

A New Computation Method for Solidification Process in a Finite, Initially Overheated Slab

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An approximate theory is presented for solidification in a finite, initially overheated slab in which one wall is insulated and the other is subject to an instantaneous temperature drop below the freezing point. The simple expressions for the position of the phase change front as a function of time are derived. The results are compared with experimental data, numerical and approximate solutions as well as exact solution presented in other literature, and good agreement is attained. It is shown that the approximate method proposed herein is also valid for phase change problems with large Stefan numbers and different boundary conditions.

Keywords: solidification, phase change, heat transfer.

INTRODUCTION

Phase-change heat transfer problems often occur in many natural phenomena, such as melting, freezing, sublimation, gas dissolution and evaporation. Analytical research on these problems has generated an extensive body of literature. These analyses were mainly concerned with the situation in which the phase change medium was assumed at its melting temperature. This limitation simplifies the mathematical model, but fails to suit some problems of practical interest.

Exact solutions are available only for a few cases. The most notable example is that of Neumann^[1] who treated a semi-infinite one-dimensional slab initially at uniform temperature. Systems with more complicated geometry, boundary and initial conditions are usually handled by numerical and/or approximate analytic methods.

Goodman and Shea^[2] applied the heat-balance method to the problem of a finite slab by a complicated analysis. Gao and Sunderland^[3] studied this problem using the same method. However, [3] has the restriction that the interface position is proportional to the square-root of time. For the case of small Stefan numbers, Weinbaum and Jiji^[4] applied the perturbation theory; Charach and Zoglin^[5] combined the heat balance integral method with a time-dependent perturbation theory. In numerical

methods, Sparrow et al^[6] employed a finite difference technique in conjunction with a formulation in which both the liquid-solid interface and the forward edge of the temperature wave were immobilized. Yuen and Kleinman^[7] applied a variable time step finite difference method to the one dimensional melting problem that takes into account the effect of subcooling.

While these approaches have been proved successful, they are mathematically difficult and have limited use for practical problems.

In this paper, an approximate analysis of solidification in a finite, initially overheated slab is developed. Our method has the same feature as the heat-balance integral, but by this method it is not required to solve the heat-balance integral equation and differential equation. The simple expressions for the position of the phase change front as a function of time are derived. The results are compared with the experimental data, numerical and approximate solutions published in other literature and good agreement is attained. It is seen that the approximate theory is also valid for phase change problems with large Stefan number and any boundary conditions, so it is a simple and direct method for practical calculation of most engineering problems involving solidification process.

FORMULATION OF THE PROBLEM

Consider a uniform liquid layer initially at the temperature T_i which is above the freezing point T_f . The

Nomenclature

<i>c</i>	specific heat
<i>d</i>	length of slab
<i>E</i>	function defined by equation (20)
<i>F</i>	function defined by equation (29)
<i>K</i>	thermal conductivity
<i>L</i>	latent heat of fusion
<i>St_L</i>	Stefan number of liquid
<i>St_s</i>	Stefan number of solid
<i>T</i>	temperature
<i>T_f</i>	fusion temperature
<i>T_i</i>	initial temperature
<i>T_w</i>	the front wall temperature
<i>S₁</i>	position of phase change front
<i>S₂</i>	position of thermal front in liquid
<i>t</i>	time
<i>x</i>	distance
<i>α</i>	thermal diffusivity

<i>δ₁</i>	dimensionless position of phase change front
<i>δ₂</i>	dimensionless position of thermal front in liquid
<i>θ_L</i>	dimensionless liquid temperature
<i>θ_s</i>	dimensionless solid temperature
<i>λ₁</i>	constant defined by equation (25)
<i>λ₂</i>	constant defined by equation (25)
<i>ρ</i>	density
<i>ν</i>	ratio of thermal diffusivities of solid and liquid
<i>τ</i>	dimensionless time, = <i>t α_L/d²</i>
<i>τ_b</i>	dimensionless time of the end of process 1
<i>δ_b</i>	dimensionless position of phase change front at end of process 1
<i>η</i>	dimensionless distance
Subscripts	
<i>L</i>	liquid
<i>S</i>	solid

thickness of the slab is *d* and its back wall is insulated. At time *t* = 0 the temperature of the front wall drops to *T_w* (< *T_f*). To analyze the problem, some assumptions are made as follows:

- (1) *T_w* is constant during the entire freezing process.
- (2) Thermophysical parameters of each phase are constant.
- (3) Convection effect is neglected, and the heat transfer is one-dimensional.

