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Vector Potential Theory on Nonsmooth Domains in $\mathbf{R}^3$ and Applications to Electromagnetic Scattering

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ABSTRACT. We study boundary value problems for the time-harmonic form of the Maxwell equations, as well as for other related systems of equations, on arbitrary Lipschitz domains in the three-dimensional Euclidean space.

The main goal is to develop the corresponding theory for L^p-integrable bounday data for optimal values of p's. We also discuss a number of relevant applications in electromagnetic scattering.

1. Statement of the Problems and Introductory Remarks

Let us consider the electromagnetic wave propagation in a homogeneous, isotropic medium that occupies the exterior of a bounded domain Ω in $\mathbf{R}^3$ and has electric conductivity $\sigma \geq 0$, electric permittivity $\epsilon > 0$, and magnetic permeability μ. If we denote by $\mathcal{E}$, $\mathcal{H}$ the electric and the magnetic fields, respectively, and if J stands for the current density, then the Maxwell equations read

$$\begin{cases} \operatorname{curl} \mathcal{E}(X,t) + \mu \frac{\partial \mathcal{H}}{\partial t}(X,t) = 0 & \text{in } (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \\ \operatorname{curl} \mathcal{H}(X,t) - \epsilon \frac{\partial \mathcal{E}}{\partial t}(X,t) = J(X,t) & \text{in } (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}. \end{cases}$$

Also, in an isotropic conductor, the electric field satisfies Ohm's law $\sigma\mathcal{E} = J$. An excellent exposition of this material can be found in [24, Vol. I]; cf. also [15].

We assume time-harmonic dependency for $\mathcal{E}$ and $\mathcal{H}$, that is, that for some time-independent vector fields E, H the following separation of variables holds:

$$\begin{aligned} \mathcal{E}(X,t) &= \left(\epsilon + \frac{i\sigma}{\omega}\right)^{-\frac{1}{2}} E(X)\, e^{-i\omega t}, \\ \mathcal{H}(X,t) &= \mu^{-\frac{1}{2}} H(X)\, e^{-i\omega t}, \end{aligned}$$

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where $\omega > 0$ is the frequency. Summarizing these assumptions and eliminating the time dependency, we arrive at the stationary (or reduced, time-harmonic) Maxwell equations

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ \operatorname{curl} H + ikE = 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \end{cases} \qquad \text{(Maxwell)}$$

where the wave number $k \in \mathbf{C}$, $\operatorname{Im} k \geq 0$, is given by $k^2 := (\omega\epsilon + i\sigma)\mu\omega$. Note that the stationary Maxwell system is equivalent to the eigenvalue problem $\mathcal{M}(E, H) = k\,(E, H)$, where

$$\mathcal{M} := \begin{pmatrix} 0 & i \operatorname{curl} \\ -i \operatorname{curl} & 0 \end{pmatrix}$$

is the so-called Maxwell operator. We want to stress that the operator curl, and with it the operator $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla\operatorname{div}$ and the Maxwell operator $\mathcal{M}$, are not elliptic (the rank of the 6×6 characteristic matrix for the latter operator is 4).

Suppose now that $\partial\Omega$ is perfectly conducting, and fix a direction $d \in S^2 \subseteq \mathbf{R}^3$ and a polarization $p \in \mathbf{R}^3$. Also, let n denote the outward unit normal to Ω. The direct scattering problem for the system (Maxwell) consists in the determination of the scattered wave (E^s, H^s) for which the Silver–Müller radiation condition

$$\lim_{|X|\to\infty} \left\{ H^s(X) \times (X/|X|) - |X| E^s(X) \right\} = 0$$

is fulfilled and such that

$$E(X) := ik^{-1} \operatorname{curl}\operatorname{curl} \left(p\, e^{ik\langle X, d\rangle} \right) + E^s(X),$$

$$H(X) := \operatorname{curl} \left(p\, e^{ik\langle X, d\rangle} \right) + H^s(X),$$

is a solution of (Maxwell) satisfying the (perfect conductor, total reflection) homogeneous boundary conditions

$$\begin{cases} n \times E = 0 & \text{on } \partial\Omega, \\ \langle n, H \rangle = 0 & \text{on } \partial\Omega \end{cases}$$

(cf., e.g., [24, Vol. I, (4.47), p. 75]).

Redenoting E^s, H^s once again by E and H respectively, the above considerations lead us to the following formulation of the (direct) exterior Maxwell boundary value problem

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ \operatorname{div} E = 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ \operatorname{curl} H + ikE = 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ \operatorname{div} H = 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ E, H \text{ satisfy the radiation condition,} \\ E^*, H^* \in L^p(\partial\Omega), \\ n \times E = A \in L^p(\partial\Omega), \\ \langle n, H \rangle = f \in L^p(\partial\Omega). \end{cases} \qquad (\mathcal{M}_e)$$

Here $1 < p < \infty$ and $(\cdot)^*$ is the usual nontangential maximal operator (precise definitions will be given later). The interior Maxwell boundary value problem $(\mathcal{M}_i)$ in Ω has a similar formulation but with the radiation condition excluded. Note that the divergence-free conditions are superfluous unless $k = 0$, in which case $(\mathcal{M}_{e,i})$ decouple into separate boundary value problems for E and H. Also, the boundary data A, f must satisfy certain compatibility conditions. First, obviously, A needs to be *tangential*, that is,

$$\sum_j n_j A_j = 0 \quad \text{a.e. on } \partial\Omega, \qquad (1.1)$$

but a more subtle constraint, observed by integrations by parts and which reflects the over-determination of (Maxwell), is that

$$\sum_{l,j} n_l(n_l\partial_j - n_j\partial_l)A_l = -ik\,f \quad \text{on } \partial\Omega \tag{1.2}$$

(note that this also implies that $\int_{\partial\Omega} f\,d\sigma = 0$, that is, $f \in L^p_0(\partial\Omega)$).

It is not too difficult to see that once (1.1), (1.2) are assumed and if $k \neq 0$, the last boundary condition in $(\mathcal{M}_{\mathrm{i,e}})$ becomes superfluous and, hence, may be omitted in the formulation of these problems. This suggests the alternative of completely eliminating the magnetic field H and restricting attention only to the electric field E. More concretely, in doing so we arrive at the (exterior) *electric* boundary value problem

$$\begin{cases} (\Delta + k^2)E = 0 \quad \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ \operatorname{div} E = 0 \quad \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ E \text{ satisfies the radiation condition}, \\ E^* \in L^p(\partial\Omega), \\ n \times E = A \in L^p(\partial\Omega). \end{cases} \tag{$\mathcal{E}_{\mathrm{e}}$}$$

Of course, this time A only needs to satisfy (1.1). There is a similar interior version $(\mathcal{E}_{\mathrm{i}})$ as well, with the radiation condition dropped.

As evidenced by the huge number of research articles and monographs (see., e.g., the books [68, 52, 53, 14, 77, 76, 15, 24, 87] and the references therein), the treatment of the direct and inverse acoustic and electromagnetic scattering problems have always enjoyed center stage in mathematicalphysics. Beside the theoretical importance of these problems, this interest is also motivated by their fundamental applications to many areas of science and technology.

Without trying to discuss the history of the problem exhaustively, let us mention that from the work of C. Müller [67], A. P. Calderón [8], H. Weyl [96], W. K. Saunders [81] in the early 1950s, the treatment of the problems $(\mathcal{M}_{\mathrm{i,e}})$, $(\mathcal{E}_{\mathrm{i,e}})$ in the case in which the domain Ω has a smooth boundary (e.g., $\partial\Omega \in C^3$ in [8]) is well understood. More recently, their techniques have been refined to treat domains with C^2 boundaries [14] and also domains with C^1 boundaries [66]. The approach of these authors is based on boundary integral equations, but the method of proof is not constructive since it employs "soft arguments"; that is, it relies entirely on compactness and the closed graph theorem. In particular, one cannot solve the corresponding boundary value problems on a less smooth domain by approximating it with smooth ones, since in such an approximation the constants entering the crucial estimates depend unfavorably on the smoothness. Another basic shortcoming of the classical approach is that the boundary data have to be smooth (typically in a space of Hölder continuous functions). However, the need for realistic modeling of engineering problems naturally leads to considering domains with irregularities (e.g., with "edges" and "corners") and discontinuous boundary data. In this context, the boundary value problem associated with the Maxwell equations have been much less understood. The investigation of problems as such has a rather rich history, and below we briefly explain the main, dominant trends of this theme and describe some of the major difficulties encountered.

One of the early recognized directions, originating in the pioneering work of Carleman [11] and Radon [75] at the turn of the century, was to allow domains with piecewise smooth boundaries (for such domains the singularities are local). In more recent years, this important direction has been brought to the general attention by the influential work of V. G. Maz'ya, V. A. Kondrat'ev, J. Král and their collaborators. See, for example, the excellent surveys [58, 50, 51] and the extensive literature cited there.

Another direction, which is a central part of the so-called Calderón program (cf. [10]), has been amply substantiated in the seminal work of B. Dahlberg, D. Jerison, E. Fabes, C. Kenig, and G. Verchota on the Laplacian, the Lamé system of elasticity, and the linearized Stokes system [17, 40, 93, 20, 29, 22]. They showed that the Harmonic Analysis techniques are particularly well-suited for obtaining sharp results for elliptic boundary value problems on Lipschitz domains. Let us point out here that allowing such nonsmooth domains and "rough" boundary data drastically changes the nature of the problem since it affects the compactness of the boundary integral operators. In fact, even proving the very boundedness of these operators becomes a fundamentally harder problem. Of course, in the context of general Lipschitz boundaries, a major ingredient is the deep theorem of R. Coifman, A. McIntosh, and Y. Meyer on the boundedness of the Cauchy integral operator on Lipschitz curves [12]. Another basic idea, going back to Rellich [78] (and which has been reinvented independently by a number of authors since; cf., e.g., [70, 71, 40]) is to use the quantitative version of some appropriate integral identities to overcome the lack of compactness of the boundary integral operators on Lipschitz boundaries. Related material is contained in, for example, [2, 23, 74, 47, 82, 31, 56, 54]. See also [46] for a more up-to-date survey of developments in this very active field of research.

The Rellich type identities that are relevant for the Maxwell system on arbitrary Lipschitz domains in $\mathbf{R}^3$ have first been devised in [64]. Together with certain spectral theoretical arguments, these have been used to develop a L^2 theory for the problems $(\mathcal{M}_{i,e})$ in this setting. Subsequently, this theory has been extended to arbitrary Lipschitz domains in higher dimensions in [39], whereas the parabolic form of the Maxwell equations on Lipschitz cylinders has been treated in [65] (cf. also [63]). Furthermore, a comprehensive treatment of these problems based on a systematic use of the Clifford Algebra framework can be found in [59].

Here we continue the work initiated in [64]. Once again, our main concern is the smoothness of the domain (i.e., allowing Lipschitz boundaries) and of the boundary data (assumed to be in appropriate subspaces of $L^p(\partial\Omega)$). This time, however, our aim is to present a rather complete L^p theory for the Maxwell boundary value problems $(\mathcal{M}_{i,e})$, $(\mathcal{E}_{i,e})$, on arbitrary Lipschitz domains in $\mathbf{R}^3$ for sharp ranges of p's.

The solvability range for the problems $(\mathcal{M}_{i,e})$ turns out to be $1 < p \leq 2 + \epsilon$, for some $\epsilon = \epsilon(\partial\Omega) > 0$ (see §6). Its optimality has been conjectured in [64] and shows that, in some sense, $(\mathcal{M}_{i,e})$ behaves more like Regularity and Neumann type problems (cf. the results in [20] for the Laplace operator). In fact, this is most visible in dimension two, where the Maxwell boundary value problems reduce to the interior/exterior Neumann problems for the Helmholtz operator $\Delta + k^2$. The sharpness of this range follows from the counterexamples supplied in §7 which, as far as we are aware, are the first of this kind for systems of equations (compare to [20, 45, 73, 21, 83]). Our main result in this regard (Theorem 6.1) asserts that if Ω is an arbitrary, bounded Lipschitz domain in $\mathbf{R}^3$, then there exists $\epsilon > 0$ depending only on Ω such that, for each $1 < p \leq 2 + \epsilon$ and $k \in \mathbf{C} \setminus \{0\}$, the compatibility conditions (1.1), (1.2) are necessary and sufficient for the unique solvability of $(\mathcal{M}_e)$. The solution is expressed in the layer potential form and optimal a priori estimates are obtained. As alluded before, this theorem is sharp. A similar result is valid for the interior problem, although this time one must take into account the possibility that the wave number k is a so-called Maxwell eigenvalue for Ω. In this latter situation, existence holds only if the boundary data satisfy some further (necessary) compatibility conditions, whereas uniqueness holds only modulo a finite linear space. The endpoint case $p = 1$ for the Maxwell boundary value problems $(\mathcal{M}_{i,e})$ is discussed in §8, where the structure of the vector space of boundary data is identified in terms of certain atomic Hardy spaces (cf. Theorem 8.4).

To show the well-posedness of these direct problems in the range $1 < p \leq 2 + \epsilon$ it is not clear how to follow directly the more familiar route (originally discovered in [20] and then successfully used in several other important cases, as in for example [21, 3, 73, 83]), which consists of interpolating between the atomic results and the L^2 results for the problem at hand. This is essentially because of the complicated structure of the spaces of boundary data in the limiting case $p = 1$ (for instance, individual "atoms" do not belong to these spaces; see §8). Our idea is first to reduce matters to the potential theoretic case $k = 0$ when the Maxwell system decouples and then, further, to that of the Laplace equation. It is at this stage that we shall make use of the sharp results in [20]. In doing so, we are naturally led to introducing and studying certain vector-valued layer potential operators as well as appropriate spaces of vector fields on the boundaries of Lipschitz domains in $\mathbf{R}^3$ (cf. Theorem 5.1 and Theorem 5.3). A prominent role in our analysis is played by the surface divergence operator considered in a weak sense (see §3). Much of this theory appears to be new even for domains with smooth boundaries.

The main vehicles allowing us to relate the boundary integral operators that are used to solve $(\mathcal{M}_{\rm i,e})$ to the classical double-layer potential operator for the Laplacian on $\partial\Omega$ are some operator identities that we study in §5. As a byproduct of this approach, we are also able to present sharp L^p results for a number of boundary value problems for harmonic vector fields in Lipschitz domains in $\mathbf{R}^3$. A complete L^2 theory of the boundary value problems for harmonic differential forms of arbitrary degree in Lipschitz domains in $\mathbf{R}^m$, $m \geq 3$, has been developed in [61].

Another important connection between these problems is that the electric field tends, as the wave number k approaches zero, to the corresponding electrostatic field. This is commonly referred to as the *principle of limiting absorption* and shows how the resolvent that we constructed behaves near the spectrum. Here we prove the validity of this principle for bounded domains in $\mathbf{R}^3$ having connected, Lipschitz continuous boundaries of topological genus zero.

The electric problems $(\mathcal{E}_{\rm i,e})$ are studied in §9. They are shown to be well-posed in the range $2 - \epsilon \leq p \leq 2 + \epsilon$ for any bounded Lipschitz domain Ω in $\mathbf{R}^3$ (for the interior problem, in the case in which k is a Maxwell eigenvalue for Ω, this should be modified accordingly). See Theorem 9.1 for a precise statement. A considerable amount of analysis goes into the proof of this result. In fact, for the indicated range of p's, this is an extension of the theory for the Maxwell boundary value problem, as we show that $(\mathcal{E}_{\rm i,e})$ reduces precisely to $(\mathcal{M}_{\rm i,e})$ whenever the boundary data is sufficiently regular, that is, when $\sum_{l,j} n_l(n_l\partial_j - n_j\partial_l)A_l \in L^p(\partial\Omega)$. Thus, by contrast, it seems natural to refer to $(\mathcal{M}_{\rm i,e})$ and $(\mathcal{E}_{\rm i,e})$ as the *regularity* and the *nonregularity*, respectively, boundary value problems for the Maxwell system.

The basic difficulty in dealing with the electric problems is the lack of control of curl E to the boundary. For this reason, no useful estimates can be immediately derived from the Rellich identities of [64] (see (5.1)) and new ideas are required. We present two proofs of the invertibility of the boundary integral operators corresponding to these problems. One is algebraic and relies on duality arguments; the other one is based on devicing some new Rellich type idetities that are suited for the situation at hand. In this connection, let us also point out the remarkable fact that the natural boundary integral operators used to solve $(\mathcal{E}_{\rm i,e})$ may fail to be invertible on the space of pth power integrable tangential fields on $\partial\Omega$ except for $p = 2$. Also, the key element in the proof of the uniqueness part is the continuous dependence of the solution of the regular problem on the boundary of the domain.

In §11 we undertake a systematic study of the so-called Maxwell eigenvalues and Maxwell poles of a Lipschitz domain Ω. The former represent the collection of all wave numbers k such that there exist two divergencefree vector fields $E, H \in L^2(\Omega)$, not identically zero in Ω (called Maxwell eigenfields) for which there holds

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \Omega, \\ \operatorname{curl} H + ikE = 0 & \text{in } \Omega, \\ n \wedge E = 0 & \text{on } \partial\Omega, \\ n \cdot H = 0 & \text{on } \partial\Omega. \end{cases} \qquad \text{(Eigenvalue)}$$

For the case of bounded smooth domains, this study goes back to [69] and we utilize both integral equation and Hilbert space techniques to extend the classical theory to this more general setting.

Difficulties may arise here due to the lack of coerciveness of the Maxwell operator $\mathcal{M}$. In particular, the classical inequality of Friedrichs

$$\|u\|_{W^{1,2}(\Omega)} \leq C(\|\mathrm{curl}\, u\|_{L^2(\Omega)} + \|\mathrm{div}\, u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{1.3}$$

for $u \in W^{1,2}(\Omega)$ with $n \wedge u = 0$, which is valid if $\partial\Omega$ is smooth, fails for general Lipschitz domains. We overcome this obstacle by using an appropriate version of (1.3) for vector fields in $W^{\frac{1}{2},2}(\Omega)$ and then invoking a regularity result for harmonic functions with L^2 boundary traces [40, 26, 41]. Another ingredient, which is in fact interesting in its own right, concerns the solvability of the nonhomogeneous boundary value problems for the equations of static electromagnetism (cf. [24, Vol. I, p. 87]). Our approach is constructive and relies on the invertibility results from §5. As a corollary of this, by employing Hodge-like decomposition results for L^p vector fields in C^1 and Lipschitz domains, we are able to solve the nonhomogeneous version of (Eigenvalue).

The aforementioned results have considerable impact in virtually all aspects of the well-established, classical mathematical theory of the acoustic and electromagnetic scattering by obstacles with smooth boundaries. As an illustration, we include an application to inverse scattering. Specifically, based on our results for the direct electromagnetic scattering problem from the previous sections, in §12 we prove a uniqueness theorem for the inverse obstacle problem for scatterers having only Lipschitz continuous boundaries. This improves upon a theorem of A. Kirsch and R. Kress ([49]; cf. also [15]) where the scatterers are assumed to have (C^2) smooth boundaries.

Finally, in §13 we outline several directions of further research and formulate a number of relevant open problems and questions.

2. Definitions, Notation, and Preliminary Results

Recall that for a domain Ω in $\mathbf{R}^m$ (for our purposes $m = 2, 3$), the Sobolev–Besov space $W^{s,p}(\Omega)$, $0 \leq s \leq 1$, $1 < p < \infty$, is the collection of all distributions on Ω such that (with dV standing for the Lebesgue measure in $\mathbf{R}^3$)

$$\|u\|_{W^{s,p}(\Omega)} := \left\{ \sum_{|\alpha|\leq s} \int\!\!\int_\Omega |\partial^\alpha u|^p \, dV \right\}^{1/p}$$

if s is an integer and such that

$$\|u\|_{W^{s,p}(\Omega)} := \left\{ \sum_{|\alpha|\leq[s]} \int\!\!\int_\Omega |\partial^\alpha u|^p + \sum_{|\alpha|=[s]} \int\!\!\int_\Omega \int\!\!\int_\Omega \frac{|\partial^\alpha u(X) - \partial^\alpha u(Y)|^p}{|X-Y|^{m+p(s-[s])}} \right\}^{\frac{1}{p}}$$

if $s - [s] > 0$ (here $[s]$ stands for the largest integer, which is $\leq s$). Furthermore, if $W_0^{s,p}(\Omega)$ stands for the closure of $C^\infty_{\mathrm{comp}}(\Omega)$ in the $\|\cdot\|_{W^{s,p}(\Omega)}$ norm, then we also set $W^{-s,p}(\Omega)$ for the dual space of $W_0^{s,q}(\Omega)$, where $0 \leq s \leq 1$ and $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

Now consider Ω a bounded, connected open subset of $\mathbf{R}^3$ with connected boundary (note that this automatically implies that Ω and $\mathbf{R}^3 \setminus \overline{\Omega}$ are connected). We shall call Ω a *bounded Lipschitz domain* if for each $P \in \partial\Omega$ there exist $r, h > 0$; a coordinate system $\{y_0, y_1, y_2\}$ in $\mathbf{R}^3$ that is isometric to the usual one and has origin at P; and a Lipschitz continuous function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that if $C(r, h)$ denotes the cylinder $\{(y_1, y_2)\,;\, |y_j| < r,\ j = 1, 2\} \times (0, h) \subset \mathbf{R}^3$, then

$$\Omega \cap C(r,h) = \{Y = (y_0, y_1, y_2)\,;\ |y_j| < r,\ 0 \leq j \leq 2,\ y_0 > \varphi(y_1, y_2)\},$$

$$\partial\Omega \cap C(r,h) = \{Y = (y_0, y_1, y_2)\,;\ |y_j| < r,\ 0 \leq j \leq 2,\ y_0 = \varphi(y_1, y_2)\}.$$

We shall call $\partial\Omega \cap C(r, h)$ a *coordinate patch* for $\partial\Omega$. Since $\partial\Omega$ is compact, it is always possible to cover $\partial\Omega$ with finitely many coordinate patches. Each such a collection is called an *atlas* for $\partial\Omega$.

Also, we denote by $g(\partial\Omega)$ the *topological genus* of the surface $\partial\Omega$ (i.e., the number of "handles" of Ω).

Let Ω be a Lipschitz domain in $\mathbf{R}^3$, and let $(U_\nu, \varphi_\nu)_\nu$ be an atlas for $\partial\Omega$ (here φ_ν are the corresponding Lipschitz functions describing $\partial\Omega$ in U_ν). Also, fix $(\theta_\nu)_\nu$ a partition of unity subordinated to the finite, open covering $(U_\nu)_\nu$ of $\partial\Omega$. A distribution u on $\partial\Omega$ of order ≤ 1 is said to belong to the Sobolev–Besov space $W^{s,p}(\partial\Omega)$ with $|s| \le 1$, $1 < p < \infty$, if

$$(\theta_\nu u) \circ z_\nu \in W^{s,p}(\mathbf{R}^2)$$

after extension with zero outside the support, for each ν, where $z_\nu(y_1, y_2) := (\varphi_\nu(y_1, y_2), y_1, y_2)$. We endow this space with the norm

$$\|u\|_{W^{s,p}(\partial\Omega)} := \sum_\nu \|(\theta_\nu u) \circ z_\nu\|_{W^{s,p}(\mathbf{R}^2)}.$$

Clearly, with $d\sigma$ standing for the canonical surface area on $\partial\Omega$, the spaces $W^{0,p}(\partial\Omega)$ and $W^{1,p}(\partial\Omega)$ can be identified (algebraically and topologically) with $L^p(\partial\Omega)$, the Banach space of measurable, complex-valued functions that are p-integrable with respect to $d\sigma$ on $\partial\Omega$, and with the space of functions $f \in L^p(\partial\Omega)$ for which $\sum_\nu |\nabla[(\theta_\nu f) \circ z_\nu]|$ belongs to $L^p(\mathbf{R}^2)$, respectively. We endow this latter space with the natural norm $\|f\|_{W^{1,p}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_\nu \|\nabla[(\theta_\nu f) \circ z_\nu]\|_{L^p(\mathbf{R}^2)}$. It is not difficult to check that, for $0 < s < 1$, $1 < p < \infty$, an equivalent norm on $W^{s,p}(\partial\Omega)$ is given by

$$\|u\|_{W^{s,p}(\partial\Omega)} := \left(\int_{\partial\Omega} |u|^p \, d\sigma + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(P) - u(Q)|^p}{|P - Q|^{2+ps}} \, d\sigma(P) \, d\sigma(Q)\right)^{1/p}.$$

It is a well-known fact that the Sobolev–Besov spaces with fractional index also arise as interpolation spaces. Specifically, for any $1 < p < \infty$ and $0 < s < 1$, real interpolation techniques give that

$$[L^p(\partial\Omega), W^{1,p}(\partial\Omega)]_{s,p} = W^{s,p}(\partial\Omega).$$

We shall frequently use the following lemma.

Lemma 2.1.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$. For any $|s| \le 1$, $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, one has $(W^{s,p}(\partial\Omega))^* = W^{-s,q}(\partial\Omega)$.

The proof can be adapted from that of [57, Theorem 7.6, p. 36] with only minor alterations and is omitted.

If we now introduce $W_0^{s,p}(\partial\Omega) := \{u \in W^{s,p}(\partial\Omega);\ \langle u, 1\rangle = 0\}$, then for $|s| \le 1$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we also have $W_0^{-s,p}(\partial\Omega) = (W^{s,p}(\partial\Omega)/\mathbf{C})^*$. We shall occasionally write $L_0^p(\partial\Omega)$ in place of $W_0^{0,p}(\partial\Omega)$. A useful observation is that the collection of all Lipschitz continuous functions on $\partial\Omega$ is dense in $W^{s,p}(\partial\Omega)$, $|s| \le 1$, $1 < p < \infty$.

Next, let γ denote the restriction to the boundary operator initially defined on $C^\infty(\overline{\Omega})$, say. The well-known lemma of Gagliardo asserts that for each $1 < p < \infty$ the application γ extends as a bounded operator from $W^{1,p}(\Omega)$ into $W^{1-\frac{1}{p},p}(\partial\Omega)$ and has a bounded right inverse. In fact, γ maps $W^{s,p}(\Omega)$ boundedly into $W^{s-\frac{1}{p},p}(\partial\Omega)$ for $1 < p < \infty$ and $\frac{1}{p} < s \le 1 + \frac{1}{p}$ (cf. [34, Theorem 1.5.1.2]).

Let Ω be a Lipschitz domain in $\mathbf{R}^3$ and set $\Omega_+ := \Omega$, $\Omega_- := \mathbf{R}^3 \setminus \overline{\Omega}$. For a complex-valued function (or vector field) u defined in $\Omega_\pm$, the *nontangential maximal function* u^* is given by $u^*(X) := \sup_{Y \in \Gamma_\pm(X)} |u(Y)|$, where $|\cdot|$ refers to the absolute value or Euclidean norm in $\mathbf{R}^3$. Here $\Gamma_\pm(X)$ denote the interiors of the two components (in Ω_+ and in Ω_-) of a regular family of circular, doubly truncated cones $\{\Gamma(X);\ X \in \partial\Omega\}$, with vertex at X, as defined in, for example, [93]. Also, the boundary trace $u|_{\partial\Omega_\pm}$ of a function (or vector field) defined in $\Omega_\pm$ is assumed to be taken as the nontangential limit almost everywhere with respect to surface measure on the boundary (whenever this exists).

In the sequel, we shall find it useful to approximate, in a suitable sense, a given Lipschitz domain with a sequence of C^∞ domains. More specifically, we note the following Nečas–Verchota type result.

Lemma 2.2.

For any Lipschitz domain Ω in $\mathbf{R}^3$, there exist two families of C^∞ domains $\Omega_j \subset \Omega$ and $\Omega'_j \supset \Omega$, respectively, approximating Ω in the following sense.

- **i.** *There exists a covering of $\partial\Omega$ with finitely many coordinate cylinders that also form a family of coordinate cylinders for $\partial\Omega_j$, for each j. Moreover, for each such cylinder $C(r,h)$, if φ and φ_j are the corresponding Lipschitz functions whose graphs describe the boundaries of Ω and Ω_j, respectively, in $C(r,h)$, then $\|\nabla\varphi_j\|_{L^\infty} \leq \|\nabla\varphi\|_{L^\infty}$ and $\nabla\varphi_j \to \nabla\varphi$ pointwise a.e.*
- **ii.** *There exist two sequences of Lipschitz diffeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$ and $\Lambda'_j : \partial\Omega \to \partial\Omega'_j$ such that the Lipschitz constants of Λ_j, Λ_j^{-1}, Λ'_j, ${\Lambda'_j}^{-1}$ are uniformly bounded in j.*
- **iii.** *For all j and all $Q \in \partial\Omega$, $\Lambda_j(Q) \in \Gamma_+(Q)$, $\Lambda'_j(Q) \in \Gamma_-(Q)$, and $\sup_{Q\in\partial\Omega}(|Q-\Lambda_j(Q)| + |Q-\Lambda'_j(Q)|) \leq C/j$.*
- **iv.** *There exist positive functions $\omega_j : \partial\Omega \to \mathbf{R}_+$, bounded away from zero and infinity uniformly in j, such that for any measurable set $F \subset \partial\Omega$, $\int_F \omega_j d\sigma = \int_{\Lambda_j(F)} d\sigma_j$, where $d\sigma_j$ denotes the surface measure on $\partial\Omega_j$. In addition, $\omega_j \to 1$ a.e. and in every $L^p(\partial\Omega)$, $1 \leq p < \infty$. A similar statement with Λ'_j in place of Λ_j is also valid.*
- **v.** *If n_j and n'_j are the outward unit normal vectors to $\partial\Omega_j$ and to $\partial\Omega'_j$, respectively, then $n_j(\Lambda_j(\cdot))$ and $n'_j(\Lambda'_j(\cdot))$ converge a.e. and in every $L^p(\partial\Omega)$, $1 \leq p < \infty$, to $n(\cdot)$, the outward unit normal to $\partial\Omega$.*
- **vi.** *There exists a real-valued, smooth, compactly supported vector field Θ in $\mathbf{R}^3$ and $\kappa > 0$ such that $\langle\Theta(\Lambda_j(P)), n_j(\Lambda_j(P))\rangle \geq \kappa > 0$, at almost every $P \in \partial\Omega$, for all j. A similar statement with Λ'_j in place of Λ_j is also valid.*

A proof can be found in [70, 92]. These approximating sequences of domains will be denoted by $\Omega_j \uparrow \Omega$ and $\Omega'_j \downarrow \Omega$, respectively. The various constants appearing in the statement of this lemma will, in short, be referred to as the *Lipschitz character of* Ω. The above lemma is particularly useful for, for example, performing integrations by parts that would normally require boundaries smoother than Lipschitz. However, the reader is warned that many times in what follows this will be used only tacitly, with no specific mention.

Now consider Ω a bounded Lipschitz domain in $\mathbf{R}^3$ and recall that n stands for the outward unit normal to Ω, $d\sigma$ stands for the usual surface measure on $\partial\Omega$, and $\times$ is the usual vector product in $\mathbf{R}^3$. Let f, g be two Lipschitz functions in $\mathbf{R}^3$ such that f is scalar-valued and g is vector-valued. Straightforward integrations by parts then yield the formula

$$\int_{\partial\Omega} \langle n \times \nabla f\,,\, g\rangle\, d\sigma = -\int_{\partial\Omega} f\,\langle n,\, \operatorname{curl} g\rangle\, d\sigma. \tag{2.1}$$

Furthermore, if both f and g are scalar-valued, then

$$\int_{\partial\Omega} g\,(n \times \nabla f)\, d\sigma = -\int_{\partial\Omega} f\,(n \times \nabla g)\, d\sigma. \tag{2.2}$$

Note that $n \times \nabla := (n_2\partial_3 - n_3\partial_2, n_3\partial_1 - n_1\partial_3, n_1\partial_2 - n_2\partial_1)^T$ and $\langle n, \operatorname{curl}\rangle := \langle (n_2\partial_3 - n_3\partial_2, n_3\partial_1 - n_1\partial_3, n_1\partial_2 - n_2\partial_1)\,,\,\cdot\rangle$ contain only tangential derivatives and, hence, can be thought of as operators on $\partial\Omega$. In fact, by (2.1), (2.2), duality and interpolation, we see that these operators map $W^{s,p}(\partial\Omega)$

boundedly into $W^{s-1,p}(\partial\Omega)$, for $0 \le s \le 1$, $1 < p < \infty$. Also, the integration by parts formulas (2.1), (2.2) naturally extend to the case in which $f \in W^{s,p}(\partial\Omega)$ and $g \in W^{1-s,q}(\partial\Omega)$, for $0 \le s \le 1$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Another simple but useful observation is that $f \in L^p(\partial\Omega)$ has $(n \times \nabla) f = 0$ on $\partial\Omega$ if and only if f equals a constant a.e. on $\partial\Omega$.

