The Journal of Fourier Analysis and Applications
Volume 3, Number 2, 1997

The Banach Algebra A^* and Its Properties

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ABSTRACT. Beurling's algebra $A^* = \{f : \sum_{k=0}^{\infty} \sup_{k\le|m|} |\hat{f}(m)| < \infty\}$ is considered. A^* arises quite naturally in problems of summability of the Fourier series at Lebesgue points, whereas Wiener's algebra A of functions with absolutely convergent Fourier series arises when studying the norm convergence of linear means. Certainly, both algebras are used in some other areas. A^* has many properties similar to those of A, but there are certain essential distinctions. A^* is a regular Banach algebra, its space of maximal ideals coincides with $[-\pi,\pi]$, and its dual space is indicated. Analogs of Herz's and Wiener–Ditkin's theorems hold. Quantitative parameters in an analog of the Beurling–Pollard theorem differ from those for A. Several inclusion results comparing the algebra A^* with certain Banach spaces of smooth functions are given. Some special properties of the analogous space for Fourier transforms on the real axis are presented. The paper ends with a summary of some open problems.

1. Introduction

1.1.

The algebra that we are going to consider is closely related to Wiener's algebra A, which is well studied (see, e.g., Kahane's book [K]). In what follows we will consider continuous functions f on $\mathbf{R} = (-\infty, \infty)$ that are the (inverse) Fourier transforms of integrable functions $\hat{f}$:

$$f(x) = (2\pi)^{-1} \int_{\mathbf{R}} \hat{f}(u) e^{ixu}\, du.$$

As is well known, not every continuous function is such. Frequently the (direct) Fourier transform $\hat{f}$ can be reconstructed from f as

$$\hat{f}(u) = \int_{\mathbf{R}} f(x) e^{-iux}\, dx.$$

So in the case of $\mathbf{R}$ the Wiener algebra is defined by

Math Subject Classifications. Primary 42A28, 42A24, 42A16
Keywords and Phrases. Banach algebra, absolute convergence, synthesis
Acknowledgements and Notes. The research of the first author was supported by the Israeli Ministry of Science and the Arts through the Ma'agara program for absorption of immigrant mathematicians at the Technion Israel Institute of Technology. The second author acknowledges the support of the Minerva Foundation in Germany through the Emmy Noether Mathematics Institute in Bar-Ilan University. The authors thank H. Feichtinger and G. Zimmermann for helpful discussions and valuable remarks. We are indebted to the referee for various remarks which definitely made the presentation better.

ISSN 1069-5869

$$\mathbf{A}(\mathbf{R}) = \left\{ f : \|f\|_{\mathbf{A}(\mathbf{R})} = \int\limits_{\mathbf{R}} |\hat{f}(u)|\, du < \infty \right\}.$$

In the case of 2π-periodic continuous functions we have

$$\mathbf{A}(\mathbf{T}) = \left\{ f : \|f\|_{\mathbf{A}(\mathbf{T})} = \sum_{m=-\infty}^{\infty} |\hat{f}(m)| < \infty \right\},$$

where $\mathbf{T} = [-\pi, \pi)$ and

$$\hat{f}(m) = (2\pi)^{-1} \int\limits_{\mathbf{T}} f(u) e^{-imu}\, du$$

is the mth Fourier coefficient of f. This means that integrability of the Fourier transform and absolute convergence of the sequence of Fourier coefficients, respectively, define these spaces. Let us introduce the regularized integrability of a function and the regularized absolute convergence of a number series as

$$L^*(\mathbf{R}) = \left\{ g : ||f||_{L^*(\mathbf{R})} = \int\limits_0^{\infty} \operatorname*{ess\,sup}_{x \le |u| < \infty} |g(u)|\, dx < \infty \right\},$$

$$l^* = \left\{ d = \{d_k\} : ||d||_{l^*} = \sum_{k=0}^{\infty} \sup_{k \le |m| < \infty} |d_m| < \infty \right\},$$

and consider two spaces of continuous functions

$$\mathbf{A}^*(\mathbf{R}) = \left\{ f : \|f\|_{\mathbf{A}^*(\mathbf{R})} = ||\hat{f}||_{L^*(\mathbf{R})} = \int\limits_0^{\infty} \operatorname*{ess\,sup}_{x \le |u| < \infty} |\hat{f}(u)|\, dx < \infty \right\}, \tag{1}$$

$$\mathbf{A}^*(\mathbf{T}) = \left\{ f : \|f\|_{\mathbf{A}^*(\mathbf{T})} = ||\hat{f}(m)||_{l^*} = \sum_{k=0}^{\infty} \sup_{k \le |m| < \infty} |\hat{f}(m)| < \infty \right\}. \tag{2}$$

The norm $\|f\|_{\mathbf{A}^*(\mathbf{T})}$ is equivalent to

$$\sum_{k=0}^{\infty} 2^k \sup_{2^k \le |m| < 2^{k+1}} |\hat{f}(m)|. \tag{3}$$

It will be very convenient to compare properties of $\mathbf{A}^*$ with the corresponding properties of $\mathbf{A}$. Evidently a function from $\mathbf{A}^*$ belongs also to $\mathbf{A}$. While $\mathbf{A}$ arises naturally when studying the norm convergence of linear means (see, e.g., [T2]), $\mathbf{A}^*$ arises analogously in problems of summability at Lebesgue points.

1.2.

The spaces $\mathbf{A}^*$ were introduced by Beurling for establishing contraction properties of functions, namely [Be, Theorem 5], *Let*

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \qquad a_0 = 0,$$

be an absolutely convergent Fourier series such that $|a_{\pm n}| \le a_n^*$, $n \ge 1$, where $\{a_n^*\}$ is a nonincreasing sequence of numbers with a finite sum. Then if

$$g(t) \sim \sum_{n=-\infty}^{\infty} b_n e^{int}, \qquad b_0 = 0,$$

is a contraction of $f(t)$—that is, for any pair of arguments t_1, t_2 the inequality $|g(t_1) - g(t_2)| \le |f(t_1) - f(t_2)|$ holds—the Fourier series of $g(t)$ also converges absolutely, and, setting $a_{-n}^* = a_n^*$, $a_0^* = 0$, we have

$$\sum_{n=-\infty}^{\infty} |b_n| \le 5 \sum_{n=-\infty}^{\infty} a_n^*.$$

In [Be] an analogous result is established for $\mathbf{A}^*(\mathbf{R})$ as well.

It turned out that the consideration of summability of Fourier series by linear methods at Lebesgue points leads to the same spaces of functions (see, e.g., [SW, Chapter I]; [T1]).

Let us mention two other papers where $\mathbf{A}^*$ appeared in connection with certain problems of summability as well. These are the paper of D. Borwein [Bo] (some results from this paper were independently obtained in [BT1], but they were applied to other problems) and the paper of Telyakovskii [Te] (in this paper one condition of Sidon is given in the terms of $\mathbf{A}^*$). This space, even in the multidimensional case, appears in the paper by H. Feichtinger as $E^\wedge$(see [Fe, Theorem 3]).