The overall phenomenon is divided into two processes as shown in Fig.1. They are described by different equations and boundary conditions, and the primary process determines the initial conditions for the subsequent process. During the process 1, liquid–solid

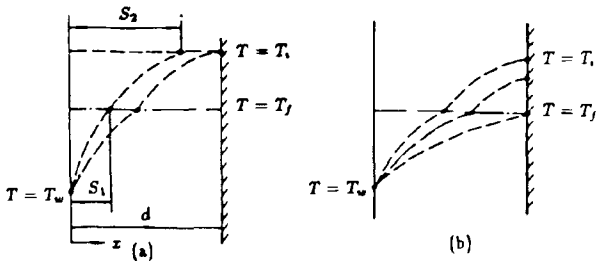


Fig.1 Solidification processes

interface *S₁* and temperature penetration depth *S₂* increase from zero, and the temperature of liquid drops in the distance *S₂*. This process terminates when *S₂* = *d*. During the process 2, liquid–solid interface *S₁* continues to increase and the temperature of insulated wall (*x* = *d*) begins to drop from *T_i*. When the temperature of the insulated wall equals *T_f* (i.e. the

position of the liquid–solid interface *S₁* = *d*) this process ends.

The governing equations for each phase in the entire process are:

$$\frac{\partial T_s}{\partial t} = a_s \frac{\partial^2 T_s}{\partial x^2}, \quad \frac{\partial T_L}{\partial t} = a_L \frac{\partial^2 T_L}{\partial x^2} \tag{1}$$

$$K_s \frac{\partial T_s}{\partial x} \Big|_{x=S_1} - K_L \frac{\partial T_L}{\partial x} \Big|_{x=S_1} = \rho L \frac{dS_1}{dt} \tag{2}$$

$$T_s(x, t) \Big|_{x=S_1} = T_L(x, t) \Big|_{x=S_1} = T_f \tag{3}$$

$$T_s(x, t) \Big|_{x=0} = T_w \tag{4}$$

The boundary conditions for different process are:
Process 1:

$$T_L(x, t) \Big|_{x=S_2} = T_i, \quad \frac{\partial T_L}{\partial x} \Big|_{x=S_2} = 0 \tag{5}$$

$$T_L(x, 0) = T_i, \quad S_1(0) = 0, \quad S_2(0) = 0 \tag{6}$$

Process 2:

$$\frac{\partial T_L}{\partial x} \Big|_{x=d} = 0 \tag{7}$$

For convenience of analysis and solution of the problem, the following dimensionless variables and parameters are introduced:

$$\theta_s = c_s(T_s - T_f)/L, \quad \theta_L = C_L(T_L - T_f)/L$$

$$\delta_1 = S_1/d, \quad \delta_2 = S_2/d, \quad \eta = x/d, \quad \tau = t a_L/d^2$$

$$v = a_s/a_L, \quad St_s = c_s(T_f - T_w)/L, \quad St_L = c_L(T_i - T_f)/L$$

Equations (1)–(7) can be written as:

$$\frac{\partial \theta_s}{\partial \tau} = v \frac{\partial^2 \theta_s}{\partial \eta^2}, \quad \frac{\partial \theta_L}{\partial \tau} = \frac{\partial^2 \theta_L}{\partial \eta^2} \tag{8}$$

$$v \frac{\partial \theta_s}{\partial \eta} \Big|_{\eta=\delta_1} - \frac{\partial \theta_L}{\partial \eta} \Big|_{\eta=\delta_1} = \frac{d\delta_1}{d\tau} \quad (9)$$

$$\theta_s(\eta, \tau) \Big|_{\eta=\delta_1} = \theta_L(\eta, \tau) \Big|_{\eta=\delta_1} = 0 \quad (10)$$

$$\theta_s(\eta, \tau) \Big|_{\eta=0} = -St_s \quad (11)$$

Process 1:

$$\theta_L(\eta, \tau) \Big|_{\eta=\delta_2} = St_L, \quad \frac{\partial \theta_L}{\partial \eta} \Big|_{\eta=\delta_2} = 0, \quad (12)$$

$$\theta_L(\eta, 0) = St_L, \quad \delta_1(0) = 0, \quad \delta_2(0) = 0 \quad (13)$$