The usual div and curl operators will be considered acting in the distribution sense for vector fields defined in an open domain Ω in $\mathbf{R}^3$. For any vector field $u \in L^1(\Omega)$ such that $\operatorname{curl} u \in L^1(\Omega)$, we define the (vector-valued) distribution $n \wedge u$ in $\mathbf{R}^3$ (which is actually supported on $\partial\Omega$) by

$$\langle n \wedge u, \varphi \rangle := \iint_\Omega \langle \operatorname{curl} u, \varphi \rangle \, dV - \iint_\Omega \langle u, \operatorname{curl} \varphi \rangle \, dV \tag{2.3}$$

for each test vector field φ in $\mathbf{R}^3$. Similarly, for a vector field $u \in L^1(\Omega)$ with $\operatorname{div} u \in L^1(\Omega)$, we define the distribution $n \cdot u$ by

$$\langle n \cdot u, \varphi \rangle := \iint_\Omega \varphi \operatorname{div} u \, dV + \iint_\Omega \langle u, \nabla\varphi \rangle \, dV, \tag{2.4}$$

where φ is an arbitrary test function in $\mathbf{R}^3$. Obviously, $n \cdot u$ is once again supported on $\partial\Omega$. Also, note that if, for example, $u \in C^1(\overline{\Omega})$, then $n \wedge u$ and $n \cdot u$ coincide with $n \times u$ and $\langle n, u \rangle$, respectively.

Lemma 2.3.

Consider Ω a bounded Lipschitz domain in $\mathbf{R}^3$, and let $1 < p < \infty$. If the vector field $u \in L^p(\Omega)$ is such that $\operatorname{curl} u \in L^p(\Omega)$, then $n \wedge u \in W^{-\frac{1}{p},p}(\partial\Omega)$ and there exists a positive constant C depending only on the Lipschitz character of $\partial\Omega$ so that

$$\|n \wedge u\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \le C \left(\|u\|_{L^p(\Omega)} + \|\operatorname{curl} u\|_{L^p(\Omega)} \right).$$

Also, if $u \in L^p(\Omega)$ is such that $\operatorname{div} u \in L^p(\Omega)$, then $n \cdot u \in W^{-\frac{1}{p},p}(\partial\Omega)$ and

$$\|n \cdot u\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \le C \left(\|u\|_{L^p(\Omega)} + \|\operatorname{div} u\|_{L^p(\Omega)} \right)$$

for some positive C depending only on the Lipschitz character of $\partial\Omega$.

Proof. Let q be the conjugate exponent of p so that $W^{-\frac{1}{p},p}(\partial\Omega) = \left(W^{1-\frac{1}{q},q}(\partial\Omega) \right)^*$. By the classical Gagliardo lemma, any $v \in W^{1-\frac{1}{q},q}(\partial\Omega)$ is the boundary trace of some $\omega \in W^{1,q}(\Omega)$ with $\|\omega\|_{W^{1,q}(\Omega)} \le C \|v\|_{W^{1-\frac{1}{q},q}(\partial\Omega)}$. Thus, we may extend $n \wedge u$ as a linear functional on $W^{1-\frac{1}{q},q}(\partial\Omega)$ by setting

$$\langle n \wedge u, v \rangle := \iint_\Omega \langle \operatorname{curl} u, \omega \rangle \, dV - \iint_\Omega \langle u, \operatorname{curl} \omega \rangle \, dV.$$

Since $W^{1,q}(\Omega)/W_0^{1,q}(\Omega)$ is isomorphic (algebraically and topologically) to $W^{1-\frac{1}{q},q}(\partial\Omega)$ (cf. [70]), it follows that this definition is correct, agrees with the old one, and the norm of the functional $n \wedge u$ does not exceed a (fixed) multiple of $\|u\|_{L^p(\Omega)} + \|\operatorname{curl} u\|_{L^p(\Omega)}$. The first part of the lemma follows.

The proof of the second part, based on (2.4), is similar and, hence, omitted. □

3. Single-Layer, Double-Layer and Newtonian-like Potential Operators

For each $k \in \mathbf{C}$, we let Φ_k stand for the standard radial fundamental solution for the Helmholtz operator $\Delta + k^2$ in $\mathbf{R}^3$,

$$\Phi_k(X) := -\frac{e^{ik|X|}}{4\pi|X|}, \qquad X \ne 0.$$

In particular, Φ_0 is the usual fundamental solution for the Laplacian in $\mathbf{R}^3$.

The single-layer acoustic potential operator with density f is defined by

$$\mathcal{S}_k f(X) := \int_{\partial\Omega} \Phi_k(X-Q) f(Q)\, d\sigma(Q), \qquad X \in \mathbf{R}^3 \setminus \partial\Omega.$$

For any f in $L^p(\partial\Omega)$ we have that $\mathcal{S}_k f$ is a vector field in $\mathbf{R}^3 \setminus \partial\Omega$ that solves the Helmholtz equation $(\Delta + k^2)\mathcal{S}_k f = 0$ in $\mathbf{R}^3 \setminus \partial\Omega$ and, if Im $k \geq 0$, satisfies $\|(\mathcal{S}_k f)^*\|_{L^p(\partial\Omega_\pm)} + \|(\nabla\mathcal{S}_k f)^*\|_{L^p(\partial\Omega_\pm)} \leq C\|f\|_{L^p(\partial\Omega)}$ for any $1 < p < \infty$ [12]. The (interior/exterior) nontangential boundary traces of $\mathcal{S}_k f$ are given by

$$\lim_{\substack{X \longrightarrow P \\ X \in \Gamma_+(P)}} \mathcal{S}_k f(X) = \lim_{\substack{X \longrightarrow P \\ X \in \Gamma_-(P)}} \mathcal{S}_k f(X) = S_k f(P), \qquad P \in \partial\Omega,$$

where

$$S_k f(P) := -\frac{1}{4\pi} \int_{\partial\Omega} \frac{e^{ik|Q-P|}}{|Q-P|} f(Q)\, d\sigma(Q), \qquad P \in \partial\Omega.$$

The action of the operators $\mathcal{S}_k$, S_k on vector fields is defined componentwise.

Note that $S_k : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega)$ is a compact operator for any $1 < p < \infty$. In addition, at almost any $P \in \partial\Omega$,

$$\lim_{\substack{X \longrightarrow P \\ X \in \Gamma_\pm(P)}} \frac{\partial \mathcal{S}_k f}{\partial n}(X) := \lim_{\substack{X \longrightarrow P \\ X \in \Gamma_\pm(P)}} \langle n(P), \nabla \mathcal{S}_k f(X)\rangle =: \left(\mp\tfrac{1}{2} I + K_k^*\right) f(P),$$

where K_k^* is the formal transpose of the principal value integral operator

$$K_k f(P) := \text{p.v.} \frac{1}{4\pi} \int_{\partial\Omega} \frac{\langle n(Q), Q-P\rangle}{|Q-P|^3} e^{ik|Q-P|}(1 - ik|Q-P|) f(Q)\, d\sigma(Q),$$

the so-called (singular) double-layer acoustic potential operator. In fact, if we set $\mathcal{K}_k f := -\text{div}\, \mathcal{S}_k(nf)$, then clearly

$$\lim_{\substack{X \longrightarrow P \\ X \in \Gamma_\pm(P)}} \mathcal{K}_k f(X) = \left(\pm\tfrac{1}{2} I + K_k\right) f(P),$$

at almost any $P \in \partial\Omega$.

Combining the techniques of [28] with the results in [12] we can infer that for any vector field A in $L^p(\partial\Omega)$, $1 < p < \infty$, at almost any $P \in \partial\Omega$ we have

$$\lim_{\substack{X \to P \\ X \in \Gamma_\pm(P)}} \text{div}\, \mathcal{S}_k A(X) = \mp\frac{1}{2} \langle n, A\rangle (P) + \text{p.v.} \int_{\partial\Omega} \text{div}_P \{\Phi_k(P-Q)A(Q)\}\, d\sigma,$$

and

$$\lim_{\substack{X \to P \\ X \in \Gamma_\pm(P)}} \text{curl}\, \mathcal{S}_k A(X) = \mp\frac{1}{2} (n \times A)(P) + \text{p.v.} \int_{\partial\Omega} \text{curl}_P \{\Phi_k(P-Q)A(Q)\}\, d\sigma.$$

Another basic boundary integral operator for us here is the so-called *magnetic dipole operator* defined by

$$M_k A := n \times (\text{p.v.}\, \text{curl}\, S_k A)$$

for vector-valued densities A on $\partial\Omega$. Set

$$L^p_{\tan}(\partial\Omega) := \{A : \partial\Omega \to \mathbf{C}^3;\ A \in L^p(\partial\Omega),\ \langle n, A\rangle = 0 \text{ a.e. on } \partial\Omega\}.$$

Once again relying on the results of [12] one can show that, for each $1 < p < \infty$, the operator M_k is a bounded mapping of $L^p_{\tan}(\partial\Omega)$. Also, from the above discussion and since for $A \in L^p_{\tan}(\partial\Omega)$ one has $n \times (n \times A) = -A$, we see that

$$\lim_{\substack{X \to P \\ X \in \Gamma_{\pm}(P)}} n(P) \times \operatorname{curl} \mathcal{S}_k A(X) = \left(\pm\tfrac{1}{2}I + M_k\right) A(P)$$

at almost any $P \in \partial\Omega$.

Finally, we shall also work with the Newtonian potential type operator

$$L_k f(X) := \iint_{\Omega} \Phi_k(X - Y) f(Y)\, dY, \qquad X \in \Omega$$

(note that for $k = 0$ this is precisely the more-familiar Newtonian potential operator on Ω).

Lemma 3.1.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$. For any $1 < p < \infty$, $k \in \mathbf{C}$, the operator $\mathcal{S}_k$ maps $W^{-\frac{1}{p},p}(\partial\Omega)$ boundedly into $W^{1,p}(\Omega)$. In particular, $n \cdot \nabla\mathcal{S}_k = -\frac{1}{2}I + K_k^*$ on $W^{-\frac{1}{p},p}(\partial\Omega)$.

Also, under the same assumptions, the operator $\mathcal{K}_k$ maps $W^{1-\frac{1}{p},p}(\partial\Omega)$ boundedly into $W^{1,p}(\Omega)$ and, moreover, $\gamma \circ \mathcal{K}_k = \frac{1}{2}I + K_k$ on $W^{1-\frac{1}{p},p}(\partial\Omega)$.

Proof. If q is the conjugate exponent of p and if f, g are two scalar- and vector-valued, respectively, Lipschitz functions in $\mathbf{R}^3$, then

$$\begin{aligned}
\left|\iint_{\Omega} \langle \nabla\mathcal{S}_k f, g\rangle\, dV\right| &= \left|\int_{\partial\Omega} f\,(\operatorname{div} L_k g)\, d\sigma\right| \\
&\le C\|\operatorname{div} L_k g\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \|f\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \\
&\le C\|\operatorname{div} L_k g\|_{W^{1,q}(\Omega)} \|f\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \\
&\le C\|g\|_{L^q(\Omega)} \|f\|_{W^{-\frac{1}{p},p}(\partial\Omega)}
\end{aligned}$$

The last inequality follows from the boundedness of the operator L_k from $L^q(\Omega)$ into $W^{2,q}(\Omega)$, which, in turn, is a direct consequence of the classical Calderón–Zygmund inequality. Thus, by density, $\nabla\mathcal{S}_k$ extends as a bounded operator between $W^{-\frac{1}{p},p}(\partial\Omega)$ and $L^p(\Omega)$. Similarly, $\mathcal{S}_k$ maps $W^{-\frac{1}{p},p}(\partial\Omega)$ boundedly into $L^p(\Omega)$, and this completes the proof of the boundedness of $\mathcal{S}_k$.

Next, we note that Lemma 2.3 and the above reasoning imply that $f \mapsto n \cdot \nabla\mathcal{S}_k f$ is a bounded mapping of $W^{-\frac{1}{p},p}(\partial\Omega)$. Since, by the techniques in [28], the conclusion in the lemma obviously holds if f belongs to, for example, $L^p(\partial\Omega)$, an easy density argument finishes the proof of the first part of the lemma.

To see the second part, let $f \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and let $u \in W^{1,p}(\Omega)$ be such that $\gamma(u) = f$ with $\|u\|_{W^{1,p}(\Omega)} \le C\|f\|_{W^{1-\frac{1}{p},p}(\Omega)}$. Green's integral formula (cf. [70]) then yields

$$u = \mathcal{K}_k f + \operatorname{div} L_k(\nabla u) + k^2 L_k u.$$

Consequently, $\mathcal{K}_k f \in W^{1,p}(\Omega)$ and $\|\mathcal{K}_k f\|_{W^{1,p}(\Omega)} \le C\|f\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}$. In particular, by Gagliardo's lemma, $\gamma \circ \mathcal{K}_k$ is a bounded mapping of $W^{1-\frac{1}{p},p}(\partial\Omega)$. Now the conclusion easily follows from this and a density argument. $\square$

Remark. It is easy to see that the boundary trace map γ extends to a bounded operator between $W^{2,p}(\Omega)$ and $W^{1,p}(\partial\Omega)$ for any $1 < p < \infty$. Using this, the fact that L_k maps $L^p(\Omega)$ boundedly into $W^{2,p}(\Omega)$ and a duality argument, it follows that $\mathcal{S}_k$ also maps $W^{-1,p}(\partial\Omega)$ boundedly into $L^p(\Omega)$ for each $1 < p < \infty$. $\square$

Next we present a basic integral representation formula (a version of the so-called fundamental theorem of vector analysis).

Theorem 3.2.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$, $k \in \mathbf{C}$ and $1 < p < \infty$. If $u \in L^p(\Omega)$ is such that $\operatorname{curl} u \in L^p(\Omega)$ *and* $\operatorname{div} u \in L^p(\Omega)$, *then at almost any point in Ω one has*

$$u = k^2 L_k u - \operatorname{curl} L_k(\operatorname{curl} u) + \nabla L_k(\operatorname{div} u) - \nabla \mathcal{S}_k(n \cdot u) + \operatorname{curl} \mathcal{S}_k(n \wedge u).$$

Proof. Let u be as in the statement of the theorem and extend u, $\operatorname{curl} u$, and $\operatorname{div} u$ with zero outside Ω. First, we claim that there exists a sequence $(u_j)_j$ of functions in $C^\infty(\overline{\Omega})$ with u_j, $\operatorname{curl} u_j$, and $\operatorname{div} u_j$ converging in $L^p(\Omega)$ to u, $\operatorname{curl} u$, and $\operatorname{div} u$, respectively. To see this, using a partition of unity, there is no loss of generality to assume that $\operatorname{supp} u \cap \partial\Omega$ lies in a coordinate patch of $\partial\Omega$. Hence it is possible to construct an open, upright cone Γ centered at the origin of $\mathbf{R}^3$ and such that

$$\Gamma + (\partial\Omega \cap \operatorname{supp} u) \subseteq \mathbf{R}^3 \setminus \overline{\Omega}. \tag{3.1}$$

Let φ be a smooth, compactly supported function in $\mathbf{R}^3$ having integral one and such that $\operatorname{supp} \varphi \subseteq \Gamma$. Also, let $\varphi_\epsilon := \epsilon^{-3}\varphi(\cdot\epsilon^{-1})$ for $\epsilon > 0$. Clearly, $u * \varphi_\epsilon$, $(\operatorname{curl} u) * \varphi_\epsilon$, and $(\operatorname{div} u) * \varphi_\epsilon$ are smooth in $\mathbf{R}^3$ and converge in $L^p(\Omega)$ to u, $\operatorname{curl} u$, and $\operatorname{div} u$, respectively.

Next, $\operatorname{curl}(u * \varphi_\epsilon) = (\operatorname{curl} u) * \varphi_\epsilon + v * \varphi_\epsilon$, where v is a distribution supported on $\partial\Omega \cap \operatorname{supp} u$. Thus, by (3.1),

$$\operatorname{supp}(v * \varphi_\epsilon) \subseteq \Gamma + (\partial\Omega \cap \operatorname{supp} u) \subseteq \mathbf{R}^3 \setminus \overline{\Omega}.$$

Consequently, we have $\operatorname{curl}(u * \varphi_\epsilon) \to \operatorname{curl} u$ in $L^p(\Omega)$. Similarly, we also obtain $\operatorname{div}(u * \varphi_\epsilon) \to \operatorname{div} u$ in $L^p(\Omega)$ and the claim is proved.

For arbitrary smooth u and f, the following identities are easily verified:

$$\begin{aligned} \operatorname{div} L_k u &= L_k(\operatorname{div} u) - \mathcal{S}_k(\langle n, u\rangle), \\ \operatorname{curl} L_k u &= L_k(\operatorname{curl} u) - \mathcal{S}_k(n \times u), \\ \nabla L_k f &= L_k(\nabla f) - \mathcal{S}_k(nf). \end{aligned}$$

Now we observe that by straightforward integration by parts based on the above identities the corresponding statement of the theorem for functions in $C^\infty(\overline{\Omega})$ holds true. On account of Lemma 2.3 and Lemma 3.1, the proof of the theorem is now concluded by a simple limiting argument. □

A frequently used consequence of Theorem 3.2 is a Green type integral representation formula for divergencefree vector fields with *metaharmonic* components (i.e., the components are annihilated by the Helmholtz operator $\Delta + k^2$).

Corollary 3.3.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$, and consider a vector field $u \in L^p(\Omega)$, $1 < p < \infty$, that is divergencefree, satisfies the Helmholtz equation $(\Delta + k^2)u = 0$ in Ω and such that $\operatorname{curl} u \in L^p(\Omega)$.

Then at almost any point in Ω we have

$$u = \operatorname{curl} \mathcal{S}_k(n \wedge u) - \nabla \mathcal{S}_k(n \cdot u) + \mathcal{S}_k(n \wedge \operatorname{curl} u). \tag{3.2}$$

Proof. Using the fact that $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla\operatorname{div}$, simple integrations by parts give

$$\operatorname{curl} L_k(\operatorname{curl} u) - k^2 L_k u = -L_k((\Delta + k^2)u) - \mathcal{S}_k(n \wedge \operatorname{curl} u),$$

so that everything follows from Theorem 3.2. □

4. The Surface Divergence and Related Function Spaces and Operators

Let Ω be a bounded domain in $\mathbf{R}^3$ that, for the moment, we assume to have a smooth boundary. A basic invariant associated to a (smooth, complex-valued) tangential vector field $A = (A_j)_j$ on $\partial\Omega$ by tangential differentiation is the (scalar-valued) function

$$\mathrm{Div} A := \sum_{j,l} n_l(n_l\partial_j - n_j\partial_l)A_j \tag{4.1}$$

which is called the *surface divergence* of A.

An intrinsic definition of Div A can be obtained as follows. If A is supported in a surface patch where $\partial\Omega$ has the parametric representation $\partial\Omega \ni P = \vec{r}\,(t_1, t_2)$ and $A = a_1\partial_1\vec{r} + a_2\partial_2\vec{r}$, then

$$\mathrm{Div}\, A = \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial t_1}(\sqrt{g}a_1) + \frac{\partial}{\partial t_2}(\sqrt{g}a_2)\right),$$

where $g := \det(\langle\partial_j\vec{r}, \partial_l\vec{r}\,\rangle)_{j,l}$ (cf., e.g., [68, p. 154]).

For the following calculation assume that the tangential field A has been smoothly extended to all $\mathbf{R}^3$. Since, by (4.1), $\mathrm{Div}\, A = \mathrm{div}\, A - \langle n, (n\cdot\nabla)A\rangle = \langle n, \mathrm{curl}\,(n\times A)\rangle$, using for example (2.1) it is not difficult to see that

$$\int_{\partial\Omega}\langle\nabla\varphi\,,\ A\rangle\, d\sigma = -\int_{\partial\Omega}\varphi\, \mathrm{Div}\, A\, d\sigma \tag{4.2}$$

for any (eventually complex-valued) $\varphi \in C^\infty(\mathbf{R}^3)$.

The above formula (4.2) is the departure point for extending the definition of the surface divergence operator Div to the case when the boundary of the domain is only Lipschitz continuous. For the rest of this section, we fix an arbitrary Lipschitz domain Ω in $\mathbf{R}^3$. To be more specific, if $1 < p < \infty$ and $A \in L^p_{\tan}(\partial\Omega)$, we define Div A as the functional

$$\langle \mathrm{Div}\, A, f\rangle := -\int_{\partial\Omega}\langle A, \nabla_{\tan} f\rangle\, d\sigma, \tag{4.3}$$

where f is an arbitrary Lipschitz continuous function on $\partial\Omega$. Since A is tangential, this definition is in agreement with (4.2) in the case when f extends smoothly in all $\mathbf{R}^3$. Also, if $\frac{1}{p}+\frac{1}{q}=1$,

$$|\langle \mathrm{Div}\, A, f\rangle| \le \|A\|_{L^p(\partial\Omega)}\|\nabla_{\tan} f\|_{L^q(\partial\Omega)} \le \|A\|_{L^p(\partial\Omega)}\|f\|_{W^{1,q}(\partial\Omega)}.$$

Thus, by density, Div A extends to an element in $(W^{1,q}(\partial\Omega))^* = W^{-1,p}(\partial\Omega)$. In fact, the surface divergence operator

$$\mathrm{Div}\ : L^p_{\tan}(\partial\Omega) \to W^{-1,p}(\partial\Omega)$$

is well-defined, linear, and bounded and we have the integration by parts formula

$$\int_{\partial\Omega} f\, \mathrm{Div}\, A\, d\sigma = -\int_{\partial\Omega}\langle n\times A, (n\times\nabla)f\rangle\, d\sigma, \tag{4.4}$$

for any $A \in L^p_{\tan}(\partial\Omega)$, $f \in W^{1,q}(\partial\Omega)$, where $1 < p, q < \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Going further, another important observation is that

$$\mathrm{Div}\,(n\times A) = -\langle n, \mathrm{curl}\, A\rangle \tag{4.5}$$

for any vector field $A \in L^p(\partial\Omega)$ (note that $n\times A \in L^p_{\tan}(\partial\Omega)$ and that the operator $\langle n, \mathrm{curl}\,\cdot\rangle$ in the right-hand side has to be interpreted as an intrinsic tangential derivative operator in the sense discussed in the previous section). This follows immediately from (4.4) and (2.1).

Lemma 4.1.

Let $1 < p < \infty$ and $u \in L^p(\Omega)$ such that $\operatorname{curl} u \in L^p(\Omega)$. *If $n \wedge u \in L^p(\partial\Omega)$, then in fact $n \wedge u \in L^p_{\tan}(\partial\Omega)$ and* $\operatorname{Div}(n \wedge u) = -n \cdot \operatorname{curl} u$. *In particular,* $\operatorname{Div}(n \wedge u) \in W^{-\frac{1}{p},p}(\partial\Omega)$.

Proof. Using a partition of unity and working in local coordinates it is possible to construct a sequence of smooth function ψ_j in $\mathbf{R}^3$ such that $\psi_j|_{\partial\Omega} \to 0$ in $W^{1,q}(\partial\Omega)$ and $(\nabla\psi_j)|_{\partial\Omega} \to \rho\, n$ in $L^q(\partial\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$ for some positive scalar-valued ρ that is bounded away from zero and infinity. Fix an arbitrary smooth function φ in $\mathbf{R}^3$. Then

$$\begin{aligned}\int_{\partial\Omega} \langle n \wedge u, n\rangle\, \rho\, \varphi\, d\sigma &= \lim_j \int_{\partial\Omega} \langle n \wedge u, \nabla(\varphi\psi_j)\rangle\, d\sigma \\ &= \lim_j \iint_{\Omega} \langle \operatorname{curl} u, \nabla(\varphi\psi_j)\rangle\, dV \\ &= \lim_j \int_{\partial\Omega} (n \cdot \operatorname{curl} u)\, \varphi\psi_j\, d\sigma = 0.\end{aligned}$$

Hence $\langle n \wedge u, n\rangle = 0$ a.e. on $\partial\Omega$ so that $n \wedge u \in L^p_{\tan}(\partial\Omega)$.

To calculate the surface divergence of $n \wedge u$, we use (4.2) to write

$$\begin{aligned}\int_{\partial\Omega} \varphi \operatorname{Div}(n \wedge u)\, d\sigma &= -\int_{\partial\Omega} \langle n \wedge u, \nabla\varphi\rangle\, d\sigma = -\iint_{\Omega} \langle \operatorname{curl} u, \nabla\varphi\rangle\, dV \\ &= -\int_{\partial\Omega} (n \cdot \operatorname{curl} u)\, \varphi\, d\sigma.\end{aligned}$$

Therefore the first part of the lemma follows from a simple limiting argument. The last part of the lemma is immediately seen from Lemma 2.3, and this concludes the proof. □

Next we study the action of the surface divergence operator in connection with the boundary integral operators introduced in the previous section.

Lemma 4.2.

For $k \in \mathbf{C}$, $1 < p < \infty$, and $A \in L^p_{\tan}(\partial\Omega)$, we have

$$\operatorname{div} \mathcal{S}_k A = \mathcal{S}_k(\operatorname{Div} A)$$

in $\mathbf{R}^3 \setminus \partial\Omega$. Moreover, a similar identity is valid by interpreting the operators in the principal value sense on $\partial\Omega$.

Proof. For a fixed point $X \in \mathbf{R}^3 \setminus \partial\Omega$, we have

$$\begin{aligned}(\operatorname{div} \mathcal{S}_k A)(X) &= -\int_{\partial\Omega} \langle (\nabla\Phi_k)(X - Y), A(Y)\rangle\, d\sigma(Y) \\ &= \int_{\partial\Omega} \Phi_k(X - Y)(\operatorname{Div} A)(Y)\, d\sigma(Y) \\ &= \mathcal{S}_k(\operatorname{Div} A)(X),\end{aligned}$$

which proves the first part of the lemma. To see the second part, we shall use a limiting argument. First, let us note that the vector space

$$\{n \times \varphi|_{\partial\Omega}\, ;\, \varphi \in C^\infty_{\text{comp}}(\mathbf{R}^3)\}$$

is densely embedded into $L^p_{\tan}(\partial\Omega)$. Indeed, any $A \in L^p_{\tan}(\partial\Omega)$ can be written as $A = -n \times (n \times A)$ and, if $\{\varphi_j\}_j$ is a sequence of functions in $C^\infty_{\text{comp}}(\mathbf{R}^3)$ such that $\varphi_j|_{\partial\Omega} \to -n \times A$ in $L^p(\partial\Omega)$, then $n \times \varphi_j \to A$ in $L^p(\partial\Omega)$.

Next, $\varphi_j \in C^{\infty}_{\text{comp}}(\mathbf{R}^3)$ so that $A_j := n \times \varphi_j|_{\partial\Omega}$ converges to A in $L^p(\partial\Omega)$. In particular, $\operatorname{Div} A_j \to \operatorname{Div} A$ in $W^{-1,p}(\partial\Omega)$. Now, for each fixed j and at almost every $P \in \partial\Omega$, we have that

$$\begin{aligned} S_k(\operatorname{Div} A_j)(P) &= \lim_{\substack{X \to P \\ X \in \Gamma_+(P)}} \mathcal{S}_k(\operatorname{Div} A_j)(X) = \lim_{\substack{X \to P \\ X \in \Gamma_+(P)}} (\operatorname{div} \mathcal{S}_k A_j)(X) \\ &= (\operatorname{div} S_k A_j)(P). \end{aligned}$$

The second equality is provided by the first part of the lemma, and the third equality follows from the jump relations for the derivatives of the single-layer potential operator (cf. §3) and the results in [12].

Thus, by the continuity properties of S_k and $\operatorname{div} S_k$ (cf. also the remark after Lemma 3.1), we have

$$S_k(\operatorname{Div} A) = \lim_j S_k(\operatorname{Div} A_j) = \lim_j \operatorname{div} S_k A_j = \operatorname{div} S_k A$$

in $L^p(\partial\Omega)$. □

Lemma 4.3.

For any $k \in \mathbf{C}$, $1 < p < \infty$, and any scalar-valued function $f \in L^p(\partial\Omega)$, we have

$$\operatorname{curl} \mathcal{S}_k(nf) = -\mathcal{S}_k(n \times \nabla f)$$

in $\mathbf{R}^3 \setminus \partial\Omega$. A similar result holds on $\partial\Omega$ too.

Proof. As in the proof of the Lemma 4.2 it suffices to prove the identity in $\mathbf{R}^3 \setminus \partial\Omega$ and for $f \in W^{1,p}(\partial\Omega)$. In this case, fixing an arbitrary point $X \in \mathbf{R}^3 \setminus \partial\Omega$, from (2.2) we have that

$$\begin{aligned} \mathcal{S}_k(n \times \nabla f)(X) &= \int_{\partial\Omega} \Phi_k(X - Y)(n \times \nabla) f(Y)\, d\sigma(Y) \\ &= -\int_{\partial\Omega} [(n \times \nabla)\Phi_k(X - \cdot)](Y) f(Y)\, d\sigma(Y) \\ &= \int_{\partial\Omega} n(Y) \times (\nabla\Phi_k)(X - Y) f(Y)\, d\sigma(Y) \\ &= -\operatorname{curl} \mathcal{S}_k(nf)(X), \end{aligned}$$

and the conclusion follows. □

The following lemma will also be used many times in the sequel.

Lemma 4.4.

For each $k \in \mathbf{C}$, $A \in L^p_{\tan}(\partial\Omega)$, and $1 < p < \infty$, we have that

$$\operatorname{Div} M_k A = -k^2 \langle n, S_k A \rangle - K_k^*(\operatorname{Div} A)$$

in $W^{-1,p}(\partial\Omega)$. In particular, for each $1 < p < \infty$, the diagram

$$\begin{array}{ccc} L^p_{\tan}(\partial\Omega) & \xrightarrow{M_0} & L^p_{\tan}(\partial\Omega) \\ {\scriptstyle \operatorname{Div}}\downarrow & & \downarrow{\scriptstyle \operatorname{Div}} \\ W_0^{-1,p}(\partial\Omega) & \xrightarrow{-K_0^*} & W_0^{-1,p}(\partial\Omega) \end{array} \tag{4.6}$$

is commutative.

Proof. If A is of the form $n \times B$ with B a smooth vector field in $\mathbf{R}^3$, then the above identity is a corollary of Lemma 4.1, Lemma 4.2, and the fact that $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla\operatorname{div}$. The general case follows from this and the density argument employed in the proof of the Lemma 4.2. □

We conclude this section by introducing some Banach spaces of boundary vector fields that will play a fundamental role in the sequel. First, for each $1 < p < \infty$, we set

$$L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) := \{A \in L^p_{\tan}(\partial\Omega)\,;\ \mathrm{Div}\,A \in L^p(\partial\Omega)\}.$$

By Lemma 4.4 and the results in [12], we see that M_k is a bounded mapping of $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, for each $1 < p < \infty$, $k \in \mathbf{C}$. We shall also use the principal value singular integral operator N_k, called the *electric dipole operator*, defined by

$$N_k A := n \times \mathrm{curl}\,\mathrm{curl}\, S_k A = k^2 n \times S_k A + n \times \nabla S_k(\mathrm{Div}\,A)$$

for $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. Since, by (4.5), we have $\mathrm{Div}\,N_k A = -k^2\langle n, \mathrm{curl}\,S_k A\rangle$, it follows that N_k is a bounded mapping of $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for each $1 < p < \infty$. In particular, note that $N_0 A = n \times \nabla S_0(\mathrm{Div}\,A)$ for $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$.

Second, introducing

$$L^{p,0}_{\tan}(\partial\Omega) := \{A \in L^p_{\tan}(\partial\Omega)\,;\ \mathrm{Div}\,A = 0\},$$

then once again by Lemma 4.4 and the results in [12] we have that M_0 is a bounded mapping of $L^{p,0}_{\tan}(\partial\Omega)$ for each $1 < p < \infty$. We remark that from (4.5) the operator $n \times \nabla S_0$ maps $L^p(\partial\Omega)$ boundedly into $L^{p,0}_{\tan}(\partial\Omega)$ for each $1 < p < \infty$. In fact, the tangential derivative operator $n \times \nabla$ maps $W^{1,p}(\partial\Omega)$ boundedly into $L^{p,0}_{\tan}(\partial\Omega)$, $1 < p < \infty$. It follows that N_0 also maps $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ boundedly into $L^{p,0}_{\tan}(\partial\Omega)$ for $1 < p < \infty$.

A Sobolev-like scale of spaces in which $L^{p,0}_{\tan}(\partial\Omega)$ can be incorporated is obtained by setting

$$V^{-s,p}_{\tan}(\partial\Omega) := \{(n \times \nabla)f\,;\ f \in W^{1-s,p}(\partial\Omega)\} \subseteq W^{-s,p}(\partial\Omega) \tag{4.7}$$

for $1 < p < \infty$, $0 \le s \le 1$. This is because, as we shall see in the next section (cf., e.g., Corollary 5.2) one has $V^{0,p}_{\tan} = L^{p,0}_{\tan}(\partial\Omega)$. These spaces are complete when equipped with the natural norm $\|A\|_{V^{-s,p}_{\tan}(\partial\Omega)}$, which we take to be

$$\inf\{\|\lambda + f\|_{W^{1-s,p}(\partial\Omega)}\,;\ \lambda \in \mathbf{C},\ f \in W^{1-s,p}(\partial\Omega),\ (n \times \nabla)f = A\}.$$

Note that, with this notation, the operator $n \times \nabla S_0$ maps $W^{-s,p}(\partial\Omega)$ boundedly into $V^{-s,p}_{\tan}(\partial\Omega)$ for $1 < p < \infty$ and $0 \le s \le 1$.