1.3.

In this work properties of $\mathbf{A}^*$ as an algebra of functions are studied, some necessary (and, separately, sufficient) conditions of belonging to $\mathbf{A}^*$ are found, and criteria of summability at Lebesgue points are given.

We will concentrate on $\mathbf{A}^*(\mathbf{T})$ and, after detailed study of it, give a certain comparison with $\mathbf{A}^*(\mathbf{R})$ as well as some special properties of the latter.

The same letter, say C, will be used to denote different universal constants in different parts of the text.

We give the proofs both of unpublished results and of some results already published but not in the accessible literature.

2. Properties of $\mathbf{A}^*(\mathbf{T})$ as an Algebra

2.1.

We begin with the following proposition . Recall that the local property means that a space can be characterized by local membership to this class; that is, for each point there exists a neighborhood on which the given function coincides with a function from this space. The definitions of the main notions and facts from the theory of the Banach algebras can be found, for example, in [GRS]; for instance, the radical can be identified with the intersection of all maximal ideals.

Proposition 1.

The following statements hold.

- **i.** $\mathbf{A}^*$ *is a Banach algebra with the local property.*
- **ii.** *The space of maximal ideals of* $\mathbf{A}^*$ *coincides with* T.
- **iii.** $\mathbf{A}^*$ *is a regular Banach algebra with trivial radical.*
- **iv.** *If $f \in \mathbf{A}^*$ and $F(z)$ is defined and analytic on a neighborhood of the set of values of the function f, then $F \circ f \in \mathbf{A}^*$ (in particular, if f does not vanish anywhere, then $1/f \in \mathbf{A}^*$).*

Proof. **i.** It is obvious that $\mathbf{A}^*$ is a normed linear space over the field of complex numbers. Completeness may be proved by the usual argument. Let us prove that $\mathbf{A}^*$ is an algebra. It suffices to prove the numerical inequality

$$\sum_{m=0}^{N} \sup_{m\le|n|\le N} \sum_{k=-N}^{N} |a_k||b_{n-k}| \le \left(\sum_{|k|\le N/2} |a_k| \sum_{0\le m\le (N+1)/2} \sup_{m\le|n|\le 3N/2} |b_n| + \sum_{|n|\le 2N} |b_n| \sum_{0\le m\le (N+1)/2} \sup_{m\le|k|\le N} |a_k| \right). \tag{4}$$

Let us decompose the inner sum on the left-hand side of (4) into two summands corresponding to $|k| \le m/2$ and to $|k| \ge (m+1)/2$, respectively. We have

$$\begin{aligned}\sum_{m=0}^{N} \sup_{m\le|n|\le N} \sum_{|k|\le m/2} |a_k||b_{n-k}| &\le \sum_{m=0}^{N} \sup_{m/2\le|n|\le 3N/2} |b_n| \sum_{|k|\le m/2} |a_k| \\ &\le \sum_{|k|\le N/2} |a_k| \sum_{m=0}^{N} \sup_{m/2\le|n|\le 3N/2} |b_n| \\ &\le 2 \sum_{|k|\le N/2} |a_k| \sum_{0\le m\le (N+1)/2} \sup_{m\le|n|\le 3N/2} |b_n|.\end{aligned}$$

The remaining part is estimated analogously.

Passing to the limit as $N \to \infty$ in (4), we obtain for each $f, g \in \mathbf{A}^*$

$$\|fg\|_{\mathbf{A}^*(\mathbf{T})} \le 2\left(\|f\|_{\mathbf{A}(\mathbf{T})}\|g\|_{\mathbf{A}^*(\mathbf{T})} + \|g\|_{\mathbf{A}(\mathbf{T})}\|f\|_{\mathbf{A}^*(\mathbf{T})}\right) \le 4\|f\|_{\mathbf{A}^*(\mathbf{T})}\|g\|_{\mathbf{A}^*(\mathbf{T})}. \tag{5}$$

Hence, $\mathbf{A}^*(\mathbf{T})$ is an algebra with respect to the usual product of functions.

The local property may be proved by repeating the argument for the analogous Wiener's theorem for $\mathbf{A}$ (see, e.g., [K, Chapter II]).

ii. This statement can be proved by standard argument (see, e.g., [GRS]) taking into account that

$$\|e^{inx}\|_{\mathbf{A}^*(\mathbf{T})} = |n| + 1, \qquad n = \pm 1, \pm 2, \ldots.$$

iii. Since every maximal ideal in $\mathbf{A}^*$ is the set of functions $f \in \mathbf{A}^*$ that are vanishing at some $x_0 \in \mathbf{T}$, the radical of $\mathbf{A}^*$ is trivial.

The regularity of $\mathbf{A}^*$ follows trivially from the fact that all the functions with two continuous derivatives are in $\mathbf{A}^*$.

iv. This statement is a direct analog of the Wiener–Levy theorem for $\mathbf{A}$. It follows immediately from ii, but Wiener's proof may be repeated to the letter as well (see, e.g., [K, Chapters I and V]). □

2.2.

Let us describe the dual space of $\mathbf{A}^*(\mathbf{T})$.

Proposition 2.

A space $\mathbf{PM}^*$ *of all the sequences* $d = \{d_k\}_{k=-\infty}^{\infty}$ *with the finite norm*

$$\|d\|_{\mathbf{PM}^*} = \sup_{n\ge 0} \frac{1}{n+1} \sum_{k=-n}^{n} |d_k|$$

is the dual space of $\mathbf{A}^*(\mathbf{T})$.

Proof. The proof of this result is based on the following lemma.

Lemma 1.

For any two sequences $x = \{x_k\}_{k\geq 1}$, $y = \{y_k\}_{k\geq 1}$ *the inequality*

$$\left|\sum_{k=1}^{\infty} x_k y_k\right| \leq \|x\|_{l^*}\|y\|_{\mathbf{PM}^*} \tag{6}$$

holds. Moreover,

$$\sup_{\|x\|_{l^*}\leq 1}\left|\sum_{k=1}^{\infty} x_k y_k\right| = \|y\|_{\mathbf{PM}^*}, \tag{7}$$

$$\sup_{\|y\|_{\mathbf{PM}^*}\leq 1}\left|\sum_{k=1}^{\infty} x_k y_k\right| = \|x\|_{l^*}. \tag{8}$$

Proof. Define $x_k^* = \sup_{m\geq k}|x_m|$. Then

$$\left|\sum_{k=1}^{\infty} x_k y_k\right| \leq \sum_{k=1}^{\infty} x_k^*|y_k|.$$

Applying Abel's transformation, we obtain, for each N,

$$\begin{aligned}\sum_{k=1}^{N} x_k^*|y_k| &= \sum_{k=1}^{N-1} k\Delta x_k^* \frac{1}{k}\sum_{m=1}^{k}|y_m| + Nx_N^*\frac{1}{N}\sum_{m=1}^{N}|y_m| \\ &\leq \|y\|_{\mathbf{PM}^*}\left(\sum_{k=1}^{N-1} k\Delta x_k^* + Nx_N^*\right) = \|y\|_{\mathbf{PM}^*}\sum_{k=1}^{N} x_k^*.\end{aligned}$$

The last equality follows from the monotonicity of x_k^*. It remains now to pass to the limit as $N \to \infty$, and (6) is proved.