Process 2:

$$\frac{\partial \theta_L}{\partial \eta} \Big|_{\eta=1} = 0 \quad (14)$$

Equations (8)-(14) can not be solved exactly except for the case $St_L = 0$.

SOLUTION OF THE PROBLEM

Since the two processes have different boundary and initial conditions, the solution is divided into two processes.

1. Process 1.

During this process, the liquid-solid interface δ_1 and temperature penetration depth δ_2 increase from zero, and the temperature of the liquid phase drops in the interval $\delta_1 \leq \eta \leq \delta_2$ and there is no heat flux in $\eta \geq \delta_2$. This process terminates when $\delta_2 = 1$.

To illustrate Goodman's heat balance integral method, the form of the temperature profile for each phase can be written as:

$$\theta_L(\eta, \tau) = \begin{cases} A_L(\eta - \delta_1) + B_L(\eta - \delta_1)^2 & \delta_1 \leq \eta \leq \delta_2 \\ St_L & \delta_2 \leq \eta \leq 1 \end{cases} \quad (15)$$

$$\theta_s(\eta, \tau) = A_s(\eta - \delta_1) + B_s(\eta - \delta_1)^2, \quad 0 \leq \eta \leq \delta_1 \quad (16)$$

where $A_L, B_L, A_s, B_s, \delta_1$ and δ_2 are functions of τ to be determined by relevant boundary conditions and the boundary condition at the liquid-solid interface (10) is automatically satisfied.

Using the boundary condition (9), (10) and (12) we obtain

$$A_L = \frac{2St_L}{\delta_2 - \delta_1}, \quad B_L = -\frac{St_L}{(\delta_2 - \delta_1)^2} \quad (17)$$

$$A_s = \frac{1}{v} \left[\frac{d\delta_1}{d\tau} + \frac{2St_L}{\delta_2 - \delta_1} \right]$$

$$B_s = -\frac{1}{2v^2} \left[\frac{d\delta_1}{d\tau} + \frac{2St_L}{\delta_2 - \delta_1} \right] \frac{d\delta_1}{d\tau} \quad (18)$$

Thus the temperature profiles for the two phases are:

$$\theta_L(\eta, \tau) = \begin{cases} St_L \frac{\eta - \delta_1}{\delta_2 - \delta_1} \left[2 - \frac{\eta - \delta_1}{\delta_2 - \delta_1} \right] & \delta_1 \leq \eta \leq \delta_2 \\ St_L & \delta_2 \leq \eta \leq 1 \end{cases} \quad (19)$$

$$\theta_s(\eta, \tau) = E(\delta_1, \delta_2) \left[2v(\eta - \delta_1) - \frac{d\delta_1}{d\tau}(\eta - \delta_1)^2 \right] \quad 0 \leq \eta \leq \delta_1 \quad (20)$$

where

$$E(\delta_1, \delta_2) = \frac{1}{2v^2} \left[\frac{d\delta_1}{d\tau} + \frac{2St_L}{\delta_2 - \delta_1} \right]$$

By the method of Goodman's heat balance integral, the relation between δ_1 , and δ_2 can be obtained:

$$2 \frac{d\delta_1}{d\tau} + \frac{d\delta_2}{d\tau} = \frac{6}{\delta_2 - \delta_1} \quad (21)$$

In the mean time, the simple relation between δ_1 and δ_2 can be derived by differentiating Eq.(10) with respect to τ , and using Eq.(19) there results

$$\frac{d\delta_1}{d\tau} = \frac{1}{\delta_2 - \delta_1} \quad (22)$$

Substituting Eq.(22) into Eq.(19), the temperature profile of the solid phase can be expressed as

$$\theta_s(\eta, \tau) = \frac{(2St_L + 1)}{2v^2} \frac{d\delta_1}{d\tau} \left[2v(\eta - \delta_1) - \frac{d\delta_1}{d\tau}(\eta - \delta_1)^2 \right] \quad 0 \leq \eta \leq \delta_1 \quad (23)$$

Now, the entire problem has been transformed into the solution in the solid phase.