5. Inverting Boundary Integral and Boundary Derivative Operators

In this section we discuss the L^p invertibility of vector potential and tangential derivative operators on Lipschitz boundaries in $\mathbf{R}^3$. The main results in this direction are Theorem 5.1 and Theorem 5.3. Recall that $g(\partial\Omega)$ stands for the topological genus of $\partial\Omega$.

Theorem 5.1.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then there exists ϵ positive, depending only on $\partial\Omega$ such that the following operators are isomorphisms between the indicated spaces.

i. $\langle n, \mathrm{curl}\,S_0\rangle : L^{p,0}_{\tan}(\partial\Omega) \to L^p_0(\partial\Omega)$ *for each* $1 < p \le 2+\epsilon$;

ii. $n \times \nabla S_0 : L^p_0(\partial\Omega) \to L^{p,0}_{\tan}(\partial\Omega)$ *for each* $1 < p \le 2+\epsilon$;

iii. $T : L^p_0(\partial\Omega) \times L^{p,0}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, *where* $T(f, A) := n \times \nabla S_0 f + n \times S_0 A$, *for each* $1 < p \le 2+\epsilon$;

iv. $\pi \circ (n \times S_0) : L^{p,0}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega)$, *where* π *is the canonical projection operator of* $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *onto* $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega)$ *for each* $1 < p \le 2+\epsilon$;

v. $n \times \nabla : W^{1,p}(\partial\Omega)/\mathbf{C} \to L^{p,0}_{\tan}(\partial\Omega)$ *for each* $1 < p < \infty$ *(in fact* $n \times \nabla : W^{1-s,p}(\partial\Omega)/\mathbf{C} \to V^{-s,p}_{\tan}(\partial\Omega)$ *for each* $0 \le s \le 1$, $1 < p < \infty$*)*;

vi. $\langle n, \operatorname{curl} S_0\rangle : V^{-s,p}_{\tan}(\partial\Omega) \to W^{-s,p}_0(\partial\Omega)$ *for each* $0 \le s \le 1$ *and* $2 - \epsilon \le p \le 2 + \epsilon$;

vii. $\langle n, \operatorname{curl} S_0\rangle : V^{-1,p}_{\tan} \to W^{-1,p}_0(\partial\Omega)$ *for each* $2 - \epsilon \le p < \infty$;

viii. $\operatorname{Div} : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to L^p_0(\partial\Omega)$ *for each* $1 < p < \infty$;

ix. $\langle n, \operatorname{curl}\rangle : \left(n \times L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)\right) / \left(n \times L^{p,0}_{\tan}(\partial\Omega)\right) \to L^p_0(\partial\Omega)$ *for each* $1 < p < \infty$;

x. $\operatorname{div} S_0 : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to W^{1,p}(\partial\Omega)/\mathbf{C}$ *for each* $1 < p \le 2 + \epsilon$;

xi. $n \times \operatorname{curl} S_0 : n\left(W^{1,p}(\partial\Omega)/\mathbf{C}\right) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega)$ *for each* $1 < p \le 2 + \epsilon$;

xii. $N_0 : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to L^{p,0}_{\tan}(\partial\Omega)$ *for each* $1 < p \le 2 + \epsilon$;

xiii. $\operatorname{Div} : L^p_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to W^{-1,p}_0(\partial\Omega)$ *for each* $1 < p < \infty$;

xiv. $\langle n, \operatorname{curl}\rangle : L^p_{\tan}(\partial\Omega)/\left(n \times L^{p,0}_{\tan}(\partial\Omega)\right) \to W^{-1,p}_0(\partial\Omega)$ *for each* $1 < p < \infty$;

xv. $\operatorname{div} S_0 : L^p_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to L^p(\partial\Omega)/\mathbf{C}$ *for each* $2 - \epsilon \le p < \infty$;

xvi. $\pm\frac{1}{2}I + M_0 : L^{p,0}_{\tan}(\partial\Omega) \to L^{p,0}_{\tan}(\partial\Omega)$ *for each* $1 < p \le 2 + \epsilon$ *(and, in fact, on* $V^{-s,p}_{\tan}(\partial\Omega)$ *for each* $0 \le s \le 1$, $1 < p \le 2 + \epsilon$*)*;

xvii. $\pm\frac{1}{2}I + M_0 : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for each* $1 < p \le 2 + \epsilon$;

xviii. $\pm\frac{1}{2}I + M_0 : L^p_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to L^p_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega)$ *for each* $2 - \epsilon \le p < \infty$;

xix. $\pm\frac{1}{2}I + M_0 : L^p_{\tan}(\partial\Omega) \to L^p_{\tan}(\partial\Omega)$ *for each* $2 - \epsilon \le p \le 2 + \epsilon$;

xx. $\pm\frac{1}{2}I + M_0$ *acting on* $\{A \in L^p_{\tan}(\partial\Omega);\ \operatorname{Div} A \in W^{-s,p}(\partial\Omega)\}$ *for each* $2 - \epsilon \le p \le 2 + \epsilon$ *and* $0 \le s \le 1$.

In the class of Lipschitz domains these results are sharp. If the domain Ω *has in fact a* C^1 *boundary, then the same results are valid for* $1 < p < \infty$.

An immediate corollary of Theorem 5.1 regarding the L^p-cohomology of the Lipschitz manifold $\partial\Omega$ is the following (recall first the usual tangential gradient $\nabla_{\tan} := -n \times (n \times \nabla)$).

Corollary 5.2.

Let Ω *be a bounded Lipschitz domain in* $\mathbf{R}^3$ *with* $g(\partial\Omega) = 0$. *Then for each* $1 < p < \infty$, *the sequences of boundary derivative operators*

$$0 \longrightarrow \mathbf{C} \xrightarrow{\ \iota\ } W^{1,p}(\partial\Omega) \xrightarrow{\ n\times\nabla\ } L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \xrightarrow{\ \mathrm{Div}\ } L^p_0(\partial\Omega) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbf{C} \xrightarrow{\ \iota\ } W^{1,p}(\partial\Omega) \xrightarrow{\ \nabla_{\tan}\ } n \times L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \xrightarrow{\ \langle n,\mathrm{curl}\rangle\ } L^p_0(\partial\Omega) \longrightarrow 0$$

are exact.

Before we state our next result, let us recall that the application taking the tangential component of the electric field E into the tangential component of the magnetic field H, for each pair (E, H) satisfying the Maxwell equations (with wave number $k \in \mathbf{C} \setminus \{0\}$, $\operatorname{Im} k \ge 0$) in $\Omega_\pm$, is called the voltage-to-current map and is denoted by $\Lambda^\pm_k$. Also, recall the definition of the Maxwell eigenvalues for a domain Ω in the first section.

Theorem 5.3.

Let Ω *be a bounded Lipschitz domain in* $\mathbf{R}^3$ *with* $g(\partial\Omega) = 0$. *Then there exists* ϵ *positive, depending only on* $\partial\Omega$ *such that, if* $k \in \mathbf{C} \setminus \{0\}$, $\operatorname{Im} k \ge 0$, *is not a Maxwell eigenvalue for* Ω, *then the following operators are isomorphisms between the indicated spaces.*

i. $\pm\frac{1}{2}I + M_k : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for each* $1 < p \leq 2+\epsilon$;
ii. $N_k : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for each* $1 < p \leq 2+\epsilon$;
iii. $\Lambda^{\pm}_k : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for each* $1 < p \leq 2+\epsilon$;
iv. $\pm\frac{1}{2}I + M_k : L^{p}_{\tan}(\partial\Omega) \to L^{p}_{\tan}(\partial\Omega)$ *for each* $2-\epsilon \leq p \leq 2+\epsilon$;
v. $\pm\frac{1}{2}I + M_k$ *acting on* $\{A \in L^p_{\tan}(\partial\Omega);\ \mathrm{Div}\, A \in W^{-s,p}(\partial\Omega)\}$ *for each* $2-\epsilon \leq p \leq 2+\epsilon$ *and* $0 \leq s \leq 1$;
vi. $\pm\frac{1}{2}I + M_k : L^{p}_{\tan}(\partial\Omega)/L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \to L^{p}_{\tan}(\partial\Omega)/L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for each* $2-\epsilon \leq p \leq 2+\epsilon$, *and here we may take* $k \in \mathbf{C}$ *arbitrary.*

For an arbitrary bounded Lipschitz domain Ω *in* $\mathbf{R}^3$ *(i.e., not necessarily having topological genus zero), the same operators are still invertible provided* $2-\epsilon \leq p \leq 2+\epsilon$. *Also, for a general* $k \in \mathbf{C}$, *all the above operators (with the exception of that in* **iii**, *which may no longer be well defined) are Fredholm with index zero.*

If the domain Ω *has in fact a* C^1 *boundary, then the same results are valid for* $1 < p < \infty$.

The rest of this section is devoted to presenting the proofs of Theorem 5.1 and Theorem 5.3. First, let us recall the corresponding results for the scalar-valued layer potential operators for the Laplacian. The various parts of the following theorem are due to B. Dahlberg, C. Kenig, G. Verchota, E. Fabes, M. Jodeit and N. Rivière [20, 93, 28].

Theorem 5.4.

For any bounded Lipschitz domain Ω *in* $\mathbf{R}^3$ *there exists* $\epsilon > 0$, *which depends only on* $\partial\Omega$, *such that for each* $1 < p \leq 2+\epsilon$ *the operator* S_0 *is an invertible mapping of* $L^p(\partial\Omega)$ *onto* $W^{1,p}(\partial\Omega)$ *and of* $L^p_0(\partial\Omega)$ *onto* $W^{1,p}(\partial\Omega)/\mathbf{C}$, $\frac{1}{2}I + K_0$ *is an isomorphism of* $W^{1,p}(\partial\Omega)$, *and the operators* $\pm\frac{1}{2}I + K_0^*$ *are isomorphisms of* $L^p_0(\partial\Omega)$.

Furthermore, for each $2-\epsilon \leq p < \infty$ *the operator* $\frac{1}{2}I + K_0$ *is an isomorphism of* $L^p(\partial\Omega)$, *whereas* $-\frac{1}{2}I + K_0$ *is an isomorphism of* $L^p(\partial\Omega)/\mathbf{C}$.

For general Lipschitz domains these ranges are sharp. If, however, $\partial\Omega \in C^1$, *then the above invertibility results are valid in the full range* $1 < p < \infty$.

Extensions of this theorem to more general Sobolev–Besov spaces have been recently obtained in [60]. For the convenience of the reader, below we record a particular case of the main result in [60] that suffices for the applications we have in mind.

Theorem 5.5.

For any bounded Lipschitz domain Ω *in* $\mathbf{R}^3$ *there exists* $\epsilon > 0$, *which depends only on* $\partial\Omega$, *such that for each* $\frac{3}{2}-\epsilon < p \leq 2+\epsilon$ *the operators* $\pm\frac{1}{2}I + K_0$ *are invertible on* $W^{1-\frac{1}{p},p}(\partial\Omega)/\mathbf{C}$, *and the operators* $\pm\frac{1}{2}I + K_0^*$ *are isomorphisms of* $W_0^{-\frac{1}{p},p}(\partial\Omega)$.

In particular, for each $\frac{3}{2}-\epsilon \leq p \leq 2+\epsilon$, *the Poisson equation for the Laplace operator with homogeneous Dirichlet boundary conditions*

$$\begin{cases} u \in W_0^{1,p}(\Omega), \\ \Delta u = q \in W^{-1,p}(\Omega) \end{cases} \tag{$*$}$$

as well as the Poisson equation for the Laplace operator with homogeneous Neumann boundary conditions

$$\begin{cases} u \in W^{1,p}(\Omega); \\ \Delta u = q \in (W^{1,p'}(\Omega))^*, \qquad \frac{1}{p}+\frac{1}{p'}=1; \\ \frac{\partial u}{\partial n} = 0, \end{cases} \tag{$**$}$$

have unique solutions. Moreover, these solutions satisfies natural a priori estimates.

If $\partial\Omega \in C^1$, then the above invertibility results are valid for the the full range $1 < p < \infty$.

For (*) see [41] and especially [42] where a complete analysis of this problem can be found. A unified treatment of both (*) and (**) is contained in [60].

Let us also note that, since $\Phi_k(X) - \Phi_0(X) = -\frac{ik}{4\pi}\int_0^1 e^{ikt|X|}\,dt$, the operator $K_k - K_0$ is only weakly singular and hence, compact (cf. [5]). Consequently, except for a discrete set of real values of k, similar results are valid for the operators $\pm\frac{1}{2}I + K_k$, as well as for $\pm\frac{1}{2}I + K_k^*$, S_k, for example (cf. also §11).

Another corollary of Theorem 5.5 (for a more complete statement see [60]) that is of interest for us is the following Hodge type decomposition result.

Theorem 5.6.

Let Ω be a bounded Lipschitz domain Ω in $\mathbf{R}^3$. Then there exists $\epsilon > 0$, which depends only on $\partial\Omega$ such that for $\frac{3}{2} - \epsilon < p \leq 2 + \epsilon$ each vector field $u \in L^p(\Omega)$ can be decomposed as $u = \nabla v + w$, with $v \in W^{1,p}(\Omega)$, $w \in L^p(\Omega)$, $\operatorname{div} w = 0$, and either

i. $\gamma(v) = 0$, *or*
ii. $n \cdot w = 0$.

If $\partial\Omega \in C^1$, then the above decomposition results are valid for the the full range $1 < p < \infty$.

A key ingredient employed by B. Dahlberg, C. Kenig, and G. Verchota in the proof of the Theorem 5.4 is a certain integral identity of Rellich [78] (see, e.g., [93] for more details and the history of this formula). This identity has been further refined in [64] (cf. also [63, 37] for some related material) where it has been shown that for any bounded Lipschitz domain Ω in $\mathbf{R}^3$ and any vector fields $E \in C^\infty(\bar{\Omega}, \mathbf{C}^3)$ and $\Theta = (\Theta_j)_{j=1}^3 \in C^\infty_{\text{comp}}(\mathbf{R}^3, \mathbf{R}^3)$ one has

$$\begin{aligned}\int_{\partial\Omega} &\left\{\tfrac{1}{2}|E|^2\langle\Theta, n\rangle - \operatorname{Re}\langle\bar{E}, \Theta\rangle\langle E, n\rangle\right\} d\sigma \\ &= \operatorname{Re}\left(\iint_\Omega \tfrac{1}{2}|E|^2 \operatorname{div}\Theta - \langle\bar{E}, \Theta\rangle \operatorname{div} E - \langle\bar{E}, (\nabla\Theta)E\rangle + \langle\Theta \times \bar{E}, \operatorname{curl} E\rangle\right).\end{aligned} \tag{5.1}$$

Here $(\nabla\Theta)E$ is the matrix $\{\partial_i\Theta_j\}_{i,j}$ acting on the vector E. By taking $\Theta \in C^\infty_{\text{comp}}(\mathbf{R}^3, \mathbf{R}^3)$ in (5.1) such that $\langle n, \Theta\rangle \geq \kappa > 0$ on $\partial\Omega$ (cf. the point **vi** in Lemma 2.2), it has been shown in [64] that

$$\begin{aligned}\int_{\partial\Omega}|E|^2 \;\;\leq\;\; & C\min\left\{\int_{\partial\Omega}|n \times E|^2, \int_{\partial\Omega}|\langle n, E\rangle|^2\right\} \\ & + C\iint_\Omega |E|^2 + |\operatorname{curl} E|^2 + |\operatorname{div} E|^2\end{aligned} \tag{5.2}$$

for some positive constant C depending exclusively on the Lipschitz character of Ω. This was used in [64] to prove the following theorem.

Theorem 5.7.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$. There exists $\epsilon > 0$ depending only on $\partial\Omega$ such that for each $k \in \mathbf{C}$ with $\operatorname{Im} k > 0$ and each $2 - \epsilon \leq p \leq 2 + \epsilon$, the operators $\pm\frac{1}{2}I + M_k$ are isomorphisms of $L^{p,\text{Div}}_{\tan}(\partial\Omega)$.

Next, we prove some operator identities. In Lemmas 5.6–5.10 we assume that Ω is a fixed, arbitrary bounded Lipschitz domain in $\mathbf{R}^3$.

Lemma 5.8.

For each $1 < p < \infty$, we have that

$$M_0(n \times \nabla S_0) = (n \times \nabla S_0)K_0^* \tag{5.3}$$

and

$$\langle n, \operatorname{curl} S_0\rangle(n \times \nabla S_0) = \left(\tfrac{1}{2}I + K_0^*\right)\left(-\tfrac{1}{2}I + K_0^*\right) \tag{5.4}$$

on $L^p(\partial\Omega)$.

Proof. For an arbitrary $f \in L^p(\partial\Omega)$, Corollary 3.3 applied to the curlfree, divergence-free vector field $u := \nabla S_0 f$ in Ω gives

$$\operatorname{curl} S_0(n \times \nabla S_0 f) = \nabla S_0\left(\left(\tfrac{1}{2}I + K_0^*\right) f\right)$$

in Ω. Going (nontangentially) to the boundary and then taking $n\times$ of both sides yields (5.3), whereas taking $\langle n, \cdot\rangle$ of both sides yields (5.4). □

Lemma 5.9.

For each $1 < p < \infty$, *we have that*

$$K_0^*(\langle n, \operatorname{curl} S_0\rangle) = \langle n, \operatorname{curl} S_0\rangle M_0 \tag{5.5}$$

and

$$(n \times \nabla S_0)\langle n, \operatorname{curl} S_0\rangle = \left(\tfrac{1}{2}I + M_0\right)\left(-\tfrac{1}{2}I + M_0\right) \tag{5.6}$$

on $L^{p,0}_{\tan}(\partial\Omega)$.

Proof. For $A \in L^{p,0}_{\tan}(\partial\Omega)$, we use the identity (3.2) from Corollary 3.3 for the divergence-free vector field $u := \operatorname{curl} S_0 A$ and arrive at

$$\operatorname{curl} S_0\left(\left(-\tfrac{1}{2}I + M_0\right) A\right) = \nabla S_0(\langle n, \operatorname{curl} S_0 A\rangle).$$

Again going to the boundary and taking $\langle n, \cdot\rangle$, $n\times$ of both sides yields (5.5) and (5.6), respectively. □

Recall now the (interior and exterior) voltage-to-current mappings $\Lambda_k^{\pm}$ taking $n \times E$ into $n \times H$, where (E, H) are solutions of the interior and exterior, respectively, boundary value problem for the Maxwell system. From the results in [64], we know that $\Lambda_k^{\pm}$ are well-defined mappings of $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ whenever k is not a so-called Maxwell eigenvalue for Ω (in fact, as we shall see momentarily, the same holds for $1 < p \le 2 + \epsilon$). For the time being, we note the following lemma.

Lemma 5.10.

Assume that k is not a Maxwell eigenvalue for Ω. *Then we have that*

$$\left(\Lambda_k^{\pm}\right)^2 = -I, \tag{5.7}$$

$$N_k = ik\,\Lambda_k^{\pm}\left(\pm\tfrac{1}{2}I + M_k\right) = ik\,\left(\mp\tfrac{1}{2}I + M_k\right)\Lambda_k^{\pm}, \tag{5.8}$$

and

$$ik\,N_k\left(\Lambda_k^{-} - \Lambda_k^{+}\right) = ik\,\left(\Lambda_k^{-} - \Lambda_k^{+}\right)N_k = I \tag{5.9}$$

on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. *Also, for any* $1 < p < \infty$ *and any* $k \in \mathbf{C}$,

$$N_k^2 = k^2\left(\tfrac{1}{2}I + M_k\right)\left(-\tfrac{1}{2}I + M_k\right) \tag{5.10}$$

on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$.

Proof. The first identity is an easy consequence of the fact that if (E, H) is a solution of the Maxwell system, then so is $(H, -E)$.

To show the second identity, let $A \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ and set $E := \operatorname{curl} \mathcal{S}_k A$, $H := \frac{1}{ik}\operatorname{curl} E$ in $\Omega_\pm$. On $\partial\Omega_\pm$ we have that $n \times E = (\pm\frac{1}{2}I + M_k)A$ so that

$$\begin{aligned}\Lambda_k^\pm(\pm\tfrac{1}{2}I + M_k)A &= \Lambda_k^\pm(n \times E) = n \times H \\ &= -ik^{-1}n \times \operatorname{curl}\operatorname{curl} S_k A = -ik^{-1}N_k A.\end{aligned}$$

Thus, the first equality in (5.8) follows. To complete the proof of (5.8) we note that by (5.7) it suffices to show that $N_k\Lambda_k^\pm = -ik\,(\mp\frac{1}{2}I + M_k)$. To this effect, keeping the notation previously introduced, we have that

$$\begin{aligned}N_k\Lambda_k^\pm(n \times E) &= N_k(n \times H) = k^2\, S_k(n \times H) + ik\, n \times \nabla S_k(\langle n, E\rangle) \\ &= -ik\,(\mp\tfrac{1}{2}I + M)(n \times E).\end{aligned}$$

The next-to-the-last equality comes from the definition of N_k and direct calculation, whereas the last equality is a consequence of the Green type formula described in Corollary 3.3. Hence, the proof of (5.8) is finished.

Note that the identity (5.9) is an immediate consequence of (5.7) and (5.8). Finally, an easy way (for us here) to show (5.10) is once again to rely on (5.7), (5.8) and a density argument. □

Lemma 5.11.

For any $f \in W^{1,p}(\partial\Omega)$, $1 < p < \infty$, we have that

$$(n \times \nabla)K_k f = k^2 n \times S_k(nf) + M_k(n \times \nabla f) \tag{5.11}$$

so that, in particular,

$$(n \times \nabla)K_0 = M_0(n \times \nabla) \tag{5.12}$$

on $W^{1,p}(\partial\Omega)$. It follows that M_0 is a well-defined, bounded mapping of $V^{-s,p}_{\tan}(\partial\Omega)$ for $0 \le s \le 1$, $1 < p < \infty$.

Proof. For $f \in W^{1,p}(\partial\Omega)$, we have

$$\begin{aligned}\nabla\mathcal{K}_k f &= -\nabla\operatorname{div}\mathcal{S}_k(nf) = -(\Delta + \operatorname{curl}\operatorname{curl})\mathcal{S}_k(nf) \\ &= k^2\,\mathcal{S}_k(nf) + \operatorname{curl}\mathcal{S}_k(n \times \nabla f),\end{aligned}$$

where the last equality uses Lemma 4.3. Going to the boundary and taking $n\times$ of both sides yield (5.11). Now (5.12) follows from this by making $k = 0$. □

Lemma 5.12.

For any $A \in L^p_{\tan}(\partial\Omega)$, $1 < p < \infty$, we have that

$$\operatorname{div} \mathcal{S}_0 M_0 A = -K_0 \operatorname{div} \mathcal{S}_0 A.$$

Proof. Using Lemma 4.2 and Lemma 4.4 we may write

$$\operatorname{div} \mathcal{S}_0 M_0 A = \mathcal{S}_0\mathrm{Div}\, M_0 A = -\mathcal{S}_0 K_0^*\mathrm{Div}\, A = -K_0\mathcal{S}_0\mathrm{Div}\, A = -K_0 \operatorname{div} \mathcal{S}_0 A. \quad □$$

Before we present the proof of the Theorem 5.1, we note two more basic results.

Lemma 5.13.

Assume that Ω is a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then for each $1 < p < \infty$, the operators $\pm\frac{1}{2}I + M_0 : L^{p,0}_{\tan}(\partial\Omega) \to L^{p,0}_{\tan}(\partial\Omega)$ are injective.

Proof. Assume that $A \in L^{p,0}_{\tan}(\partial\Omega)$ is such that $(-\frac{1}{2}I + M_0)A = 0$, and set $E := \operatorname{curl}\mathcal{S}_0 A$ in $\mathbf{R}^3 \setminus \partial\Omega$. Clearly E is divergencefree, curlfree, $n \times E|_{\partial\Omega_-} = 0$, and moreover, E decays at infinity. Consequently, Green's formula for E gives

$$\pm E = \operatorname{curl}\mathcal{S}_0(n \times E|_{\partial\Omega_\pm}) - \nabla\mathcal{S}_0(\langle n, E|_{\partial\Omega_\pm}\rangle) \tag{5.13}$$

in $\Omega_\pm$. In particular, going to the boundary in Ω_-, using $n \times E|_{\partial\Omega_-} = 0$ and taking $\langle n, \cdot\rangle$ of both sides yield $(-\frac{1}{2}I + K_0^*)(\langle n, E|_{\partial\Omega_-}\rangle) = 0$. Note that, since $\langle n, E\rangle$ does not jump across $\partial\Omega$, the divergence theorem gives

$$\int_{\partial\Omega} \langle n, E|_{\partial\Omega_-}\rangle \, d\sigma = \int_{\partial\Omega} \langle n, E|_{\partial\Omega_+}\rangle \, d\sigma = \iint_{\Omega_+} \operatorname{div} E \, dV = 0;$$

hence, $\langle n, E|_{\partial\Omega_-}\rangle \in L_0^p(\partial\Omega)$. By Theorem 5.4 it follows that $\langle n, E|_{\partial\Omega_-}\rangle = \langle n, E|_{\partial\Omega_+}\rangle = 0$. Returning to (5.13), we observe that this implies $E = 0$ in Ω_-.

Next, we analyze E in Ω_+. We fix a point $X_0 \in \Omega_+$ and define the potential

$$u(X) := \int_{C(X_0,X)} \langle t, E\rangle \, ds, \qquad X \in \Omega_+,$$

where $C(X_0, X)$ is a smooth curve inside Ω_+ joining X_0 and X, t is the unit tangent vector to $C(X_0, X)$, and ds is the arc-length measure on $C(X_0, X)$. Since for an arbitrary smooth surface Σ in Ω_+, with ν standing for the "right-handed" normal to the surface Σ, the classical Stokes formula (see, e.g., [48]) gives

$$\int_{\partial\Sigma} \langle t, E\rangle \, ds = \int_{\Sigma} \langle \nu, \operatorname{curl} E\rangle \, d\sigma = 0;$$

and since $g(\partial\Omega) = 0$, we infer that u is a well-defined, smooth function in Ω_+. Furthermore, it is not difficult to check that $\nabla u = E$. In particular, $(\nabla u)^* \in L^p(\partial\Omega)$ and $\Delta u = \operatorname{div} \nabla u = \operatorname{div} E = 0$. Moreover, $\partial u/\partial n = \langle \nabla u, n\rangle = \langle n, E\rangle = 0$. Thus, u solves the interior homogeneous L^p-Neumann problem for the Laplace operator in Ω. Consequently, from the corresponding uniqueness theorem (cf. [20]), u must be a constant; hence, E vanishes identically in Ω_+, too. Finally, $A = n \times E|_{\partial\Omega_+} - n \times E|_{\partial\Omega_-} = 0$. This proves the injectivity of the operator $-\frac{1}{2}I + M_0$ on $L_{\tan}^{p,0}(\partial\Omega)$.

Now consider $A \in L_{\tan}^{p,0}(\partial\Omega)$ such that $(\frac{1}{2}I + M_0)A = 0$ and, once again, set $E := \operatorname{curl} \mathcal{S}_0 A$ in $\mathbf{R}^3 \setminus \partial\Omega$. This time, using (5.13), $n \times E|_{\partial\Omega_+} = 0$, that the operator $\frac{1}{2}I + K_0^*$ is injective on $L^p(\partial\Omega)$ for each $1 < p < \infty$, and proceeding as before, we arrive at $E = 0$ in Ω_+ and $\langle n, E|_{\partial\Omega_-}\rangle = \langle n, E|_{\partial\Omega_+}\rangle = 0$. To show that E also vanishes identically in Ω_- for a fixed $X_0 \in \Omega_-$, we study the potential

$$u(X) := \int_{C(X_0,X)} \langle t, E\rangle \, ds + \lambda_0, \qquad X \in \Omega_-.$$

Here $C(X_0, X)$ is a smooth curve inside Ω_- joining X_0 with X and λ_0 is a suitable complex number (yet to be fixed). Since $g(\partial\Omega) = 0$, as before, u is a well-defined harmonic function in Ω_- with $\nabla u = E$. The idea is to choose λ so that u becomes harmonic at infinity. Because $E = \mathcal{O}(|X|^{-2})$ at infinity, we may define

$$\lambda_0 := \int_{C(X_0,\infty)} \langle t, E\rangle \, ds,$$

where $C(X_0, \infty)$ is a smooth curve inside Ω_- joining X_0 with ∞. For this choice, it is easy to check that $u(X) = \mathcal{O}(|X|^{-1})$ as $|X| \to \infty$, uniformly in all directions in $\mathbf{R}^3$; that is, u is harmonic at infinity.

As $(\nabla u)^* \in L^p(\partial\Omega)$ and u decays at infinity, it is not difficult to check that $u^* \in L^p(\partial\Omega)$ and that the usual Green's formula for u holds in Ω_-. In fact, since $\frac{\partial u}{\partial n} = \langle n, E|_{\Omega_-}\rangle = 0$, setting $f := u|_{\partial\Omega_-} \in L^p(\partial\Omega)$, Green's formula for the harmonic function u in Ω_- reduces to $-u = \mathcal{K}_0(f)$. From this, going to the boundary, we immediately obtain that $(\frac{1}{2}I + K_0)f = 0$ in $L^p(\partial\Omega)$. Next we observe that $(n \times \nabla)f = (n \times \nabla)u = n \times (\nabla u)|_{\partial\Omega_-} \in L^p(\partial\Omega)$ as $(\nabla u)^* \in L^p(\partial\Omega)$. Thus, $f \in W^{1,p}(\partial\Omega)$ and Theorem 5.4 implies $f = 0$. Consequently, u and, hence, E vanish in Ω_-. Finally, $A = 0$ as before, and this proves the injectivity of the operator $\frac{1}{2}I + M_0$ on $L_{\tan}^{p,0}(\partial\Omega)$. □

Corollary 5.14.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then, for each $1 < p < \infty$, the operators $\pm\frac{1}{2}I + M_0 : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ are injective.

Proof. If $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ is such that $(\pm\frac{1}{2}I + M_0)A = 0$, then by Lemma 4.4 we have $(\pm\frac{1}{2}I + K_0^*)(\mathrm{Div}\, A) = 0$. Since $\mathrm{Div}\, A \in L^p_0(\partial\Omega)$, Theorem 5.4 gives that $\mathrm{Div}\, A = 0$ so that $A \in L^{p,0}_{\tan}(\partial\Omega)$. Consequently, the conclusion follows from Lemma 5.13. □

We are now ready to present the proof of the Theorem 5.1.

Proof of Theorem 5.1. From (5.6) and Lemma 5.13 it follows that the operator $\langle n, \mathrm{curl}\, S_0\rangle$ is injective on $L^{p,0}_{\tan}(\partial\Omega)$ for any $1 < p < \infty$. By Theorem 5.4, the operator in the right-hand side of (5.4) is an isomorphism of $L^p_0(\partial\Omega)$ for $1 < p \le 2+\epsilon$. Thus, in particular, $\langle n, \mathrm{curl}\, S_0\rangle$ also maps $L^{p,0}_{\tan}(\partial\Omega)$ onto $L^p_0(\partial\Omega)$ for $1 < p \le 2+\epsilon$.

At this point we have shown that $\langle n, \mathrm{curl}\, S_0\rangle$ is an isomorphism between $L^{p,0}_{\tan}(\partial\Omega)$ and $L^p_0(\partial\Omega)$ for $1 < p \le 2+\epsilon$. Based on this, (5.4), and Theorem 5.4 we readily obtain that $n \times \nabla S_0$ is an isomorphism between $L^p_0(\partial\Omega)$ and $L^{p,0}_{\tan}(\partial\Omega)$ for $1 < p \le 2+\epsilon$.

Let us now consider **iii**. We first note that $\mathrm{Div}\, T(f, A) = \langle n, \mathrm{curl}\, S_0 A\rangle$ so that, by **i** and **ii**, T is injective. To prove surjectivity, fix $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. Since $\mathrm{Div}\, A \in L^p_0(\partial\Omega)$ and assuming that $1 < p \le 2+\epsilon$, we have from **i** that there exists $B \in L^{p,0}_{\tan}(\partial\Omega)$ such that $\langle n, \mathrm{curl}\, S_0 B\rangle = \mathrm{Div}\, A$. Hence, by (4.5), we see that $A - n \times S_0 B \in L^{p,0}_{\tan}(\partial\Omega)$, so there exists $f \in L^p_0(\partial\Omega)$ with $n \times \nabla S_0 f = A - n \times S_0 B$. This implies that $T(f, B) = A$, hence T is an isomorphism. Now, **iv** is readily seen from **iii** and **i**.