Let us go on to (7). For the rest of the proof we can restrict, without loss of generality, to nonnegative sequences. By considering the particular sequence $x_k = 1/n$ for $1 \leq k \leq n$ and $x_k = 0$ for $k > n$, we obtain

$$\sup_{\|x\|_{l^*}\leq 1}\left|\sum_{k=1}^{\infty} x_k y_k\right| \geq \frac{1}{n}\sum_{k=1}^{n} y_k,$$

and (7) holds because of arbitrarity of n.

If $\sum_{k=1}^{\infty} x_k = \infty$, then (8) is obvious with $y_k = 1$ for all k. Otherwise we build an extremal sequence as follows. Let n_1 be the largest number for which $x_n = x_1^*$, n_p $(p \geq 2)$ is the largest number for which $x_n = x_{n_{p-1}+1}^*$. Put $y_{n_1} = n_1$, $y_{n_p} = n_p - n_{p-1}$ $(p \geq 2)$, $y_k = 0$ when $k \neq n_p$, $p = 1, 2, \ldots$. Then $\|y\|_{\mathbf{PM}^*} = 1$ for this sequence and

$$\sum_{k=1}^{\infty} x_k y_k = x_1^* n_1 + \sum_{p=2}^{\infty} x_{n_{p-1}+1}^*(n_p - n_{p-1}) = \sum_{k=1}^{\infty} x_k^*,$$

and the proof is complete. □

Since the vectors $e_j = \{0, \ldots, 0, 1, 0, \ldots, 0\}$ form the basis in the space $\mathbf{A}^*$, every functional f can be represented in the form

$$\langle f, x\rangle = \sum x_k y_k$$

for a sequence $\{y_k\}$. Proposition 2 follows from this and Lemma 1 immediately. □

Remark 1. The space $\mathbf{PM}^*$ is *not separable*. The following argument was suggested to the authors by M. Dveirin.

Let α be a real number, $1 \leq \alpha \leq 2$. Consider, for all such α, the set of sequences $c^{(\alpha)} = \{c_j^{(\alpha)}\}$ with

$$c_j^{(\alpha)} = \begin{cases} 2^{k\alpha}, & j = [2^{k\alpha}], \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $\{c_j^{(\alpha)}\} \in \mathbf{PM}^*$ for each α. When $\alpha' \neq \alpha''$, then $\|c^{(\alpha')} - c^{(\alpha'')}\|_{\mathbf{PM}^*} > 1/2$. Therefore we have continuum different elements in $\mathbf{PM}^*$ distant more than $1/2$ one from another. This just means that the space $\mathbf{PM}^*$ is not separable. □

Remark 2. This idea, namely, take $c_j^{(\alpha)} = 1$ for $j = [2^{k\alpha}]$ and 0 otherwise, also gives a proof, different from the common one, of nonseparability of the classical space m of all bounded sequences. □

Remark 3. We have that the space $\mathbf{A}^*$ *is not reflexive* as well as $\mathbf{A}$. Indeed, it is known (see, e.g., [Ro, Theorem 2.5.13]) that if the dual space to a normed space X is separable, then X itself is separable. □

Remark 4. D. Borwein [Bo, Theorem 1] proved that $\mathbf{A}^*$ is dual of the separable subspace of $\mathbf{PM}^*$ defined as follows. For each sequence $d = \{d_k\}$ there is a number $l = l_d$ such that

$$\sum_{k=-N}^{N} |d_k - l| = o(N). \qquad \square$$

2.3.

Let us now give some spectral properties of $\mathbf{A}^*$.

The following result on approximation by piecewise linear functions in the $\mathbf{A}^*$-norm is an $\mathbf{A}^*$-analogue of Herz's theorem [K, Chapter V].

Let f_N be a function, coinciding with f at each $2\pi k/N$ point, where k, $N > 0$ are integers, $k \leq N$, and f_N is linear on intervals.

Proposition 3.

Let $f \in \mathbf{A}^*(\mathbf{T})$. Then

$$\lim_{N\to\infty} \|f - f_N\|_{\mathbf{A}^*(\mathbf{T})} = 0.$$

Proof. Let us calculate the mth Fourier coefficient of f_N. Integrating by parts the piecewise linear function f_N we obtain

$$\begin{aligned}\hat{f}_N(m) &= (2\pi)^{-1} \int_{\mathbf{T}} f_N(t) e^{-imt}\, dt \\ &= (2\pi i m)^{-1} \int_{\mathbf{T}} f_N'(t) e^{-imt}\, dt \\ &= (2\pi m^2)^{-1} \sum_{k=0}^{N-1} \left(\frac{N}{2\pi}\right) \left(f\left(\frac{2\pi(k+1)}{N}\right) - f\left(\frac{2\pi k}{N}\right)\right) \left(e^{-2\pi i(k+1)m/N} - e^{-2\pi i k m/N}\right).\end{aligned}$$

Taking into account that

$$\sum_{k=0}^{N-1} e^{2\pi i k/N} = \begin{cases} N, & \dfrac{k}{N} \text{ is integer,} \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$\hat{f}_N(m) = \frac{\sin^2(\pi m/N)}{(\pi m/N)^2} \sum_{k=-\infty}^{\infty} \hat{f}(m+kN).$$

Then

$$\begin{aligned}\|f - f_N\|_{A^*(T)} &= \sum_{n=0}^{\infty} \sup_{n\le|m|<\infty} \left|\hat{f}(m) - \frac{\sin^2(\pi m/N)}{(\pi m/N)^2} \sum_{k=-\infty}^{\infty} \hat{f}(m+kN)\right| \\ &\le \sum_{n=0}^{\infty} \sup_{n\le|m|<\infty} |\hat{f}(m)| \left(1 - \frac{\sin^2(\pi m/N)}{(\pi m/N)^2}\right) \\ &\quad + \sum_{n=0}^{\infty} \sup_{n\le|m|<\infty} \frac{\sin^2(\pi m/N)}{(\pi m/N)^2} \left|\sum_{k\ne 0} \hat{f}(m+kN)\right| = I'_N + I''_N.\end{aligned}$$

Let us show that $I'_N, I''_N \to 0$ as $N \to \infty$. Let M be a positive integer.

$$I'_N \le \sum_{0\le n\le M} \sup_{n\le|m|<\infty} |\hat{f}(m)| \left(1 - \frac{\sin^2(\pi m/N)}{(\pi m/N)^2}\right) + \sum_{n>M} \sup_{n\le|m|<\infty} |\hat{f}(m)|.$$

In view of the inequality

$$1 - \frac{\sin^2(\pi m/N)}{(\pi m/N)^2} \le \frac{\pi^2 m^2}{3N^2}$$

we have $(M < N)$

$$I'_N \le C\left(\sum_{0\le n\le M} \sup_{n\le|m|\le M} \frac{m^2}{N^2} |\hat{f}(m)| + \sum_{0\le n\le M} \sup_{M\le|m|<\infty} |\hat{f}(m)| + \sum_{n>M} \sup_{n\le|m|<\infty} |\hat{f}(m)|\right).$$

The second sum is equal to

$$2 \sum_{M/2\le n\le M} \sup_{M\le|m|<\infty} |\hat{f}(m)|$$

and does not exceed

$$2 \sum_{n>M/2} \sup_{n\le|m|<\infty} |\hat{f}(m)|.$$

Therefore

$$I'_N \le C\left(\frac{M}{N}\|f\|_{A^*(T)} + \sum_{n>M/2} \sup_{n\le|m|<\infty} |\hat{f}(m)|\right).$$

For M large the second sum is small as a remainder of the convergent series. Then choosing $N > M$ sufficiently large we get that I'_N is small.

