 3.1 and 3.2 again reflect the algebraic contexts of the problems used so far in the project, in which they proved to be gatekeepers to progress. (As before we would expect this suite of skills to be extended as the range of model types expands). Activity 3.8 arose later in solution processes and was usually activated by students who felt their existing calculations did not provide the amount of assurance or consistency they sought. Activities 3.3 to 3.7 are the action counterparts of the technology decision-making that occurs in transition 2; that is, as well as knowing what operations a calculator or computer can perform, it is necessary to be able to do them. For all models involving multiple data, the ability to set up and use the LISTs facility of a graphing calculator was essential to progress; as was the ability to use both spreadsheets and the function graphing capacity of a calculator. Inability here induced fatal blockages requiring specific intervention by teachers or capable peers.

The interpretive phase of the modelling process (transition 4) featured different requirements that varied from the very simple to the sophisticated. The simplest interpretative requirement is to achieve a one to one matching between a mathematical outcome and the real world entity that corresponds to it (as in a particular distance value associated with a corresponding run length). Others require greater depth: for example distinguishing between the needs of a question that can be resolved by means of a single calculation, and one that requires the comparison of several values that must first be computed. Sometimes the process is circular, as when an interpretive act is required to support a further modelling effort that builds upon it. And finally, the depth of interpretation varies from unsupported guesses devoid of mathematical support, to carefully integrated arguments in which mathematical outcomes are the central stimulus. In summary we have found this transition to be more complex than one built on the idea of interpretation as simply a translation between mathematical results and values of real world quantities. Correspondingly, the variety of demands described, provide a number of traps that block progress when unrecognised or underestimated.

The final transition is critical, for it embodies decisions that either accepts a modelling exercise as satisfying requirements, or rejects the model as unsatisfactory and revisits the problem solving process. With successful modellers not all model evaluation occurs at the end of course. Competent modellers constantly review mathematical outcomes for believability in terms of both expected values from known mathematical operations, and in terms of their knowledge of the real situation. Both these monitoring processes cause adaptive procedures to be activated if necessary.

In general terms evaluation of model output generally requires decisions at two levels: 1) Locally, do the individual results make separate sense in terms of the problem context? 2) Globally, does the model succeed in answering *all* the questions posed by the problem statement? A blockage at this point would be the inability to carry out some or all checking procedures, or more generally to accept a 'solution' from an inadequate model, so blocking the possibility of a better outcome. Constraints on time in the school contexts mean that we have not at this point been able to look at more than one cycle of the modelling process, so evaluation has depended essentially on local criteria in the sense described above.

7. Applications for the Framework

We conclude by outlining ways in which we see the current work within the wider field of applications and mathematical modelling in education. Firstly, a direct application derives from the way the research has been conducted. This is to identify specifically, activities with which modellers need to have competence, in order to successfully apply mathematics. The Framework (a dynamic structure) is our attempt to begin to document these. As the elements in the framework were identified by observing students working (and in particular wrestling with blockages to progress), there are two immediate potential applications.

First are the insights obtained into student learning, and how these can inform our understanding of the ways that students act when faced with application problems. Closely allied to this, are associated didactical insights. By identifying difficulties with generic properties, the possibility arises to predict where, in given problems, blockages of different types might be expected. This understanding will then contribute to the planning of teaching, in particular the identification of necessary prerequisite knowledge and skills, preparation of interventions for introduction at key points if required, and the scaffolding of significant learning episodes. In our work it became very clear that aspects of technology use, and the modelling process, mutually interfered with each other at various times. This occurred, for example, when students needed a special tutorial on the use of LISTs (diverting attention from the modelling process), and also emerged when almost no students were able to verify a model by drawing a function graph. Clearly cognitive demand is increased when attention must be divided between activating the modelling cycle, and puzzling about technical aspects of technology (or indeed by-hand mathematics). A lesson for didactics from our project is the importance of ensuring the prior competence of students with both the mathematics that will be involved in a model, and an understanding of, and facility with, technical procedures involved in using appropriate technology. These all have implications for the design and organisation of learning.

Other potential uses for the framework are in the design of modelling tasks, and in assessment. Taking account of the activities in the respective transitions should enable tasks to be designed so that all phases of the modelling cycle are adequately represented. It also highlights in an implementation when they are not. Regarding assessment, a potential means is provided for examining modelling reports, to see just what characterises them analytically, and how those recognised as qualitatively different might be compared. Finally of course, the framework when refined and stabilised may be useful as an instrument in classroom based research. The current development and then testing in these two settings described in this paper has taken us some way to achieving our goal.

8. References

Barab, S., & Squire, K.. (2004). Design-based research: Putting a Stake in the ground. *The Journal of the Learning Sciences, 13*(1), 1—14.

