Plastic Relaxation of the Transformation Strain Energy of a Misfitting Spherical Precipitate: Ideal Plastic Behavior

J. K. LEE, Y. Y. EARMME, H. I. AARONSON, AND K. C. RUSSELL

Assuming ideal plastic behavior for an isotropic matrix containing a misfitting spherical precipitate, the total amount of work expended during elasto-plastic deformation is calculated and compared with the total strain energy in the corresponding pure elastic state. For precipitates larger than one micron (μm), the effective yield stress is taken as the macroscopic yield stress while for smaller precipitates, size-dependent yield stresses are obtained from the Ashby-Johnson model. In the case of coherent submicron precipitates, the effective yield stress becomes the theoretical yield strength and thus plastic relaxation is not possible unless the transformation stress is extremely large. For incoherent submicron precipitates, the effective yield stress is approximately inversely proportional to the precipitate radius, r. Hence plastic relaxation again is not possible when $r < 10$ nm, but when $r \simeq 100$ nm the strain energy can decrease by $10 \sim 40$ pct at a misfit of 3 pct. For supra-micron particles, however, the ratio of the effective yield stress to the shear modulus becomes 10^{-3} or less, and plastic relaxation can reduce the strain energy by factors of 3 to 15 at misfits of 1 to 3 pct. Under this circumstance, the plastic zone becomes wide, its radius ranging from 3 to 5 r.

THE strain energy associated with a solid state phase transformation diminishes the net driving force and hence the rate of reaction. In some cases the strain energy is greater than the thermodynamic driving force, precluding the transformation itself. Plastic relaxation of the transformation stress and strain energy associated with the formation of misfitting precipitates has been thought to assist the progress of the reaction. Examples of such relaxation in the matrix phase include the observation by Kinsman, Sprys and Asaro[1] of dislocation tangles in the Cu matrix surrounding Fe precipitates and the finding by Rigsbee and VanderArend[2] of a high density of dislocations in the ferrite regions near small martensitic regions in dual-phase steels.

Because of the absence of basic laws governing plasticity, as compared with Hooke's law in elasticity, exact details of the plastic relaxation process remain to be worked out, and are furthermore likely to be material-dependent. Nevertheless, considering the important nature of the problem in relation to phase transformations, precipitation hardening, and so forth, it seems worthwhile to estimate the extent of plastic relief of the transformation stress by assuming a simple stress-strain behavior. For mathematical tractability we assume perfect plastic behavior for the matrix phase and neglect the crystallographic nature of plastic flow. Thus the matrix phase is considered to yield under the condition of a constant yield stress* and the yielding is assumed independent of the orientation of the stress axis. However, a relationship between the precipitate size and yield stress will be considered based on the model of Ashby and Johnson.[4]

* Strain-hardening behavior of the matrix phase will be considered in a follow-up paper.[3]

We shall first obtain the stress, strain and strain energy associated with a misfitting spherical precipitate in the absence of plastic relaxation, *i.e.*, under purely elastic conditions, by the method of Christian.[5] Then, employing the foregoing assumptions and the solution[6,7] of a classical problem in plasticity theory, the elasto-plastic deformation of a thick hollow sphere under internal pressure, the total strain energy is determined for a misfitting spherical precipitate when plastic relaxation takes place in the matrix with a particle size-dependent yield stress. These results will be compared with their counterparts in the elastic case.

THEORY

A. Pure Elastic State

Let a misfitting spherical precipitate, whose radius is $a(1 + \epsilon)$ in the absence of constraints, be introduced into a spherical hole of radius a in an infinite matrix with shear modulus μ and Poisson's ratio ν. The solution of the misfitting sphere model has been obtained and discussed by many investigators[5,8,9] but is reformulated here in the manner of Christian[5] for comparison with the elastoplastic case in a later section. We use spherical coordinates whose origin is at the precipitate center. Suppose that the effective radius of the precipitate is $a(1 + \beta\epsilon)$ under the constraint of the matrix. Because of symmetry, the tangential displacements as well as the shear stresses and shear strains are

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all zero and the radial displacement u is a function of the radial distance r. Further, the equilibrium equations in the absence of body forces reduce to

$$\frac{d\sigma_r}{dr} + \frac{2(\sigma_r - \sigma_\theta)}{r} = 0, \quad [1]$$

where σ_r and σ_θ are radial and tangential stress components, respectively. The strains are related to the radial displacement u by

$$e_r = \frac{du}{dr}, \quad e_\theta = \frac{u}{r}. \quad [2]$$

For the third diagonal components, $\sigma_\phi = \sigma_\theta$ and $e_\phi = e_\theta$ are implicitly assumed. Hooke's law provides

$$e_r = \frac{1}{2\mu(1+\nu)}\{\sigma_r - 2\nu\sigma_\theta\},$$

$$e_\theta = \frac{1}{2\mu(1+\nu)}\{-\nu\sigma_r + (1-\nu)\sigma_\theta\}. \quad [3]$$

Substituting Eqs. [2] into Eqs. [3] and combining with Eq. [1] leads to:

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{2u}{r^2} = 0,$$

whose general solution is:

$$u = C_1 r + C_2/r^2. \quad [4]$$

The boundary conditions are

$$u_{r=a} = u_a = \beta\epsilon a, \; u_{r=0} = u_{n=\infty} = 0, \quad [5]$$

if the initial state is taken to be the introduction of the compressed precipitate into the hole of radius a. The displacements are thus

$$u = \beta\epsilon r; \qquad r \leq a \quad [6(a)]$$

$$u = \beta\epsilon a^3/r^2; \qquad r \geq a. \quad [6(b)]$$

Equation [6(b)] represents the displacement in the matrix relative to the unstressed state. However, since the initial state chosen for the precipitate is when it has been stressed from radius $a(1+\epsilon)$ to a, we have the displacement $u = (\beta - 1)\epsilon r$ relative to the unstressed state. Substituting $u = (\beta - 1)\epsilon r$ into Eqs. [2] yields the elastic strains within the precipitate:

$$e_r^I = e_\theta^I = (\beta - 1)\epsilon. \quad [7]$$

Again employing Hooke's law when the precipitate has a shear modulus μ^* and Poisson's ratio ν^* furnishes the elastic stress components,

$$\sigma_r^I = \sigma_\theta^I = 2\mu^* \frac{1+\nu^*}{1-2\nu^*}(\beta - 1)\epsilon. \quad [8]$$

Similarly, substitution of Eq. [6(b)] into Eqs. [2] gives the elastic strains in the matrix phase as:

$$e_r^M = -2e_\theta^M = -2\beta\epsilon\left(\frac{a}{r}\right)^3. \quad [9]$$

Again, application of Hooke's law (Eqs. [3] to Eq. [9]) provides the elastic stress components in the matrix as

$$\sigma_r^M = -2\sigma_\theta^M = -4\mu\beta\epsilon\left(\frac{a}{r}\right)^3. \quad [10]$$

Since the matrix and precipitate are in equilibrium, the radial stress component σ_r^M must be equal to the internal hydrostatic stress σ_r^I at the matrix: precipitate interface, $r = a$. This gives an expression for β:

$$\beta = \frac{\alpha\gamma}{\alpha(\gamma - 1) + 1}, \quad [11]$$

where $\alpha = (1+\nu)/3(1-\nu)$ and γ is the ratio of the bulk modulus K^* of the precipitate to that of the matrix, K, *i.e.*, $\gamma = K^*/K = \mu^*(1+\nu^*)(1-2\nu)/\mu(1+\nu)(1-2\nu^*)$. Therefore, the total elastic strain energy associated with a misfitting spherical precipitate becomes, per unit volume of the precipitate,

$$E_{el} = \frac{1}{2}(3\sigma_r^I e_r^I) + \frac{1}{2}\left(\frac{4\pi a^3}{3}\right)^{-1}\int_a^\infty (\sigma_r^M e_r^M + 2\sigma_\theta^M e_\theta^M) \times 4\pi r^2 dr = \frac{6\mu\alpha\gamma\epsilon^2}{\alpha(\gamma - 1) + 1}. \quad [12]$$

B. Plastic Deformation

When an effective elastic stress exceeds the yield stress of a metallic crystal, the crystal will undergo plastic deformation rather than remain in a purely elastic state. As shown by Eq. [8], the stress state within the spherical precipitate is hydrostatic. Since a hydrostatic stress state does not induce yielding,[10] the precipitate is considered to remain in a purely elastic state. Hence plastic deformation may be taken to occur only in the matrix adjacent to the precipitate, as shown schematically in Fig. 1, where r_p is the radius of the plastic zone.