Let us pass to the estimation of I''_N. Without loss of generality we may consider $m \ge 0$. Any $m \ge 1$ can be written in the form $m = rN + j$, $r = 0, 1, 2, \ldots$, $j = 1, \ldots, N-1$. Taking into account that $\sin(\pi m/N) = 0$ for $j = 0$ we obtain

$$\sum_{n=0}^{\infty} \sup_{n\le m<\infty} \frac{\sin^2(\pi m/N)}{(\pi m/N)^2} \left| \sum_{k\ne 0,-r,-r-1} \hat f(m+kN) \right|$$

$$\le \pi^{-2} \sum_{n=0}^{\infty} \sup_{\substack{r:n<Nr+j\\ j=1,\dots,N-1}} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} \sum_{k\ne 0,-r,-r-1} |\hat f((r+p)N+j)|$$

$$\le \pi^{-2} \sum_{q=0}^{\infty} \sum_{n=Nq}^{N(q+1)} \sup_{\substack{r:n<Nr+j\\ j=1,\dots,N-1}} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} \sum_{k=1}^{\infty} \sup_{Nk\le|p|<\infty} |\hat f(p)|$$

$$\le \frac{2N}{\pi^{-2}} \sum_{q=1}^{\infty} q^{-2} \sum_{k=1}^{\infty} \sup_{Nk\le|p|<\infty} |\hat f(p)| \le C \sum_{k>N} \sup_{k\le|p|<\infty} |\hat f(p)|,$$

and so this value is as small as we please, provided N is sufficiently large. It remains to consider $k = -r, -r-1$. In this case we have to estimate

$$\sum_{n=0}^{\infty} \sup_{\substack{r:n<Nr+j\\ j=1,\dots,N-1}} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} (|\hat f(j)| + |\hat f(j-N)|)$$

$$= \sum_{p=0}^{\infty} \sum_{n=pN}^{(p+1)N-1} \sup_{\substack{r:n<Nr+j\\ j=1,\dots,N-1}} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} (|\hat f(j)| + |\hat f(j-N)|).$$

It is obvious enough to prove estimates for only one case, $\hat f(j)$ or $\hat f(j-N)$; the other is analogous.

$$\sum_{p=0}^{\infty} \sum_{n=pN}^{(p+1)N-1} \sup_{\substack{r:n<Nr+j\\ j=1,\dots,N-1}} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} |\hat f(j-N)|$$

$$\le \sum_{n=0}^{N} \sup_{\substack{r:n<Nr+j\\ j=1,\dots,N-1}} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} |\hat f(j-N)|$$

$$+\ N \sum_{p=1}^{\infty} \sup_{p\le r<\infty} \sup_{j=1,\dots,N-1} \frac{\sin^2(\pi j/N)}{(r+j/N)^2} |\hat f(j-N)|$$

$$\le 2N \sup_{0<j<N} \sin^2(\pi j/N)\, |\hat f(-j)| \le \frac{2\pi^2}{N} \sup_{0<|j|<N} j^2 |\hat f(j)|$$

$$\le \frac{2\pi^2}{N} \sup_{0<j<N} 2j \sum_{j/2\le n\le j} \sup_{n\le|m|<\infty} |\hat f(m)|$$

$$\le \frac{4\pi^2}{N} \left(\sup_{0<j<M} + \sup_{M<j<N} \right) j \sum_{n>j/2} \sup_{n\le|m|<\infty} |\hat f(m)|$$

$$\le 4\pi^2 \|f\|_{A^*(\mathbf{T})} \frac{M}{N} + 4\pi^2 \sum_{n>M} \sup_{n\le|m|<\infty} |\hat f(m)|,$$

and the right-hand side is small by the argument like above. □

Let $\Delta_\varepsilon(t) = (1-|t|/\varepsilon)_+$ for $t \in \mathbf{T}$ and $\varepsilon \in (0, \pi/2)$, $\Delta_\varepsilon(t+2\pi) = \Delta_\varepsilon(t)$, and $V_\varepsilon = 2\Delta_{2\varepsilon} - \Delta_\varepsilon$. Δ_ε is a so-called triangular function, and the graph of V_ε on $\mathbf{T}$ is a trapezoid with the unit height.

Proposition 4.
Let $f \in \mathbf{A}^*(\mathbf{T})$ *and* $f(0) = 0$. *Then*

$$\lim_{\varepsilon \to 0} \| f V_\varepsilon \|_{\mathbf{A}^*(\mathbf{T})} = 0.$$

Proof. This result is a direct $\mathbf{A}^*$ analogue of the Wiener–Ditkin theorem for A [K, Chapter V] and means that f is a limit in $\mathbf{A}^*$ of the functions $f(1 - V_\varepsilon)$, which are vanishing in the ε-neighborhood of the origin. It is not difficult to calculate that

$$\hat{V}_\varepsilon(m) = \frac{\cos m\varepsilon - \cos 2m\varepsilon}{\pi \varepsilon m^2}, \qquad m \neq 0,$$

and $\hat{V}_\varepsilon(0) = 3\varepsilon/(2\pi)$. Thus, $|\hat{V}_\varepsilon(m)| \le 3\varepsilon/(2\pi)$ for each m. Let us estimate $\|V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})}$. We have

$$\begin{aligned} \|V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})} &\le \sum_{0 \le k \le 1/\varepsilon} \sup_{k \le m < \infty} \left| \frac{2 \sin(m\varepsilon/2) \sin(3m\varepsilon/2)}{\pi \varepsilon m^2} \right| \\ &+ \sum_{k > 1/\varepsilon} \sup_{k \le m < \infty} \frac{2}{\pi \varepsilon m^2} \le \frac{3\varepsilon}{2\pi} \frac{1}{\varepsilon} + \frac{2}{\pi\varepsilon} \varepsilon = \frac{7}{2\pi}. \end{aligned}$$