To show **v**, we first restrict attention to the case $1 < p \le 2$. This is easily seen by using the fact that $\pi \circ S_0$ is an isomorphism between $L^p_0(\partial\Omega)$ and $W^{1,p}(\partial\Omega)/\mathbf{C}$ for $1 < p \le 2+\epsilon$ ($\pi =$ the canonical projection) and **ii**. Next we consider the case $2 < p < \infty$. Fix $A \in L^{p,0}_{\tan}(\partial\Omega)$. Then $A \in L^{2,0}_{\tan}(\partial\Omega)$ and, by the L^2-case, there exists a unique (modulo constants) function $f \in W^{1,2}(\partial\Omega)$ such that $(n \times \nabla)f = A$. Poincaré's inequality then gives that $f \in L^p(\partial\Omega)$ so that, in fact, $f \in W^{1,p}(\partial\Omega)$. The proof of **v** is finished.

Next, the identity (5.4) in Lemma 5.8 together with **v** and Theorem 5.4 account for both **vi** and **vii**. Going further, for $1 < p \le 2$, **viii** is immediate from **i** and (4.5) (or **iv**). If $2 < p < \infty$ and $f \in L^p_0(\partial\Omega)$, then by the L^2-theory for **i**, there exists $A \in L^{2,0}_{\tan}(\partial\Omega)$ such that $\langle n, \mathrm{curl}\, S_0 A\rangle = f$. Now $B := n \times S_0 A$ belongs to $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ by the Sobolev embedding theorem and has $\mathrm{Div}\, B = f$. Clearly, such a B is a unique modulo vector field in $L^{p,0}_{\tan}(\partial\Omega)$. This concludes the proof of **viii**. Note that **ix** is immediate from this and (4.5). Also, the operator in **x** is the composition of that in **viii** with the isomorphism $S_0 : L^p_0(\partial\Omega) \to W^{1,p}(\partial\Omega)/\mathbf{C}$, **xi** follows from **v** and **iv**, whereas the operator in **xii** is the composition of those in **xi** and **v**.

Coming now to the proof of **xiii**, we debut with a basic remark, namely, that for any $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ one has

$$\left(L^{p,0}_{\tan}(\partial\Omega)\right)^* \equiv L^q_{\tan}(\partial\Omega)/\left(n \times L^{q,0}_{\tan}(\partial\Omega)\right). \tag{5.14}$$

To prove this, we introduce the linear mapping

$$\Phi : L^q_{\tan}(\partial\Omega)/\left(n \times L^{q,0}_{\tan}(\partial\Omega)\right) \to \left(L^{p,0}_{\tan}(\partial\Omega)\right)^*$$

by setting $\Phi(\hat{A})(B) := \int_{\partial\Omega}\langle A, B\rangle\, d\sigma$ for each $\hat{A} \in L^q_{\tan}(\partial\Omega)/\left(n \times L^{q,0}_{\tan}(\partial\Omega)\right)$ and $B \in L^{p,0}_{\tan}(\partial\Omega)$. Recall from **v** that $L^{p,0}_{\tan}(\partial\Omega) = (n \times \nabla)W^{1,p}(\partial\Omega)$, thus the correctness of this definition is seen by checking that

$$\int_{\partial\Omega} \langle (n \times \nabla) f, A \rangle \, d\sigma = 0$$

for any $f \in W^{1,p}(\partial\Omega)$ and $A \in n \times L^{q,0}_{\tan}(\partial\Omega)$. However, this is immediate from (4.4). We also introduce

$$\Psi : \left(L^{p,0}_{\tan}(\partial\Omega)\right)^* \to L^q_{\tan}(\partial\Omega) / \left(n \times L^{q,0}_{\tan}(\partial\Omega)\right).$$

In order to explain how Ψ acts on functionals, let $l \in (L^{p,0}_{\tan}(\partial\Omega))^*$ and, by the Hahn–Banach theorem, extend l to an element in $\left(L^p_{\tan}(\partial\Omega)\right)^* = L^q_{\tan}(\partial\Omega)$. Thus, there exists $A \in L^q_{\tan}(\partial\Omega)$ such that $l(B) = \int_{\partial\Omega} \langle A, B \rangle \, d\sigma$ for any $B \in L^{p,0}_{\tan}(\partial\Omega)$. Then we set $\Psi(l) := \hat{A} \in L^q_{\tan}(\partial\Omega)/(n \times L^{q,0}_{\tan}(\partial\Omega))$. Invoking once again **v** and (4.4), it is not difficult to see that Ψ is well defined, linear, and bounded. Now a routine calculation shows that Φ and Ψ are inverse to each other so that (5.14) is proved.

Next, dualizing **v**, we get that $(n \times \nabla)^* : (L^{p,0}_{\tan}(\partial\Omega))^* \to W_0^{-1,q}(\partial\Omega)$ is an isomorphism for any $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If we now remark that, by (4.4), the diagram

$$\begin{array}{ccc}
L^q_{\tan}(\partial\Omega)/L^{q,0}_{\tan}(\partial\Omega) & \xrightarrow{\ \text{Div}\ } & W_0^{-1,q}(\partial\Omega) \\
{\scriptstyle n\times\cdot}\downarrow & & \uparrow{\scriptstyle (n\times\nabla)^*} \\
L^q_{\tan}(\partial\Omega)/\left(n \times L^{q,0}_{\tan}(\partial\Omega)\right) & \xrightarrow{\ \Phi\ } & (L^{p,0}_{\tan}(\partial\Omega))^*
\end{array}$$

is commutative, then **xiii** follows. Furthermore, **xiv** is a direct consequence of **xiii** and (4.5); while **xv** is seen from **xiii**, Lemma 4.2, and Theorem 5.4.

We now turn our attention to the operators $\pm\frac{1}{2}I + M_0$. First, the fact that $\pm\frac{1}{2}I + M_0$ are isomorphisms of $L^{p,0}_{\tan}(\partial\Omega)$ (and of $V^{-s,p}_{\tan}(\partial\Omega)$, $0 \le s \le 1$) for $1 < p \le 2 + \epsilon$ can be seen from **v**, (5.12), and Theorem 5.4 as the diagram

$$\begin{array}{ccc}
W^{1,p}(\partial\Omega)/\mathbf{C} & \xrightarrow{\ \pm\frac{1}{2}I+K_0\ } & W^{1,p}(\partial\Omega)/\mathbf{C} \\
{\scriptstyle n\times\nabla}\downarrow & & \downarrow{\scriptstyle n\times\nabla} \\
L^{p,0}_{\tan}(\partial\Omega) & \xrightarrow{\ \pm\frac{1}{2}I+M_0\ } & L^{p,0}_{\tan}(\partial\Omega)
\end{array}$$

is commutative (alternatively, we may invoke **i**, **ii**, and (5.6)).

Before going any further we digress and note a functional analytic result that will be relevant for us here (the second part of it will be used later).

Lemma 5.15.

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces and consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longrightarrow & 0
\end{array} \tag{5.15}$$

where all arrows are linear and bounded and the horizontal sequences are exact. Then the following hold:

a. *If two vertical arrows are isomorphisms, then so is the third one.*

b. *If two vertical arrows are Fredholm operators, then so is the third. Moreover, the index of the middle vertical arrow is the sum of the indexes of the other two vertical arrows.*

Now, Lemma 4.4 gives that the diagram

$$\begin{array}{ccc} L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I+M_0} & L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \\ {\scriptstyle\mathrm{Div}}\downarrow & & \downarrow{\scriptstyle\mathrm{Div}} \\ L^p_0(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I-K^*_0} & L^p_0(\partial\Omega) \end{array} \tag{5.16}$$

is commutative; whereas if $1 < p \leq 2+\epsilon$, then **viii** and Theorem 5.4 give that all arrows but the top are isomorphisms. Hence, we also obtain that

$$\pm\tfrac{1}{2}I + M_0 : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \to L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega) \tag{5.17}$$

are isomorphisms for each $1 < p \leq 2+\epsilon$. Based on this, **xvi** and the first part of Lemma 5.15 (with $\mathcal{X} := L^{p,0}_{\tan}(\partial\Omega)$, $\mathcal{Y} := L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, $\mathcal{Z} := L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)/L^{p,0}_{\tan}(\partial\Omega)$, and all vertical arrows manifestations of the operators $\pm\frac{1}{2}I + M_0$), **xvii** follows. Furthermore, using once again that Div intertwines $\pm\frac{1}{2}I + M_0$ acting on $L^p_{\tan}(\partial\Omega)$ with $\pm\frac{1}{2}I - K^*_0$ acting on $W^{-1,p}_0(\partial\Omega)$ (cf. (4.6)), we see that **xviii** follows from Theorem 5.4 and **xiii**. Alternatively, one may equally use **xv** and Lemma 5.12.

Also, **xix** is seen from **xvi**, **xviii**, and the functional analytic result from above. Finally, **xx** follows from **xix** and Theorem 5.4. □

Remark. Perhaps the best way to understand the nature of the range $2-\epsilon \leq p \leq 2+\epsilon$ for which $\pm\frac{1}{2}I + M_0$ are isomorphism of $L^p_{\tan}(\partial\Omega)$ is via the points **xviii**, **xvi** in Theorem 5.1 and the point **b** in Lemma 5.15. □

Next we present a constructive proof of the invertibility of the operators $\pm\frac{1}{2}I+M_0$ on $L^p_{\tan}(\partial\Omega)$, $2-\epsilon \leq p \leq 2+\epsilon$, for arbitrary bounded Lipschitz domains in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. This alternative approach is based on a new Rellich type identity for harmonic vector fields and has first been observed in connection with the work in [55].

An alternative proof of xix in Theorem 5.1. It suffices to treat the case $p = 2$ as the techniques in [9, 20, 22] may then be employed to boost the range to $2-\epsilon \leq p \leq 2+\epsilon$ for some small, positive ϵ.

The idea of proof is to derive an estimate of the form

$$\int_{\partial\Omega} |\mathrm{div}\,\mathcal{S}_0 A|^2\, d\sigma + \int_{\partial\Omega} |\langle n, \mathrm{curl}\,\mathcal{S}_0 A\rangle|^2\, d\sigma \approx \int_{\partial\Omega} |n \times \mathrm{curl}\,\mathcal{S}_0 A|^2\, d\sigma \tag{5.18}$$

uniformly for $A \in L^2_{\tan}(\partial\Omega)$. Since, for A tangential, the quantities $\mathrm{div}\,\mathcal{S}_0 A$ and $\langle n, \mathrm{curl}\,\mathcal{S}_0 A\rangle$ do not jump across $\partial\Omega$, whereas $(n \times \mathrm{curl}\,\mathcal{S}_0 A)|_{\partial\Omega_\pm} = (\pm\frac{1}{2}I + M_0)A$, the estimate (5.18) would imply that

$$\|(-\tfrac{1}{2}I + M_0)A\|_{L^2(\partial\Omega)} \approx \|(\tfrac{1}{2}I + M_0)A\|_{L^2(\partial\Omega)}$$

and, ultimately, that

$$\|A\|_{L^2(\partial\Omega)} \leq C\|(\pm\tfrac{1}{2}I + M_0)A\|_{L^2(\partial\Omega)}$$

uniformly for $A \in L^2_{\tan}(\partial\Omega)$. This gives that $\pm\frac{1}{2}I + M_0$ are injective with closed ranges on $L^2_{\tan}(\partial\Omega)$, which is the "hardest" part in showing that they are actually isomorphisms of $L^2_{\tan}(\partial\Omega)$.

Turning now to the specific calculation, assume that Ω is an arbitrary, fixed, bounded Lipschitz domain in $\mathbf{R}^3$, and let $\Theta \in C^\infty_{\mathrm{comp}}(\mathbf{R}^3, \mathbf{R}^3)$ and $U \in C^\infty(\overline{\Omega}, \mathbf{C}^3)$ be such that $\Delta U = 0$ in Ω. We claim that the following Rellich type identity is valid:

$$\begin{aligned}
&\int_{\partial\Omega} \tfrac{1}{2}|n \times \operatorname{curl} U|^2 \langle n, \Theta\rangle - \tfrac{1}{2}|\langle n, \operatorname{curl} U\rangle|^2 \langle n, \Theta\rangle - \tfrac{1}{2}|\operatorname{div} U|^2 \langle n, \Theta\rangle \\
&\qquad + \int_{\partial\Omega} \operatorname{Re}\langle n \times \operatorname{curl} \bar{U}, \Theta\rangle \operatorname{div} U - \operatorname{Re}\langle n \times \operatorname{curl} \bar{U}, n \times \Theta\rangle \langle n, \operatorname{curl} U\rangle \\
&\qquad = \iint_{\Omega} \tfrac{1}{2}|\operatorname{curl} U|^2 \operatorname{div}\Theta - \operatorname{Re}\langle \operatorname{curl}\bar{U}, (\nabla\Theta)\operatorname{curl} U\rangle \\
&\qquad - \iint_{\Omega} \operatorname{Re}\langle \operatorname{curl}\bar{U}, \operatorname{curl}\Theta\rangle \operatorname{div} U + \tfrac{1}{2}|\operatorname{div} U|^2 \operatorname{div}\Theta.
\end{aligned} \tag{5.19}$$

The departure point is to write the identity (5.1) for Θ and the divergencefree vector field $E := \operatorname{curl} U$ in Ω as

$$\begin{aligned}
&\int_{\partial\Omega} \left\{\tfrac{1}{2}|\operatorname{curl} U|^2 \langle n, \Theta\rangle - \operatorname{Re}\langle \operatorname{curl}\bar{U}, \Theta\rangle \langle n, \operatorname{curl} U\rangle\right\} d\sigma \\
&\qquad = \operatorname{Re}\iint_{\Omega} \tfrac{1}{2}|\operatorname{curl} U|^2 \operatorname{div}\Theta - \big\langle \operatorname{curl}\bar{U}, (\nabla\Theta)\operatorname{curl} U\big\rangle \\
&\qquad + \operatorname{Re}\iint_{\Omega} \big\langle \Theta \times \operatorname{curl}\bar{U}, \operatorname{curl}\operatorname{curl} U\big\rangle.
\end{aligned} \tag{5.20}$$

Next, we observe that $\operatorname{curl}\operatorname{curl} U = (-\Delta + \nabla\operatorname{div})U = \nabla\operatorname{div} U$ and successively integrate by parts in the term containing two derivatives on U:

$$\begin{aligned}
&\operatorname{Re}\iint_{\Omega} \big\langle \Theta \times \operatorname{curl}\bar{U}, \nabla\operatorname{div} U\big\rangle \\
&\quad = -\operatorname{Re}\iint_{\Omega} \operatorname{div} U \operatorname{div}(\Theta \times \operatorname{curl}\bar{U}) + \operatorname{Re}\int_{\partial\Omega} \langle n, \Theta \times \operatorname{curl}\bar{U}\rangle \operatorname{div} U \\
&\quad = -\operatorname{Re}\iint_{\Omega} \operatorname{div} U \,\langle \operatorname{curl}\bar{U}, \operatorname{curl}\Theta\rangle + \operatorname{Re}\iint_{\Omega} \operatorname{div} U \,\langle \Theta, \operatorname{curl}\operatorname{curl}\bar{U}\rangle \\
&\quad - \operatorname{Re}\int_{\partial\Omega} \langle n \times \operatorname{curl}\bar{U}, \Theta\rangle \operatorname{div} U \\
&\quad = -\operatorname{Re}\iint_{\Omega} \operatorname{div} U \,\langle \operatorname{curl}\bar{U}, \operatorname{curl}\Theta\rangle - \iint_{\Omega} \tfrac{1}{2}|\operatorname{div} U|^2 \operatorname{div}\Theta \\
&\quad + \int_{\partial\Omega} \tfrac{1}{2}|\operatorname{div} U|^2 \langle n, \Theta\rangle - \operatorname{Re}\textstyle\int_{\partial\Omega} \langle n \times \operatorname{curl}\bar{U}, \Theta\rangle \operatorname{div} U.
\end{aligned}$$

If we now observe that the left-hand side of (5.20) can be rewritten in the form

$$\begin{aligned}
&\int_{\partial\Omega} \left\{\tfrac{1}{2}|\operatorname{curl} U|^2 \langle n, \Theta\rangle - \operatorname{Re}\langle \operatorname{curl}\bar{U}, \Theta\rangle \langle n, \operatorname{curl} U\rangle\right\} d\sigma \\
&\qquad = -\int_{\partial\Omega} \tfrac{1}{2}|\langle n, \operatorname{curl} U\rangle|^2 \langle n, \Theta\rangle + \tfrac{1}{2}|n \times \operatorname{curl} U|^2 \langle n, \Theta\rangle \\
&\qquad - \int_{\partial\Omega} \operatorname{Re}\langle n \times \operatorname{curl}\bar{U}, n \times \Theta\rangle \langle n, \operatorname{curl} U\rangle,
\end{aligned}$$

then (5.19) follows.

By choosing Θ such that $\langle n, \Theta\rangle \geq \kappa > 0$ a.e. on $\partial\Omega$ (cf. Lemma 2.2), simple inspection of (5.19) shows that for some positive $C = C(\partial\Omega)$ one has

$$\|\langle n, \operatorname{curl} U\rangle\|_{L^2(\partial\Omega)} + \|\operatorname{div} U\|_{L^2(\partial\Omega)} \leq C\left(\|n \times \operatorname{curl} U\|_{L^2(\partial\Omega)} + \|\nabla U\|_{L^2(\Omega)}\right), \tag{5.21}$$

and

$$\|n \times \operatorname{curl} U\|_{L^2(\partial\Omega)} \leq C\left(\|\langle n, \operatorname{curl} U\rangle\|_{L^2(\partial\Omega)} + \|\operatorname{div} U\|_{L^2(\partial\Omega)} + \|\nabla U\|_{L^2(\Omega)}\right). \tag{5.22}$$

Now let $A \in L^2_{\tan}(\partial\Omega)$ and set $U := \mathcal{S}_0 A$ and $E := \operatorname{curl}\mathcal{S}_0 A$ in $\mathbf{R}^3 \setminus \partial\Omega$. Furthermore, let $\Omega_j \uparrow \Omega$ and Θ be as in Lemma 2.2. For each fixed j, we write the estimates (5.21) and (5.22) for Ω_j in place of Ω and, by letting $j \to \infty$, obtain that

$$\|\langle n, E\rangle\|_{L^2(\partial\Omega)} + \|\operatorname{div} \mathcal{S}_0 A\|_{L^2(\partial\Omega)} \leq C\left(\|n \times E\|_{L^2(\partial\Omega)} + \|\operatorname{Comp}(A)\|_{L^2(\Omega)}\right) \tag{5.23}$$

and

$$\|n \times E\|_{L^2(\partial\Omega)} \leq C\left(\|\langle n, E\rangle\|_{L^2(\partial\Omega)} + \|\operatorname{div} \mathcal{S}_0 A\|_{L^2(\partial\Omega)} + \|\operatorname{Comp}(A)\|_{L^2(\Omega)}\right), \tag{5.24}$$

where Comp stands for generic compact operators. Moreover, similar estimates are valid for E regarded as a vector field in the exterior domain Ω_-. Note that these two estimates are the rigorous formulation of (5.18). They reduce precisely to (5.18) when Ω is the domain above the graph of a Lipschitz function, in which case Θ can be chosen to be a constant vector field.

Using the facts that $n \times E|_{\partial\Omega_\pm} = (\pm\frac{1}{2}I + M_0)A$ and that $\operatorname{div}\mathcal{S}_0 A$, $\langle n, E\rangle$ do not jump across the boundary, (5.23), (5.24), and the triangle inequality give us

$$\|A\|_{L^2(\partial\Omega)} \leq C\|(\pm\tfrac{1}{2}I + M_0)A\|_{L^2(\partial\Omega)} + \|\operatorname{Comp}(A)\|. \tag{5.25}$$

Next we prove that, in the above estimate, $\pm\frac{1}{2}$ can in fact be replaced by any $\lambda \in \mathbf{R}$ with $|\lambda| \geq \frac{1}{2}$. To this effect, fix a real λ with $|\lambda| > \frac{1}{2}$ and let $A \in L^2_{\tan}(\partial\Omega)$. Set $E := \operatorname{curl}\mathcal{S}_0 A$ and write the identity (5.19) for $U := \mathcal{S}_0 A$ in $\Omega_\pm$:

$$\begin{aligned}
&\int_{\partial\Omega_\pm} \tfrac{1}{2}|n \times E|^2\langle n, \Theta\rangle - \tfrac{1}{2}|\langle n, E\rangle|^2\langle n, \Theta\rangle - \operatorname{Re}\langle n \times \bar{E}, n \times \Theta\rangle\langle n, E\rangle \\
&\quad + \int_{\partial\Omega_\pm} \operatorname{Re}\langle n \times \bar{E}, \Theta\rangle \operatorname{div}\mathcal{S}_0 A - \tfrac{1}{2}|\operatorname{div}\mathcal{S}_0 A|^2\langle n, \Theta\rangle \\
&\quad = \iint_{\Omega_\pm} \tfrac{1}{2}|E|^2 \operatorname{div}\Theta - \operatorname{Re}\langle \bar{E}, (\nabla\Theta)E\rangle - \operatorname{Re}\langle \bar{E}, \operatorname{curl}\Theta\rangle \operatorname{div}\mathcal{S}_0 A \\
&\quad - \iint_{\Omega_\pm} \tfrac{1}{2}|\operatorname{div}\mathcal{S}_0 A|^2 \operatorname{div}\Theta.
\end{aligned}$$

Multiplying the corresponding identities on $\partial\Omega_\pm$ by $\pm\lambda + \frac{1}{2}$, respectively, and then adding them, we arrive after a straightforward calculation at

$$\begin{aligned}
&\left(\lambda^2 - \tfrac{1}{4}\right)\int_{\partial\Omega} |A|^2\langle n, \Theta\rangle\, d\sigma + \frac{1}{2}\int_{\partial\Omega} \left(|\langle n, E\rangle|^2 + |\operatorname{div}\mathcal{S}_0 A|^2\right)\langle n, \Theta\rangle\, d\sigma \\
&\quad = \mathcal{O}(\|(\lambda + M_0)A\|_{L^2(\partial\Omega)}\|A\|_{L^2(\partial\Omega)}) + \|\operatorname{Comp}(A)\|.
\end{aligned}$$

Thus, the above estimate and (5.25) yield that for any $\lambda \in \mathbf{R}$ with $|\lambda| \geq \frac{1}{2}$ there exists some $C = C(\partial\Omega, \lambda) > 0$ such that

$$\|A\|_{L^2} \leq C\|(\lambda + M_0)A\|_{L^2} + \|\operatorname{Comp}(A)\|, \tag{5.26}$$

uniformly for $A \in L^2_{\tan}(\partial\Omega)$. This proves that $\lambda + M_0$ is semi-Fredholm on $L^2_{\tan}(\partial\Omega)$ for any $\lambda \in \mathbf{R}$ with $|\lambda| \geq \frac{1}{2}$. Since the index of semi-Fredholm operators is homotopic invariant and since, obviously, $\lambda + M_0$ depends continuously on λ and is invertible for a sufficiently large λ, it follows that $\pm\frac{1}{2}I + M_0$ are Fredholm with index zero on $L^2_{\tan}(\partial\Omega)$. Finally, by, for example, Lemma 4.4 and Theorem 5.4, $\pm\frac{1}{2}I + M_0$ are also seen to be one-to-one, so they are in fact isomorphisms of $L^2_{\tan}(\partial\Omega)$. □

Proof of Theorem 5.3. We begin by observing that the fact that $k \in \mathbf{C}$, $\operatorname{Im} k \geq 0$, is not a Maxwell eigenvalue for Ω implies that the operators $\pm\frac{1}{2}I + M_k$ are injective on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for each $1 < p < \infty$. Indeed, this follows from the corresponding invertibility result for $\pm\frac{1}{2}I + M_0$ in Theorem 5.1; the fact that $M_k - M_0$ maps $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ boundedly into $L^{p+\epsilon,\mathrm{Div}}_{\tan}(\partial\Omega)$ for some fixed, positive ϵ; the corresponding L^2 result (cf. also §11); and (successive applications of) the fractional integration theorem.

With this at hand, Theorem 5.3 **i** follows. Furthermore, relying on Lemma 5.10, it is clear that **ii** and **iii** are direct corollaries of **i**. Also, **iv** has a similar proof. Furthermore, **v** is a direct consequence of **iv**, Lemma 4.4, and Theorem 5.4; whereas **vi** follows directly **i** and **iv** via Lemma 5.15.

Next we consider the corresponding invertibility statements for $2- \leq p \leq 2+\epsilon$ but this time with the simple connectivity assumption dropped. First we deal will **iv**. To this end, let $k \in \mathbf{C}$ be an arbitrary, fixed complex number and $2-\epsilon \leq p \leq 2+\epsilon$. We claim that if $A \in L^p_{\tan}(\partial\Omega)$ is such that either $(\frac{1}{2}I + M_k)A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ or $(-\frac{1}{2}I + M_k)A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, then actually A belongs to $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. Indeed, since for a fixed $k_0 \in \mathbf{C}$ with $\operatorname{Im} k_0 > 0$ the operator $M_k - M_{k_0}$ maps $L^p_{\tan}(\partial\Omega)$ boundedly into $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for any $1 < p < \infty$, it suffices to prove the claim for $k = k_0$. In this latter case, we may use Theorem 5.4 and the fact that, by Lemma 4.4, $\operatorname{Div} A \in W^{-1,p}_0(\partial\Omega)$ has

$$(\pm\tfrac{1}{2}I + K^*_{k_0})(\operatorname{Div} A) = -k_0^2\langle n, S_{k_0}A\rangle - \operatorname{Div}[(\mp\tfrac{1}{2}I + M_{k_0})A] \in L^p(\partial\Omega)$$

to infer that $\operatorname{Div} A \in L^p(\partial\Omega)$, so that $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. Thus, the claim is now seen from Theorem 5.7. This also shows that the null spaces of the operators $\pm\frac{1}{2}I + M_k$ on $L^p_{\tan}(\partial\Omega)$ and on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ are the same.

Assume now that k is not a Maxwell eigenvalue for Ω so that, by the above discussion, $\pm\frac{1}{2}I + M_k$ are injective on $L^p_{\tan}(\partial\Omega)$. If we now recall from the alternative proof of Theorem 5.1 **xix** that these operators are also Fredholm with index zero (this proof did not require the domain to have topological genus zero), then **iv** follows.

The reasoning for the remaining operators can be seen from **iv** and Theorem 5.7. The proof of Theorem 5.3 is therefore completed. □

An alternative proof of Theorem 5.3 iv. Let Ω be an arbitrary, bounded Lipschitz domain in $\mathbf{R}^3$ (not necessarily having $g(\partial\Omega) = 0$). Here we indicate another proof of the fact that if $2-\epsilon \leq p \leq 2+\epsilon$, then for any $k \in \mathbf{C}\setminus\{0\}$ that is not a Maxwell eigenvalue for Ω, the operators $\pm\frac{1}{2}I + M_k$ are isomorphisms of $L^p_{\tan}(\partial\Omega)$. This alternative approach employs a localization lemma observed in [59] (compare with [30]). To state it, first recall that a boundary integral operator on $\partial\Omega \subseteq \mathbf{R}^3$ is *singular* if its kernel $K(X,Y)$ satisfies $K(X,Y) = \mathcal{O}(|X-Y|^2)$ on $\partial\Omega$. Also, if X denotes a Banach space and $T : X \to X$ is a bounded, linear operator, we let $\sigma_\mu(T; X)$ stand for the collection of all complex λ so that $\lambda - T$ does not have a closed range or a finite-dimensional kernel.

Lemma 5.16.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$, and let $\{\Omega_j;\ 1 \leq j \leq N\}$, be a family of bounded Lipschitz domains in $\mathbf{R}^3$ such that $\{\partial\Omega \cap \partial\Omega_j\}_j$ is an open finite covering of $\partial\Omega$. Assume that $\mathcal{K}$, $\{\mathcal{K}_j\}_j$ are some given singular integral operators on $\partial\Omega$ and $\partial\Omega_j$, respectively, with the property that, for each j, the kernels of $\mathcal{K}$ and $\mathcal{K}_j$ coincide when restricted to $\partial\Omega \cap \partial\Omega_j$.

Then, for each $1 < p < \infty$,

$$\sigma_\mu(\mathcal{K}; L^p(\partial\Omega)) \subseteq \bigcup_{j=1}^{N} \sigma_\mu(\mathcal{K}_j; L^p(\partial\Omega_j)).$$

Using the above lemma for an arbitrary bounded Lipschitz domain Ω and $\{\Omega_j\}_j$ bounded Lipschitz domains with $g(\partial\Omega_j) = 0$ for each j, we deduce from Theorem 5.1 **xix** that $\pm\frac{1}{2}I + M_k$ have closed ranges on $L^p_{\tan}(\partial\Omega)$ for each $2 - \epsilon \le p \le 2 + \epsilon$. Moreover, recall that from the claim made in the last part of the proof of Theorem 5.3, they are also one-to-one on $L^p_{\tan}(\partial\Omega)$. Now, this and the readily verified identity

$$\int_{\partial\Omega} \langle (\tfrac{1}{2}I + M_k)A_1, A_2\rangle \, d\sigma = \int_{\partial\Omega} \langle A_1, n \times (-\tfrac{1}{2}I + M_k)(n \times A_2)\rangle \, d\sigma, \tag{5.27}$$

for any $A_1 \in L^p_{\tan}(\partial\Omega)$, $A_2 \in L^q_{\tan}(\partial\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$, also give that $\pm\frac{1}{2}I + M_k$ have dense ranges on $L^p_{\tan}(\partial\Omega)$ for each $2 - \epsilon \le p \le 2 + \epsilon$. The conclusion then follows. □

Before we conclude this section, we remark that our results can also be used to obtain information about the spectrum of the operator M_0. For instance, it follows directly from (5.3) (or (5.5)) and Theorem 5.1 that

$$\sigma(M_0; L^{p,0}_{\tan}(\partial\Omega)) = \sigma(K_0^*; L^p_0(\partial\Omega))$$

for $1 < p \le 2 + \epsilon$. It is also possible to show that $\sigma(M_0; L^p_{\tan}(\partial\Omega)) \subseteq \sigma(K_0^*; L^p_0(\partial\Omega))$ for $2 - \epsilon \le p \le 2 + \epsilon$. Hence, the spectral radius of K_0^* on $L^p_0(\partial\Omega)$ is the same as that of M_0 on $L^{p,0}_{\tan}(\partial\Omega)$ and the same as that of M_0 on $L^p_{\tan}(\partial\Omega)$ for each $2 - \epsilon \le p \le 2 + \epsilon$. In particular, by the results in [30] and [25] we have that the spectral radii of M_0 on $L^{2,0}_{\tan}(\partial\Omega)$ and on $L^2_{\tan}(\partial\Omega)$ are $< \frac{1}{2}$ for any bounded convex or polyhedral domain Ω in $\mathbf{R}^3$. In fact, the same conclusion remains valid if M_0 is regarded as an operator on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. We also refer to [38] for various related results.

6. L^p-Theory for the Maxwell Equations and Related Systems

Fix $k \in \mathbf{C} \setminus \{0\}$ with $\operatorname{Im} k \ge 0$, and let E be a vector field E defined in a neighborhood of infinity in $\mathbf{R}^3$ such that $(\Delta + k^2)E = 0$. We shall call E *radiating* if

$$(\operatorname{curl} E)(X) \times \frac{X}{|X|} + (\operatorname{div} E)(X)\frac{X}{|X|} - ikE(X) = o(|X|^{-1}), \tag{6.1}$$

as $|X| \to \infty$, uniformly for all directions $X/|X|$ in $\mathbf{R}^3$. It is possible to prove that (6.1) is actually equivalent to the requirement that the individual components of E satisfy the classical Sommerfeld radiation condition

$$\frac{\partial u}{\partial r} - iku = o(r^{-1}) \quad \text{as } r := |X| \to \infty \tag{6.2}$$

(cf. also [14, Corollary 4.14, p. 120]). It is also known that (6.2) can be relaxed to

$$\int_{|X|=R} |u|^2 \, d\sigma = \mathcal{O}(1) \quad \text{as } R \to \infty. \tag{6.3}$$

To understand the delicate balance of radiating conditions, we recall that by the famous Rellich lemma, if $k \in \mathbf{R} \setminus \{0\}$, then any (scalar-valued) radiating metaharmonic function u defined in a neighborhood of infinity such that

$$\int_{|X|=R} |u|^2 \, d\sigma = o(1) \quad \text{as } R \to \infty \tag{6.4}$$

must vanish identically in its domain (this is in fact true for any $k \in \mathbf{C} \setminus \{0\}$).

We single out several properties of radiating metaharmonic vector fields that are going to be of importance for us. To this end, assume that Ω is a bounded Lipschitz domain in $\mathbf{R}^3$. First, the operators $\mathcal{S}_k$, $\operatorname{div}\mathcal{S}_k$, and $\operatorname{curl}\mathcal{S}_k$ map $L^p(\partial\Omega)$ boundary densities into radiating vector fields.