For mathematical simplicity two additional assumptions are included in the model for the plasic behavior of the matrix. First we assume that the flow stress-strain behavior of the matrix phase is independent of the strain rate and of the stress orientation. Secondly, we neglect strain-hardening, so that the matrix phase is taken to be a perfectly plastic material whose flow stress-strain curve is horizontal at the yield stress, σ_y.

1. Elasto-plastic Shell. Based on the previous assumptions, we can now use the solution to the problem of the elasto-plastic deformation of a thick

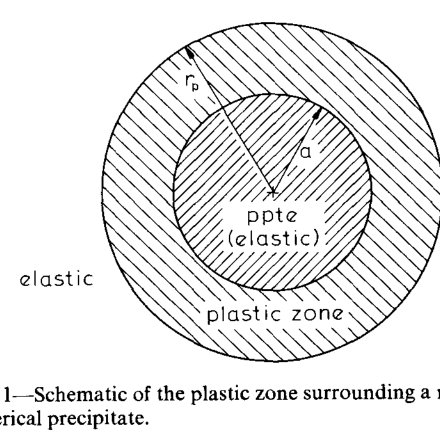

Fig. 1—Schematic of the plastic zone surrounding a misfitting spherical precipitate.

hollow sphere under internal pressure.[6,7] Consider the deformed region of the matrix to be the thick hollow sphere whose internal surface of radius a is subjected to a pressure p by the misfitting spherical precipitate. We note from Eqs. [8], [10] and [11] that by identifying p as $4\mu\beta\epsilon$, the stress components in the matrix can be expressed in a *pure elastic state* as

$$\sigma_r = -2\sigma_\theta = -p\left(\frac{a}{r}\right)^3. \qquad [13]$$

We first let the misfit ϵ be positive and add later a minor change for the case of $\epsilon < 0$.

We adopt the von Mises yielding criterion[11], namely that yielding occurs when an equivalent stress σ_e exceeds the yield stress σ_y, where σ_e is given by*

* Tresca's maximum shear stress theory[11] furnishes the same result. The yield criterion can also predict that no plastic deformation occurs within the spherical precipitate under a hydrostatic stress state.

$$\sigma_e = \frac{1}{\sqrt{2}}[(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_\phi)^2 + (\sigma_\phi - \sigma_r)^2]^{1/2}$$
$$= \sigma_\theta - \sigma_r. \qquad [14]$$

Substitution of Eqs. [13] into Eq. [14] yields $\sigma_e = (3/2)p(a/r)^3$ in the purely elastic state. Therefore, yielding will start at the matrix: precipitate interface when the pressure p reaches the critical value $2\sigma_y/3$. As the internal pressure p increases beyond the critical value, a plastic zone develops adjacent to the matrix: precipitate interface, extending to a radius depending upon the magnitude of p. Consider an instantaneous condition, when the plastic front of radius r_p separates the inner plastic shell from the outer elastic region. The equilibrium equation (Eq. [1]) for the plastic shell, in conjunction with the yield condition, $\sigma_\theta - \sigma_r = \sigma_y$, becomes:

$$\frac{d\sigma_r}{dr} - \frac{2\sigma_y}{r} = 0. \qquad [15]$$

Integrating Eq. [15] and using the boundary condition $\sigma_r = -p$ at $r = a$:

$$\sigma_r = \sigma_\theta - \sigma_y = 2\sigma_y \ln\left(\frac{r}{a}\right) - p; \; a \leq r \leq r_p. \qquad [16]$$

By substituting the critical value $2\sigma_y/3$ for p and r_p for a in Eq. [13], the stress components in the elastic region of the matrix are obtained as

$$\sigma_r = -2\sigma_\theta = -\frac{2\sigma_y}{3}\left(\frac{r_p}{r}\right)^3; \; r_p \leq r. \qquad [17]$$

Since the stresses must be continuous at the plastic front, the plastic zone radius r_p is obtained by equating Eqs. [16] and [17] at $r = r_p$. This yields an expression for r_p:

$$r_p = a \exp\left\{\frac{p}{2\sigma_y} - \frac{1}{3}\right\}. \qquad [18]$$

By applying the relationship between the displacement and the radial stress in the purely elastic state given in Eqs. [6(*b*)] and [10] to the radial stress given in Eq. [17], the displacement *outside the plastic zone* is found to be:

$$u = \frac{\sigma_y r_p^3}{6\mu r^2}; \; r_p \leq r. \qquad [19]$$

Substituting Eq. [19] into Eqs. [2]:

$$e_r = -2e_\theta = -\frac{\sigma_y}{3\mu}\left(\frac{r_p}{r}\right)^3; \; r_p \leq r. \qquad [20]$$

Within the plastic zone, the strains are the sum of the plastic and elastic strains. Since the elastic strains are related to stresses by Hooke's law, we may write

$$e_r = \frac{du}{dr} = \frac{1}{2\mu(1+\nu)}\{\sigma_r - 2\nu\sigma_\theta\} + e_r^p, \qquad [21]$$

and

$$e_\theta = \frac{u}{r} = \frac{1}{2\mu(1+\nu)}\{-\nu\sigma_r + (1-\nu)\sigma_\theta\} + e_\theta^p. \qquad [22]$$

Here, e_r^p and e_θ^p denote plastic strain components in the plastic zone. In order to obtain the displacement u in the plastic zone, we use the incompressibility condition for plastic strains, $e_r^p + 2e_\theta^p = 0$. Multiplying Eq. [22] by 2, adding the result to Eq. [21] and using Eqs. [16] yields:

$$\frac{du}{dr} + \frac{2u}{r} = \frac{2\sigma_y}{K}\ln\left(\frac{r}{a}\right) + \frac{2\sigma_y - 3p}{3K}, \qquad [23]$$

for which the general solution is

$$u = \frac{2\sigma_y}{3K} r \ln\left(\frac{r}{a}\right) - \frac{p}{3K} r + \frac{C}{r^2}. \qquad [24]$$

Substituting Eq. [19] into Eq. [24] with $r = r_p$ and rearranging the result with Eq. [18] furnishes an expression for u:

$$u = \frac{2\sigma_y}{3K} r \ln\left(\frac{r}{a}\right) - \frac{p}{3K} r + \frac{\sigma_y}{6\alpha\mu}\frac{r_p^3}{r^2}; \; a \leq r \leq r_p. \qquad [25]$$

From Eqs. [21], [22] and [25], the strains in the plastic zone are finally given as

$$\left.\begin{aligned} e_r &= \frac{2\sigma_y}{3K}\left[\ln\left(\frac{r}{a}\right) + 1\right] - \frac{p}{3K} - \frac{\sigma_y}{3\alpha\mu}\left(\frac{r_p}{r}\right)^3; \\ e_\theta &= \frac{2\sigma_y}{3K}\ln\left(\frac{r}{a}\right) - \frac{p}{3K} + \frac{\sigma_y}{6\alpha\mu}\left(\frac{r_p}{r}\right); \\ e_r^p &= -2e_\theta^p = \frac{\sigma_y}{3\alpha\mu}\left\{1 - \left(\frac{r_p}{r}\right)^3\right\}; \end{aligned}\right] \; a \leq r \leq r_p. \qquad [26]$$

Equations [16] to [20], [25] and [26] completely describe the stress, strain, and displacement in the matrix provided that the internal pressure p is known. We note that if the incompressibility condition for the total strains, *i.e.*, $e_r + 2e_\theta = 0$ is assumed, the solutions given in Eqs. [25] and [26] would reduce to those of Eqs. [19] and [20] because $K = \infty$ and $\alpha = 1$ in that situation.