Further, let $N \le \pi/\varepsilon < N + 1$ for some integer N. We have

$$\begin{aligned} \|f V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})} &\le \|(f - f_N) V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})} + \|f_N V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})} \\ &\le 4 \|V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})} \|f - f_N\|_{\mathbf{A}^*(\mathbf{T})} + \|f_N V_\varepsilon\|_{\mathbf{A}^*(\mathbf{T})}, \end{aligned}$$

where f_N is the same as in Proposition 3. Proposition 3 gives the estimate needed for the first summand on the right-hand side. Let us now estimate the numbers $c_m = \widehat{f_N V_\varepsilon}(m)$. Since $f(0) = 0$, on the interval $(-2\varepsilon, 2\varepsilon)$ the function f_N consists of two lines connecting the origin and the points $(2\pi/N, f(2\pi/N))$ and $(-2\pi/N, f(-2\pi/N))$, respectively. Estimates for positive and negative parts are similar. Thus,

$$\begin{aligned} &\left| \frac{N}{2\pi} f\left(\frac{2\pi}{N}\right) \left\{ \int_0^\varepsilon x e^{-imx}\, dx + \int_\varepsilon^{2\varepsilon} x \left(2 - \frac{x}{\varepsilon}\right) e^{-imx}\, dx \right\} \right| \\ &\le \frac{N}{2\pi} \left| f\left(\frac{2\pi}{N}\right) \right| \min \left\{ \frac{5\varepsilon^2}{3}, \frac{1}{m^2} |1 - 2e^{-2im\varepsilon} + e^{-im\varepsilon}| + \frac{2}{\varepsilon m^3} |e^{-2im\varepsilon} - e^{-im\varepsilon}| \right\} \\ &\le \frac{N}{2\pi} \left| f\left(\frac{2\pi}{N}\right) \right| \min \left\{ \frac{5\varepsilon^2}{3}, \frac{6}{m^2} \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{\infty} \sup_{k \le |m| < \infty} |c_m| &\le \sum_{k=0}^{N} \sup_{k \le |m| < N} |c_m| + \sum_{k=0}^{N} \sup_{N \le |m| < \infty} |c_m| + \sum_{k > N} \sup_{k \le |m| < \infty} |c_m| \\ &\le \max \left| f\left(\pm \frac{2\pi}{N}\right) \right| \left(5\frac{N^2 \varepsilon^2}{6\pi} + \frac{3N^2}{\pi N^2} + \sum_{k > N} \frac{N}{3\pi k^2} \right), \end{aligned}$$

and $\max |f(\pm 2\pi/N)| \to 0$ as $N \to \infty$ while the expression in parantheses is bounded. □

Remark 5. It is not difficult to see that something more was obtained when proving these two propositions. Indeed, the following quantitative estimates hold:

$$\|f - f_N\|_{\mathbf{A}^*(\mathbf{T})} \leq \inf_{0<M<N} \left\{ \frac{M}{N} + \sum_{n>M} \sup_{n\leq|m|<\infty} |\hat{f}(m)| \right\}, \tag{9}$$

$$\|V_\varepsilon f\|_{\mathbf{A}^*(\mathbf{T})} \leq C|f(\pm 2\varepsilon)| + \inf_M \left\{ M\varepsilon + \sum_{n>M} \sup_{n\leq|m|<\infty} |\hat{f}(m)| \right\}. \quad \square \tag{10}$$

2.4.

Let us go on to some problems of synthesis for $\mathbf{A}^*(\mathbf{T})$. For the classical case of synthesis in **A** we can refer a reader to several well-known books [Bd, Part 3.2]; [GM, Chapter 3]; [K, Chapter V]; [Kz, Chapter VIII]. We introduce several definitions similar to those for **A**.

Definition 1. A function $f \in \mathbf{A}^*$ admits synthesis, denoted $f \in S$, if it is limit in $\mathbf{A}^*$ of a sequence of functions that are vanishing in a neighborhood of the set $N(f)$ of zeros of f. □

Definition 2. A closed subset of **T**, say E, is a set of synthesis if the closed ideal consisting of functions that vanish on E is equal to the closed ideal generated by functions that vanish on a neighborhood of E. □

The so-called Ditkin sets may be defined in just the same way as for **A** (see, e.g., [K, Chapter V], and every such set will be a set of synthesis in $\mathbf{A}^*$. Let us define another type of sets.

Definition 3. We say that a closed set E satisfies Herz's condition if there exists a sequence of integers $N_\nu \to \infty$ such that for each ν every point of type $2\pi k/N_\nu$ is either in E or is distant less than $2\pi/N_\nu$ from E. □

Proposition 3 has the consequence that a set E satisfying Herz's condition is a set of synthesis.

Let E_ε be the set of points distant less than ε from E and **V** be the space of functions of bounded variation.

Proposition 5.

If $f \in \mathbf{A}^*$, then the following statements hold.

- **i.** *If $f \in \mathrm{Lip}\,1$, then $f \in S$.*
- **ii.** *If $f \in \mathbf{V} \cap \mathrm{Lip}\,\alpha$ $(\alpha > 1/2)$, then $f \in S$.*
- **iii.** *If $f \in \mathrm{Lip}\,\alpha$, $\alpha > 2/3$, then $f(t) - x \in S$ for almost all x.*
- **iv.** *If $\sup_{t\in E_\varepsilon} |f(t)| = O(\varepsilon)$ $(E = N(f))$, then $f \in S$.*
- **v.** *If $\lim_{\varepsilon\to 0}((1/\varepsilon)\sup_{t\in E_\varepsilon} |f(t)|\sqrt{\mathrm{meas}(E_\varepsilon \backslash E)}) = 0$ $(E = N(f))$, then $f \in S$.*

Remark 6. Proposition 5 is an $\mathbf{A}^*$-analogue of the Beurling–Pollard theorem for **A**. Let us compare the conditions in the two theorems. We have:

- **i.** $\mathrm{Lip}(1/2)$ for **A** and $\mathrm{Lip}\,1$ for $\mathbf{A}^*$;
- **ii.** **V** for **A** and $\mathbf{V} \cap \mathrm{Lip}\,\alpha$, $(\alpha > 1/2)$ for $\mathbf{A}^*$;
- **iii.** $\alpha > 1/3$ for **A** and $\alpha > 2/3$ for $\mathbf{A}^*$;
- **iv.** $O(\sqrt{\varepsilon})$ for **A** and $O(\varepsilon)$ for $\mathbf{A}^*$;
- **v.** $1/\sqrt{\varepsilon}$ for **A** and $1/\varepsilon$ for $\mathbf{A}^*$. □

Proof. We will give a part of the proof (cf. [K, Chapter V]) that differs from that for **A**. A complete proof of ii will be given because such argument, different from Katznelson's proof, may be applied to **A** as well.