Substituting the temperature profile of the solid phase, Eq.(23), into the external boundary condition Eq.(11), the differential equation for δ_1 can be derived:

$$\left(\delta_1 \frac{d\delta_1}{d\tau} \right)^2 + 2v\delta_1 \frac{d\delta_1}{d\tau} - \frac{2v^2 St_L}{2St_L + 1} = 0 \quad (24)$$

The initial condition leads to the solution

$$\delta_1 = \lambda_1 \tau^{1/2}, \quad \delta_2 = \lambda_2 \tau^{1/2} \quad (25)$$

where

$$\lambda_1 = 2^{1/2} v \left[-1 + \left(1 + \frac{2St_s}{2St_L + 1} \right)^{1/2} \right]^{1/2}$$

$$\lambda_2 = \left[-\lambda_1 + (9\lambda_1^2 + 48)^{1/2} \right] / 2$$

Define τ_b as the end time of process 1 and $\delta_b = \delta_1(\tau_b)$ as the corresponding position of the solid-liquid interface, thus

$$\tau_b = \lambda_2^{-2}, \quad \delta_b = \lambda_1 \tau_b^{1/2} \tag{26}$$

2. Process 2

During the process 2, the liquid–solid interface δ_1 continues to increase and the temperature of the insulated wall ($\eta = 1$) begins to drop from T_i . Assume as the temperature of the insulated wall equals T_f (ie. the position of the liquid–solid interface $\delta_1 = 1$) this process ends.

Thus the temperature of the insulated wall, θ_0 , can be written as^[6]:

$$\theta_0 = \frac{e^{(1-\delta_1)^3} - 1}{e^{(1-\delta_b)^3} - 1} \times \frac{(1 - \delta_1)^2}{(1 - \delta_b)^2}, \quad \eta = 1$$

The method of solution for this process is the same as that of process 1. The temperature profile of each phase is:

$$\theta_L(\eta, \tau) = 2St_L F(\delta_1) \left[(\eta - \delta_1) - \frac{1}{1 - \delta_1} (\eta - \delta_1)^2 \right], \quad \delta_1 \leq \eta \leq 1 \tag{27}$$

$$\theta_s(\eta, \tau) = \frac{1}{2v^2} \left[2St_L F(\delta_1) + \frac{d\delta_1}{d\tau} \left[2v(\eta - \delta_1) - \frac{d\delta_1}{d\tau} (\eta - \delta_1)^2 \right] \right] \quad 0 \leq \eta \leq \delta_1 \tag{28}$$

Introducing the boundary condition into the temperature distribution of the solid phase, we finally obtain the following differential equation:

$$\left[\delta_1 \frac{d\delta_1}{d\tau} \right]^2 + 2v \left[1 + St_L \delta_1 F(\delta_1) \right] \delta_1 \frac{d\delta_1}{d\tau} - 2 \left[v^2 St_s - 2v St_L \delta_1 F(\delta_1) \right] = 0 \tag{29}$$

where

$$F(\delta_1) = \frac{e^{(1-\delta_1)^3} - 1}{e^{(1-\delta_b)^3} - 1} \times \frac{1 - \delta_1^2}{(1 - \delta_b)^2}$$

The initial condition leads to the solution

$$\tau(\delta) = \tau_b + \int_{\delta_b}^{\delta} \delta_1 v^{-1} \left[1 + St_L \delta_1 F(\delta_1) \right]^{-1}$$

$$\left\{ -1 + \left[1 + \frac{2 \left[St_s - 2St_L \delta_1 F(\delta_1) / v \right]^{1/2}}{\left[1 + St_L \delta_1 F(\delta_1) \right]^2} \right]^{-1} \right\}^{-1} d\delta_1 \tag{30}$$

The solidification time τ^* can be obtained by setting $\delta = 1$.