Second, if E is a smooth vector field in $\mathbf{R}^3\setminus\overline{\Omega}$ such that E^*, $(\operatorname{div}E)^*$, $(\operatorname{curl}E)^*\in L^p(\partial\Omega)$ for some $1<p<\infty$, $(\Delta+k^2)E=0$ in $\mathbf{R}^3\setminus\overline{\Omega}$; and E has (6.1), then the following Green type representation formula holds in $\mathbf{R}^3\setminus\overline{\Omega}$:

$$-E=\operatorname{curl}\mathcal{S}_k(n\times E)-\nabla\mathcal{S}_k(\langle n,E\rangle)+\mathcal{S}_k(n\times\operatorname{curl}E)-\mathcal{S}_k(n\operatorname{div}E). \tag{6.5}$$

Third, using this integral representation formula and elementary asymptotics it is not difficult to show that any metaharmonic radiating vector field E has the behavior

$$E(X)=\Phi_k(X)E_\infty(X/|X|)+\mathcal{O}(|X|^{-2})\quad\text{as } |X|\to\infty. \tag{6.6}$$

The vector field $E_\infty: S^2\to\mathbf{C}^3$ is called the *far field pattern* of E. Note that, by the Rellich lemma, the correspondence $E\mapsto E_\infty$ is one-to-one (this should be contrasted with the fact that the first term in the asymptotic expansion of a harmonic vector field at infinity does not determine it uniquely).

Fourth, if (E,H) is a pair of vector fields satisfying the Maxwell equations in $\mathbf{R}^3\setminus\overline{\Omega}$, then the same integral representation formula may be employed to show that E or H are radiating if and only if both E and H are radiating. In this case, (6.1) reduces to the more familiar Silver–Müller radiation condition

$$H(X)\times(X/|X|)-|X|E(X)=o(1)$$

or, equivalently,

$$E(X)\times(X/|X|)+|X|H(X)=o(1)$$

as $|X|\to\infty$, uniformly in all directions in $\mathbf{R}^3$.

Next we briefly analyze the case in which $k=0$. For a vector field E satisfying the Laplace equation $\Delta E=0$ in a neighborhood of infinity, the radiation condition is

$$E(X)=o(1)\quad\text{as } |X|\to\infty. \tag{6.7}$$

Once again, E admits a Green type integral representation formula and, as a corollary, (6.7) can be improved in the form

$$E(X),\ \ |X|\operatorname{div}E(X),\ \ |X|\operatorname{curl}E(X)=\mathcal{O}(|X|^{-1})\quad\text{as } |X|\to\infty.$$

We also note that if the harmonic vector field E satisfies $\operatorname{div}E=0$, $\operatorname{curl}E=0$ in a neighborhood of infinity, and is radiating, then actually $E(X)=\mathcal{O}(|X|^{-2})$ as $|X|\to\infty$.

The main result of this section is the following theorem, which extends the results in [64] and [66]. To state it, recall the interior and the exterior Maxwell boundary value problems $(\mathcal{M}_{\mathrm{i,e}})$ formulated in the introductory section.

Theorem 6.1.

Assume that Ω is a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega)=0$. Then there exists $\epsilon>0$ depending only on $\partial\Omega$ such that for each $k\in\mathbf{C}\setminus\{0\}$ with $\operatorname{Im}k\geq 0$ and each $1<p\leq 2+\epsilon$ the exterior Maxwell boundary value problem $(\mathcal{M}_{\mathrm{e}})$ is solvable if and only if $A\in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ and $\operatorname{Div}A=-ikf$. Moreover, the solution is unique, depends on k analytically in $\mathbf{R}^2_+$ and continuously on $\overline{\mathbf{R}^2_+}\setminus\{0\}$ (the principle of limiting absorption), and satisfies

$$\|E^*\|_{L^p(\partial\Omega)}+\|H^*\|_{L^p(\partial\Omega)}\leq C\|A\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)} \tag{6.8}$$

for some positive constant C depending only on k, p, and Ω.

Furthermore, if k is not a Maxwell eigenvalue for Ω, then a similar result is valid for the interior Maxwell boundary value problem $(\mathcal{M}_{\mathrm{i}})$. In the case in which k is a Maxwell eigenvalue for

Ω, then $(\mathcal{M}_i)$ is solvable if and only if $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, $\mathrm{Div}\, A = -ik\, f$, and A, f satisfy finitely many linear conditions (in which case the solution is unique modulo a finite-dimensional space).

Also, if $\partial\Omega \in C^1$, then one may take $1 < p < \infty$.

As we shall see in §7, the above result is sharp. The case when $k = 0$ is also examined later in this section. We also note that, as an immediate corollary of this theorem, the exterior voltage-to-current map Λ_k^- is well defined on (and in fact is an isomorphism of) $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for any $1 < p \le 2+\epsilon$ and any $k \in \mathbf{C}\setminus\{0\}$, $\mathrm{Im}\, k \ge 0$ (compare with **iii** in Theorem 5.3).

Proof of Theorem 6.1. Let us first deal with the exterior boundary value problem $(\mathcal{M}_e)$. The necessity of the membership of A to $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ together with the compatibility condition $\mathrm{Div}\, A = -ikf$ follow from, for example, (4.5), so we turn to the sufficiency part.

To show existence, we first remark that if k is not a Maxwell eigenvalue for Ω, then we may take $E := \mathrm{curl}\, \mathcal{S}_k B$, $H := \frac{1}{ik}\mathrm{curl}\, E$ for some $B \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ and, based on the jump relations for the derivatives of the single-layer potential operator, Theorem 5.1 **xi** may be used to conclude.

The treatment of the general case (i.e., when $\mathrm{Im}\, k \ge 0$, $k \ne 0$) requires an appropriate modification of this approach that is inspired from [15] and which we now describe. The new difficulty is that the previous boundary integral equations may no longer be uniquely solvable, and the idea is to add further source terms in order to correct this deficiency. More specifically, set $\eta := 0$ if $\mathrm{Im}\, k \ne 0$ and $\eta := 1$ otherwise. For $B \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ we consider

$$E := \mathrm{curl}\, \mathcal{S}_k B + i\eta\, \mathrm{curl}\,\mathrm{curl}\, \mathcal{S}_k(n \times S_0^2 B)$$

and

$$H := \frac{1}{ik}\mathrm{curl}\, E$$

in $\mathbf{R}^3\setminus\overline{\Omega}$. Then, by the discussion at the beginning of this section, (E, H) is a radiating solution of the Maxwell system in $\mathbf{R}^3\setminus\overline{\Omega}$. Moreover, $E^*, H^* \in L^p(\partial\Omega)$ and $n\times E = (-\frac{1}{2}I + M_k + i\eta N_k(n\times S_0^2))B$. Hence, matters are reduced to proving the invertibility of $(-\frac{1}{2}I + M_k + i\eta N_k(n\times S_0^2))$ on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for $1 < p \le 2+\epsilon$.

To this end, we note that Theorem 5.1 **xiii** gives that for any $k \in \mathbf{C}$ the operator $-\frac{1}{2}I + M_k$ is Fredholm with index zero on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for $1 < p \le 2+\epsilon$. Since, by (4.5), $n \times S_0^2$ maps $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ compactly into itself, it follows that $-\frac{1}{2}I + M_k + i\eta\, N_k(n\times S_0^2)$ is also Fredholm with index zero. Consequently, we are left with showing that this operator is injective on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. For $p = 2$ this has been proved in [15] (there it is assumed that $\partial\Omega \in C^2$, but for this particular passage essentially the same reasoning applies to the Lipschitz case equally). Consider now $B \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ such that $(-\frac{1}{2}I + M_k + i\eta N_k(n\times S_0^2))B = 0$ and fix $k_0 \in \mathbf{C}$ with $\mathrm{Im}\, k_0 > 0$. It follows that

$$B = -(-\tfrac{1}{2}I + M_{k_0})^{-1}[(M_{k_0} - M_k)B + i\eta N_k(n\times S_0^2)B].$$

The fractional integration theorem (see [85]) yields that $B \in L^{p+\delta,\mathrm{Div}}_{\tan}(\partial\Omega)$ for some $\delta > 0$. Iterating this sufficiently many times we finally arrive at $B \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$, therefore, by the L^2-theory, $B = 0$. Thus, the existence part is proved.

Next we address the uniqueness issue. We first assume that $p \ge 2$. The case $\mathrm{Im}\, k > 0$ has been considered in [64], so we restrict attention to $k \in \mathbf{R}\setminus\{0\}$. From the radiation condition we have

$$\begin{aligned} 0 &= \lim_{R\to\infty}\int_{|X|=R} |H\times n - E|^2\, d\sigma \\ &= \lim_{R\to\infty}\int_{|X|=R} \big(|H\times n|^2 + |E|^2 - 2\mathrm{Re}\,\langle n\times E, \bar{H}\rangle\big)\, d\sigma. \end{aligned} \tag{6.9}$$

Since $n \times E = 0$ on $\partial\Omega$ and $k \in \mathbf{R}$, integrating by parts it follows that

$$\mathrm{Re} \int_{|X|=R} \langle n \times E, \bar{H} \rangle \, d\sigma = \mathrm{Re} \left(ik \iint_{X \in B_R(0) \setminus \overline{\Omega}} |H|^2 - |E|^2 \right) = 0,$$

and we see from (6.9) that $\int_{|X|=R} |E|^2 \, d\sigma = o(1)$ as $R \to \infty$. The Rellich lemma then yields that E and, hence, also H vanish in $\mathbf{R}^3 \setminus \overline{\Omega}$.

We consider now the remaining case $1 < p < 2$. First, from the Green formula (3.2) for E in $\mathbf{R}^3 \setminus \overline{\Omega}$, we have that

$$E = \nabla S_k(\langle n, E \rangle) - ik \mathcal{S}_k(n \times H), \tag{6.10}$$

which, by taking the curl of both sides gives

$$H = -\mathrm{curl}\, \mathcal{S}_k(n \times H). \tag{6.11}$$

In particular, going to the boundary and applying $n\times$ to both sides yield $(\frac{1}{2}I + M_k)(n \times H) = 0$. Since, pretty much as before,

$$n \times H = -(-\tfrac{1}{2}I + M_{k_0})^{-1}[(M_k - M_{k_0})(n \times H)] \in L^{p+\delta,\mathrm{Div}}_{\tan}(\partial\Omega)$$

for some positive δ, iterating it follows that $n \times H \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. Consequently, using (6.11), we see that $H^* \in L^2(\partial\Omega)$. Finally, since $\langle n, E \rangle = -\frac{1}{ik}\mathrm{Div}\,(n \times H) \in L^2(\partial\Omega)$, it follows from (6.10) that $E^* \in L^2(\partial\Omega)$. At this point, the conclusion follows from the case $p = 2$.

For the interior boundary value problem $(\mathcal{M}_{\mathrm{i}})$, the case when k is not a Maxwell eigenvalue is immediate from Theorem 5.1 **xiv**. Assume now that k is a Maxwell eigenvalue for Ω, and set U_k for the collection of all vector fields of the form $n \times E$ where (E, H) satisfies the Maxwell equation with wave number k in Ω and $E^*, H^* \in L^p(\partial\Omega)$. Since U_k contains $(\frac{1}{2}I + M_k)L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, it follows that U_k is a finite codimensional (hence closed) subspace of $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. Clearly, the problem $(\mathcal{M}_{\mathrm{i}})$ is solvable for the boundary data A, f if and only if $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, $\mathrm{Div}\, A = -ikf$, and in fact $A \in U_k$. The proof of the theorem is therefore finished. □

Remark. Assume that $2 - \epsilon \le p \le 2 + \epsilon$ and recall the definition of U_k from above. We want to point out that if k is a Maxwell eigenvalue for Ω, then U_k is a proper subspace of $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. In fact, a more explicit description of U_k is available. Below we show that $U_k = (\frac{1}{2}I + M_k)L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \neq L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ and that $A \in U_k$ if and only if

$$\int_{\partial\Omega} \langle A, H \rangle \, d\sigma = 0 \tag{6.12}$$

for any pair of vector fields (E, H) solving the L^q homogeneous version of the $(\mathcal{M}_{\mathrm{i}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. In particular, the condition (6.12) is actually necessary and sufficient for the solvability of $(\mathcal{M}_{\mathrm{i}})$ with boundary data A ($f := -\frac{1}{ik}\mathrm{Div}\, A$).

The necessity of (6.12) is a simple consequence of the readily verified identity

$$\int_{\partial\Omega} \langle n \times E_1, H_2 \rangle \, d\sigma = \int_{\partial\Omega} \langle n \times E_2, H_1 \rangle \, d\sigma$$

valid for any two pairs of vector fields (E_j, H_j), $j = 1, 2$, solving the Maxwell equations in Ω.

Conversely, any vector field A in $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, $2 - \epsilon \le p \le 2 + \epsilon$, that satisfies (6.12) belongs to $(\frac{1}{2}I + M_k)L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)(\subseteq U_k)$. To see this, we claim that $\{n \times H;\ (E, H) \text{ as in } (6.12)\}$ coincides with the null space of the operator $-\frac{1}{2}I + M_k$ acting on $L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)$. First, we indicate how this can be used to conclude. Indeed, this claim, (5.27), and elementary functional analysis imply that A must belong to $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \cap (\frac{1}{2}I + M_k)L^p_{\tan}(\partial\Omega)$. Now, by Theorem 5.3 **vi**, this latter space coincides precisely with $(\frac{1}{2}I + M_k)L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$.

Finally, we present the proof of the claim. The left-to-right inclusion is readily seen by taking the curl of both sides in the Green formula (3.2) for E, going to the boundary, and, finally, taking $n \times$ of both sides. As for the right-to-left inclusion, let $A \in L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)$ be such that $(-\frac{1}{2}I + M_k)A = 0$ and define $H := \operatorname{curl} \mathcal{S}_k A$, $E := -\frac{1}{ik}\operatorname{curl} H$ in $\mathbf{R}^3 \setminus \partial\Omega$. Then $n \times H|_{\partial\Omega_-} = 0$, and by the uniqueness in the exterior Maxwell boundary value problem, we obtain that E, H vanish identically in Ω_-. In particular, $n \times E|_{\partial\Omega_+} = n \times E|_{\partial\Omega_-} = 0$; thus, (E, H) are as in (6.12). Since $A = n \times H|_{\partial\Omega_+} - n \times H|_{\partial\Omega_-} = n \times H|_{\partial\Omega_+}$, the claim follows. □

In the sequel, the following boundary value problem (related to $(\mathcal{M}_e)$) will also be of importance for us.

Theorem 6.2.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$, $k \in \mathbf{C} \setminus \{0\}$, $\operatorname{Im} k \geq 0$, and $1 < p \leq 2 + \epsilon$. Then the boundary value problem

$$\begin{cases} U \in C^\infty(\mathbf{R}^3 \setminus \overline{\Omega}), \\ (\Delta + k^2)U = 0 \quad \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ U^*,\ (\nabla \operatorname{div} U)^*,\ (\operatorname{curl} U)^* \in L^p(\partial\Omega), \\ (\operatorname{div} U)|_{\partial\Omega} = h \in W^{1,p}(\partial\Omega), \\ n \times U|_{\partial\Omega} = A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega), \\ U \text{ satisfies the radiation condition,} \end{cases} \qquad (\mathrm{BVP}_1)$$

has a unique solution. This solution also satisfies

$$\|U^*\|_{L^p(\partial\Omega)} + \|(\nabla \operatorname{div} U)^*\|_{L^p(\partial\Omega)} + \|(\operatorname{curl} U)^*\|_{L^p(\partial\Omega)}$$
$$\leq C(\|h\|_{W^{1,p}(\partial\Omega)} + \|A\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)}),$$

for some $C = C(p, k, \Omega) > 0$.

Furthermore, for any $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, $1 < p \leq 2+\epsilon$, the vectorfields $E := U$, $H := \frac{1}{ik}\operatorname{curl} U$ solve $(\mathcal{M}_e)$ with boundary datum A if and only if U solves (BVP_1) *for A and $h = 0$.*

Proof. Set $\eta := 0$ if $\operatorname{Im} k \neq 0$ and $\eta := 1$ otherwise, and recall from the proof of Theorem 6.1 that $-\frac{1}{2}I + M_k + i\eta N_k(n \times S_0^2)$ is an invertible operator on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for $1 < p \leq 2 + \epsilon$.

Next, we claim that $-\frac{1}{2}I + K_k - i\eta S_k S_0^2$ is also an invertible operator on $W^{1,p}(\partial\Omega)$ for $1 < p \leq 2 + \epsilon$ and any $k \in \mathbf{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$. Assuming the claim for a moment, the existence part in our lemma follows immediately from the above invertibility results and the usual jump relations by taking

$$U := \operatorname{curl} \mathcal{S}_k B + i\eta \operatorname{curl}\operatorname{curl} \mathcal{S}(n \times S_0^2 B) + \mathcal{S}_k(n\, g) - ik^{-2}\eta \nabla \mathcal{S}_k S_0^2 g,$$

in $\mathbf{R}^3 \setminus \overline{\Omega}$, for some appropriate $B \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ and $g \in W^{1,p}(\Omega)$. The uniqueness part in this lemma is a consequence of the uniqueness part for the exterior Maxwell boundary value problem discussed in Theorem 6.1.

Returning to the proof of the claim, first we note that, by Theorem 5.4 and the fact that $K_k - K_0$ is compact, the operator $-\frac{1}{2}I + K_k - i\eta S_k S_0^2$ is Fredholm with index zero on $W^{1,p}(\partial\Omega)$ for $1 < p \leq 2 + \epsilon$ and any $k \in \mathbf{C}$. Also, much in the spirit of what we did in, for example, the proof of Theorem 6.1, its null space is actually included in $W^{1,2}(\partial\Omega)$. Hence, to show invertibility, it suffices to prove that if $f \in W^{1,2}(\partial\Omega)$ has $(-\frac{1}{2}I + K_k + i\eta\, S_k S_0^2)f = 0$, then necessarily $f = 0$.

To this end, introducing $u := \mathcal{K}_k f - i\eta \mathcal{S}_k S_0^2 f$ in $\Omega_\pm$, we have that $(\nabla u)^*$, $u^* \in L^2(\partial\Omega)$, $(\Delta + k^2)u = 0$ in $\Omega_\pm$, $u|_{\partial\Omega_-} = 0$, and u radiates at infinity. Hence, $u = 0$ in Ω_- (see the discussion in §6) so that, in particular, $(\partial u/\partial n)|_{\partial\Omega_-} = 0$. Also, $u|_{\partial\Omega_+} = u|_{\partial\Omega_+} - u|_{\partial\Omega_-} = f$ and $(\partial u/\partial n)|_{\partial\Omega_+} = (\partial u/\partial n)|_{\partial\Omega_+} - (\partial u/\partial n)|_{\partial\Omega_-} = -i\eta S_0^2 f$. Consequently, Green's formula gives

$$\iint_{\Omega_+} \left(|\nabla u|^2 - k^2|u|^2\right) dV = -i\eta \int_{\partial\Omega} |S_0 f|^2 \, d\sigma.$$

We now remark that η is chosen so that the above forces $f = 0$. The proof of the first part of the theorem is now completed.

The last part in the statement of the theorem follows from the fact that $\operatorname{div} U = 0$ in $\mathbf{R}^3 \setminus \overline{\Omega}$ if and only if $h = 0$ by the uniquness part in the Regularity problem for the Helmholtz operator. □

In the second part of this section we deal with the L^p-theory of boundary value problems for harmonic vector fields in Lipschitz domains in $\mathbf{R}^3$. This extends the corresponding L^2-results first obtained in [61]. However, it should be noted that while we are restricting ourselves to $\mathbf{R}^3$, the results in [61] are valid in arbitrary space dimensions.

Theorem 6.3.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then there exists $\epsilon > 0$, depending only on $\partial\Omega$, such that for each $1 < p \leq 2 + \epsilon$ the boundary value problem

$$\begin{cases} \operatorname{div} E = 0 \quad \textit{in } \Omega, \\ \operatorname{curl} E = 0 \quad \textit{in } \Omega, \\ E^* \in L^p(\partial\Omega), \\ n \times E = A \in L^{p,0}_{\tan}(\partial\Omega) \end{cases} \tag{BVP$_2$}$$

has a unique solution. This solution satisfies

$$\|E^*\|_{L^p(\partial\Omega)} \leq C\|A\|_{L^p(\partial\Omega)}.$$

Moreover, a similar statement holds for the exterior problem

$$\begin{cases} \operatorname{div} E = 0 \quad \textit{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ \operatorname{curl} E = 0 \quad \textit{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ E^* \in L^p(\partial\Omega), \\ E \textit{ satisfies the radiation condition}, \\ n \times E = A \in L^{p,0}_{\tan}(\partial\Omega), \\ \int_{\partial\Omega} \langle n, E\rangle \, d\sigma = \zeta \in \mathbf{C}. \end{cases} \tag{BVP$_3$}$$

Also, if the domain Ω has a C^1 boundary, then we may take $1 < p < \infty$.

Proof. For the interior problem, setting $E := \operatorname{curl} S_0 B$, $B \in L^{p,0}_{\tan}(\partial\Omega)$, existence is readily seen from the jump relations for the derivatives of S_0 and Theorem 5.1 **xvi**. Uniqueness follows from the Green formula (3.2) and Theorem 5.4.

To deal with the exterior problem, let $f \in L^p(\partial\Omega)$ be the unique solution of $S_0 f = 1 \in W^{1,p}(\partial\Omega)$ (cf. Theorem 5.4). This time we look for E in the form $E := \operatorname{curl} S_0 B + \xi \, \nabla S_0 f$ for some $B \in L^{p,0}_{\tan}(\partial\Omega)$, $\xi \in \mathbf{C}$, and we proceed as before. The reason this works is that $f \notin L^p_0(\partial\Omega)$ (recall that $\pi \circ S_0 : L^p_0(\partial\Omega) \to W^{1,p}(\partial\Omega)/\mathbf{C}$ is an isomorphism) implies that $(\frac{1}{2}I + K_0^*)f \notin L^p_0(\partial\Omega)$ and, consequently, that $\int_{\partial\Omega} \langle n, \nabla S_0 f\rangle \, d\sigma \neq 0$. □

Theorem 6.4.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then there exists $\epsilon > 0$, depending only on $\partial\Omega$, such that for each $1 < p \leq 2 + \epsilon$ the boundary value problem

$$\begin{cases} \Delta E = 0 \quad \textit{in } \Omega, \\ \operatorname{div} E = 0 \quad \textit{in } \Omega, \\ E^*, \ (\operatorname{curl} E)^* \in L^p(\partial\Omega), \\ n \times E = A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \end{cases} \tag{BVP$_4$}$$

has a unique solution. This solution satisfies

$$\|E^*\|_{L^p(\partial\Omega)} + \|(\operatorname{curl} E)^*\|_{L^p(\partial\Omega)} \le C\|A\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)},$$

and $\operatorname{curl} E = 0$ *if and only if* $\operatorname{Div} A = 0$.

Moreover, a similar statement holds for the exterior problem, provided E is radiating and the charge integral $\int_{\partial\Omega}\langle n, E\rangle\, d\sigma$ *is a priori prescribed (in* $\mathbf{C}$*), that is, for*

$$\begin{cases} \Delta E = 0 \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega}, \\ \operatorname{div} E = 0 \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega}, \\ E^*,\ (\operatorname{curl} E)^* \in L^p(\partial\Omega), \\ E \text{ satisfies the radiation condition}, \\ n\times E = A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega), \\ \int_{\partial\Omega}\langle n, E\rangle\, d\sigma = \zeta \in \mathbf{C}. \end{cases} \tag{BVP$_5$}$$

Furthermore, if the domain Ω *has a* C^1 *boundary, then we may take* $1 < p < \infty$.

Proof. This is quite similar to the proof of Theorem 6.3, except that we now take $B \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ and use Theorem 5.1 **xvii**. □

Theorem 6.5.

Let Ω *be a bounded Lipschitz domain in* $\mathbf{R}^3$ *with* $g(\partial\Omega) = 0$. *Then there exists* $\epsilon > 0$, *depending only on* $\partial\Omega$, *such that for each* $1 < p \le 2+\epsilon$ *the boundary value problem*

$$\begin{cases} \Delta E = 0 \quad \text{in } \Omega, \\ \operatorname{div} E = 0 \quad \text{in } \Omega, \\ E^*,\ (\operatorname{curl} E)^* \in L^p(\partial\Omega), \\ \langle n, \operatorname{curl} E\rangle = f \in L^p_0(\partial\Omega), \\ \langle n, E\rangle = g \in L^p_0(\partial\Omega) \end{cases} \tag{BVP$_6$}$$

has a unique solution. This solution satisfies

$$\|E^*\|_{L^p(\partial\Omega)} + \|(\operatorname{curl} E)^*\|_{L^p(\partial\Omega)} \le C(\|f\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}).$$

Also, if the domain Ω *has a* C^1 *boundary, then we may take* $1 < p < \infty$.

Proof. Existence is seen by taking $E := \mathcal{S}_0 A + \nabla\mathcal{S}_0 h$, with $A \in L^{p,0}_{\tan}(\partial\Omega)$, $h \in L^p_0(\partial\Omega)$, by using the usual jump relations and Theorem 5.1 **i** together with Theorem 5.4. Uniqueness follows from (6.5) written for curl E first and then for E. □

Next we present a result about the behavior of the solutions of the problems $(\mathcal{M}_{\mathrm{i,e}})$ as the wave number $k \in \mathbf{C}\setminus\{0\}$ approaches zero, under the additional hypothesis that the topological genus of $\partial\Omega$ is zero (note that this is the only case not covered by Theorem 6.1).

Theorem 6.6.

Let Ω *be a bounded Lipschitz domain in* $\mathbf{R}^3$ *with* $g(\partial\Omega) = 0$, *and let* $\epsilon > 0$ *be as in Theorem 6.1,* $1 < p \le 2+\epsilon$, $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. *For each* $k \in \mathbf{C}\setminus\{0\}$ *with* $\operatorname{Im} k \ge 0$, *let* (E_k, H_k) *be the unique solution of the exterior Maxwell boundary value problem* $(\mathcal{M}_{\mathrm{e}})$ *with wave number k and boundary data A (and* $f := -\frac{1}{ik}\operatorname{Div} A$*).*

Then, as $k \to 0$, *the electric field* E_k *converges uniformly on compact subsets of* $\mathbf{R}^3\setminus\overline{\Omega}$ *to* E_0, *the unique solution of* (BVP$_5$) *with boundary data A, and* $\zeta = 0$.

If $\partial\Omega \in C^1$, *then we may take* $1 < p < \infty$.

Proof. Since $g(\partial\Omega) = 0$, it follows that $-\frac{1}{2}I + M_0$ is invertible on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ (cf. Theorem 5.1). Furthermore, since

$$\|M_0 - M_k\|_{\text{operator}} = \mathcal{O}(k^2)$$

as $k \to 0$, we also have that $-\frac{1}{2}I + M_k$ is invertible on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for small k and that $(-\frac{1}{2}I + M_k)^{-1}$ converges to $(-\frac{1}{2}I + M_0)^{-1}$ in the operator norm. Consequently,

$$E_k(X) := n(X) \times \operatorname{curl} \int_{\partial\Omega} \Phi_k(X-Y)[(-\tfrac{1}{2}I + M_k)^{-1}A](Y)\, d\sigma(Y)$$

converges uniformly on compact subsets of $\mathbf{R}^3 \setminus \overline{\Omega}$ to

$$E_0(X) := n(X) \times \operatorname{curl} \int_{\partial\Omega} \Phi_0(X-Y)[(-\tfrac{1}{2}I + M_0)^{-1}A](Y)\, d\sigma(Y).$$

To see that E_0 solves (BVP$_5$) we only need to check that $\int_{\partial\Omega}\langle n, E_0\rangle d\sigma = 0$. However,

$$\int_{\partial\Omega_-} \langle n, E_0\rangle\, d\sigma = \int_{\partial\Omega_+} \langle n, E_0\rangle\, d\sigma = \iint_{\Omega_+} \operatorname{div} E_0\, dV = 0,$$

and the conclusion follows. □

Our last result in this section deals with another physically relevant case, which is related to the so-called magnetic (Neumann) screen problem for electromagnetic waves (compare with, e.g., [86]).

Theorem 6.7.

Let Ω be an arbitrary, fixed, bounded Lipschitz domain in $\mathbf{R}^3$. Then there exists $\epsilon = \epsilon(\Omega) > 0$ such that for each $2-\epsilon \le p \le 2+\epsilon$ and each $k \in \mathbf{C}\setminus\{0\}$, $\operatorname{Im} k \ge 0$, the boundary value problem

$$\begin{cases} (\Delta + k^2)H = 0 \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega}, \\ H^*,\ (\operatorname{div} H)^*,\ (\operatorname{curl} H)^* \in L^p(\partial\Omega), \\ n \times \operatorname{curl} H = A \in L^p_{\tan}(\partial\Omega), \\ \langle n, H\rangle = f \in L^p(\partial\Omega), \\ H \text{ satisfies the radiation condition} \end{cases} \qquad (\mathrm{BVP}_7)$$

has a unique solution. This solution depends continuously on k in the range $\{k \in \mathbf{C}\setminus\{0\};\ \operatorname{Im} k \ge 0\}$ and satisfies

$$\|H^*\|_{L^p} + \|(\operatorname{div} H)^*\|_{L^p} + \|(\operatorname{curl} H)^*\|_{L^p} \le C\,(\|A\|_{L^p} + \|f\|_{L^p})$$

for some $C = C(\Omega, p, k) > 0$. Also, $(\operatorname{curl}\operatorname{curl} H)^* \in L^p(\partial\Omega)$ if and only if $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$.

Furthermore, if also $g(\partial\Omega) = 0$, then this solution converges uniformly on compact subsets of $\mathbf{R}^3\setminus\overline{\Omega}$, as $k \to 0$, to the unique solution of the problem

$$\begin{cases} \Delta H_0 = 0 \quad \text{in } \mathbf{R}^3\setminus\overline{\Omega}, \\ H_0^*,\ (\operatorname{div} H_0)^*,\ (\operatorname{curl} H_0)^* \in L^p(\partial\Omega), \\ n \times \operatorname{curl} H_0 = A \in L^p_{\tan}(\partial\Omega), \\ \langle n, H_0\rangle = f \in L^p(\partial\Omega), \\ H_0 \text{ satisfies the radiation condition.} \end{cases} \qquad (\mathrm{BVP}_8)$$

If $\partial\Omega \in C^1$, then we may take $1 < p < \infty$.

Proof. For $\eta \in \mathbf{R}\setminus\{0\}$ as in the proof of Theorem 6.1 and for $B \in L^p_{\tan}(\partial\Omega)$, $g \in L^p(\partial\Omega)$, consider

$$\begin{aligned} H :=\ & S_k B + i\eta \operatorname{curl} S_k(n \times S_0^2 B) + \nabla S_k g \\ & + ik^2\eta\, S_k(n S_0^2 g) + i\eta \operatorname{curl} S_k(n \times \nabla S_0(S_0 g)) \end{aligned}$$

in $\mathbf{R}^3\setminus\overline{\Omega}$. Thus, existence in (BVP$_7$) rests on the invertibility of the boundary integral operators

$$-\tfrac{1}{2}I + M_k + ik^2\eta\, n \times S_k(n \times S_0^2) - ik^2\eta\, n \times \nabla S_k(\langle n, \operatorname{curl} S_0\rangle S_0)$$

and

$$\tfrac{1}{2}I + K_k^* + ik^2\eta\,\langle n, S_k(nS_0^2)\rangle + i\eta\,\langle n, \operatorname{curl} S_k\rangle(n \times \nabla S_0)S_0$$

on $L^p_{\tan}(\partial\Omega)$ and on $L^p(\partial\Omega)$, respectively. Assuming now that $\epsilon > 0$ is as in Theorem 5.3 and since $2 - \epsilon \le p \le 2 + \epsilon$, both cases can be handled by the results in Theorem 5.3, Theorem 5.4, and Fredholm theory, much in the spirit of the approach in the proof of the Theorem 6.1.

For $\operatorname{Im} k > 0$, uniqueness in (BVP_7) is seen from the Green formula (6.5) and the invertibility results in Theorem 5.3; whereas if $k \in \mathbf{R}\setminus\{0\}$, uniqueness is seen from energy estimates and Rellich's lemma.

The reasoning for (BVP_8) is similar (here one uses Theorem 5.1), and the convergence statement is proved pretty much as in Theorem 6.6. We omit the details. □

7. Some Counterexamples

Here we shall construct counterexamples that show our L^p results on the invertibility of the boundary integral operators and on the solvability of the boundary value problems discussed in the previous sections are sharp.

Theorem 7.1.

For each $2 < p < \infty$ there exist a simply connected, bounded Lipschitz domain Ω in $\mathbf{R}^3$ and a vector field $A \in L^{p,\operatorname{Div}}_{\tan}(\partial\Omega)(\subseteq L^{2,\operatorname{Div}}_{\tan}(\partial\Omega))$ such that if $f := -\frac{1}{ik}\operatorname{Div} A \in L^p(\partial\Omega)$, then for any $k \in \mathbf{C}\setminus\{0\}$ the (unique) L^2-solution (E, H) of the boundary value problem ($\mathcal{M}_e$) corresponding to $p = 2$ and the boundary data A, f is not a L^p-solution (in the sense that the estimate (6.8) fails). In particular, for such a domain Ω and such boundary data A, f, the boundary value problem ($\mathcal{M}_e$) has no solution.