Let $f(x) = 0$ for all $x \in E$. Let $T \in \mathbf{PM}^*$, and supp $T \subset E$. The property $f \in S$ is equivalent to the fact that

$$(T, f) = \sum_n \hat{T}(n)\hat{f}(-n) = 0.$$

We have

$$(T, f) = \lim_{\varepsilon\to 0}(T_\varepsilon, f) = \lim_{\varepsilon\to 0}\sum_n \hat{T}(n)\hat{f}(-n)\frac{\sin^2(\varepsilon n/2)}{(\varepsilon n/2)^2}.$$

The function $T_\varepsilon(x) = \sum_n \hat{T}(n)(\sin^2(\varepsilon n/2)/(\varepsilon n/2)^2)e^{inx}$ is supported on E_ε. Therefore,

$$\begin{aligned}
\lim_{\varepsilon\to 0}|(T_\varepsilon, f)| &= \lim_{\varepsilon\to 0}\left|\int\limits_{E_\varepsilon\setminus E} f(x)T_\varepsilon(x)\,dx\right| \\
&\le \lim_{\varepsilon\to 0}\left\{\sup_{x\in E_\varepsilon}|f(x)|\left(\int\limits_{E_\varepsilon\setminus E}|T_\varepsilon(x)|^2\,dx\right)^{1/2}\sqrt{\operatorname{meas}(E_\varepsilon\setminus E)}\right\} \\
&\le \lim_{\varepsilon\to 0}\left\{\sup_{x\in E_\varepsilon}|f(x)|\left(\int\limits_{\mathbf{T}}|T_\varepsilon(x)|^2\,dx\right)^{1/2}\sqrt{\operatorname{meas}(E_\varepsilon\setminus E)}\right\}.
\end{aligned}$$

Let us estimate $\|T_\varepsilon\|_2$. Applying Parseval's equality, the evident inequality $|\hat{T}(n)| \le |n|$, and (6), we obtain

$$\begin{aligned}
\|T_\varepsilon\|_2 &= \left(\sum_n |\hat{T}(n)|^2\,\frac{\sin^4(\varepsilon n/2)}{(\varepsilon n/2)^4}\right)^{1/2} \\
&\le \|T\|_{\mathbf{PM}^*}\left\{\sum_{k=1}^{\infty}\sup_{k\le m<\infty} m\frac{\sin^4(\varepsilon n/2)}{(\varepsilon n/2)^4}\right\}^{1/2}.
\end{aligned}$$

The second multiplier on the right-hand side may be estimated by

$$\begin{aligned}
&\left\{\sum_{1\le k\le 1/\varepsilon}\left(\sup_{k\le m<1/\varepsilon} m + \sup_{1/\varepsilon\le m<\infty}\varepsilon^{-4}m^{-3}\right) + \sum_{k>1/\varepsilon}\sup_{k\le m<\infty}\varepsilon^{-4}m^{-3}\right\}^{1/2} \\
&\le \left\{\sum_{1\le k\le 1/\varepsilon}\left(\frac{1}{\varepsilon}+\frac{1}{\varepsilon}\right) + \sum_{k>1/\varepsilon}\varepsilon^{-4}k^{-3}\right\}^{1/2} \le \frac{C}{\varepsilon}.
\end{aligned}$$

Now the rest of the proof, say, given in [K, Chapter V, 5], may be repeated, but we will prove ii in another way.

Let $f \in \mathbf{V}$. The set $E_\varepsilon \setminus E$ is open. It may be represented as a union of at most a countable number of intervals (a_k, b_k). Each interval has a length not greater than 2ε, and there exists at least one point from E in the closure of each interval. We can consider the end points a_k of these intervals as those points from E. Then

$$\int_{E_\varepsilon\setminus E} |f(x)|\,dx = \sum_{k=1}^{\infty}\int_{a_k}^{b_k} |f(x)|\,dx = \sum_{k=1}^{\infty}\int_{a_k}^{b_k} |f(x)-f(a_k)|\,dx$$
$$= \sum_{k=1}^{\infty}\int_{a_k}^{b_k}\left|\int_{a_k}^{x} df(t)\right| dx \le \sum_{k=1}^{\infty}\int_{a_k}^{b_k} |df(t)|\int_t^{b_k} dx$$
$$\le \sum_{k=1}^{\infty}\int_{a_k}^{b_k}(b_k-a_k)|df(t)| \le 2\varepsilon \mathbf{V}_f.$$

Now ii follows from the above estimates. □

Observe that as in the case of A, Proposition 5 v yields all the others.

3. Comparison of A* with Spaces of Smooth Functions

3.1.

The following statements describe structural properties of functions in $\mathbf{A}^*(\mathbf{T})$. Let us recall some well-known notions.

$$\omega_k(g;h)_p = \sup_{|x|\le h} \|\Delta_x^k g\|_{L^p(\mathbf{T})},$$

where $\Delta_x^1 g = \Delta_x g = g(\cdot + x) - g(\cdot)$, $\Delta_x^k g = \Delta_x(\Delta_x^{k-1} g)$ is the kth difference, with step x, of the function g, $0 < h \le \pi$, defines the modulus of smoothness of order $k = 1, 2, 3, \dots$ of the function g in the L^p-space (g is considered to be 2π-periodic). We call $\omega_1 = \omega$ the modulus of continuity.

We say that $g \in B^r_{p,\theta}$, $1 \le p \le \infty$, $r > 0$ (the Besov space, see, e.g., [N]), if $g \in L^p$ and

$$\begin{aligned} &\int_0^\infty t^{-1-\theta r}\omega_{[r+1]}(g;t)_p^\theta\,dt < \infty, \qquad 1 \le \theta < \infty,\\ &\sup_{t>0} t^{-r}\omega_{[r+1]}(g;t)_p < \infty, \qquad \theta = \infty. \end{aligned} \tag{11}$$

Theorem 1.

i. *$B^1_{1,1} \subset \mathbf{A}^* \subset B^{1/2}_{2,1}$ and the embeddings are both continuous.*

ii. *If f is absolutely continuous and*

$$\int_0^1 \omega(f';t)_p \left(\ln\frac{1}{t}\right)^{-1/p'} t^{-1}\,dt < \infty \tag{12}$$

for some $p \in [1,2]$, $p' = p/(p-1)$, then $f \in \mathbf{A}^*$.

iii. *There exists a continuously differentiable function $f \notin \mathbf{A}^*$ for which*

$$\omega(f';t)_\infty = O\left(\ln\frac{1}{t}\right)^{-1/2}$$

Proof. **i.** We have (cf. (3))

$$\|f\|_{A^*(T)} \geq \sum_{k=0}^{\infty} 2^k \sup_{2^k \leq |m| < 2^{k+1}} |\hat{f}(m)| \geq \sum_{k=0}^{\infty} 2^{k/2} \left\{ \sum_{2^k \leq |m| \leq 2^{k+1}} |\hat{f}(m)|^2 \right\}^{1/2}$$
$$\geq \sum_{k=0}^{\infty} 2^{k/2} \omega\left(f; \frac{1}{2^k}\right)_2 \geq \sum_{k=1}^{\infty} k^{-1/2} \omega\left(f; \frac{1}{k}\right)_2$$
$$\geq \frac{1}{2} \sum_{k=1}^{\infty} \int_{1/(k+1)}^{1/k} k^{-3/2} \omega\left(f; \frac{1}{k}\right)_2 dx \geq \int_0^1 x^{-3/2} \omega(f; x)_2 \, dx,$$

and the right embedding is proved. For the left embedding, it is enough to take into account that

$$|\hat{f}(k)| = (8\pi)^{-1} \left| \int_{\mathbf{T}} \left[f\left(x + \frac{\pi}{k}\right) - 2f(x) + f\left(x - \frac{\pi}{k}\right) \right] e^{-ikx} \, dx \right| \leq (8\pi)^{-1} \omega_2 \left(f; \frac{\pi}{|k|}\right)_1$$

and to estimate from above the usual norm (2) as it was done for the right embedding.