Eq.(30) can be easily calculated by numerical integration.

RESULTS AND DISCUSSION

In this section the main results of our analyses are summarized and compared with other results derived by previous researchers.

Interface Motion

The instantaneous position of the interface δ_1 is determined by equations (25) and (30). For $\tau < \tau_b$, the finite size effects are negligible. The accuracy of our results for this stage can be estimated by comparing Eq.(25) with the result of Neumann solution. For the case $St_s = 0.3, St_L = 0.1$, the value of λ_1 by our method is 0.670 while the exact result is 0.657. The error is about 2%. Table 1 presents the values of λ_1 over a large range of St_s number together with the exact solution and the result obtained using Eq.(25). It can be seen that they are in very good agreement.

Table 1. Comparison of the value of δ_1 by the present solution with exact result

Ste	0.3	0.4	0.5	0.6	0.7	0.8
our λ	0.670	0.763	0.841	0.910	0.972	1.027
Exact λ	0.657	0.752	0.830	0.896	0.952	1.001
error %	2.0	1.5	1.3	1.6	2.1	2.6

For $\tau > \tau_b$, the motion of the interface can be given by Equation (30). Since no exact solution is available, the accuracy of our results can be examined by the solutions of other authors and experimental results. In Fig.2a the instantaneous location of the interface as a

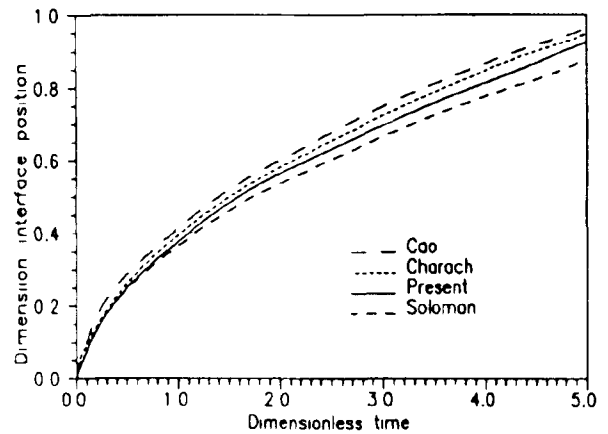


Fig.2a Position of phase change front as a function of time ($v = 1, St_s = St_L = 0.1$)

function of time is presented for the case of $St_L = 0.1, v = 1$ with the data given by Cao^[3], Charach^[5] and Soloman^[11]. The four approximate solutions are in close agreement. It is interesting to note that the curve predicted by our solution is bounded

from below and above by the curves corresponding to the Cao and Soloman solutions, respectively.

Fig.2b shows the transient velocity of phase change front for freezing of water in the case $T_i = 4.4^\circ\text{C}$, $T_w = -40.3^\circ\text{C}$ and $d = 0.005\text{ m}$. The calculated result is in good agreement with the experimental data given by Boger and Westwater^[9]. For example, at time $t = 70\text{ min}$, the theoretical and experimental results is within 5 percent.

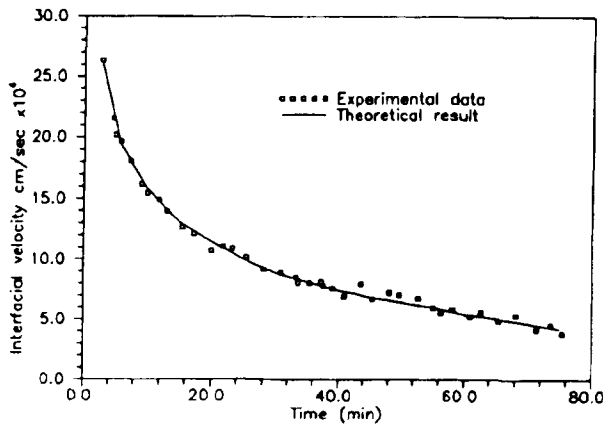


Fig.2b Comparison of experimental and theoretical interfacial velocity

The solutions given by Cao and Charach are restricted to small Stefan numbers. It is interesting to examine our solution for large Stefan numbers. For large Stefan numbers our solution and numerical solution given by Yuen^[7] are shown in Table 2, and good agreement between the two is evident.