2. Equilibrium Internal Pressure. In order to determine the equilibrium pressure p which obtains after plastic relaxation, we use the same procedure employed in the case of the pure elastic state. First, the continuity of the (constrained) displacement at the matrix: precipitate interface requires that, from Eq. [6(*a*)] and Eq. [25] with $r = a$,

$$\beta\epsilon a = -\frac{pa}{3K} + \frac{\sigma_y}{6\alpha\mu}\frac{r_p^3}{a^2}$$

or

$$\frac{6\alpha\mu}{\sigma_y}\left\{\beta\epsilon + \frac{p}{3K}\right\} = \exp\left\{\frac{3p}{2\sigma_y} - 1\right\}, \qquad [27]$$

where Eq. [18] is used. The continuity of the radial stress σ_r at the matrix: precipitate interface yields, from Eqs. [8] and [16] with $r = a$,

$$p = 3K\gamma\epsilon(1 - \beta). \qquad [28]$$

Substituting Eq. [28] into Eq. [27] for p provides an equation for β:

$$\frac{6\alpha\mu\epsilon}{\sigma_y}(\gamma + \beta - \gamma\beta) = \exp\left\{\frac{9K\gamma\epsilon}{2\sigma_y}(1 - \beta) - 1\right\}. \qquad [29]$$

Equations [28] and [29] combined with the stress, strain and displacement in the previous section solves the problem completely for a given set of material parameters, μ, ν, σ_y, ϵ and γ.

3. Strain Energy. The elastic strain energy inside the spherical precipitate, W_{ppt}, per unit volume of the precipitate is readily given as

$$W_{\text{ppt}} = \frac{1}{2}\left(3\sigma_r^I e_r^I\right) = \frac{3}{2}p(1 - \beta)\epsilon = \frac{p^2}{2\gamma K}, \qquad [30]$$

where p and β are given in Eqs. [28] and [29], respectively.

The total work consumed in the plastic zone is the sum of the plastic strain energy and the elastic strain energy. First, we seek the plastic strain energy, W_{plastic}. The plastic work per unit volume of an element located at distance r is, by definition[7]

$$\omega^p(r) = \int d\omega^p(r) = \int \sigma_{ij}de_{ij}^p$$
$$= \int_0^{e_r^p} \sigma_r de_r^p + \int_0^{e_\theta^p} 2\sigma_\theta de_\theta^p, \qquad [31]$$

where the usual tensor suffix notation is used in the first equation. From the incompressibility condition, $e_r^p + 2e_\theta^p = 0$ for the plastic strains we have $de_r^p = -2de_\theta^p$. Substituting this relationship into Eq. [31] and noting that $\sigma_r - \sigma_\theta = -\sigma_y =$ a constant, we obtain

$$\omega^p(r) = \int_0^{e_r^p}(\sigma_r - \sigma_\theta)de_r^p = -\sigma_y e_r^p$$
$$= \frac{\sigma_y^2}{3\alpha\mu}\left\{\left(\frac{r_p}{r}\right)^3 - 1\right\}^*, \qquad [32]$$

* Note that the result is path-independent.

where the expression for e_r^p in Eqs. [26] is used.

Integrating $\omega^p(r)$ over the plastic zone and dividing the result by the precipitate volume:

$$W_{\text{plastic}} = \left(\frac{4\pi a^3}{3}\right)^{-1}\int_a^{r_p}\frac{\sigma_y^2}{3\alpha\mu}\left\{\left(\frac{r_p}{r}\right)^3 - 1\right\}4\pi r^2 dr$$
$$= \frac{\sigma_y^2}{\alpha\mu}\left\{\left(\frac{r_p}{a}\right)^3\ln\left(\frac{r_p}{a}\right) - \frac{1}{3}\left(\frac{r_p}{a}\right)^3 + \frac{1}{3}\right\}. \qquad [33]$$

The elastic strain energy, W_{elastic}, *stored in the plastic zone* is given as

$$W_{\text{elastic}} = \left(\frac{4\pi a^3}{3}\right)^{-1}\int_a^{r_p}\frac{1}{2}\left\{\sigma_r\left(e_r - e_r^p\right)\right.$$
$$\left. + 2\sigma_\theta\left(e_\theta - e_\theta^p\right)\right\}4\pi r^2 dr. \qquad [34]$$

Substituting Eqs. [16] for the stresses and Eqs. [26] for the strains into Eq. [34] and rearranging,

$$W_{\text{elastic}} = \sigma_y^2\left(\frac{r_p}{a}\right)^3\left(\frac{2}{9K} + \frac{1}{6\alpha\mu}\right) - \frac{\sigma_y^2}{6\alpha\mu} - \frac{p^2}{2K}. \qquad [35]$$

Similarly, the elastic strain energy stored *outside* the plastic zone is

$$W_{\text{out}} = \left(\frac{4\pi a^3}{3}\right)^{-1}\int_{r_p}^{\infty}\frac{1}{2}$$
$$\times(\sigma_r e_r + 2\sigma_\theta e_\theta)4\pi r^2 dr, \qquad [36]$$

and substituting Eqs. [17] and [20] into Eq. [36] yields:

$$W_{\text{out}} = \frac{\sigma_y^2}{6\mu}\left(\frac{r_p}{a}\right)^3. \qquad [37]$$

Combining Eqs. [30], [33], [35] and [37], the total strain energy involved in elastoplastic deformation per unit volume of the spherical precipitate is:

$$W = W_{\text{ppt}} + W_{\text{plastic}} + W_{\text{elastic}} + W_{\text{out}}$$
$$= \frac{p^2}{2\gamma K}(1 - \gamma) + \frac{\sigma_y^2}{\alpha\mu}\left\{\frac{1}{6} + \left(\frac{r_p}{a}\right)^3\ln\left(\frac{r_p}{a}\right)\right\}. \qquad [38]$$

Finally, in the case of $\epsilon < 0$, *i.e.*, when the misfit parameter, ϵ, is negative, the only change required is to replace σ_y with $-\sigma_y$ in all the above relationships because the yield condition now becomes $\sigma_\theta - \sigma_r = -\sigma_y$.

C. Effective Yield Stress

A typical annealed crystal is known to have a dislocation density of $10^6 \sim 10^8$ cm/cm^3. If the size of a precipitate is of the order of a micron (μm) or larger, the precipitate can have a high probability of having in its neighborhood dislocations which contribute to plastic deformation induced by the transformation stress. In the case of a submicron particle, however, the availability of nearby preexisting dislocation which might contribute to the generation of further dislocations would be negligible. Therefore, the effective yield stress, σ_y, defined in the previous section, may be considered as a macroscopic yield stress for a supramicron particle, but as a size-dependent critical stress for a submicron precipitate.