A similar statement is valid for the interior problem ($\mathcal{M}_i$) as well.

Proof. We first consider the case of the interior problem ($\mathcal{M}_i$) which is somewhat easier to present. The departure point is to recall the L^p-counterexamples for the solvability of the Neumann problem from [45]. That is, for $\alpha \in (0, \frac{\pi}{2})$ we let D_α denote a bounded simply connected Lipschitz domain in $\mathbf{R}^2$ such that

$$D_\alpha \subseteq \{re^{i\theta};\, r > 0,\ |\theta| < \alpha/2\},$$

$$\partial D_\alpha \cap \{x^2 + y^2 < 1\} = \{re^{i\theta};\, r > 0,\ |\theta| < \alpha/2\} \cap \{x^2 + y^2 < 1\}$$

and with the property that $\partial D_\alpha \setminus \{0\}$ is smooth. Taking $v(x, y) := \operatorname{Re}(x + iy)^{\pi/\alpha}$, $(x, y) \in D_\alpha$, one can show that $\Delta v = 0$ in D_α, $\frac{\partial v}{\partial n} \in L^\infty(\partial D_\alpha)$ and, if s denotes the arc-length parametrization of ∂D_α, $(\nabla v)^*(s) \approx s^{-1+\pi/\alpha}$ (see [45]). In particular, for any $p > 2$ it is possible to choose $\alpha \in (0, \frac{\pi}{2})$ such that $(\nabla v)^* \notin L^p(\partial D_\alpha)$. This two dimensional counterexample can be immediately lifted to $\mathbf{R}^3$ in a wedge-like domain. Specifically, we consider the simply connected, bounded Lipschitz domain $\Omega_\alpha := D_\alpha \times (-1, 1) \subseteq \mathbf{R}^3$ and the harmonic function $u(x, y, z) := u(x, y)$, $(x, y, z) \in \Omega_\alpha$. Clearly, u is smooth in $\overline{\Omega_\alpha} \setminus \{(0, 0, z);\, z \in [-1, 1]\}$ and $\frac{\partial u}{\partial n} \in L^\infty(\partial\Omega_\alpha)$. Also, $(\nabla u)^* \in L^2(\partial\Omega_\alpha)$ for any α, but for each fixed $p > 2$ one can choose α such that $(\nabla u)^* \notin L^p(\partial\Omega_\alpha)$.

Now fix $k \in \mathbf{C}\setminus\{0\}$ and $p > 2$ and select $\alpha \in (0, \frac{\pi}{2})$ such that Ω_α and u are as above. Then the idea is to set

$$E := ik\, S_k(n \times \nabla u)$$

and

$$H := \nabla u - k^2 L_k(\nabla u) - \nabla S_k(\partial u/\partial n)$$

in Ω_α. Straightforward calculation shows that E and H are divergencefree solutions of the vector Helmholtz equation in Ω_α. In fact, as $E = -\frac{1}{ik}\operatorname{curl} H$, it follows that (E, H) is a solution of the Maxwell system in Ω_α. Also, since by the Sobolev embedding theorem $L_k(\nabla u)$ is (Hölder) continuous up to and including the boundary of Ω, we have that E^*, $H^* \in L^2(\partial\Omega_\alpha)$. Thus (E, H) is a solution of the interior problem $(\mathcal{M}_i)$ for $p = 2$ and boundary data

$$A := ik\, n \times S_k(n \times \nabla u) \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega),$$

$$f := -\frac{1}{ik}\langle n, \operatorname{curl} S_k(n \times \nabla u)\rangle \in L^2(\partial\Omega).$$

Furthermore, since S_k maps $L^2(\partial\Omega_\alpha)$ boundedly into $W^{1,2}(\partial\Omega_\alpha)$, using the Sobolev embedding theorem we see that actually $A \in L^p_{\tan}(\partial\Omega)$. Recall that $\frac{\partial u}{\partial n} \in L^p(\partial\Omega_\alpha)$ so that $f = \langle n, H\rangle \in L^p(\partial\Omega_\alpha)$. If we combine this with the fact that $\operatorname{Div} A = -ik\, f \in L^p(\partial\Omega)$, we may also conclude that $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. However, the estimate (6.8) fails because $(\nabla u)^* \notin L^p(\partial\Omega_\alpha)$ also forces $H^* \notin L^p(\partial\Omega_\alpha)$. This concludes the reasoning for the interior boundary value problem.

The reasoning for the exterior boundary value problem $(\mathcal{M}_e)$ is an adaptation of the above argument and is only slightly more involved. For some fixed $k \in \mathbf{C} \setminus \{0\}$ and $p > 2$ one departs from a bounded, simply connected Lipschitz domain Ω in $\mathbf{R}^3$ and $u \in C^\infty(\mathbf{R}^3 \setminus \overline{\Omega})$ such that $\Delta u = 0$ in $\mathbf{R}^3 \setminus \overline{\Omega}$, $(\nabla u)^* \in L^2(\partial\Omega)$, $(\nabla u)^* \notin L^p(\partial\Omega)$, $\frac{\partial u}{\partial n} \in L^p(\partial\Omega)$, and u decays at infinity. Next, we take $\varphi \in C^\infty_{\mathrm{comp}}(\mathbf{R}^3)$ such that $\varphi \equiv 1$ in a neighborhood of $\partial\Omega$ and set $v := \nabla(\Delta + k^2)(\varphi u)$ in $\mathbf{R}^3 \setminus \overline{\Omega}$. Clearly, v is curl-free, vanishes at infinity, and equals $k^2 \nabla u$ near $\partial\Omega$. Then we set

$$E := ik^{-1} S_k(n \times v),$$

$$H := \nabla(\varphi u) - L_k(v) - k^{-2}\nabla\mathcal{S}_k(\langle n, v\rangle)$$

in $\mathbf{R}^3 \setminus \overline{\Omega}$. Successive integrations by parts then give

$$\begin{aligned}
\operatorname{div} H &= \Delta(\varphi u) - \operatorname{div} L_k(\nabla(\Delta + k^2)(\varphi u)) - k^2\mathcal{S}_k(\partial u/\partial n) \\
&= \Delta(\varphi u) - \Delta L_k((\Delta + k^2)(\varphi u)) - \operatorname{div} \mathcal{S}_k(n(\Delta + k^2)(\varphi u)) - k^2\mathcal{S}_k(\partial u/\partial n) \\
&= -k^2(\varphi u) + k^2 L_k((\Delta + k^2)(\varphi u)) - k^2\mathcal{K}_k(u|_{\partial\Omega}) - k^2\mathcal{S}_k(\partial u/\partial n) \\
&= 0.
\end{aligned}$$

With this at hand, and observing that $(\Delta + k^2)H = 0$, $E = -\frac{1}{ik}\operatorname{curl} H$, we may proceed as before. □

Theorem 7.2.

The Theorems 6.3, 6.4, and 6.5 are sharp (in the sense of Theorem 7.1).

Proof. Here we adapt an argument first used in [19]. For $0 < \lambda < 1$ recall the generalized Legendre function P_λ of degree λ. In particular, P_λ satisfies

$$(1 - t^2)\frac{d^2 P_\lambda}{dt^2}(t) - 2t\frac{dP_\lambda}{dt}(t) + \lambda(\lambda + 1)P_\lambda(t) = 0$$

for $-1 < t < 1$, and we assume that P_λ is normalized such that $P_\lambda(1) = 1$. Following [19], we set $a_\lambda := \sup\{t \in (-1, 1);\ P_\lambda(t) = 0\}$, $-1 < a_\lambda < 1$, and denote by Ω_λ a bounded, simply connected Lipschitz domain in $\mathbf{R}^3$ such that $\partial\Omega_\lambda \setminus \{0\}$ is smooth and

$$\Omega_\lambda \cap \{X;\ |X| < \epsilon\} = \{X;\ X_1 > a_\lambda |X|\} \cap \{X;\ |X| < \epsilon\}$$

for some fixed, small, positive ϵ. Next, set $v_\lambda(X) := |X|^\lambda P_\lambda(\frac{X_1}{|X|})$ for $X \in \Omega_\lambda$ so that, by direct calculation,

$$|X|^{-\lambda+2}\Delta v_\lambda(X) = \left(1 - \frac{X_1^2}{|X|^2}\right)\frac{d^2P_\lambda}{dt^2}\left(\frac{X_1}{|X|}\right) - 2\frac{X_1}{|X|}\frac{dP_\lambda}{dt}\left(\frac{X_1}{|X|}\right) + \lambda(\lambda+1)P_\lambda\left(\frac{X_1}{|X|}\right) = 0.$$

Thus v_λ is harmonic in Ω_λ. Also, it is easy to check that $v_\lambda \in C^\infty(\overline{\Omega_\lambda} \setminus \{0\})$, $\nabla_{\tan} v_\lambda \in L^\infty(\partial\Omega_\lambda)$, and $(\nabla v_\lambda)^*(X) \approx |X|^{\lambda-1}$ for $X \in \partial\Omega_\lambda$. In particular, $(\nabla v_\lambda)^* \in L^2(\partial\Omega_\lambda)$ and $(\nabla v_\lambda)^* \notin L^p(\partial\Omega_\lambda)$ if $p > \frac{2}{1-\lambda}$.

Then, clearly, for any $p > 2$ it is possible to select $\lambda \in (0, 1)$ such that $E := \nabla v_\lambda$ in Ω_λ is a counterexample to the L^p-solvability of (BVP$_2$) with boundary datum $A := n \times \nabla v_\lambda \in L^{p,0}_{\tan}(\partial\Omega)$. A similar argument (as in the second part of the proof of Theorem 7.1) is valid for the exterior problem as well. Hence, Theorem 6.3 is sharp.

Finally, the sharpness of Theorems 6.4 and 6.5 is immediate from the sharpness of the results for the Neumann problem for the Laplace operator in [20] and the sharpness of Theorem 6.3. □

Theorem 7.3.

For any $p \neq 2$, there exists a bounded, simply connected Lipschitz domain Ω in $\mathbf{R}^3$ such that the operators $\pm\frac{1}{2}I + M_k$ (where $k = 0$ or $k \in \mathbf{C}$ with $\operatorname{Im} k > 0$, say) are not isomorphisms of $L^p_{\tan}(\partial\Omega)$.

Proof. Fix $k \in \mathbf{C}$ with $\operatorname{Im} k > 0$ (essentially the same reasoning applies to the case $k = 0$ too). Assume now that $2 < p < \infty$, and recall the Lipschitz domain Ω_λ and the harmonic function v_λ introduced in the proof of Theorem 7.2, where $\lambda \in (0, 1)$ is chosen such that $\frac{2}{1-\lambda} < p$. If we now set

$$E := \nabla v_\lambda - k^2 L_k(\nabla v_\lambda) + \mathcal{S}_k(nf)$$

in Ω_λ, where

$$f := (\tfrac{1}{2}I + K_k)^{-1}[S_k(\partial v_\lambda/\partial n)],$$

on $\partial\Omega_\lambda$, then clearly $(\Delta + k^2)E = 0$ and $\operatorname{div} E = 0$ in Ω_λ. Also, $n \times E \in L^\infty_{\tan}(\partial\Omega)$ but $E^* \notin L^p(\partial\Omega)$ since $(\nabla v_\lambda)^* \notin L^p(\partial\Omega)$. If the operator $\frac{1}{2}I + M_k$ would be invertible on $L^p_{\tan}(\partial\Omega)$, then by the uniqueness part of Theorem 9.1 (stated and proved in §9) for the L^2-case it would follow that

$$E = \operatorname{curl} \mathcal{S}_k(\tfrac{1}{2}I + M_k)^{-1}(n \times E) \quad \text{in } \Omega$$

and, consequently, that $E^* \in L^p(\partial\Omega)$. This contradiction shows that the operator $\frac{1}{2}I + M_k$ cannot be invertible on $L^p_{\tan}(\partial\Omega)$.

A similar construction works for the exterior electric boundary value problem (cf. also the last part in Theorem 7.1) and gives that $-\frac{1}{2}I + M_k$ is not invertible on $L^p_{\tan}(\partial\Omega)$. Finally, since, by (5.27), $\pm\frac{1}{2}I + M_k$ are invertible on some $L^p(\partial\Omega)$ if and only if $\mp\frac{1}{2}I + M_k$ are invertible on $L^q_{\tan}(\partial\Omega)$, for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the conclusion follows. □

8. Atomic Theory for the Maxwell Equations and Related Systems

In this section our aim is to present the endpoint case $p = 1$ for the the boundary value problem formulated in the previous sections.

Throughout this section we assume that Ω is a fixed, bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Recall that a scalar-valued atom a is a function supported in a surface ball $B := B(P,r) \cap \partial\Omega$ (for some $P \in \partial\Omega$ and $r > 0$) such that $\|a\|_{L^\infty} \leq \sigma(B)$ and $\int_{\partial\Omega} a\, d\sigma = 0$. The atomic Hardy space $H^1_{\rm at}(\partial\Omega)$ (cf. [13]) is then defined as the subspace of $L^1(\partial\Omega)$ consisting of elements of the form $f = \sum_j \lambda_j a_j$, where a_j's are atoms and $\sum_j |\lambda_j| < +\infty$. This space is endowed with the natural norm

$$\|f\|_{H^1_{\rm at}(\partial\Omega)} := \inf \left\{ \sum |\lambda_j|;\ f = \sum_j \lambda_j a_j,\ a_j \text{ atoms} \right\}.$$

By slightly abusing notation, we shall also denote by $H^1_{\rm at}(\partial\Omega)$ the space of all vector fields with components in $H^1_{\rm at}(\partial\Omega)$. In particular, following [20], we set

$$H^{1,1}_{\rm at}(\partial\Omega) := \{f \in L^2(\partial\Omega) :\ (n \times \nabla) f \in H^1_{\rm at}(\partial\Omega)\}.$$

Next we introduce the atomic spaces that are relevant for the Maxwell system. Specifically, we shall work with

$$H^{1,0}_{\rm tan}(\partial\Omega) := \{A \in H^1_{\rm at}(\partial\Omega);\ \langle n, A\rangle = 0,\ \ \mathrm{Div}\, A = 0\} \subseteq H^1_{\rm at}(\partial\Omega)$$

and

$$H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega) := \{A_1 + A_2;\ A_1 \in H^{1,0}_{\rm tan}(\partial\Omega),\ A_2 \in L^2_{\rm tan}(\partial\Omega), \mathrm{Div}\, A_2 \in H^1_{\rm at}(\partial\Omega)\}.$$

The latter space is equipped with the norm

$$\|A\|_{H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)} := \inf \left(\|A_1\|_{H^1_{\rm at}(\partial\Omega)} + \|A_2\|_{L^2(\partial\Omega)} + \|\mathrm{Div}\, A_2\|_{H^1_{\rm at}(\partial\Omega)} \right)$$

where the infimum is taken over all decompositions $A = A_1 + A_2$ with A_1 and A_2 as in the definition of $H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)$.

At the level of boundary operators, we have the following result.

Theorem 8.1.

The following operators are isomorphisms between the indicated spaces.

- **i.** $\langle n, \mathrm{curl}\, S_0\rangle : H^{1,0}_{\rm tan}(\partial\Omega) \to H^1_{\rm at}(\partial\Omega)$;
- **ii.** $n \times \nabla S_0 : H^1_{\rm at}(\partial\Omega) \to H^{1,0}_{\rm tan}(\partial\Omega)$;
- **iii.** $n \times \nabla : H^{1,1}_{\rm at}(\partial\Omega)/\mathbf{C} \to H^1_{\rm at}(\partial\Omega)$;
- **iv.** $\mathrm{Div} : H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)/H^{1,0}_{\rm tan}(\partial\Omega) \to H^1_{\rm at}(\partial\Omega)$;
- **v.** $\pm\frac{1}{2}I + M_0 : H^{1,0}_{\rm tan}(\partial\Omega) \to H^{1,0}_{\rm tan}(\partial\Omega)$;
- **vi.** $\pm\frac{1}{2}I + M_0$ *acting on* $\{A \in L^2_{\rm tan}(\partial\Omega);\ \mathrm{Div}\, A \in H^1_{\rm at}(\partial\Omega)\}$;
- **vii.** $\pm\frac{1}{2}I + M_0 : H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega) \to H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)$;
- **viii.** $\pm\frac{1}{2}I + M_k : H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega) \to H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)$, *provided* $k \in \mathbf{C}$, $\mathrm{Im}\, k \geq 0$, *is not a Maxwell eigenvalue for* Ω;
- **ix.** $N_k : H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega) \to H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)$, *provided* $k \in \mathbf{C} \setminus \{0\}$, $\mathrm{Im}\, k \geq 0$, *is not a Maxwell eigenvalue for* Ω;
- **x.** $\Lambda^{\pm}_k : H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega) \to H^{1,\mathrm{Div}}_{\rm tan}(\partial\Omega)$, *provided* $k \in \mathbf{C} \setminus \{0\}$, $\mathrm{Im}\, k \geq 0$, *is not a Maxwell eigenvalue for* Ω.

Note that, as a direct corollary, we obtain that the sequence of boundary derivative operators

$$0 \longrightarrow \mathbf{C} \xrightarrow{\iota} H^{1,1}_{\mathrm{at}}(\partial\Omega) \xrightarrow{n\times\nabla} H^{1,\mathrm{Div}}_{\tan}(\partial\Omega) \xrightarrow{\mathrm{Div}} H^{1}_{\mathrm{at}}(\partial\Omega) \longrightarrow 0$$

is exact.

Before we proceed to the proof of this theorem we note some facts of importance for us here. First recall that the closure of compactly supported continuous functions on $\partial\Omega$ in the BMO "norm" is $VMO(\partial\Omega)$, the space of functions of vanishing mean oscillations. It is well known that $H^1_{\mathrm{at}}(\partial\Omega) = (VMO(\partial\Omega))^*$ (cf. [13]). Furthermore, recall the Fabes-Kenig [27] form of the Varopoulos extension theorem [91], which asserts that each $b \in VMO(\partial\Omega)$ has an extension B in Ω such that $|\nabla B|dV$ is a Carleson measure in Ω with norm $\leq C\|b\|_{VMO(\partial\Omega)}$, where $C > 0$ depends only on the Lipschitz character of $\partial\Omega$.

Now if E is a curlfree vector field in Ω such that $E^* \in L^1(\partial\Omega)$ and $E|_{\partial\Omega}$ exists, then for each $b \in VMO(\partial\Omega)$ one has

$$\left|\int_{\partial\Omega} \langle n \times E, b\rangle\, d\sigma\right| = \left|\iint_{\Omega} \langle E, \operatorname{curl} B\rangle\, dV\right| \leq C\|E^*\|_{L^1(\partial\Omega)}\|b\|_{VMO(\partial\Omega)}$$

by the usual Carleson estimate. Hence, $n \times E$ belongs to $H^1_{\mathrm{at}}(\partial\Omega)$ and, furthermore, $\|n \times E\|_{H^1_{\mathrm{at}}(\partial\Omega)} \leq C\|E^*\|_{L^1(\partial\Omega)}$.

Similarly, if E is a divergencefree vector field in Ω with $E^* \in L^1(\partial\Omega)$ and such that $E|_{\partial\Omega}$ exists, then $\langle n, E\rangle \in H^1_{\mathrm{at}}(\partial\Omega)$ and $\|\langle n, E\rangle\|_{H^1_{\mathrm{at}}(\partial\Omega)} \leq C\|E^*\|_{L^1(\partial\Omega)}$. In particular, if E is both curlfree and divergencefree, has $E^* \in L^1(\partial\Omega)$, and $E|_{\partial\Omega}$ exists, then $n \times E$ belongs to $H^{1,0}_{\tan}(\partial\Omega)$. Also, Theorems 5.1 and 5.4 and the above reasoning show that $L^{2,0}_{\tan}(\partial\Omega) \subseteq H^{1,0}_{\tan}(\partial\Omega)$, $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega) \subseteq H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$, and $L^2_0(\partial\Omega) \subseteq H^1_{\mathrm{at}}(\partial\Omega)$.

Finally, from [20] it is known that S_0 maps $H^1_{\mathrm{at}}(\partial\Omega)$ isomorphically onto $H^{1,1}_{\mathrm{at}}(\partial\Omega)$ and that $\pm\frac{1}{2}I + K_0^*$ are invertible operators from $H^1_{\mathrm{at}}(\partial\Omega)$ onto itself.

We are now ready to present the proof of Theorem 8.1.

Proof of Theorem 8.1. Note that, by the results in [13, 12] and §4, these operators are well defined and bounded on the indicated spaces (cf. also the above discussion). The proof of the points **i**–**v** is the same as in the L^p case, using the above preliminary remarks and we omit it (all operator identities contained in Lemmas 5.8–5.10 have natural analogs in the atomic setting as well). Part **vi** follows from Theorem 5.1 **xix** and (the atomic version of) Lemma 4.4.

Next we prove that M_k maps $H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$ boundedly into itself for any $k \in \mathbf{C}$. Note that for $k = 0$ this follows from points **v** and **vi**, which were already proved. For a general k, we decompose $M_k = M_0 + (M_k - M_0)$ and, since $M_k - M_0$ maps $L^1(\partial\Omega)$ into $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$, the conclusion follows. With this at hand, **vii** is easily seen from **v** and **vi**, whereas **viii** follows from **vii** and Fredholm theory.

The proof of the remaining part of the theorem then proceeds as in the L^p case. □

Next we analyze the endpoint case $p = 1$ for boundary value problems for harmonic fields.

Theorem 8.2.

The necessary and sufficient condition for (BVP$_2$) *to be solvable in the case $p = 1$ is that the boundary datum A belongs to $H^{1,0}_{\tan}(\partial\Omega)$. In this case, the solution is unique and satisfies*

$$\|E^*\|_{L^1(\partial\Omega)} \leq C\|A\|_{H^1_{\mathrm{at}}(\partial\Omega)}$$

for some $C = C(\Omega) > 0$.

A similar result is valid for the exterior version (BVP$_3$) *as well.*

Proof. The necessity of the membership of A to $H^{1,0}_{\tan}(\partial\Omega)$ is immediate from the remarks preceding Theorem 8.1. Existence and uniqueness follow from Theorem 8.1 **v**. □

Theorem 8.3.

The necessary and sufficient condition for (BVP$_4$) *to be solvable in the case $p = 1$ is that the boundary datum A belongs to $H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$. In this case, the solution is unique and satisfies*

$$\|E^*\|_{L^1(\partial\Omega)} + \|(\operatorname{curl} E)^*\|_{L^1(\partial\Omega)} \leq C\|A\|_{H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)}$$

for some $C = C(\Omega) > 0$.

A similar result is valid for the exterior version (BVP$_5$) *as well.*

Proof. Assume that (BVP$_4$) is solvable in the case $p = 1$. Clearly, $A \in L^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$ and in fact $\operatorname{Div} A = -\langle n, \operatorname{curl} E\rangle \in H^1_{\mathrm{at}}(\partial\Omega)$. Next, using Theorem 8.1, let $B \in H^{1,0}_{\tan}(\partial\Omega)$ be such that $\langle n, \operatorname{curl} \mathcal{S}_0 B\rangle = \operatorname{Div} A \in H^1_{\mathrm{at}}(\partial\Omega)$. Using the simple connectivity of Ω it is possible to construct a scalar-valued u such that $\Delta u = 0$, $\nabla u = \operatorname{curl} E - \operatorname{curl} \mathcal{S}_0 B$ in Ω. It follows that $(\nabla u)^* \in L^1(\partial\Omega)$ and $\partial u/\partial n = 0$, so from the uniqueness part in the Neumann problem with atomic data [20] we infer that u is constant in Ω.

Consequently, $E - \mathcal{S}_0 B$ is curlfree, divergencefree and, by Theorem 8.2, we have that $n \times (E - \mathcal{S}_0 B) \in H^{1,0}_{\tan}(\partial\Omega)$. Since $n \times \mathcal{S}_0 B \in L^2_{\tan}(\partial\Omega)$ has $\operatorname{Div}(n \times \mathcal{S}_0 B) \in H^1_{\mathrm{at}}(\partial\Omega)$, this implies that A necessarily belongs to $H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$.

Existence and uniqueness in (BVP$_4$) with $p = 1$ for $A \in H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$ is seen from **vii** in Theorem 8.1 and Green's formula. □

We also remark that there is an analogous statement in the limiting case $p = 1$ for Theorem 6.5, in which situation the boundary data should belong to $H^1_{\mathrm{at}}(\partial\Omega)$.

We are now in a position to state and prove the main result of this section describing the atomic theory for the Maxwell boundary value problem on Lipschitz domains.

Theorem 8.4.

For each fixed $k \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$, the exterior Maxwell boundary value problem $(\mathcal{M}_e)$ is solvable for $p = 1$ if and only if $A \in H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$, $f \in H^1_{\mathrm{at}}(\partial\Omega)$, and $\operatorname{Div} A = -ikf$. The solution is unique, depends on k analytically in $\mathbf{R}^2_+$ and continuously on $\overline{\mathbf{R}^2_+} \setminus \{0\}$, and satisfies

$$\|E^*\|_{L^1(\partial\Omega)} + \|H^*\|_{L^1(\partial\Omega)} \leq C\|A\|_{H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)} \tag{8.1}$$

for some positive constant C depending only on k and Ω.

A similar result is valid for the interior Maxwell boundary value problem $(\mathcal{M}_i)$ provided k is not a Maxwell eigenvalue for Ω. In this latter case, A, f must also satisfy finitely many necessary linear conditions and the solution to $(\mathcal{M}_i)$ is unique modulo a finite-dimensional space.

Proof. First we address the necessity part of the theorem and, for simplicity, assume that we deal with the interior problem. To this end, we claim that if a divergencefree vector field E satisfies $(\Delta + k^2)E = 0$ in Ω and has E^*, $(\operatorname{curl} E)^* \in L^1(\partial\Omega)$, then $n \times E \in H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$.

To see the claim, we note that it is possible to select $f \in H^{1,1}_{\mathrm{at}}(\partial\Omega)$ such that the vector field

$$u := E + k^2 L_0 E + k^2 \mathcal{S}_0(nf)$$

is divergencefree in Ω. Indeed, $\operatorname{div} u = k^2 \mathcal{S}_0(\langle n, E\rangle) - k^2 \mathcal{K}_0 f$ and, since $\langle n, E\rangle \in H^1_{\mathrm{at}}(\partial\Omega)$, the results in [20] may be used to conclude. Since u is harmonic and has u^*, $(\operatorname{curl} u)^* \in L^1(\partial\Omega)$, Theorem 8.3 gives that $n \times u \in H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$. In particular, this shows that $n \times E \in H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$, and this completes the proof of the necessity part in the theorem.

The proof of the sufficiency follows the lines of the proof of Theorem 6.1; therefore, we shall be brief. Since the smoothing operator $n \times \mathcal{S}^2_0$ maps $H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$ compactly into $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$, using Theorem 8.1 **viii** and proceeding as before, it follows that $-\frac{1}{2}I + M_k + i\eta\, N_k(n \times \mathcal{S}^2_0)$ is an isomorphism of $H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$. This gives the existence part in $(\mathcal{M}_e)$ with boundary data $A \in H^{1,\mathrm{Div}}_{\tan}(\partial\Omega)$ (and

$f := -\frac{1}{k}\operatorname{Div} A \in H^1_{at}(\partial\Omega))$. Uniqueness follows from the above claim, Green's formula, and Theorem 8.1. □

Without further proof, as a corollary of the previous theorem, we also note the following result.

Theorem 8.5.

For each fixed $k \in \mathbf{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$, boundary value problem (BVP_1) *is solvable for $p = 1$ if and only if $h \in H^{1,1}_{at}(\partial\Omega)$ and $A \in H^{1,\mathrm{Div}}_{tan}(\partial\Omega)$. The solution is unique, depends on k analytically in $\mathbf{R}^2_+$ and continuously on $\overline{\mathbf{R}^2_+} \setminus \{0\}$, and satisfies naturally accompanying estimates.*

Remark. Using the results in [4] and proceeding analogously as in this section, it is possible to extend the results presented here to Hardy-based spaces of order less than one (depending on the Lipschitz character of the domain). □

9. L^p Theory for the Nonregularity Boundary Value Problem for the Maxwell System

In this section we study the electric boundary value problems ($\mathcal{E}_{i,e}$) stated in §1. In fact, we take the opportunity to deal with the somewhat more general problems that we formulate below. This extends a similar result proved by Calderón in [8] in which case the domain has a C^3 boundary.

Theorem 9.1.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$, and let $k \in \mathbf{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$.

I. *Then there exists $\epsilon = \epsilon(\Omega) > 0$ such that for each $2 - \epsilon \leq p \leq 2 + \epsilon$ the boundary value problem*

$$\begin{cases} E \in C^\infty(\mathbf{R}^3 \setminus \overline{\Omega}), \\ (\Delta + k^2)E = 0 \quad \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ E^*,\ (\operatorname{div} E)^* \in L^p(\partial\Omega), \\ (\operatorname{div} E)|_{\partial\Omega} = f \in L^p(\partial\Omega), \\ n \times E|_{\partial\Omega} = A \in L^p_{tan}(\partial\Omega), \\ E \text{ satisfies the radiation condition} \end{cases} \tag{BVP$_9$}$$

is solvable.

II. *The solution is unique, depends continuously on k in the indicated range (the principle of limiting absorption), and satisfies*

$$\|E^*\|_{L^p(\partial\Omega)} + \|(\operatorname{div} E)^*\|_{L^p(\partial\Omega)} \leq C(\|f\|_{L^p(\partial\Omega)} + \|A\|_{L^p(\partial\Omega)}) \tag{9.1}$$

for some $C = C(p, k, \Omega) > 0$.

III. *The following regularity results are true: $\operatorname{div} E = 0$ in $\mathbf{R}^3 \setminus \overline{\Omega}$ if and only if $f = 0$ in which situation* (BVP_9) *reduces to ($\mathcal{E}_e$) (stated in §1); also, $(\operatorname{curl} E)^* \in L^p(\partial\Omega)$ if and only if $A \in L^{p,\mathrm{Div}}_{tan}(\partial\Omega)$, and $(\operatorname{curl}\operatorname{curl} E)^* \in L^p(\partial\Omega)$ if and only if $f \in W^{1,p}(\partial\Omega)$. Moreover, there are natural accompanying estimates in each case.*

IV. *If the boundary $\partial\Omega$ has topological genus zero, then the unique solution $E = E_k$ of* (BVP_9) *converges uniformly on compact subsets in $\mathbf{R}^3 \setminus \overline{\Omega}$ to E_0, the unique solution of the boundary value problem*

$$\begin{cases} \Delta E_0 = 0 \quad \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \\ E_0^*,\ (\operatorname{div} E_0)^* \in L^p(\partial\Omega), \\ (\operatorname{div} E_0)|_{\partial\Omega} = f \in L^p(\partial\Omega), \\ n \times E_0|_{\partial\Omega} = A \in L^p_{tan}(\partial\Omega), \\ \int_{\partial\Omega} \langle n, E_0 \rangle \, d\sigma = 0, \\ E_0 \text{ satisfies the radiation condition.} \end{cases} \tag{BVP$_{10}$}$$

V. *Similar results hold true for the interior version of the above boundary value problem (when, of course, the radiation condition is dropped). More specifically, one has existence, uniqueness and the estimate* (9.1) *as long as k is neither a Maxwell eigenvalue nor a Laplace eigenvalue with homogeneous Dirichlet boundary condition for* Ω *(e.g.,* $k \in \mathbf{C}$ *with* $\operatorname{Im} k > 0$ *will do). In this latter situation, existence holds if and only if the boundary data satisfy finitely many linear conditions, whereas uniqueness holds only modulo a finite-dimensional linear space.*

VI. *Finally, in the class of Lipschitz domains this theorem is sharp. If, however, the domain* Ω *has a* C^1 *boundary, then the same results are in fact valid for each* $1 < p < \infty$.

Compared with Theorem 6.1, the new difficulties in proving this result arise from the fact that no assumptions on the behavior of curl E (i.e., the equivalent of the magnetic field H) to the boundary are made. The main idea in the proof of the uniqueness part is to use the well-posedness of the *regular* Maxwell boundary value problem together with the continuous dependence of the solution on the boundary of the domain. In this regard, we shall first prove two lemmas that are of importance for us.

Consider first Ω an arbitrary, fixed Lipschitz domain and introduce

$$Tf(X) := \text{p.v.} \int_{\partial\Omega} (\nabla\Phi_0)(X-Y) f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega. \tag{9.2}$$

Recall the approximating sequence $\Omega_j \uparrow \Omega$ described in Lemma 2.2 and set T_j for the operator similar to (9.2) corresponding to $\partial\Omega_j$.

Lemma 9.2.