ii. Let us use the equivalent norm (3). We have

$$\|f\|_{A^*(T)} \leq C \sum_{k=0}^{\infty} 2^k \sup_{2^k \leq |m| < 2^{k+1}} |\hat{f}(m)|$$
$$\leq C \sum_{k=0}^{\infty} \sup_{2^k \leq |m| < 2^{k+1}} |\hat{f'}(m)| \leq C \sum_{k=0}^{\infty} \left\{ \sum_{2^k \leq |m| < 2^{k+1}} |\hat{f'}(m)|^{p'} \right\}^{1/p'}$$
$$= C \sum_{k=0}^{\infty} \sum_{2^k \leq n < 2^{k+1}} \left\{ \sum_{2^n \leq |m| < 2^{n+1}} |\hat{f'}(m)|^{p'} \right\}^{1/p'}$$
$$\leq C \sum_{k=0}^{\infty} 2^{k/p} \left\{ \sum_{2^k \leq n < 2^{k+1}} \sum_{2^n \leq |m| < 2^{n+1}} |\hat{f'}(m)|^{p'} \right\}^{1/p'}$$
$$\leq C \sum_{k=0}^{\infty} 2^{k/p} \left\{ \sum_{2^{2^k} \leq |m| < \infty} |\hat{f'}(m)|^{p'} \right\}^{1/p'} .$$

It follows from the Hausdorff–Young inequality and usual estimate of the remainder of a series that the right-hand sum does not exceed

$$\sum_{k=0}^{\infty} 2^{k/p} \omega\left(f'; \frac{1}{2^{2^k}}\right)_p \leq C \sum_{k=1}^{\infty} k^{-1/p'} \omega\left(f'; \frac{1}{2^k}\right)_p$$
$$\leq C \int_1^{\infty} t^{-1/p'} \omega\left(f'; \frac{1}{2^t}\right)_p dt$$
$$\leq C \int_0^1 \omega(f'; t)_p \left(\ln \frac{1}{t}\right)^{-1/p'} t^{-1} \, dt.$$

The statement is proved.

iii. Let us use the following result of Banach (see, e.g., [Ba, Chapter IV, §16). *If $\{n_k\}$ is an arbitrary lacunary sequence, and numbers α_k and β_k are arbitrary, satisfying only $\sum(\alpha_k^2+\beta_k^2) < \infty$, then it is always possible to find a continuous function $f(x) \sim \sum_{k=0}^{\infty} c_k e^{ikx}$ for which $\Re c_{n_k} = \alpha_k$, $\Im c_{n_k} = \beta_k$.*

Set $n_k = 2^k$, and $\beta_k = 0, \alpha_k > 0$ such that $\sum_{k=1}^{\infty} \alpha_k^2 = \infty$. Thus, we have a continuous function f with $c_{\pm 2^k} = \alpha_k$. Let $c_0 = 0$. Then $F(x) = \sum_{k\neq 0}(c_k/ik)e^{ikx}$ is continuously differentiable, and as above

$$\begin{aligned}
\|F\|_{\mathbf{A}^*(\mathbf{T})} &= \sum_{m=0}^{\infty} \sup_{m\le |k|<\infty} \left|\frac{c_k}{k}\right| \ge \frac{1}{2}\sum_{m=0}^{\infty} \sup_{2^m\le |k|<2^{m+1}} |c_k| \\
&= \frac{1}{4}\sum_{m=0}^{\infty} 2^{-m} \sum_{2^m\le |k|<2^{m+1}} \sup_{2^m\le |k|<2^{m+1}} |c_k| \\
&\ge \frac{1}{4}\sum_{m=0}^{\infty} 2^{-m} \sum_{2^m\le |k|<2^{m+1}} |c_{2^{m+1}}| = \frac{1}{2}\sum_{m=1}^{\infty} \alpha_m = \infty.
\end{aligned}$$

Therefore, $F \notin \mathbf{A}^*$. Now consider the function

$$\varphi(x) \sim \sum_{k=2}^{\infty} (\ln k)^{-1/2} \cos kx.$$

Let us find its modulus of continuity in L^1. The series converges everywhere except 0 (see, e.g., [Ba, Chapter I]). So, we obtain, after applying Abel's transformation twice, that

$$\|\varphi(x) - \varphi(x+h)\|_1 \le \int_{\mathbf{T}} \left|\sum_{k=2}^{\infty} k\Delta_2((\ln k)^{-1/2})[\Phi_k(x) - \Phi_k(x+h)]\, dx,\right.$$

where $\Phi_k(x) = \sin^2(kx/2)/(2\pi k \sin^2(x/2))$ is the Fejer kernel. The right-hand side is not less than

$$\sum_{2\le k\le 1/h} k^{-2}(\ln k)^{-3/2} k^2 h + \sum_{k>1/h} 2k^{-1}(\ln k)^{-3/2},$$

where the first term was obtained by using Bernstein's inequality in L^1 for the Fejer kernel. It is easy to see now that $\omega(\varphi; h)_1 = O((\ln(1/h))^{-1/2})$. Consider the convolution $H = f * \varphi$. We have

$$\omega(H'; h)_\infty \le \|f'\|_C\, \omega(\varphi; h)_1 = O\left(\left(\ln\frac{1}{h}\right)^{-1/2}\right).$$

Taking $\alpha_k = k^{-1/2}(\ln k)^{-1/2-\varepsilon}$, with $0 < \varepsilon \le 1/2$, one can see that $H \notin \mathbf{A}^*$. Indeed, $\hat{H}(2^k) = k^{-1}(\ln k)^{-1/2-\varepsilon}$, and the proof is complete. □

3.2.

The following examples show that the second inclusion in Theorem 1 i is sharp and that $\mathbf{A}^*(\mathbf{T})$ does not imply bounded total variation.

Remark 7. For each $\varepsilon > 0$ there exists a function $f \notin B_{2,\infty}^{1/2+\varepsilon}$ such that $f \in \mathbf{A}^*(\mathbf{T})$.