Experimental results[12,13] have indeed shown that for certain systems plastic relaxation of a misfitting precipitate depends upon the particle size, *i.e.*, for a given misfit, there is a critical precipitate size below which no plastic deformation is observed. The existence of a critical size for a given misfit is related to the fact that the transformation stress must be larger than the stress necessary for the generation of dislocations. Several models have been developed for nucleation of dislocations at the particle interface. Brown *et al*[14,15] considered generation of prismatic dislocation loops while Ashby and Johnson[4] studied generation of shear dislocation loops. Although other investigators[16,17] have extended or modified these two models, a relationship of the form $e_c^* \simeq A a^n$ for the critical misfit, e_c^*, to just cause nucleation of a dislocation for a given particle radius, a, seems to prevail regardless of the details of the model. The theoretical value of the exponent, n, is about $0.7 \sim 0.9$ and is in good agreement with experiments.[12,13] Therefore, in order to formulate the size-dependence of the effective yield stress, we employ Ashby and Johnson's model[4] with some modifications for mathematical simplicity.

As shown in Fig. 2, the Ashby and Johnson model begins with a shear dislocation loop which resides in the plane of maximum shear stress due to the misfitting precipitate. In terms of Cartesian coordinates, the stress in the matrix in the purely elastic state (Eq. [10]) is given by[4,8]

$$\sigma_{ij}^M = 2\mu e_c a^3 \left[\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5} \right] \qquad [39]$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$, $e_c = \beta\epsilon$ and δ_{ij} is the Kronecker delta function. The maximum value of the shear stress, σ_{13}^M, occurs in the plane $x_3 = a/\sqrt{2}$. The change in energy of the system, $\Delta\Phi$, due to the introduction of a shear dislocation loop of radius, r_l, is [4]

$$\Delta\Phi = \frac{\mu b^2 r_l}{4}\left(\frac{2-\nu}{1-\nu}\right)\left[\ln\left(\frac{8r_l}{b}\right) - 2\right] + \iint \sigma_{13}^M b\, dA, \qquad [40]$$

where the first term is the self strain energy of the dislocation loop, the second term is the interaction energy between the dislocation and misfitting precipitate, the integration is over the loop area and b is the Burgers vector. Ashby and Johnson numerically integrated the interaction term by considering a rectangular loop with the same area as the circularly shaped loop. For mathematical simplicity, we use a circular loop but instead of integrating employ an average shear stress approximated by the σ_{13}^M value at the loop center. By using σ_{13}^M at the loop center (see Fig. 2), Eq. [40] becomes:

$$\Delta\Phi = \frac{\mu b^2 r_l}{4}\left(\frac{2-\nu}{1-\nu}\right)\left[\ln\left(\frac{8r_l}{b}\right) - 2\right] - 3\pi\mu b r_l^2 e_c \left\{ \frac{1+\sqrt{2}\rho}{(1+\sqrt{2}\rho+\rho^2)^{5/2}} \right\}, \qquad [41]$$

where $\rho = r_l/a$. Taking $\nu = 1/3$ and dividing Eq. [41] by μb^3:

$$\frac{\Delta\Phi}{\mu b^3} = 0.625\left(\frac{r_l}{b}\right)\left[\ln\left(\frac{8r_l}{b}\right) - 2\right] - 3\pi\left(\frac{r_l}{b}\right)^2 e_c \left\{ \frac{1+\sqrt{2}\rho}{(1+\sqrt{2}\rho+\rho^2)^{5/2}} \right\}, \qquad [42]$$

1. Without Activation Energy: Weatherly Criterion. The introduction of a dislocation loop into the system *without an activation energy barrier* requires that $\Delta\Phi$ be negative for all r_l. Assuming a minimum dislocation loop size of $r_l = 2b$* and requiring $\Delta\Phi = 0$ from Eq. [42]:

$$e_c^* = 0.0256 \left\{ \frac{(1+\sqrt{2}\rho+\rho^2)^{5/2}}{1+\sqrt{2}\rho} \right\}, \qquad [43]$$

where $\rho = 2b/a$. Since the equivalent stress σ_e is equal to $6\mu e_c (e_c = \beta\epsilon)$ at the precipitate: matrix interface before plastic deformation starts (see Eq. [14]), we may infer that

$$\sigma_y = 6\mu e_c^* = 0.154 \left\{ \frac{(1+\sqrt{2}\rho+\rho^2)^{5/2}}{1+\sqrt{2}\rho} \right\} \mu . \qquad [44]$$

Since $\rho \ll 1$ in typical cases, σ_y is essentially independent of particle size and thus approximates the theoretical yield strength, $\mu/2\pi$.[18] This is referred to as the Weatherly criterion[19] by Ashby and Johnson.

*The difficulty associated with the definition of the dislocation core limits the meaning of a dislocation loop of "minimum" size. However, the choice of $r_l = 2 \sim 4b$ does not change Eq. [43] significantly.

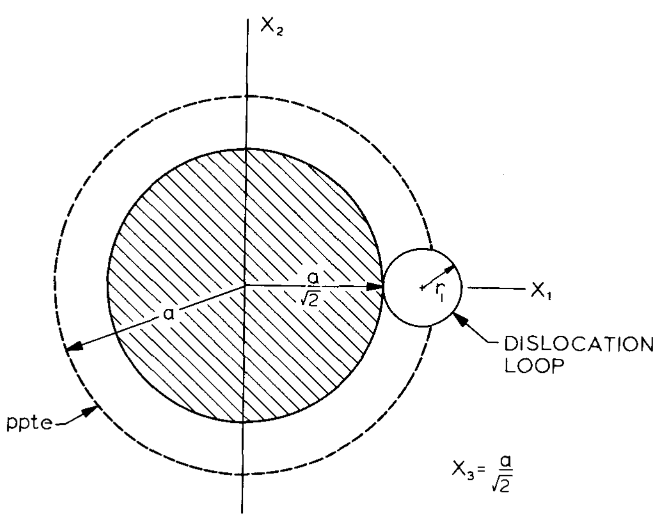

Fig. 2—A shear dislocation loop of radius r_l in the plane of $x_3 = a/\sqrt{2}$.

2. *With Activation Energy.* It can also be energetically feasible to introduce a shear dislocation into the system even when an activation energy barrier must be overcome by a mechanism such as one based upon a supersaturation of point defects.[4] In this case, the criterion for generation of a shear loop is given by two simultaneous conditions. For a given precipitate radius, a, they are $\Delta\Phi(r_l = r_l^*, e_c = e_c^*) = 0$ and $\partial\Delta\Phi(r_l = r_l^*, e_c = e_c^*)/\partial r_l = 0$.[4] These conditions yield:

$$\ln\left(\frac{8r_l^*}{b}\right) = \frac{3 + 3\sqrt{2}\rho - 7\rho^2 - 5\sqrt{2}\rho^3}{1 + \rho/\sqrt{2} - 5\rho^2 - 3\sqrt{2}\rho^3}, \qquad [45]$$

and

$$e_c^* = \frac{0.625}{3\pi}\left(\frac{b}{r_l^*}\right)\left[\ln\left(\frac{8r_l^*}{b}\right) - 2\right] \times \left[\frac{(1 + \sqrt{2}\rho + \rho^2)^{5/2}}{1 + \sqrt{2}\rho}\right], \qquad [46]$$

where $\rho = r_l^*/a$. As in the previous case, we again define σ_{yx} as

$$\sigma_y = 6\mu e_c^* = \frac{1.25}{\pi}\left(\frac{b}{r_l^*}\right)\left[\ln\left(\frac{8r_l^*}{b}\right) - 2\right] \times \left[\frac{(1 + \sqrt{2}\rho + \rho^2)^{5/2}}{1 + \sqrt{2}\rho}\right]\mu. \qquad [47]$$

For a given precipitate radius, a, the values of r_l^* and ρ are first determined from Eq. [45] and then used in Eq. [47] to evaluate σ_y.