With the above notation, for each $1 < p < \infty$ *and each* $f \in L^p(\partial\Omega)$ *one has*

$$\lim_{j\to\infty} \|[T_j(f \circ \Lambda_j^{-1})] \circ \Lambda_j - Tf\|_{L^p(\partial\Omega)} = 0.$$

Proof. Since $\sup\{\|T_j\|\}_j < +\infty$ and $\nabla\Phi_0 = \frac{\partial\Phi_0}{\partial n} n - n \times (n \times \nabla\Phi_0)$ on $\partial\Omega$, it actually suffices to show that

$$\lim_{j\to\infty} \|(T_j^l h) \circ \Lambda_j - T^l h\|_{L^p(\partial\Omega)} = 0, \qquad l = 1, 2, \tag{9.3}$$

for any Lipschitz continuous function h in $\mathbf{R}^3$, where $\{T^l\}_{l=1,2}$ are the operators corresponding to the kernels $\frac{\partial\Phi_0}{\partial n}$ and $n \times \nabla\Phi_0$, respectively.

The case of T^1 was treated in [93, p. 586]. The idea to handle T^2 is to integrate by parts, i.e., to use (2.2) to write that

$$\int_{\partial\Omega_j} (n_j \times \nabla)\Phi_0\, h\, d\sigma_j = -\int_{\partial\Omega_j} \Phi_0\, (n_j \times \nabla) h\, d\sigma_j$$

$$= -\int_{|X_j - Y_j| \geq \delta} \Phi_0\, (n_j \times \nabla) h\, d\sigma_j - \int_{|X_j - Y_j| \leq \delta} \Phi_0\, (n_j \times \nabla) h\, d\sigma_j =: I + II.$$

Now the singularity in I has been eliminated, whereas $|II| \leq C\delta \|\nabla h\|_{L^\infty}$. With this at hand, the conclusion easily follows. □

Lemma 9.3.

Let Ω *be a bounded Lipschitz domain in* $\mathbf{R}^3$ *and recall the mappings* Λ_j *introduced in Lemma* 2.2. *Assume that the kernel* $K(X, Y)$ *is continuous and such that* $|K(X,Y)| \leq C|X-Y|^{-1}$ *for* $X \neq Y$, $X, Y \in \mathbf{R}^3$.

Then, if $\{f_j\}_j$ is a sequence of functions in $L^p(\partial\Omega)$ that converges to zero weakly in $L^p(\partial\Omega)$, for some $1 < p < \infty$ we have that

$$\int_{\partial\Omega} K(\Lambda_j(X), \Lambda_j(Y)) f_j(Y)\, d\sigma(Y) \to 0 \quad \text{in } L^p(\partial\Omega).$$

Proof. Fix $\epsilon > 0$ and for each $X \in \partial\Omega$ decompose the domain of integration into $\{Y \in \partial\Omega;\ |X - Y| \geq \epsilon\}$ and $\{Y \in \partial\Omega;\ |X - Y| \leq \epsilon\}$. The first resulting term is easily seen to converge to zero in $L^p(\partial\Omega)$ by Lebesgue's dominated convergence theorem. Also, for the second term, Schur's test (or interpolation between L^1 and L^∞) yields

$$\left\| \int_{\partial\Omega} K(\Lambda_j(X), \Lambda_j(Y)) f_j(Y) \chi_{|X-Y|\leq\epsilon}\, d\sigma(Y) \right\|_{L^p(\partial\Omega)}$$
$$\leq C \left(\sup_X \int_{|X-Y|\leq\epsilon} \frac{d\sigma(Y)}{|X-Y|} \right) \sup_j \|f_j\|_{L^p(\partial\Omega)}.$$

The L^p norm of the left-hand side is $\leq C\epsilon$, and since ϵ is arbitrary, the lemma follows. □

We now turn to the proof of Theorem 9.1.

Proof of Theorem 9.1. Consider first the exterior problem. As in the proof of Theorem 6.2, we look for a solution E expressed in the form

$$E := \operatorname{curl} \mathcal{S}_k B + i\eta \operatorname{curl}\operatorname{curl} \mathcal{S}(n \times S_0^2 B) + \mathcal{S}_k(n\, g) - ik^{-2}\eta \nabla \mathcal{S}_k S_0^2 g,$$

in $\mathbf{R}^3 \setminus \overline{\Omega}$, but this time for some $B \in L^p_{\tan}(\partial\Omega)$ and $g \in L^p(\partial\Omega)$. Based on the usual jump relations, this choice leads to the question of inverting the operators $-\frac{1}{2}I + M_k + i\eta N_k(n \times S_0^2)$ and $-\frac{1}{2}I + K_k - i\eta S_k S_0^2$ on $L^p_{\tan}(\partial\Omega)$ and $L^p(\partial\Omega)$, respectively. The fact that they are Fredholm with zero index on the indicated spaces follows from Theorems 5.3 and 5.4, whereas injectivity is seen from **vi** in Theorem 5.3. The existence part follows. Also, the estimate (9.1) is a direct consequence of the explicit integral representation formula of the solution and the results in [12].

To prove uniqueness, we first claim that if $\Omega_j \downarrow \Omega$ is a sequence of smooth approximating domains (as in Lemma 2.2), then there exists $C_0 > 0$, independent of j, such that

$$\|(-\tfrac{1}{2}I + M_{k,j} + i\eta N_{k,j}(n_j \times S^2_{0,j}))^{-1} B\|_{L^p(\partial\Omega_j)} \leq C_0 \|B\|_{L^p(\partial\Omega_j)} \tag{9.4}$$

for each j and any $B \in L^p_{\tan}(\partial\Omega_j)$ and such that

$$\|(-\tfrac{1}{2}I + K_{k,j} - i\eta S_{k,j} S^2_{0,j})^{-1} g\|_{L^p(\partial\Omega_j)} \leq C_0 \|g\|_{L^p(\partial\Omega_j)} \tag{9.5}$$

for each j and any $g \in L^p(\partial\Omega_j)$ (recall that we are assuming $2 - \epsilon \leq p \leq 2 + \epsilon$).

Let us accept the claim for the moment and assume that E is a solution of the homogeneous version of (BVP$_9$). Also, fix $P \in \mathbf{R}^3 \setminus \overline{\Omega}$. Then, it follows from Theorem 6.2 that

$$|E(P)| \leq C(P, p, k, \partial\Omega, C_0)(\|\operatorname{div} E\|_{L^p(\partial\Omega_j)} + \|n_j \times E\|_{L^p(\partial\Omega_j)}). \tag{9.6}$$

On account of Lebesgue's dominated convergence theorem that the right-hand side of the above estimate tends to zero as $j \to \infty$, $E(P) = 0$, and since P was arbitrary, this concludes the proof of the uniqueness (modulo the claim).

Returning to the proof of the claim, we only give a brief outline of the proof of (9.4), as (9.5) is similar and somewhat easier. Furthermore, to lighten the exposition we assume that $p = 2$. Reasoning by contradiction, let us suppose that the uniform invertibility claim is violated. Then there exists a sequence $\{A_j\}_j$ with $A_j \in L^2_{\tan}(\partial\Omega_j)$ and such that $\|A_j\|_{L^2(\partial\Omega_j)} = 1$ for each j, whereas

$$\|(-\tfrac{1}{2}I + M_{k,j} + i\eta N_{k,j}(n_j \times S^2_{0,j}))A_j\|_{L^2(\partial\Omega_j)} \to 0$$

as $j \to \infty$. To obtain a contradiction, we proceed in a sequence of steps.

First, it is not difficult to show that there exists $A \in L^2_{\tan}(\partial\Omega)$ such that $A_j \circ \Lambda_j \to A$, weakly in $L^2(\partial\Omega)$ as $j \to \infty$.

Second, we claim that $(-\frac{1}{2}I + M_k + i\eta N_k(n \times S_0^2))A = 0$. To prove the claim we shall need that $M^*_{k,j}(n_j \times F|_{\partial\Omega_j}) \circ \Lambda_j \to M^*_k(n \times F|_{\partial\Omega})$ in $L^2(\partial\Omega)$ for any vector-valued Lipschitz function F in $\mathbf{R}^3$. This is readily seen by using Lemma 9.2. Then an argument based on this, duality, and density yields the claim.

Finally, using (5.25), for some $C = C(\partial\Omega) > 0$ independent of j we may write

$$1 = \|A_j\|_{L^2(\partial\Omega_j)} \leq C\|(-\tfrac{1}{2}I + M_{k,j} + i\eta N_{k,j}(n_j \times S^2_{0,j}))A_j\|_{L^2(\partial\Omega_j)} + \|R_j A_j\|_{L^2(\partial\Omega_j)}$$
$$=: I + II,$$

where $\{R_j\}$ are some weakly singular integral operators (with kernels as in Lemma 9.3). Now $I \to 0$ by hypothesis, while $II \to 0$ by (repeated) applications of Lemma 9.3. This leads to an obvious contradiction, hence the uniform invertibility claim follows.

In the case in which $f = 0$, we infer from the uniqueness part in (BVP$_9$) applied to the field $E - \operatorname{curl}\mathcal{S}_k[(-\frac{1}{2}I + M_k + i\eta N_k(n \times S_0^2))^{-1}A]$ that $\operatorname{div} E$ should necessarily vanish identically in $\mathbf{R}^3 \setminus \overline{\Omega}$. Also, the fact that $A \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ implies $(\operatorname{curl} E)^* \in L^p(\partial\Omega)$ is immediate from Theorem 6.1 and the uniqueness part for (BVP$_9$). Other regularity claims may be seen from Theorem 6.2.

The proof of the limiting absorption principle (point **IV** in the theorem) is similar to the proof of Theorem 6.6, but of course this time one needs to use **xix** in Theorem 5.1.

The discussion of the interior boundary value problem is similar (cf. also the proof of Theorem 6.1) and we omit it. □

Remark. Other versions of the results presented in this section are possible. For instance, there is an analogous form of Theorem 9.1 in which the boundary condition $n \times E = A \in L^p(\partial\Omega)$ is replaced by $\langle n, E\rangle = f \in L^p_0(\partial\Omega)$ (cf. **vii** in Theorem 5.3). □

10. Nonhomogeneous Boundary Value Problems for the Decoupled Maxwell Equations

Here we consider nonhomogeneous boundary value problems for the the equations of static electromagnetism, i.e., the two problems in which the Maxwell problem decouples for $k = 0$ (see, for instance, [24, Vol. I, pp. 87–91; 80]).

Our first theorem deals with the system of electrostatics.

Theorem 10.1.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then there exists a small, positive $\epsilon = \epsilon(\Omega)$ such that if $\frac{3}{2} - \epsilon \leq p \leq 2 + \epsilon$, then a necessary and sufficient set of conditions for the boundary value problem

$$\begin{cases} u \in L^p(\Omega), \\ \operatorname{div} u = q \in W^{-1,p}(\Omega), \\ \operatorname{curl} u = j \in L^p(\Omega), \\ n \wedge u = A \in L^p(\partial\Omega) \end{cases} \qquad (\mathrm{BVP}_{11})$$

to be solvable is

$$\begin{cases} \operatorname{div} j = 0 \text{ in } \Omega, \\ A \in L^p_{\tan}(\partial\Omega), \\ \operatorname{Div} A = -n \cdot j \in W^{-\frac{1}{p},p}(\partial\Omega). \end{cases} \qquad (*)$$

In addition, the solution is unique and satisfies

$$\|u\|_{L^p(\Omega)} \leq C\left(\|q\|_{W^{-1,p}(\Omega)} + \|j\|_{L^p(\Omega)} + \|A\|_{L^p(\partial\Omega)}\right) \qquad (10.1)$$

for some $C = C(p, \Omega) > 0$.

Moreover, if actually $q \in L^p(\Omega)$, then the unique solution of (BVP$_{11}$) *also satisfies $n \cdot u \in L^p(\partial\Omega)$ and the estimate*

$$\|u\|_{L^p(\Omega)} + \|n \cdot u\|_{L^p(\partial\Omega)} \le C\left(\|q\|_{L^p(\Omega)} + \|j\|_{L^p(\Omega)} + \|A\|_{L^p(\partial\Omega)}\right) \tag{10.2}$$

for some $C = C(p, \Omega) > 0$.

If, however, $\partial\Omega \in C^1$, then all the above are valid for $1 < p < \infty$.

As it will be apparent from the proof of this theorem, if the conditions $(*)$ are fulfilled and if $q \in L^p(\Omega)$, then for each $1 < p \le 2 + \epsilon$ there exists a solution u of (BVP$_{11}$) for which $\|u\|_{L^p(\Omega)} \le C(\|q\|_{L^p(\Omega)} + \|j\|_{L^p(\Omega)} + \|A\|_{L^p(\partial\Omega)})$.

Proof of Theorem 10.1. Clearly $\operatorname{div} j = \operatorname{div} \operatorname{curl} u = 0$. Also, by Lemma 4.1, we have that A belongs to $L^p_{\tan}(\partial\Omega)$ and

$$\operatorname{Div} A = \operatorname{Div}(n \wedge u) = -n \cdot \operatorname{curl} u = -n \cdot j \in W^{-\frac{1}{p},p}(\partial\Omega).$$

This shows the necessity of the conditions $(*)$ (in fact for any $1 < p < \infty$).

Conversely, assume that the conditions $(*)$ hold and let $\epsilon = \epsilon(\Omega) > 0$ be as in Theorem 5.1. First we consider the case when $q \in L^p(\Omega)$ for which we shall prove existence in the range $1 < p \le 2 + \epsilon$. We look for u expressed in the form

$$u := -\operatorname{curl} L_0 j + \nabla L_0 q + \operatorname{curl} \mathcal{S}_0 A + \operatorname{curl} \mathcal{S}_0 B, \tag{10.3}$$

for some $B \in L^{p,0}_{\tan}(\partial\Omega)$ to be specified later. Clearly, $u \in L^p(\Omega)$, $\operatorname{div} u = q$, and $\operatorname{curl} u = j + \nabla\mathcal{S}_0(n \cdot j) + \nabla\mathcal{S}_0(\operatorname{Div} A) = j$ by $(*)$. Also, the boundary condition $n \wedge u = A$ amounts to

$$-n \wedge \operatorname{curl} L_0 j + n \wedge \nabla L_0 q + (\tfrac{1}{2}I + M_0)B + (\tfrac{1}{2}I + M_0)A = A.$$

Thus, based on **xii** in Theorem 5.1, this boundary integral equation is solvable for $B \in L^{p,0}_{\tan}(\partial\Omega)$, $1 < p \le 2 + \epsilon$, if and only if

$$-n \wedge \operatorname{curl} L_0 j + n \wedge \nabla L_0 q + (-\tfrac{1}{2}I + M_0)A \in L^{p,0}_{\tan}(\partial\Omega). \tag{10.4}$$

Indeed, by Lemma 4.1, we see that $n \wedge \nabla L_0 q \in L^{p,0}_{\tan}(\partial\Omega)$. Furthermore, $\operatorname{curl}\operatorname{curl} L_0 j = (-\Delta + \nabla\operatorname{div})L_0 j = -j - \nabla\mathcal{S}_0(n \cdot j)$; hence, we may infer that

$$\begin{aligned}\operatorname{Div}(-n \wedge \operatorname{curl} L_0 j) &= -(\tfrac{1}{2}I + K_0^*)(n \cdot j) = (\tfrac{1}{2}I + K_0^*)(\operatorname{Div} A)\\ &= -\operatorname{Div}[(-\tfrac{1}{2}I + M_0)A]\end{aligned}$$

in $W^{-\frac{1}{p},p}(\partial\Omega)$. Thus (10.4) follows and the proof of the existence part for $q \in L^p(\Omega)$ is therefore complete. Also, the fact that the solution u constructed above has $n \cdot u \in L^p(\partial\Omega)$ and satisfies the estimate (10.2) is seen from the integral representation (10.3) of u.

In the proof of uniqueness, let us assume that u solves (BVP$_{11}$) for $q = 0$, $j = 0$, $A = 0$. By Theorem 3.2 we have that $u = \nabla v$, where $v := -\mathcal{S}_0(n \cdot u) \in W^{1,p}(\Omega)$. Hence, $(n \times \nabla)\gamma(v) = n \wedge \nabla v = n \wedge u = 0$. Consequently, eventually after subtracting a suitable constant from v, we may assume that $v \in W^{1,p}_0(\Omega)$ and the conclusion follows from Theorem 5.5.

Finally, in the case when $q \in W^{-1,p}(\Omega)$, we apply the above considerations to the vector field $u - \nabla u'$, where $u' \in W^{1,p}_0(\Omega)$ is the (unique) solution of $\Delta u' = q$ (cf. the Theorem 5.5). □

Next, we treat the nonhomogeneous boundary value problem for the system of magnetostatics.

Theorem 10.2.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^3$ with $g(\partial\Omega) = 0$. Then there exists $\epsilon = \epsilon(\Omega) > 0$ such that, for each $\frac{3}{2} - \epsilon \le p \le 2 + \epsilon$, a necessary and sufficient set of conditions for the boundary value problem

$$\begin{cases} u \in L^p(\Omega), \\ \operatorname{div} u = q \in (W^{1,p'}(\Omega))^*, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \\ \operatorname{curl} u = j \in L^p(\Omega), \\ n \cdot u = f \in W^{-\frac{1}{p},p}(\partial\Omega), \end{cases} \tag{BVP$_{12}$}$$

to be solvable is that

$$\begin{cases} \operatorname{div} j = 0 \quad in\ \Omega, \\ \langle q, 1\rangle = \langle f, 1\rangle. \end{cases} \tag{$**$}$$

Also, the solution is unique and satisfies

$$\|u\|_{L^p(\Omega)} \le C\left(\|q\|_{(W^{1,p'}(\Omega))^*} + \|j\|_{L^p(\Omega)} + \|f\|_{W^{-\frac{1}{p},p}(\partial\Omega)}\right) \tag{10.5}$$

for some positive constant $C = C(p, \Omega)$.

Moreover, if $f \in L^p(\partial\Omega)$ and $q \in L^p(\Omega)$, then the unique solution u of (BVP$_{12}$) *also has $n \wedge u \in L^p_{\tan}(\partial\Omega)$ and*

$$\|n \wedge u\|_{L^p(\partial\Omega)} \le C\left(\|q\|_{L^p(\Omega)} + \|j\|_{L^p(\Omega)} + \|f\|_{L^p(\partial\Omega)}\right). \tag{10.6}$$

If, in fact, $\partial\Omega \in C^1$, then we may take $1 < p < \infty$.

Proof. The necessity of $(**)$ is easily checked, so we assume that $\epsilon = \epsilon(\Omega) > 0$ is as in Theorem 5.1 and discuss sufficiency. To begin with, uniqueness follows from the simple connectivity of the domain Ω and the uniqueness for the Laplace equation $\Delta v = 0$, $v \in W^{1,p}(\Omega)$, $\frac{3}{2} - \epsilon \le p \le 2 + \epsilon$, with homogeneous Neumann boundary conditions.

To prove existence, we first treat the case when $q \in L^p(\Omega)$ and look for u expressed in the form

$$u := -\operatorname{curl} L_0 j + \nabla L_0 q - \operatorname{curl} L_0(\operatorname{curl} \mathcal{S}_0 A) - \nabla L_0(\operatorname{div} \mathcal{S}_0 A) + \mathcal{S}_0 A + \nabla \mathcal{S}_0 g, \tag{10.7}$$

where the scalar-valued function $g \in W_0^{-\frac{1}{p},p}(\partial\Omega)$ and the vector field $A \in W^{-\frac{1}{p},p}(\partial\Omega)$ are to be chosen later. Note that, by Lemma 3.1, $u \in L^p(\Omega)$ and that $\operatorname{div} u = q$.

First, we recall from **vi** in Theorem 5.1 that $\langle n, \operatorname{curl} S_0\rangle$ is one-to-one from $V_{\tan}^{-s,p}(\partial\Omega)$ onto $W_0^{-s,p}(\partial\Omega)$ for each $2 - \epsilon \le p \le 2 + \epsilon$ and $0 \le s \le 1$. In particular, for $s = \frac{1}{p}$, there exists $A \in V_{\tan}^{-\frac{1}{p},p}(\partial\Omega) \subseteq W^{-\frac{1}{p},p}(\partial\Omega)$ such that $n \cdot \operatorname{curl} S_0 A = -n \cdot j \in W_0^{-\frac{1}{p},p}(\partial\Omega)$ and

$$\|A\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \le C\|j\|_{L^p(\Omega)} \tag{10.8}$$

for some $C = C(p, \Omega) > 0$. With this choice of A, straightforward calculation based on integrations by parts gives that $\operatorname{curl} u = j$ for any $g \in W^{-\frac{1}{p},p}(\partial\Omega)$.

Therefore, we are left with checking the boundary condition for u. This leads to the following boundary integral equation for $g \in W^{-\frac{1}{p},p}(\partial\Omega)$:

$$\begin{aligned} (-\tfrac{1}{2}I + K_0^*)g = f &+ n \cdot \operatorname{curl} L_0 j - n \cdot \nabla L_0 q + n \cdot \operatorname{curl} L_0(\operatorname{curl} \mathcal{S}_0 A) \\ &+ n \cdot \nabla L_0(\operatorname{div} \mathcal{S}_0 A) - n \cdot S_0 A. \end{aligned} \tag{10.9}$$

It is not difficult to verify that the second compatibility condition in $(**)$ is equivalent to the fact that the distribution in the right-hand side of (10.9) actually annihilates constants so that it actually belongs to $W_0^{-\frac{1}{p},p}(\partial\Omega)$. Thus, the solvability of (10.9) is a consequence of Theorem 5.4. Furthermore, the estimate (10.5) follows from the corresponding estimate for g, (10.8), and (10.7).

In the case in which the boundary datum f actually belongs to $L^p(\partial\Omega)$, then (10.9) is in fact solvable for g in $L_0^p(\partial\Omega)$. Everything then follows from (10.7).

Finally, for a general $q \in (W^{1,p'}(\Omega))^*$, we apply the same pattern of reasoning to the field $u - \nabla u'$, where $u' \in W^{1,p}(\Omega)$ is the unique solution to the Poisson equation with data q and homogeneous Neumann boundary conditions (cf. Theorem 5.5). □

Remark. Relying on the results of [42] and using the explicit integral representation formulas for the solutions of (BVP_{11}), (BVP_{12}), it is possible to obtain sharp Sobolev–Besov interior estimates for these solutions (the case $p = 2$ is treated in detail later). □

Next we present a useful corollary of Theorems 10.1 and 10.2. For $p = 2$ and Ω simply connected, this has also been observed in [16] with a different proof. Recall first the Sobolev–Besov type spaces (cf., e.g., [1])

$$B_\alpha^{p,q}(\Omega) := [L^p(\Omega), W^{1,p}(\Omega)]_{\alpha,q},$$

obtained by real interpolation for $1 < p, q < \infty$, $0 < \alpha < 1$, and

$$L_\theta^p(\Omega) := [L^p(\Omega), W^{1,p}(\Omega)]_\theta,$$

obtained by complex interpolation for $1 < p < \infty$ and $0 < \theta < 1$.

Corollary 10.3.

Let Ω be an arbitrary bounded Lipschitz domain in $\mathbf{R}^3$, and consider a vector field $u \in L^p(\Omega)$ such that $\operatorname{div} u \in L^p(\Omega)$ and $\operatorname{curl} u \in L^p(\Omega)$ for some $1 < p < \infty$. Then there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that the following hold.

a. *If $\frac{3}{2} - \epsilon \le p \le 2 + \epsilon$ and $n \wedge u \in L^p(\partial\Omega)$, then also $n \cdot u \in L^p(\partial\Omega)$ and*

$$\|n \cdot u\|_{L^p(\partial\Omega)} \le C\left(\|n \wedge u\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\Omega)} + \|\operatorname{div} u\|_{L^p(\Omega)} + \|\operatorname{curl} u\|_{L^p(\Omega)}\right)$$

for some $C = C(\partial\Omega, p) > 0$.

b. *If $\frac{3}{2} - \epsilon \le p \le 2 + \epsilon$ and $n \cdot u \in L^p(\partial\Omega)$, then also $n \wedge u \in L^p(\partial\Omega)$ and*

$$\|n \wedge u\|_{L^p(\partial\Omega)} \le C\left(\|n \cdot u\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\Omega)} + \|\operatorname{div} u\|_{L^p(\Omega)} + \|\operatorname{curl} u\|_{L^p(\Omega)}\right)$$

for some $C = C(\partial\Omega, p) > 0$.

c. *If both $n \cdot u \in L^p(\partial\Omega)$ and $n \wedge u \in L^p(\partial\Omega)$, then for $1 < p \le 2$ we have that $u \in B_{1/p}^{p,2}(\Omega)$, whereas for $2 \le p < \infty$ we have $u \in L_{1/p}^p(\Omega)$. In particular, for $p = 2$ we have that $u \in W^{\frac{1}{2},2}(\Omega)$. Furthermore, there are accompanying estimates in each case.*

d. *In the case when $\partial\Omega \in C^1$, we may take $1 < p < \infty$ in* **a** *and* **b**.

Proof. If the topological genus of $\partial\Omega$ were zero, then the points **a** and **b** would immediately follow from Theorem 10.1 and Theorem 10.2. However, the general case can be readily reduced to the one above by using a partition of unity subordinated to an open, finite cover $\{U_j\}_j$ of $\overline{\Omega}$ such that, for each j, $U_j \cap \Omega$ is a starlike bounded Lipschitz domain.

Finally, the last point is seen from the integral representation formula in Theorem 3.2 and the results in [42]. □

11. Maxwell Eigenvalues and Poles

Let Ω be a fixed domain in $\mathbf{R}^3$. The new phenomenon which occurs when the wave number k is real is that the interior Maxwell boundary value problem may not be solvable or that the solution may not be unique (see [68] for such an example when the domain is a ball). In this section we consider the problem of characterizing the the collection of all wave numbers having this property. We also study the (larger) set of k's for which the operators $\pm\frac{1}{2}I + M_k$ are not invertible on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ or $L^2_{\tan}(\partial\Omega)$. Our central result in this respect is the following theorem.

Theorem 11.1.

Let Ω be a bounded, Lipschitz domain in $\mathbf{R}^3$; and let $\epsilon > 0$ be as in Theorem 5.7. For $k \in \mathbf{C}$ consider the following assertions.

i. $-\frac{1}{2}I + M_k$ *is not injective on* $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for* $p = 2$ *or, more generally, for some* $2 - \epsilon \le p < \infty$.

ii. $\frac{1}{2}I + M_k$ *is not injective on* $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ *for* $p = 2$ *or, more generally, for some* $2 - \epsilon \le p < \infty$.

iii. $-\frac{1}{2}I + M_k$ *is not injective on* $L^{p}_{\tan}(\partial\Omega)$ *for* $p = 2$ *or, more generally, for some* $2 - \epsilon \le p < \infty$.

iv. $\frac{1}{2}I + M_k$ *is not injective on* $L^{p}_{\tan}(\partial\Omega)$ *for* $p = 2$ *or, more generally, for some* $2 - \epsilon \le p < \infty$.

v. *There exists a pair of divergencefree vector fields (E, H) that are smooth and not identically zero in Ω and such that*

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \Omega, \\ \operatorname{curl} H + ikE = 0 & \text{in } \Omega, \\ E^*,\ H^* \in L^p(\partial\Omega), \\ n \times E = 0 & \text{on } \partial\Omega, \\ \langle n, H \rangle = 0 & \text{on } \partial\Omega \end{cases}$$

for $p = 2$ or, more generally, for some $2 - \epsilon \le p < \infty$.

vi. *There exist two divergencefree vector fields $E, H \in L^p(\Omega)$, not identically zero in Ω and such that*

$$\begin{cases} \operatorname{curl} E - ikH = 0 & \text{in } \Omega, \\ \operatorname{curl} H + ikE = 0 & \text{in } \Omega, \\ n \wedge E = 0 & \text{on } \partial\Omega, \\ n \cdot H = 0 & \text{on } \partial\Omega \end{cases}$$

for $p = 2$ or, more generally, for some $2 - \epsilon \le p < \infty$.

Then **i–iv** *are equivalent and so are* **v–vi**. *Also, the last two assertions imply the first four and, in fact, for* $\operatorname{Im} k \ge 0$ *all six assertions are equivalent.*

The set of values $\{\zeta_j\}_j$ of k's belonging to the lower half plane $\operatorname{Im} k < 0$ for which the assertions **i–iv** *are fulfilled is discrete and symmetric about the imaginary axis. Also, the mappings $k \mapsto (\pm\frac{1}{2}I + M_k)^{-1}$ have meromorphic continuations to the lower half plane and the numbers $\{\zeta_j\}_j$ coincide precisely their poles in $\operatorname{Im} k < 0$.*

The set of all $k \in \mathbf{C}$ for which the conditions **v** *and* **vi** *are satisfied is of the form $\{\pm k_j\}_{j=1}^{\infty}$, where $k_j \in \mathbf{R}$, $k_j \ge 0$ for each j, and $\lim_j |k_j| = \infty$. Furthermore, for each j, the pairs of vector fields (E, H) as in* **v** *or as* **vi** *corresponding to $k = k_j$ form a finite-dimensional subspace of $W^{\frac{1}{2},2}(\Omega) \times W^{\frac{1}{2},2}(\Omega)$.*

If $\partial\Omega \in C^1$, then we may take $1 < p < \infty$.

The numbers $\{\zeta_j\}_j \subseteq \{k \in \mathbf{C};\ \operatorname{Im} k < 0\}$ (for $p = 2$) will be referred to as *Maxwell poles* for Ω, whereas the numbers $\{\pm k_j\}_j$ are called *Maxwell eigenvalues* for Ω. Note that zero is a Maxwell eigenvalue for Ω if and only if Ω ismultiply connected. Also, the pairs of vector fields (E, H) as in **vi** for $p = 2$ are called *Maxwell eigenfields* corresponding to the eigenvalue k (cf. also [24, Vol. III, p. 265], where such eigenvalues are called eigenpulsations, and the corresponding eigenfields eigenmodes).

First we record an important consequence of the above theorem.

Corollary 11.2.

The points **i**, **ii**, **iv**, *and* **v** *in Theorem* 5.3 *are also valid for any k in the lower half plane that is not a Maxwell pole for Ω.*

Before we present the proof of Theorem 11.1, recall the Corollary 10.3. In fact, its converse is also true and is contained in our next theorem. For the convenience of the exposition we state it and prove it for $p = 2$, as we shall actually use it later, although appropriate versions are valid for p near 2 (cf. the results in [42]).

Theorem 11.3.

Let Ω be an arbitrary bounded, Lipschitz domain in $\mathbf{R}^3$; and let $u \in W^{\frac{1}{2},2}(\Omega)$ be a vector field such that $\operatorname{div} u \in L^2(\Omega)$ *and* $\operatorname{curl} u \in L^2(\Omega)$. *Then $n \cdot u \in L^2(\partial\Omega)$ and $n \wedge u \in L^2(\partial\Omega)$. Moreover, there holds the estimate*

$$\begin{aligned}\|u\|_{W^{\frac{1}{2},2}(\Omega)} \leq C \min\{\|n \wedge u\|_{L^2(\partial\Omega)}, \|n \cdot u\|_{L^2(\partial\Omega)}\} \\ + C\|\operatorname{curl} u\|_{L^2(\Omega)} + C\|\operatorname{div} u\|_{L^2(\Omega)},\end{aligned} \tag{11.1}$$

where the constant C depends exclusively on the Lipschitz character of Ω.

If $u \in L^2(\Omega)$ has $\operatorname{div} u \in L^2(\Omega)$, $\operatorname{curl} u \in L^2(\Omega)$, *and $\Delta u \in L^2(\Omega)$, then $u^* \in L^2(\partial\Omega)$ if and only if $n \cdot u \in L^2(\partial\Omega)$ or $n \wedge u \in L^2(\partial\Omega)$.*

If Ω is actually convex or $\partial\Omega \in C^2$, then we can replace $W^{\frac{1}{2},2}(\Omega)$ by $W^{1,2}(\Omega)$ (cf. [80, 24]), but as simple examples show, in general this is not true for arbitrary Lipschitz domains. Thus, one may think of (11.1) as the natural version of (1.3) for arbitrary Lipschitz domains. We also remark that this estimate leads to an improvement in (5.2).

Perhaps the main difficulty in proving the above theorem is the lack of a trace theorem from $W^{\frac{1}{2},2}(\Omega)$ into $L^2(\partial\Omega)$ (since, in fact, $C^\infty_{\text{comp}}(\Omega)$ is densely embedded into $W^{\frac{1}{2},2}(\Omega)$). Instead, we shall rely on a basic result due to Fabes, Jodeit, Lewis and Jerison, Kenig, which we now recall.

Theorem 11.4.

Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$ and u a harmonic function in Ω. Then $u^* \in L^2(\partial\Omega)$ if and only if $u \in W^{\frac{1}{2},2}(\Omega)$.