Blum , W., & Leiß, D. (2006). How do students and teachers deal with mathematical modelling problems? The example “Sugarloaf“ and the DISUM project. In C. Haines, P. Galbraith, W. Blum, & S. Khan, (Eds.), *Mathematical modelling (ICTMA12): Education, engineering and economics*. Chichester, UK: Horwood (to appear).

Charmaz, K. (2005). Grounded theory in the 21st century. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (3rd ed., pp. 507—536). Thousand Oaks, CA: Sage.

Edwards, D., & Hamson, M. (1996). *Mathematical modelling skills.* Basingstoke, UK: Macmillan.

Fishman, B., Marx, R. W., Blumenfeld, P., & Krajcik, J. (2004). Creating a framework for research on systemic technology innovations. *The Journal of the Learning Sciences, 13*(1), 43—76.

Galbraith, P. (2006). Dreaming a possible dream: More windmills to conquer. In C. Haines, P. Galbraith, W. Blum, & S. Khan, (Eds.), *Mathematical modelling (ICTMA12): Education, engineering and economics*. Chichester, UK: Horwood Press (to appear).

Galbraith, P., Goos, M., Renshaw, P., & Geiger, V. (2003). Technology enriched classrooms: Some implications for teaching applications and modelling. In Qi-Xiao Ye, W. Blum, K. Houston, & Qi-Yuan, Jiang (Eds.), *Mathematical modelling in education and culture: ICTMA 10* (pp. 111—125). Chichester, UK: Horwood.

Galbraith, P., Stillman, G., Brown, J., & Edwards, I. (2006). Facilitating middle secondary modelling competencies. In C. Haines, P. Galbraith, W. Blum, & S. Khan, (Eds.), *Mathematical modelling (ICTMA12): Education, engineering and economics*. Chichester, UK: Horwood Press (to appear).

Gellert, U., Jablonka, E., & Keitel, C. (2001). Mathematical literacy and common sense in mathematics education. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research on mathematics education: An international perspective*. Mahwah, NJ: Lawrence Erlbaum.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning, 1* (2), 155—177.

Julie, C. (2002). Making relevance relevant in mathematics teacher education. In I. Vakalis, D. Hughes Hallett, D. Quinney, & C. Kourouniotis (Compilers). *Proceedings of 2nd International Conference on the Teaching of Mathematics: [ICTM-2]*. New York: Wiley [CD-ROM].

Pollak, H. (1997). Solving problems in the real world. In L. A. Steen (Ed.), *Why numbers count: Quantitative literacy for tomorrow's America.* New York: College Entrance Examination Board.

Pollak, H. (2006). Mathematical modelling – A conversation with Henry Pollak. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss, (Eds.), *Modelling and applications in mathematics education, New ICMI Studies Series no. 10*. New York: Springer (to appear).

QSR (1997). Non-numerical Unstructured Data*Indexing Searching and Theorizing (NUD*IST) v.4 [Computer Software]. Melbourne, Australia: QSR.

Stillman, G. (1998). The emperor's new clothes? Teaching and assessment of mathematical applications at the senior secondary level. In P. Galbraith, W. Blum, G. Booker, & I. D. Huntley (Eds.), *Mathematical modelling: Teaching and assessment in a technology-rich world* (pp. 243—264). Chichester, UK: Horwood.

Stillman, G. (2002). Students' models, methods, limitations and assumptions for an applications task in a CAS calculator environment. In M. Goos & T. Spencer (Eds.), *Mathematics ~ making waves* (pp. 226—233). Adelaide, Australia: Australian Association of Mathematics Teachers.

Stillman, G. (2006). The role of challenge in engaging lower secondary students in investigating real world tasks. In E. Barbeau & P. Taylor (Eds.), *Proceedings of the ICMI Study 16: Mathematical Challenges*, Trondheim, Norway, Pre-Conference Vol. (to appear).

Victorian Curriculum and Assessment Authority (VCAA). (2005). *Victorian essential learning Standards: Discipline-based learning Mathematics*. Melbourne, Australia: Author.

Zbiek, R., & Conner, A. (2006). Beyond

motivation: Exploring mathematical modeling as a context for deepening students' understanding of curricular mathematics. *Educational Studies in Mathematics* (to appear).

Acknowledgement: The authors acknowledge the assistance of the teachers and Jill Brown in the classes where data were collected and in designing the tasks.

Authors

Peter Galbraith
Faculty of Education
The University of Queensland
St Lucia
Queensland
Australia, 4072
Email: p.galbraith@uq.edu.au

Gloria Stillman
Science and Mathematics Education
Faculty of Education
The University of Melbourne
Parkville
Victoria
Australia, 3010
Email: g.stillman@unimelb.edu.au