3. *Brooks Criterion.* Clearly, a necessary condition for a dislocation loop to generate at the interface of a misfitting precipitate is that the displacement must be larger than Burgers vector, b This is known as the Brooks criterion[20] which may be expressed as

$$ae_c \geq b. \qquad [48]$$

Again employing the relationship $\sigma_y = 6\mu e_c$, we have, for the yield stress,

$$\sigma_y = 6\mu b/a. \qquad [49]$$

As we will see later, when a precipitate is large ($a \gg b$), the Brooks criterion is usually satisfied. However, when the particle is small, say, less than $40b$ in radius, this criterion cannot be met, and the system remains in a purely elastic state.

RESULTS

A. Large Precipitates (Supramicron Radii)

As pointed out earlier, the effective yield stress, σ_y, can be considered as a macroscopic yield stress for precipitates of about a micron or above in radius. This is due to the likelihood that preexisting dislocations will be available in its immediate neighborhood. The yield stress is, therefore, particle size independent. We now calculate the stresses, strains, and strain energy as functions of the stress-free transformation strain for two different yield stresses and for various ratios of the precipitate to the matrix bulk moduli, γ. The yield stresses are selected so as to represent low and high temperature behavior; the variations in moduli exemplify the difference in behavior between hard and soft precipitates.

Figure 3(*a*) and (*b*) portray the ratio of the total strain energy with plastic relaxation to the total strain energy in the corresponding pure elastic state as a function of the misfit, ϵ, for three different γ's. In Fig. 3(*a*) the value of σ_y/μ is taken as 10^{-3} while 10^{-5} is used in Fig. 3(*b*). Note that the energy scale in the latter case is enlarged by a factor of 10 to portray the greater plastic relaxation at the lower yield stress. The amount of plastic relaxation is seen to increase as the bulk modulus of the precipitate increases relative to that of the matrix.

When $\gamma \gg 1$, *i.e.* when the precipitate is elastically much harder than the matrix, all of the elastic strains are accomodated within the matrix phase (see Eqs. [7], [9], [11]). In the reverse case of $\gamma \simeq 0$, the elastic strains are fully accommodated within the precipitate. Hence when γ is large, since the pressure, *i.e.*, the radial stress at the matrix: precipitate interface (see Eqs. [10] and [16]), is proportional to the magnitude of the misfit, the extent of relaxation increases with misfit as shown in Figs. 3(*a*) and (*b*).

Figure 4 shows that the fraction of the transformation strain energy dissipated as plastic work and the plastic zone size both increase with transformation strain, ϵ. This reflects the limited ability of the matrix to accommodate large strains elastically. Figure 5 presents the normalized stress as a function of the normalized radial distance for $\sigma_y = 10^{-3}\mu$, $\gamma = 5$ and $\epsilon = 0.2$ pct. The value unity for r/a corresponds to the matrix: precipitate interface. The stress distributions in the purely elastic state are represented by dashed curves while those of the elastoplastic state are displayed by solid curves. We note that the difference in stress state between the two cases are substantial within the precipitate as well as in the plastic zone, but not in the elastic region of the matrix. Inside the plastic zone, the tangential stress, σ_θ, is quite different from its counterpart of the pure elastic case in that its sign is reversed.

Figure 6 exhibits the total strains (e_r, e_θ) as well as the plastic strains (e_r^p, e_θ^p) as a function of reduced radius. We note that the plastic strains disappear at the plastic front, *i.e.*, at $r = r_p$. As in the case of the stress, the elastic strain within the precipitate is reduced by a factor of ca. 4 by plastic relaxation in this example.

As mentioned in Introduction, the assumption of ideally plastic behavior is a severe approximation. However, in a study of the deformation of tungsten single crystals, Rose *et al*[21] found a fairly broad plateau in the stress-strain curve for a [110] tensile axis, and a less broad one for a [100] axis at room temperature. Their results give an effective $\sigma_y/\mu \simeq 5 \times 10^{-3}$. In addition, the model requires an average yield stress taking into acount all the orientations of the tensile axes. In view of these arguments, the results of Fig. 3(*a*), obtained with $\sigma_y = 10^{-3}\mu$, is considered to represent better the actual situation at room temperature. On the other hand, $\sigma_y = 10^{-5}\mu$ (Fig. 3(*b*)) may be applicable to

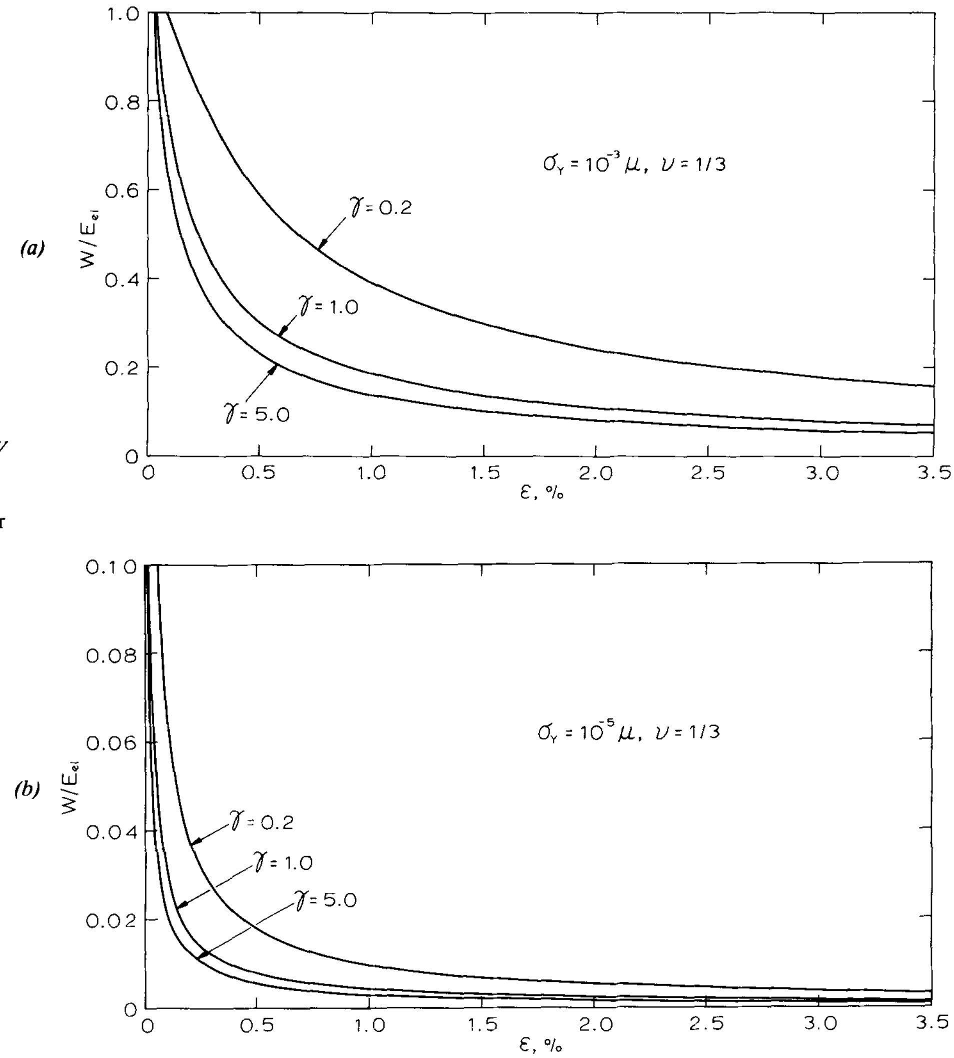

Fig. 3—The ratio of the total strain energy W with plastic relaxation to the total strain energy E_{el} in the corresponding pure elastic state *vs* the misfit ϵ. Three different values for the bulk modulus ratio γ are considered in each case. (*a*) $\sigma_y = 10^{-3}\mu$ and (*b*) $\sigma_y = 10^{-5}\mu$.

some high temperature situations, say, above the recrystallization temperature, at which there is practically no strain hardening. Note that at a misfit of 1 pct, the total strain energy W is $\sim$40 times less at $\sigma_y = 10^{-5}\mu$ than at $10^{-3}\mu$.