Proofs of this theorem based on the area theorem of Dahlberg [18] can be found in [26, 42] (cf. also [40]). However, we would like to take this opportunity to fill in a gap that occurs in [26, pp. 68–69], when estimating the term

$$I := \int_0^\infty t \iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\cap\{|x-y|>\eta t\}} \frac{|v(x,t) - v(y,t)|^2}{|x-y|^{n+1}} dx\, dy\, dt$$

(we want to thank Fabes and Brown for also calling this to our attention). This is because, unfortunately, the Besov–Sobolev type estimate

$$\int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} \frac{|v(x,t) - v(y,t)|^2}{|x-y|^{(n-1)+2s}} dx\, dy \leq C_s \int_{\mathbf{R}^{n-1}} |\nabla^s v(x,t)|^2 dx$$

breaks down for the critical exponent $s = 1$. To circumvent this, using a trace theorem and a change of variables we may write

$$I \leq C \int_0^\infty \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} \frac{|v(x,t)-v(y,t)|^2}{|x-y|^n} dx\,dy\,dt$$

$$\leq C \int_0^\infty \|v(\cdot,t)\|^2_{W^{\frac{1}{2},2}(\mathbf{R}^{n-1})} dt \leq C \int_0^\infty \|v(\cdot,t+\cdot)\|^2_{W^{1,2}(\mathbf{R}^n_+)} dt$$

$$\leq C \int_0^\infty \int_0^\infty \int_{\mathbf{R}^{n-1}} \left\{|v(x,t+\lambda)|^2 + |(\nabla v)(x,t+\lambda)|^2\right\} dx\,dt\,d\lambda$$

$$\leq C \int_0^\infty \int_{\mathbf{R}^{n-1}} r \left\{|v(x,r)|^2 + |(\nabla v)(x,r)|^2\right\} dx\,dr.$$

With this at hand, the proof proceeds as in [26].

An essentially well-known corollary of this theorem and the results in [93] is that the single-layer potential operator $\mathcal{S}_0$ maps $L^2(\partial\Omega)$ boundedly into $W^{\frac{3}{2},2}(\Omega)$.

Proof of Theorem 11.3. Since $u \in W^{\frac{1}{2},2}(\Omega)$, we have by Theorem 3.2 that u can be written in the form $u = v + w$, where $v \in W^{1,2}(\Omega)$ and w is a harmonic vector-valued function in Ω. It follows that $w \in W^{\frac{1}{2},2}(\Omega)$ and, by Theorem 11.4, $w^* \in L^2(\partial\Omega)$. Together with the results in [19], this implies that both $n \cdot u$ and $n \wedge u$ belong to $L^2(\partial\Omega)$. The estimate (11.1) follows directly from the integral representation formula in Theorem 3.2 and the corresponding estimates in Corollary 10.3

To see the last statement of the theorem, we observe that the right-to-left implication is an immediate consequence of Theorem 11.4 (applied to $u - L_0 u$) as soon as we have proved that $u \in W^{\frac{1}{2},2}(\Omega)$. However, this easily follows from Corollary 10.3, the integral representation formula in Theorem 3.2, and the regularity of the operators $L_0, \mathcal{S}_0$. The left-to-right implication is essentially well known, and this completes the proof of the theorem. □

We are now ready to present the proof of Theorem 11.1.

Proof of Theorem 11.1. To start, we remark that it suffices to consider only the case $p = 2$. Indeed, since $\pm\frac{1}{2}I + M_k$ is invertible on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ if $2 - \epsilon \leq p \leq 2 + \epsilon$ and $\operatorname{Im} k > 0$, for **i** and **ii** this follows from an argument based on the fractional integration theorem and iteration (which has been used in, for example, the proof of Theorem 6.1), whereas for **iii** and **iv** the second part of Theorem 5.3 may be invoked. Finally, for **v** and **vi** this follows from Green's formula (3.2), Theorem 10.1, Theorem 10.2, and, once again, repeated applications of the fractional integration theorem.

Now **vi** in Theorem 5.3 gives that, in fact, **iii** and **iv** are equivalent to **i** and **ii**, respectively. Also, (5.27) together with Theorem 5.3 readily imply that **iii** and **iv** are equivalent. Hence, at this point we have proved that the conditions **i**–**iv** are equivalent.

Next, obviously, **v** implies **vi**. To see the opposite implication we note that if (E, H) are as in **vi** (for $p = 2$), then Theorem 11.3 gives that $E^*, H^* \in L^2(\partial\Omega)$; in other words, (E, H) are as in **v** (for $p = 2$). To show that **v** implies **i** we reason by contradiction. Suppose that E and H are as in **vi** (with $p = 2$) and write the Green formula (3.2) for E in Ω. Applying curl to both sides, going to the boundary, and taking $n\times$, we finally arrive at $(-\frac{1}{2}I + M_k)(n \times H) = 0$. Since we assume that $-\frac{1}{2}I + M_k$ is injective on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$, this yields $n \times H = 0$. Thus, if $k \neq 0$ we may write $\langle n, E\rangle = \frac{1}{ik}\mathrm{Div}\,(n \times H) = 0$, and from Green's formula we see that E and H vanish identically in Ω. The same conclusion is reached in the case in which $k = 0$. This contradiction yields the implication **v** $\to$ **i**. Thus, the last two conditions are equivalent and imply any of the first four.

Assume now that $\operatorname{Im} k \geq 0$ and that **ii** holds, i.e., $\frac{1}{2}I + M_k$ is not injective on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. Let $A \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$, $A \neq 0$, such that $(\frac{1}{2}I + M_k)A = 0$; and set $E := \mathcal{S}_k A$, $H := \frac{1}{ik}\mathrm{curl}\,E$ in $\mathbf{R}^3 \setminus \partial\Omega$. We claim that E and H cannot vanish identically in Ω. Indeed, since $n \times H$ does not jump across

$\partial\Omega$, the opposite assumption would lead to the conclusion that $(-H, E)$ is a radiating solution of the homogeneous Maxwell exterior boundary value problem and, by Theorem 6.1, we infer that E, H vanish identically in the exterior of Ω as well. Now $A = n \times E|_{\partial\Omega_+} - n \times E|_{\partial\Omega_-} = 0$, contradicting the initial assumption. Consequently, the claim follows, so that **ii** implies **v**. Thus, for $\operatorname{Im} k \geq 0$ all six conditions are equivalent.

Let us now turn our attention to the second part of the theorem and deal first with the Maxwell eigenvalues of Ω. It has been proved in [64] that there are no Maxwell eigenvalues in the upper half plane $\operatorname{Im} k > 0$, and combining this with the observation that if (E, H) satisfies **v** for some $k \in \mathbf{C}$ then $(E, -H)$ satisfies **v** for $-k$, we infer that the collection of all Maxwell eigenvalues is a subset of the real axis. Symmetry with respect to the origin also follows from the above observation. Furthermore, the fact that all eigenfields belong to $W^{\frac{1}{2},2}(\Omega)$ also follows from the above discussion and Theorem 11.3 (cf. also Corollary 10.3).

To see that these eigenvalues form a discrete subset of $\mathbf{R}$, we fix a complex number k_0 with a strictly positive imaginary part and for each $k \in \mathbf{C}$ write

$$\tfrac{1}{2}I + M_k = (\tfrac{1}{2}I + M_{k_0})[I - (\tfrac{1}{2}I + M_{k_0})^{-1}(M_k - M_{k_0})].$$

The important thing is that $\mathbf{C} \ni k \mapsto \mathcal{A}(k) := (\frac{1}{2}I + M_{k_0})^{-1}(M_k - M_{k_0})$ is an analytic application into the Banach space of all bounded operators on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ and takes on values compact operators on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. Also, $\mathcal{A}(0) = 0$. By the analytic Fredholm theorem (cf., e.g., [44]) we infer that $I - \mathcal{A}(k)$ has a bounded inverse on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ except at isolated poles in $\mathbf{C}$, which, in fact, are the poles of the meromorphic function $(I - \mathcal{A}(k))^{-1}$. Thus, the conclusion follows.

We now prove the finite dimensionality of the space of Maxwell eigenfields corresponding to a fixed Maxwell eigenvalue k for Ω. To this effect, let $V_k := \operatorname{span}\{(E, H);\ (E, H) \text{ as in } \mathbf{v}\}$, so we need to show that

$$\dim V_k < \infty. \tag{11.2}$$

This can be accomplished in several ways. For instance, we may use Theorem 6.1 or (11.1) and Rellich's selection theorem to infer (11.2). Another solution, which actually gives more, can be observed as follows. Recall from the remark at the end of the proof of Theorem 6.1 that $\{n \times H;\ (E, H) \in V_k\}$ coincides with the null space of the operator $-\frac{1}{2}I + M_k$ acting on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. The idea is that, by Theorem 5.3, the operator $-\frac{1}{2}I + M_k$ is Fredholm and, hence, has a finite-dimensional null space. Since, using Green's formula (3.2), it is easy to check that $V_k \ni (E, H) \mapsto n \times H \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ is injective, (11.2) follows.

Finally, we discuss the structure of the set of Maxwell poles of Ω. Discreteness follows from the analytic Fredholm theorem mentioned above and so does the meromorphic continuation of $(\pm\frac{1}{2}I + M_k)^{-1}$ to the lower half plane. We are left with showing symmetry about the imaginary axis. However, this is a simple consequence of the fact that $\overline{M_k A} = M_{-\bar{k}}\overline{A}$ for any $k \in \mathbf{C}$ and $A \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$. $\square$

Let Ω be an arbitrary bounded Lipschitz domain in $\mathbf{R}^3$. Because of **vi** in Theorem 11.1, perhaps a more illuminating way of looking at the Maxwell eigenvalues of Ω is via the Maxwell operator

$$\begin{pmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{pmatrix}.$$

For $1 < p < \infty$ we consider the closed subspace

$$\mathcal{H}_p := \{(E, H) \in L^p(\Omega) \times L^p(\Omega);\ \operatorname{div} E = \operatorname{div} H = 0,\ n \cdot H = 0\}$$

of $L^p(\Omega) \times L^p(\Omega)$. Set

$$\mathcal{D}(\mathcal{M}_p) := \{(E, H) \in \mathcal{H}_p;\ \mathrm{curl}\, E,\ \mathrm{curl}\, H \in L^p(\Omega),\ n \wedge E = 0\}.$$

Then $\mathcal{D}(\mathcal{M}_p)$ is densely included into $\mathcal{H}_p$ and $\mathcal{M}_p : \mathcal{D}(\mathcal{M}_p) \to \mathcal{H}_p$ defined by $\mathcal{M}_p(E, H) := (-i \operatorname{curl} H, i \operatorname{curl} E)$ becomes a closed, unbounded operator on $\mathcal{H}_p$ with domain $\mathcal{D}(\mathcal{M}_p)$ for any $1 < p < \infty$.

First we claim that for $2-\epsilon \le p, q \le 2+\epsilon$, $\frac{1}{p}+\frac{1}{q} = 1$, one has that $\mathcal{H}_p^* = \mathcal{H}_q$ and $\mathcal{M}_p^* = \mathcal{M}_q$. This follows from the Hahn–Banach extension theorem and Theorem 5.6. In particular, $\mathcal{M}_2$ is a selfadjoint operator on the Hilbert space $\mathcal{H}_2$ (cf. also [24, Vol. III, p. 267]). Consequently $\sigma(\mathcal{M}_2; \mathcal{H}_2)$, the spectrum of the operator $\mathcal{M}_2$ on $\mathcal{H}_2$, is real. Note that $\mathcal{M}_2$ is not lower semibounded and that $\sigma(\mathcal{M}_2; \mathcal{H}_2)$ accumulates both at $+\infty$ and $-\infty$. In fact, it is a simple corollary of the block structure of the operator $\mathcal{M}_2$ that its spectrum is symmetric with respect to the origin. Note that if $g(\partial\Omega) = 0$, then we also have that $0 \notin \sigma(\mathcal{M}_2; \mathcal{H}_2)$.

Next, we note that $\mathcal{D}(\mathcal{M}_2)$ equipped with the usual graph norm is compactly embedded into $\mathcal{H}_2$. This follows immediately from Corollary 10.3 and the classical Rellich selection theorem (but also from more familiar compactness results as in, e.g., [53, 95, 94, 72]). It follows that $\mathcal{M}_2$ has a compact resolvent, i.e., $(\lambda - \mathcal{M}_2)^{-1}$ is a compact operator on $\mathcal{H}_2$ for each $\lambda \notin \sigma(\mathcal{M}_2; \mathcal{H}_2)$. In particular (cf., e.g., [24, Vol. III, Theorem 6, p. 38]), we see that $\sigma(\mathcal{M}_2; \mathcal{H}_2)$ is nonempty, discrete, and accumulates only at infinity. Also, $\sigma(\mathcal{M}_2; \mathcal{H}_2)$ contains only eigenvalues of $\mathcal{M}_2$; the corresponding eigenspaces are finite dimensional, pairwise orthogonal, and span $\mathcal{H}_2$.

Furthermore, proceeding as in the proof of the Theorem 11.1, it is not difficult to check that much of the above analysis remains true for the operator $\mathcal{M}_p$ for $2 - \epsilon \le p \le 2 + \epsilon$. In fact, one has that $\sigma(\mathcal{M}_p; \mathcal{H}_p) = \sigma(\mathcal{M}_2; \mathcal{H}_2)$ for each $2 - \epsilon \le p \le 2 + \epsilon$. Our final remark is that (E, H) is as in **vi** of Theorem 11.1 if and only if $(E, H) \in \mathcal{D}(\mathcal{M}_p)$ and $\mathcal{M}_p(E, H) = k\,(E, H)$.

Summarizing, we have proved the following theorem.

Theorem 11.5.

Let Ω be an arbitrary bounded Lipschitz domain in $\mathbf{R}^3$. Then there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that, for each $2 - \epsilon \le p \le 2 + \epsilon$, the spectrum $\sigma(\mathcal{M}_p; \mathcal{H}_p)$ consists precisely of all Maxwell eigenvalues of Ω. If $\partial\Omega \in C^1$, then we may take $1 < p < \infty$.

In particular, we remark that this theorem shows that there are actually Maxwell eigenvalues.

We conclude this section by discussing the nonhomogeneous boundary value problem for the Maxwell system in C^1 and Lipschitz domains in $\mathbf{R}^3$.

Theorem 11.6.

Let Ω be a bounded, Lipschitz domain in $\mathbf{R}^3$; and assume that $k \in \mathbf{C} \setminus \{0\}$ with $\operatorname{Im} k \ge 0$ is not a Maxwell eigenvalue for Ω. Then there exists $\epsilon = \epsilon(\partial\Omega) > 0$ such that, for each $2-\epsilon \le p \le 2+\epsilon$, a necessary and sufficient condition for the boundary value problem

$$\begin{cases} E, H \in L^p(\Omega), \\ \operatorname{curl} E - ikH = K \in L^p(\Omega), \\ \operatorname{curl} H + ikE = J \in L^p(\Omega), \\ n \wedge E = A \in L^p(\partial\Omega) \end{cases} \qquad (\mathrm{BVP}_{13})$$

to be solvable is that $A \in L^p_{\tan}(\partial\Omega)$ has $\operatorname{Div} A \in W^{-\frac{1}{p},p}(\partial\Omega)$. Moreover, the solution is unique and there exists $C = C(p, k, \Omega) > 0$ such that

$$\|E\|_{L^p(\Omega)} + \|H\|_{L^p(\Omega)} \le C(\|K\|_{L^p(\Omega)} + \|J\|_{L^p(\Omega)} + \|A\|_{L^p(\partial\Omega)} + \|\operatorname{Div} A\|_{W^{-\frac{1}{p},p}(\partial\Omega)}).$$

If $\partial\Omega \in C^1$ we may take $1 < p < \infty$.

Our result generalizes the problem that was considered in [24] (cf. the problem (4.98), p. 92 in Vol. I and Remark 4, p. 259 in Vol. III) where variational methods are used to obtain a similar result in the case in which the domain Ω is smooth, $p = 2$, $K = 0$, and $A = 0$. See also [95].

Proof of Theorem 11.6. The necessity of $A \in L^p_{\tan}(\partial\Omega)$ and $\operatorname{Div} A \in W^{-\frac{1}{p},p}(\partial\Omega)$ follows directly from Lemma 4.1. To show that this is also sufficient for the solvability of (BVP_{13}), we use the Hodge-type decompositions for vector fields in Ω discussed in Theorem 5.6. More concretely, we may write

$$K = -ik^{-1}\nabla v + w, \qquad J = -ik^{-1}\nabla v' + w',$$

where v, w and v', w' are as **ii** and **i** of Theorem 5.6, repectively. Then, working with $E - \nabla v'$ and $H - \nabla v$ in place of E and H, respectively, it follows that it suffices to prove the solvability of (BVP_{13}) for K and J replaced by w and w', respectively. Note that the pair (w', w) belongs to the space $\mathcal{H}_p$ (introduced before the statement of Theorem 11.5). Thus, by Theorem 11.5 and the fact that k is not a Maxwell eigenvalue for Ω, it follows that we can further reduce the problem to that of solving (BVP_{13}) for $K = 0$, $J = 0$, and $A \in L^p(\partial\Omega)$. Consequently, introducing $E := \operatorname{curl} \mathcal{S}_k B$ and $H := k^2 \mathcal{S}_k B + \nabla \mathcal{S}_k(\operatorname{Div} B)$, for some $B \in L^p_{\tan}(\partial\Omega)$ with $\operatorname{Div} B \in W^{-\frac{1}{p},p}(\partial\Omega)$, everything follows from **v** in Theorem 5.3 with $s = \frac{1}{p}$ and from Lemma 3.1.

Uniqueness is immediate from the fact that k is not a Maxwell eigenvalue for Ω and Theorem 11.1. $\square$

12. An Application to Inverse Electromagnetic Scattering

Let $k \in \mathbf{C} \setminus \{0\}$ be a fixed wave number, and consider the incident electromagnetic plane waves

$$E_k^{\mathrm{inc}}(X; d, p) := ik^{-1} \operatorname{curl} \operatorname{curl} \left(p\, e^{ik\langle X, d\rangle}\right), \qquad X \in \mathbf{R}^3;$$

$$H_k^{\mathrm{inc}}(X; d, p) := \operatorname{curl} \left(p\, e^{ik\langle X, d\rangle}\right), \qquad X \in \mathbf{R}^3.$$

Here $d \in S^2 \subseteq \mathbf{R}^3$ describes the direction of propagation, and $p \in \mathbf{R}^3$ gives the polarization.

Let Ω be a fixed, bounded Lipschitz obstacle in $\mathbf{R}^3$. For each d, p we have the incident wave $(E_k^{\mathrm{inc}}(\cdot; d, p), H_k^{\mathrm{inc}}(\cdot; d, p))$ and, by the theory for the (say, L^2) direct problem (developed in §6), we have a unique radiating solution (E, H) corresponding to the exterior Maxwell boundary value problem for Ω (see also the discussion in the first section). Going further, to the first component of this radiating solution (E, H) there corresponds the electric far-field pattern E_∞ (cf. §6).

In fact, a similar set-up is valid for the exterior electric boundary value problem ($\mathcal{E}_e$) (cf. §9).

The inverse obstacle problem in inverse electromagnetic scattering is concerned with the determination of the shape of the scatterer Ω from the knowledge of the electric far-field patterns. Excellent surveys of recent progress in inverse problems can be found in, for example, [89, 90, 36]. In this section we shall address the uniqueness part of this inverse obstacle problem. Specifically, we shall prove the following (compare with the main result in [49] where $\partial\Omega \in C^2$).

Theorem 12.1.

Assume that Ω_1 and Ω_2 are two bounded scatterers with Lipschitz boundaries in $\mathbf{R}^3$ such that, for a fixed wave number $k \in \mathbf{C} \setminus \{0\}$, $\operatorname{Im} k \geq 0$, the electric far-field patterns of the solutions for the corresponding exterior Maxwell boundary value problems coincide for a countable set $\{d_j\}_j \subseteq S^2$ of distinct incident directions and for three linearly independent polarizations. Then $\Omega_1 = \Omega_2$.

In fact, a similar statement is valid considering the electric far-field patterns of the solutions for the corresponding exterior electric boundary value problems.

Proof. We first consider the case of the exterior electric boundary value problems. As in [49] (which builds on some earlier work in [35]), the idea is to reason by contradiction and then to construct an incident field (or, rather, a sequence of incident fields) diffracted by Ω_1, Ω_2 in the same way but giving rise to boundary data on $\partial\Omega_1$ and $\partial\Omega_2$ of disproportionate size. This and our results on the well-posedness of the direct problem in the context of Lipschitz domains and rough boundary data then yield a contradiction (note that what one prescribes on the boundary is $-n \times E_k^{\rm inc}$, which has no smoothness as $n \in L^\infty(\partial\Omega)$ only).

Since the far-field pattern depends linearly on the polarization p and analytically on the direction d, we may assume that the electric far-field patterns corresponding to Ω_1 and Ω_2 coincide for each $p \in \mathbf{R}^3$ and each $d \in S^2$.

Let G denote the unbounded component of $\mathbf{R}^3 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ and fix an arbitrary point $X_0 \in G$. Also, fix $p_0 \in \mathbf{R}^3 \setminus \{0\}$ and let $\mathcal{E}_j$ be the (unique) solution of the exterior electric boundary value problem in $\mathbf{R}^3 \setminus \overline{\Omega_j}$ with boundary data $n_j \times \operatorname{curl}\operatorname{curl}(p_0\Phi_k(\cdot - X_0))$ on $\partial\Omega_j$, $j = 1, 2$. We claim that

$$\mathcal{E}_1 = \mathcal{E}_2 \quad \text{in } G. \tag{12.1}$$

To see this, we note that by the well-posedness of the direct problem (Theorem 9.1) and the fact that the boundary data depend analytically on X_0 in G, it follows that $\mathcal{E}_1$ and $\mathcal{E}_2$ depend analytically on X_0 in G. Hence, it suffices to show that they coincide in G for $X_0 \in G$ with $|X_0|$ very large. To this effect, let D be a bounded, Lipschitz domain in $\mathbf{R}^3$ with $\overline{\Omega_1} \cup \overline{\Omega_2} \subseteq D$, $X_0 \notin D$, and such that k is not a Maxwell eigenvalue for D nor is k^2 an eigenvalue for the Laplace operator on D with homogeneous Dirichlet boundary condition. It is then essentially well known that

$$V_k := \operatorname{span}\left\{\operatorname{curl}\operatorname{curl}\left(p\, e^{ik\langle\cdot,d\rangle}\right);\ p \in \mathbf{R}^3,\ d \in S^2\right\}$$

is such that $n \times V_k$ is densely included into $L^2_{\tan}(\partial D)$. Let $\mathcal{V}_l \in V_k$ be such that

$$n \times \mathcal{V}_l \to n \times ik^{-1}\operatorname{curl}\operatorname{curl}(p_0\,\Phi_k(\cdot - X_0)) \quad \text{as } l \to \infty$$

in $L^2_{\tan}(\partial D)$. Hence, by Theorem 9.1,

$$\mathcal{V}_l \to ik^{-1}\operatorname{curl}\operatorname{curl}(p_0\,\Phi_k(\cdot - X_0)) \quad \text{as } l \to \infty \tag{12.2}$$

uniformly on compact subsets of D. Let $\mathcal{V}^s_{l,j}$ be the corresponding scattered electric fields for the incident electric fields $\mathcal{V}_l$ in the exterior domains $\mathbf{R}^3 \setminus \overline{\Omega_j}$, $j = 1, 2$. Note that, in particular, for each l

$$n_j \times \mathcal{V}^s_{l,j} = n_j \times \mathcal{V}_l \quad \text{on } \partial\Omega_j,\ j = 1, 2. \tag{12.3}$$

Since each $\mathcal{V}_l$ is a superposition of plane waves, the hypotheses of the theorem, the classical Rellich lemma, and unique continuation results allow us to infer that

$$\mathcal{V}^s_{l,1} = \mathcal{V}^s_{l,2} \quad \text{in } G. \tag{12.4}$$

Now, using (12.3) and (12.2) it follows that

$$n_j \times \mathcal{V}^s_{l,j} \to n_j \times ik^{-1}\operatorname{curl}\operatorname{curl}(p_0\,\Phi_k(\cdot - X_0)) = n_j \times \mathcal{E}_j, \tag{12.5}$$

as $l \to \infty$, for $j = 1, 2$. Thus, once again by the well-posedness of the electric exterior boundary value problem, we have that $\mathcal{V}^s_{l,j}$ converges to $\mathcal{E}_j$ as $l \to \infty$, uniformly on compact subsets of $\mathbf{R}^3 \setminus \overline{\Omega_j}$, $j = 1, 2$. Then the claim (12.1) is provided by this observation and (12.4).

Next, reasoning by contradiction and assuming that $\Omega_1 \neq \Omega_2$, there is no loss of generality to assume that $\Omega_1 \setminus \Omega_2$ is not empty. Pick now $X_0 \in \partial G \cap \partial\Omega_1$, $\delta > 0$ such that $B_{2\delta}(X_0)$ does not intersect $\overline{\Omega_2}$; and select a sequence $X_l \in G$ with $X_l \to X_0$ as l goes to infinity. Finally, let $\mathcal{E}_{l,j}$, $j = 1, 2$, be the two solutions to the exterior electric boundary value problems in $\mathbf{R}^3 \setminus \Omega_j$ with boundary data $n_j \times \operatorname{curl}\operatorname{curl}(p_0\,\Phi_k(\cdot - X_l))$, $j = 1, 2$.

From the first part of the proof we know that $\mathcal{E}_{l,1} = \mathcal{E}_{l,2}$ in G. In particular, this implies

$$\begin{aligned}\|n_1 \times \mathcal{E}_{l,1}\|_{L^\infty(B_{2\delta}(X_0)\cap\partial\Omega_1)} &\le \|\mathcal{E}_{l,2}\|_{L^\infty(B_{2\delta}(X_0))} \\ &\le C\|n_2 \times \operatorname{curl}\operatorname{curl}(p_0\Phi_k(\cdot - X_0))\|_{L^{2,\mathrm{Div}}_{\tan}(\partial\Omega_2)} \\ &\le C < +\infty.\end{aligned}$$

If we now recall that $n_1 \times \mathcal{E}_{l,1} = n_1 \times \operatorname{curl}\operatorname{curl}(p_0\Phi_k(\cdot - X_l))$ we see, by letting $l \to \infty$, that the above estimate implies

$$\|n_1 \times \operatorname{curl}\operatorname{curl}(p_0\Phi_k(\cdot - X_0))\|_{L^\infty(B_{2\delta}(X_0)\cap\partial\Omega_1)} \le C < +\infty,$$

which is a contradiction.

Finally, the case of the Maxwell exterior boundary value problem follows from the above discussion and Theorem 9.1 as the all incident electromagnetic plane waves induce sufficiently regular boundary data. This concludes the proof of the Theorem 12.1. □

13. Some Open Problems and Questions

Here we present some possible directions for further research on related topics.

1. Although we have confined our attention only to the three-dimensional case, part of the results presented here is valid in the higher dimensional case as well. In fact, a L^2 theory for the Maxwell equations in arbitrary Lipschitz domains in $\mathbf{R}^m$ has been developed in [39] (cf. also [61]). A natural question is to consider the optimal L^p theory in the higher dimensional case too.

2. For numerical purposes, it is important to have estimates for the spectral radius of the operator M_0 on the spaces $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$, $L^{2,0}_{\tan}(\partial\Omega)$, $L^2_{\tan}(\partial\Omega)$, where Ω is a Lipschitz domain in $\mathbf{R}^3$. It has been shown in [64] that the spectral radius of M_0 acting on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ is $< \frac{1}{2}$ if the domain Ω is convex; so, in particular, $(\pm\frac{1}{2}I + M_0)^{-1}$ can be expanded in a strongly convergent Neumann series. More progress in the location of the spectrum of the operator M_0 for a general Lipschitz domain has also been made in [59] (cf. also the remark at the end of §5 and the results in [84] for the Laplacian). However, the general question as to whether the spectral radius of M_0 on any of the above mentioned spaces is $< \frac{1}{2}$ for an arbitrary Lipschitz domain remains open. The situation is even less clear for the operator M_k with $k \neq 0$.

3. An important direction is that of extending the techniques devised here to the case of nonhomogeneous, imperfectly conducting, and anisotropic media as well as to other types of boundary conditions like, for instance, for impedance, conductive, resistive, interface, or transmission problems.

4. One may expect a finer analysis of the Maxwell system (like, for instance, asymptotics near singularities, spectral radius and eigenvalue estimates, and semi-Fredholmness) if more specific information about the geometry of the Lipschitz domain is available. For practical purposes it is particularly important to carry out such an analysis for, for example, polyhedral domains or domains with conical singularities (cf. [58]).

5. Shortly after settling a famous problem raised by Hilbert at the turn of the century regarding the asymptotics of the spectrum of the Laplacian [96], Weyl also established [97] the principal term of the asymptotic expansion for the distribution function of the eigenvalues of the Maxwell operator $\mathcal{M}$ (cf. §10). Specifically, introducing

$$N(\lambda) := \sum_{0<k_j<\lambda} 1,$$

then if $\partial\Omega \in C^\infty$ Weyl showed that

$$N(\lambda) = (3\pi^2)^{-1}\operatorname{vol}(\Omega)\,\lambda^3\,(1 + o(1)) \quad \text{as } \lambda \to +\infty. \tag{13.1}$$

As alluded to in the introductory section, a major ingredient in Weyl's proof was the fact that for a smooth domain the inequality (1.3) is valid. However, we have seen that for an arbitrary Lipschitz domain $W^{1,2}(\Omega)$ needs to be replaced by $W^{\frac{1}{2},2}(\Omega)$; and, in general, this is best possible. Nonetheless, this difficulty has been overcome in [6] (cf. also [7]; we are indebted to W. Littman for pointing out these papers to us), where it has been proved that (13.1) continues to hold for arbitrary Lipschitz domains as well.

The next natural step is to try to the determine a more precise form of the residue in (13.1), i.e., the second term of the asymptotic expansion of $N(\lambda)$. If the domain Ω is smooth, it has been shown in [79] that $o(1)$ can be replaced by $\mathcal{O}(\lambda^{-1})$, but the situation for general Lipschitz domains remains wide open.

6. A problem that is interesting both in its own right and for its potential applications (to, for example, inverse scattering) is that of establishing estimates for the first (positive) Maxwell eigenvalue of a Lipschitz domain Ω in $\mathbf{R}^3$. A solution has been announced in [43], but unfortunately it is flawed; hence, the problem is still open. The error occurs because of the failure of the estimate

$$\iint_{\mathbf{R}^3\setminus\overline{\Omega}} |\nabla u|^2\, dV \geq \iint_{\Omega} |\nabla u|^2\, dV \tag{13.2}$$

for arbitrary Lipschitz domains Ω in $\mathbf{R}^3$ and $u := \mathcal{S}_0 f$, with arbitrary $f \in L^2(\partial\Omega)$. Indeed, (13.2) would imply

$$-2\int_{\partial\Omega} \mathcal{S}_0 f K_0^* f\, d\sigma = \iint_{\mathbf{R}^3\setminus\overline{\Omega}} |\nabla u|^2\, dV - \iint_{\Omega} |\nabla u|^2\, dV \geq 0;$$

and, since $-\int_{\partial\Omega} f\, \mathcal{S}_0 f\, d\sigma = \iint_{\mathbf{R}^3} |\nabla u|^2\, dV \geq 0$, it would follow that all eigenvalues of the operator K_0^* on $L^2(\partial\Omega)$ are positive. However, this can be checked to be false on certain (even smooth) domains Ω.

7. A natural conjecture is that of the existence of infinitely many Maxwell poles for each bounded Lipschitz domain in $\mathbf{R}^3$ (cf. the case of acoustic scattering).

8. Consider the inverse spectral problem for the Maxwell operator. Can two nonisometric obstacles in $\mathbf{R}^3$ be isospectral with respect to the Maxwell operator $\mathcal{M}$? More concretely, if two bounded (say, Lipschitz) domains Ω_1, Ω_2 in $\mathbf{R}^3$ are such that their associated sequences of Maxwell eigenvalues coinscide, decide what metric characteristics should coincide.

9. A longstanding open problem is to determine if only one incoming plane wave for one single direction and one fixed wave number determines the scattering obstacles completely (compare with Theorem 12.1). Let us point out that in the electromagnetic scattering theory this is not known even if some additional a priori information is available (cf. [15, Theorem 5.2, p. 107]).

10. It is likely that the techniques we have developed so far are useful for dealing with other systems of equations on Lipschitz domains (M. Taylor, personal communication). For instance, the results in §10 could be used in connection with the variational treatment of the linearized Navier–Stokes system as in [33] or [88].

Another important problem is to extend these results to the higher dimensional setting (e.g., to produce Hodge type decompositions for L^p differential forms on nonsmooth domains).

11. Recall that we have always assumed that our domains have connected boundary and that for a number of results, this assumption was of crucial importance. However, it is interesting both from a theoretical and from a practical points of view to fully develop the corresponding theory for domains with disconnected boundaries as well. In fact, it is reasonable to expect that a substantial part of our analysis can be extended to this situation (for the case of the Laplacian on smooth domains see, e.g., [32]).

12. The behavior of the electromagnetic fields (E, H) solving $(\mathcal{M}_e)$ as the wave number k ($k \in \mathbf{C}$, $\operatorname{Im} k \geq 0$) approaches zero was established under the additional hypothesis that the Lipschitz domain Ω has $g(\partial\Omega) = 0$. Nonetheless, it would be quite desirable to clarify this issue in the case of Lipschitz domains with arbitrary genus as well.

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