Table 2. Comparison of the Position of the change-phase interface by the present method with numerical result for large Stefan number ($v = 1, St_s = St_L = 1$)

τ	Present model	exact result
0.00746	0.06591	0.0653
0.02862	0.1291	0.1278
0.06374	0.1922	0.1903
0.1120	0.2540	0.2528

Solidification Time

The solidification time τ^* is defined by equation (30) with $\delta = 1$. The comparison of various solutions is presented in Fig.3 which gives the solidification time for the case $v = 1, St_s = 0.3$ and St_L varying from 0 to 0.3. For $St_L = 0$, our solution is slightly larger than those predicted by other methods. For $St_L = 0.3$, our result is larger than those given by [2] and [3] and smaller than those given by [5] and [6]. The deviation is about 10%.

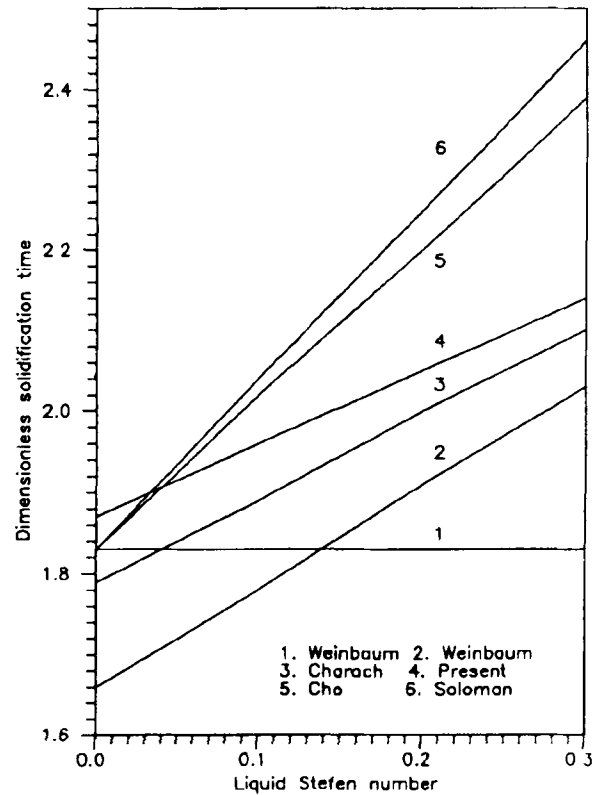


Fig.3 Solidification time as a function of the liquid Stefan number ($v = 1, St_s = 0.3$)

For the case $St_L = 0$ (without initial overheating), the position of solid-liquid interface as a function of time by our solution is derived as

$$\delta = P\tau^{1/2}$$

where

$$\frac{P}{2} = \left[\frac{(1 + 2St_s)^{1/2} - 1}{2} \right]^{1/2}$$

The above equation is the same as the result given by Goodman using heat balance integral method. Goodman compared his equation with exact solution and showed that the error is very small^[10].

Effect of Overheating

As expected, the results show that initial overheating has major effects on the solidification process. Initial overheating slows down the rate of propagation of the solidification interface.

The solidification time and position of phase change interface depend heavily on solid Stefan number St_s . Solidification time for large St_s is shorter than for small St_s number. When solid Stefan number St_s is constant, solidification time is longer for large liquid Stefan number St_L (large initial overheating) than for small St_L . In the case of $St_s = 0.3$, the solidification time for $St_L = 0.3$ increases 12% compared with that without initial overheating ($St_L = 0$).

CONCLUSION

In this paper an approximate theory of solidification in a finite, initially overheated slab is presented for the case of small and large Stefan numbers. The results of the present method are compared with those of exact and other approximate solutions as well as experimental data, and good agreement is obtained. It is found that the solidification rate of the system is faster than that corresponding to the semi-finite slab with the same initial overheating, but slower than in a slab without overheating.

One of the main purposes of the present work is to derive a simple expression for calculating the solidification time as given by Eq.(30). This expression can be used for predicting practical processes simply and effectively.

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