B. Small Precipitates (Submicron)

The general argument presented in the previous section is still valid for a small precipitate, but the yield stress must be considered to be dependent on the precipitate size and is thus higher than the macroscopic yield stress of $\sim 10^{-3} - 10^{-5}\mu$. In Fig. 7, the nearly-horizontal line (broken) represents the yield stress *without* the need to overcome an activation energy (Eq. [44]). This is the Weatherly criterion[19] and is essentially particle size-independent except for very small precipitates. The yield stress is basically equal to theoretical yield strength, $\mu/2\pi$. Weatherly[12,19] suggested that *coherent* precipitates such as Ni_3Al in Ni-Al alloy* may require a transformation stress higher than the theoretical yield strength in order to have plastic relaxation. The transformation stress in the Ni-based creep resistant alloys is actually less than the yield stress and thus there should be no plastic deformation around the Ni_3 (Al, Ti, Si) particles, which is consistent with experimental observations.

An *incoherent boundary* is a source of point defects as well as dislocations. Therefore, for a given transformation stress an incoherent precipitate will be able to undergo a plastic deformation with greater ease than the corresponding coherent case. The yield stress *with activation energy* given in Eq. [47] represents this case and is indicated by a solid line in Fig. 7. The yield stress is now strongly dependent on the particle radius, a. For comparison with actual observations, several experimental results[12-14,22,23] are given in Fig. 7. Although the experimental data are somewhat scattered, probably due to the variations in experimental technique, they are in a good overall agreement with the yield stress as predicted by Eq. [47]. Since σ_y and a are linearly related on a log-log scale, we may approximate the relationship between them as

$$\sigma_y = 3.13\mu/a^{0.794}, \qquad [50]$$

where a is in units of the Burgers vector, b. The

* Strictly speaking, the present analysis is not valid for non-spherical precipitates. However, for a given misfit, it should be easier for cuboidal precipitates such as Ni_3Al to have plastic relaxation than for spherical precipitates, because of the large stress concentrations at the corners and edges.[24]

exponent $n = 0.794$ is in excellent agreement with the Ashby and Johnson value[4] of 0.79 and in quite good agreement with the Brown and Woolhouse value of 0.86.[15]

Figure 7 also displays the Brooks criterion (Eq. [49]) as the broken-dashed line in the upper left-hand corner. When the particle is very small, say, less than $40b$ ($\simeq 10$ nm), the necessary and sufficient condition for plastic deformation becomes the Brooks criterion. Therefore, it is unlikely that a precipitate of less than 10 nm would have in its vicinity dislocation tangles due to a transformation strain of the usual magnitude, *i.e.*, 2 ~ 5 pct. The Brooks criterion may indicate that there should be no substantial plastic relaxation associated with G.P. zone of 5 nm in radius even though their stress-free transformation strains are often quite large.[25]

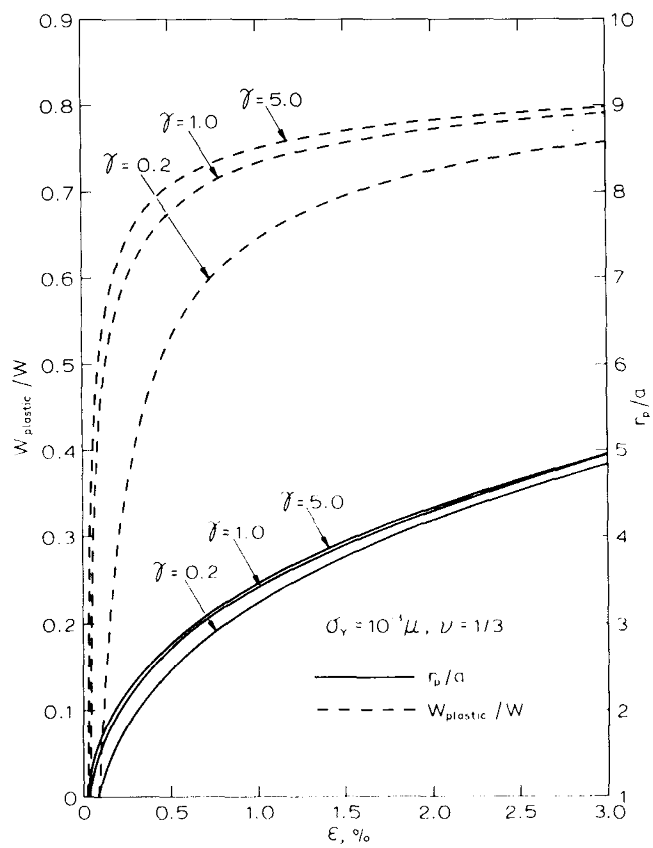

Fig. 4—The plastic zone radius r_p *vs* the misfit ϵ and the ratio $W_{plastic}/W$ *vs* the misfit ϵ. a is the precipitate radius and $W_{plastic}$ is the plastic strain energy.

Finally, Fig. 8 displays the ratio of the total strain energy with plastic relaxation to the total strain energy in the corresponding pure elastic state as a function of precipitate radius for three different bulk modulus ratios, γ. Here, the misfit ϵ is taken to be 3 pct, and the Brooks criterion (Eq. [49]) is used for the yield stress requirement when $a < 24b$ and Eq. [50] is employed for $a \geq 24b$. The corresponding plastic zone radii are also plotted with broken curves in Fig. 8. We note that when the particle size is less than $50b$ ($\simeq 13$ nm), plastic relaxation is not energetically favorable compared to the purely elastic state, even if the precipitate is three times harder than the matrix phase ($\gamma = 3$). For this misfit of 3 pct, Fig. 8 also shows that the plastic zone size is about 1 to 4 times larger than the precipitate radius and is not sensitive to the bulk modulus ratio.

SUMMARY

The degree of plastic relaxation is strongly dependent on particle size. When a precipitate is of the order of micron or above, the effective yield stress is identified as a macroscopic yield stress while for smaller particles we deduced a size-dependent yield stress from the Ashby-Johnson model.[4] In the latter case, there are two effective yield stresses which depend upon the coherency of the matrix: precipitate interface. One is essentially independent of the precipitate size and equal to theoretical yield strength, $\mu/2\pi$. For coherent precipitates, the transformation stress must overcome this high yield stress in order to have plastic relaxation.[19]

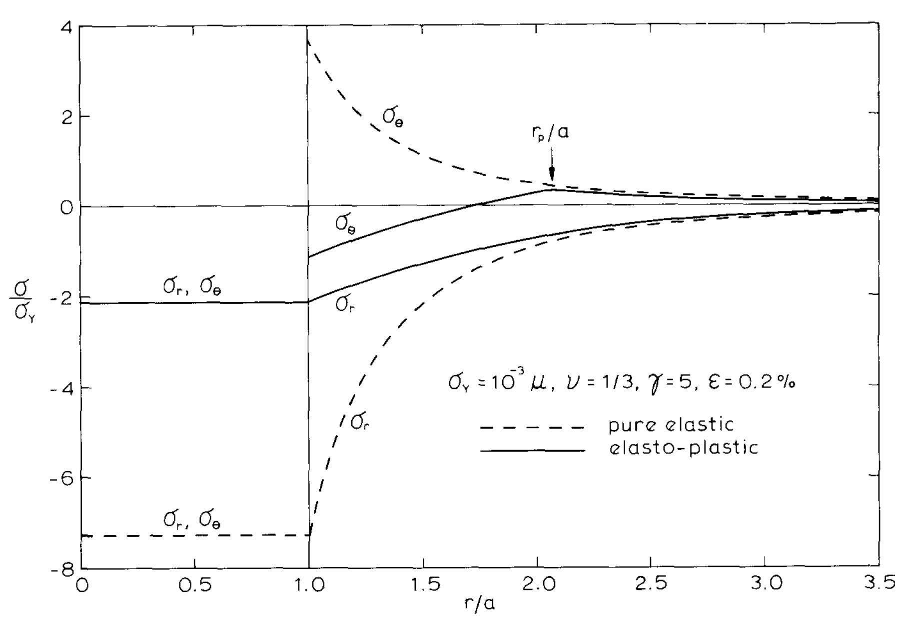

Fig. 5—The stress distribution within and without a spherical precipitate when $\sigma_y = 10^{-3}\mu$, $\gamma = 5$ and $\epsilon = 0.2$ pct. The matrix: precipitate interface lies at $r/a = 1$. The solid curves represent the stresses in the case of elasto-plastic deformation while the broken curves exhibit the stresses in the corresponding pure elastic state. The stresses are normalized with respect to σ_y.

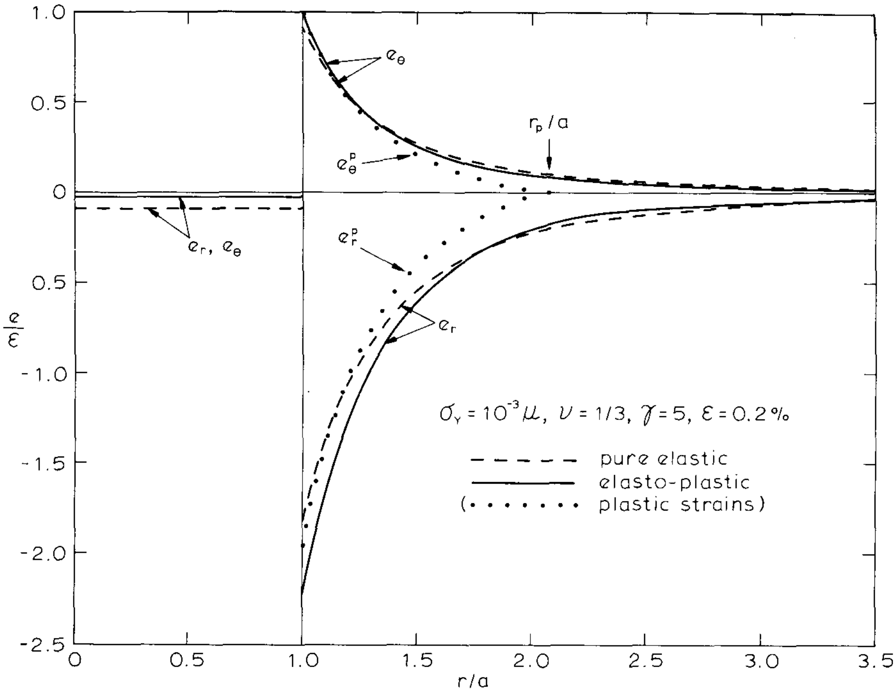

Fig. 6—The strain distribution within and without a spherical precipitate when $\sigma_y = 10^{-3}\mu$, $\gamma = 5$ and $\epsilon = 0.2$ pct. The plastic strains are included as dotted curves. All strains are normalized with respect to the misfit ϵ.

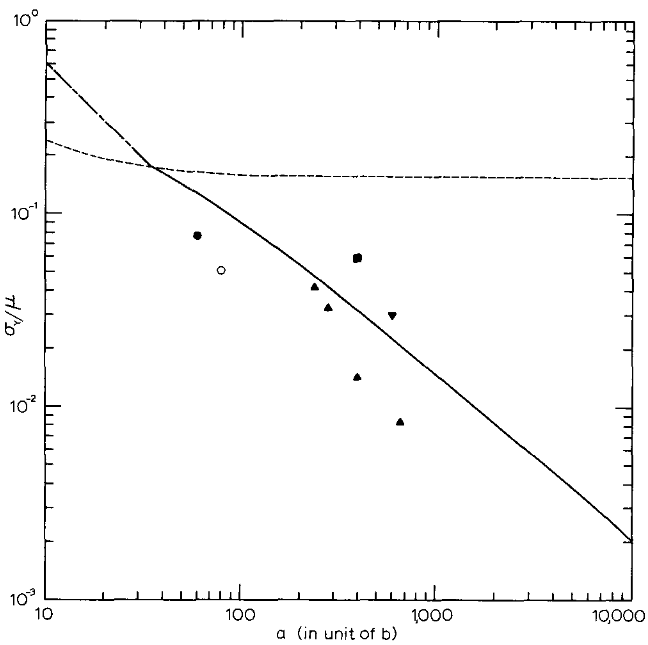

Fig. 7—Effective yield stress σ_y *vs* precipitate radius a.--- Weatherly criterion, —·—·— Brooks criterion, —— for incoherent precipitates. ▲; Cu-SiO_2 Ashby *et al*,[13] ▼; Cu-SiO_2 Weatherly,[12] ■; Fe-NbC Weatherly[12] ○; Cu-Co Philips[22] and Brown *et al*,[14] ●; Easterling.[23]

The other effective yield stress is a strong function of precipitate radius. It approaches the low-temperature macroscopic yield stress, $10^{-3}\mu$, when the precipitate size becomes of the order of a few microns. It applies to misfitting incoherent precipitates and is shown to be in good agreement with the published experimental data. With this yield criterion, the strain energy associated with a spherical precipitate of $r = 100$ nm ($\simeq 400b$) is found to decrease by $10 \sim 40$ pct at the misfit value of $\epsilon = 0.03$ through plastic relaxation. The theoretical plastic zone size is shown to be $1.5 \sim 1.8$ precipitate radii, and is in a fair agreement with the plastic zone size of $2 \sim 4$ precipitate radii observed in the work of Ashby, Gelles and Tanner in the Cu-SiO_2 system.[13] When the precipitate becomes much smaller, the effective yield stress increases beyond the theoretical yield strength because of the Brooks criterion,[20] and thus practically no plastic deformation can occur for small precipitates of $r < 10$ nm unless their transformation strains are exceptionally large.

These results are relevant to nucleation kinetics, since, from the present analysis, there would be no plastic relaxation for small nuclei of < 10 nm whether they are coherent or incoherent. Since interfacial energy is in most cases the dominant term during nucleation and coherent interfaces have a lower energy than incoherent interfaces,[26] a precipitate can be considered coherent or at least partially coherent during its early stages of nucleation and growth. Therefore, the coherency of the interface dictated by the nucleation barrier may further delay the onset of plastic deformation until the precipitate grows to substantial size. As the precipitate becomes large, the interface may change from a fully coherent to a partially coherent structure by punching out either prismatic or shear dislocation loops provided that the transformation stress exceeds the theoretical yield strength. On the other hand, if the structures of the precipitate and matrix phases are much different from each other, the maintenance of coherency will be difficult even in the early stages of precipitation and plastic relaxation will occur easily as long as the transformation stress is larger than the corresponding size-dependent yield stress.

When coherency is maintained during growth and the transformation stress is less than the theoretical yield strength, the particle may grow without plastic relaxation until it reaches a size of a micron or above. Preexisting dislocation sources in the neighborhood of

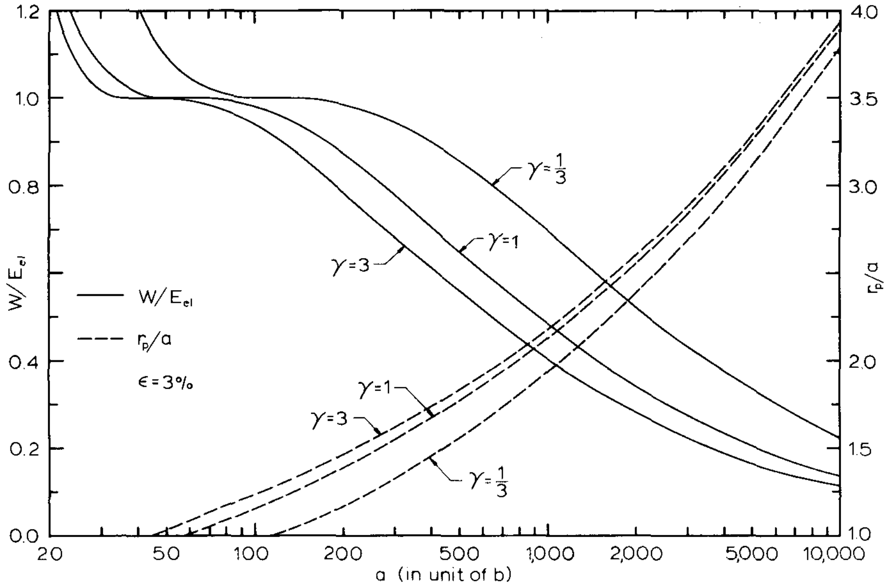

Fig. 8—W/E_{el} *vs* precipitate radius a(—) and plastic radius r_p *vs* precipitate radius a (----) for ϵ = 3 pct. Three different values for the bulk modulus ratio γ are considered.

the precipitate may then become active in generating dislocations since the macroscopic yield stress can be easily surpassed by the transformation stress. Under this circumstance, the strain energy associated with a misfitting spherical precipitate of supra-micron size can decrease by a factor of 3 to 15 at misfit of 1 ~ 3 pct through plastic relaxation. The plastic zone becomes wide, its radius ranging from 3 to 5 precipitate radii.

When plastic relaxation is possible, the stress and strain distributions are altered quite drastically in the elasto-plastic state relative to those in the purely elastic state. Plastic relaxation can markedly reduce the stresses in both the precipitate and the plastic zone and in addition causes the tangential stress component to change sign in the latter region. Outside the plastic zone, however, the stress and strain distributions are similar in both cases. We admit that the present model has rather considerable limitations resulting from the assumption of elastic isotropy, ideal plastic behavior and a spherical precipitate morphology, which were made in order to make the problem mathematically tractable. To achieve wider applicability anisotropic elasticity effects must be considered, and strain-hardening of the matrix and deviation from sphericity in shape should be taken into account. Although these factors may change some of the present results, the general aspects obtained with the present simple model may be indicative of the behavior to be expected in the more general case.

NOMENCLATURE

a	spherical precipitate radius
e_r	radial strain component
e_r^p	radial plastic strain component
e_θ	tangential strain component
e_θ^p	tangential plastic strain component
K	bulk modulus of the matrix phase
K^*	bulk modulus of the precipitate phase
r_p	plastic zone radius
u	radial displacement
α	equal to $(1 + \nu)/3(1 - \nu)$
β	constrained displacement parameter
γ	equal to K^*/K
ϵ	misfit parameter
μ	shear modulus of the matrix phase
μ^*	shear modulus of the precipitate phase
ν	Poisson's ratio of the matrix phase
ν^*	Poisson's ratio of the precipitate phase
σ_e	equivalent stress
σ_r	radial stress component
σ_y	yield stress of the matrix phase
σ_θ	tangential stress component

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REFERENCES

1. K. R. Kinsman, J. W. Sprys, and R. J. Asaro: *Acta Metall.*, 1975, vol. 23, p. 1431.
2. J. M. Rigsbee and P. J. VanderArend: *Formable HSLA and Dual-Phase Steels*, A. T. Davenport, ed., p. 58, TMS-AIME, New York, NY.
3. Y. Y. Earmme, W. C. Johnson, and J. K. Lee: unpublished research, Michigan Technological University, Houghton, MI, 1980.
4. M. F. Ashby and L. Johnson: *Philos. Mag.*, 1969, vol. 20, p. 1009.
5. J. W. Christian: *The Theory of Transformations in Metals and Alloys—Part I*, 2nd ed., p. 200, Pergamon Press, Oxford, 1975.
6. R. Hill: *The Mathematical Theory of Plasticity*, p. 97, Oxford University Press, London, 1950.
7. A. Mendelson: *Plasticity—Theory and Application*, p. 135, MacMillan, New York, 1968.
8. J. D. Eshelby: *Prog. Solid Mech.*, 1961, vol. 2, p. 89.

9. D. M. Barnett: *Scr. Metall.*, 1971, vol. 5, p. 261.
10. P. W. Bridgman: *J. Appl. Phys.*, 1947, vol. 18, p. 246.
11. G. E. Dieter: *Mechanical Metallurgy*, p. 58, McGraw-Hill, New York, 1961.
12. G. C. Weatherly: *Met. Sci.*, 1968, vol. 2, p. 237.
13. M. F. Ashby, S. H. Gelles, and L. E. Tanner: *Philos. Mag.*, 1969, vol. 19, p. 757.
14. L. M. Brown, G. R. Woolhouse, and U. Valdré: *Philos. Mag.*, 1968, vol. 17, p. 781.
15. L. M. Brown and G. R. Woolhouse: *Philos. Mag.*, 1970, vol. 21, p. 329.
16. J. W. Matthews and S. Mader: *Scr. Metall.*, 1972, vol. 6, p. 1195.
17. K. Jagannadham and E. G. Ramachandran: *J. Appl. Phys.*, 1974, vol. 45, p. 2406.
18. F. A. McClintock and A. S. Argon: *Mechanical Behavior of Materials*, p. 118, Addison-Wesley, Reading, 1966.
19. G. C. Weatherly: *Philos. Mag.*, 1968, vol. 17, p. 791.
20. H. Brooks: *Metal Interfaces*, p. 20, ASM, Metals Park, OH, 1952.
21. R. M. Rose, D. P. Ferriss, and J. Wulff: *Trans. TMS-AIME*, 1962, vol. 224, p. 981.
22. V. A. Philips: *Trans. TMS-AIME*, 1964, vol. 230, p. 967.
23. K. E. Easterling: *Acta Polytech. Scand.*, 1967, chapt. 61.
24. J. K. Lee and W. C. Johnson: *Scr. Metall.*, 1977, vol. 11, p. 477.
25. A. Kelly and R. B. Nicholson: *Prog. Mater. Sci.*, 1963, vol. 10, p. 149.
26. H. I. Aaronson, J. K. Lee, and K. C. Russell: *Precipitation Processes in Solids*, p. 31, K. C. Russell and H. I. Aaronson, eds., TMS-AIME, New York, NY, 1978.