For this, consider the function $f(x) = x^\alpha \sin(\pi^2/x)$ when $0 < x \le \pi$ and $1/2 < \alpha \le 1$, odd and defined at zero by continuity. Integration by parts yields

$$\hat{f}(u) = 2\int_0^\pi x^\alpha \sin\frac{\pi^2}{x}\sin ux\,dx$$

$$= -u^{-1}\cos(ux)x^\alpha \sin\frac{\pi^2}{x}\Big|_0^\pi + u^{-1}\int_0^\pi \alpha x^{\alpha-1}\sin\frac{\pi^2}{x}\cos ux\,dx$$

$$-\pi^2 u^{-1}\int_0^\pi x^{\alpha-2}\cos\frac{\pi^2}{x}\cos ux\,dx$$

$$= \alpha u^{-1}\int_0^\infty x^{\alpha-1}\sin\frac{\pi^2}{x}\cos ux\,dx$$

$$-\pi^2 u^{-1}\int_0^\infty x^{\alpha-2}\cos\frac{\pi^2}{x}\cos ux\,dx + O(u^{-2}).$$

The values of integrals are calculated explicitly via Bessel functions in [BE, Chapter I]. It is enough for us that $\hat{f}(u) \sim u^{-3/4-\alpha/2}$ for large u. Hence, $f \in \mathbf{A}^*$ for the range of α claimed. In view of Parseval's equality

$$\omega\left(f;\frac{1}{N}\right)_2 \ge \left\{N^{-2}\sum_{k=1}^N k^2|\hat{f}(k)|^2\right\}^{1/2} \ge \left\{N^{-2}\sum_{k=1}^N k^2 k^{-2-\alpha+1/2}\right\}^{1/2} \ge CN^{-\alpha/2-1/4}.$$

The statement follows now from (11). □

Remark 8. There exists a function $f \in \mathbf{A}^*(\mathbf{T})$ that is not a function of bounded variation. Indeed, the example from Remark 7 gives such a function even for $\alpha = 1$:

$$f(x) = x\sin\frac{\pi^2}{x}.$$

This example shows that a summability method (generated by f) may be regular at every Lebesgue point, but not regular in Toeplitz sense. □

3.3.

The following theorem says that ii is sharp for $p = 1$.

We call ω a modulus of continuity if it satisfies all the usual properties sufficient for ω to be the modulus of continuity of a continuous function (see, e.g., [L, Chapter 3]): It must be a positive, nondecreasing, subadditive function, which is approaching zero at the origin.

Theorem 2.

Let ω be a concave modulus of continuity. If

$$\int_0^1 t^{-1}\omega(t)\,dt = \infty, \tag{13}$$

then there exists a function f such that $\omega(f';t)_1 \le \omega(t)$ and $f \notin \mathbf{A}^*(\mathbf{T})$.

Proof. Consider the function $\mu(t) = \omega(t^{-2})$ and build the convex function

$$\mu^*(t) = \inf_{1\le x<t<y} \frac{(t-x)\mu(y) + (y-t)\mu(x)}{y-x},$$

which is a minimal convex majorant of the modulus of continuity ω (such a construction may be found, e.g., in [L, Chapter 3]). It is evident that $\mu^*(t) \le \mu(t)$. On the other hand,

$$\begin{aligned}
\mu^*(t) &= \inf_{1\le x<t<y} \frac{(t-x)\omega(t^{-2}(t^2/y^2)) + (y-t)\omega(t^{-2}(t^2/x^2)}{y-x} \\
&\ge \inf_{1\le x<t<y} \frac{(t-x)t^2/(y^2+t^2) + (y-t)t^2/(x^2+t^2)}{y-x}\omega(t^{-2}) \\
&= \inf_{1\le x<t<y} \left\{\frac{1}{x^2+t^2} - \frac{(t-x)(y+x)}{(y^2+t^2)(x^2+t^2)}\right\} t^2\omega(t^{-2}) \\
&\ge t^2\omega(t^{-2}) \inf_{1\le x\le t} \left\{\frac{1}{x^2+t^2} - \sup_{t\le y<\infty} \frac{(t-x)(y+x)}{(y^2+t^2)(x^2+t^2)}\right\}.
\end{aligned}$$

The expression under the sign of the least upper bound decreases as a function of y. Therefore, we obtain

$$\begin{aligned}
\mu^*(t) &\ge t^2\omega(t^{-2}) \inf_{1\le x\le t} \left\{\frac{1}{x^2+t^2} - \frac{t-x}{x^2+t^2}\cdot\frac{t+x}{2t^2}\right\} \\
&= t^2\omega(t^{-2}) \inf_{1\le x\le t} \frac{1}{2}\, t^{-2} = \frac{1}{2}\,\omega(t^{-2}) = \frac{\mu(t)}{2}.
\end{aligned}$$

Consider a function

$$\varphi \sim \sum_{k=1}^{\infty} \mu^*(k)\cos kx.$$

Since $\{\mu^*(k)\}$ is convex and monotone decreasing to zero, φ is a function whose Fourier series is integrable and converges to the function everywhere except zero (see, e.g., [Ba, Chapter I]). Let us estimate its modulus of continuity in L^1. As in iii of Theorem 1 we have

$$\|\varphi(x) - \varphi(x+h)\|_1 \le \sum_{k=1}^{\infty} k\Delta^2\mu^*(k) \int_{\mathbf{T}} |\Phi_k(x+h) - \Phi_k(x)|\,dx,$$

where $\Phi_k(x)$ is the Fejer kernel. Thus, denoting $1 + [1/h] = M$, we obtain

$$\begin{aligned}
\|\varphi(x) - \varphi(x+h)\|_1 &\le \sum_{1\le k\le M} k\Delta^2\mu^*(k) \int_{\mathbf{T}} dx \int_0^h |\Phi'_k(x+u)|\,du \\
&\quad + \sum_{k\ge M} k\Delta^2\mu^*(k) \int_{\mathbf{T}} (|\Phi_k(x+h)| + |\Phi_k(x)|)\,dx \\
&\le \frac{1}{M}\sum_{1\le k\le M} k^2\Delta^2\mu^*(k) + 2\sum_{k\ge M} k\Delta^2\mu^*(k) = \frac{S_1}{M} + 2S_2.
\end{aligned}$$

Let us estimate firstly the second sum as

$$S_2 = \sum_{k\ge M} k(\Delta\mu^*(k) - \Delta\mu^*(k+1)) = \mu^*(M+1) + M\Delta\mu^*(M).$$

Because of convexity,

$$M\Delta\mu^*(M) \le 2\sum_{M/2\le k\le M}\Delta\mu^*(M) \le 2\sum_{M/2\le k\le M}\Delta\mu^*(k) \le 2\left(\mu^*\left(\frac{M}{2}\right)+\mu^*(M+1)\right).$$

So the final estimate for S_2 is

$$S_2 \le 3\mu^*\left(\frac{M}{2}\right) \le 3\omega\left(\frac{4}{M^2}\right) \le 3\omega(4h^2).$$

Let us go on to the estimation of S_1:

$$\begin{aligned}\sum_{1\le k\le M} k^2\Delta^2\mu^*(k) &= \sum_{1\le k\le M} k^2\Delta\mu^*(k) - \sum_{1\le k\le M} k^2\Delta\mu^*(k+1)\\ &= \sum_{1\le k\le M} k^2\Delta\mu^*(k) - \sum_{k=2}^{M+1}(k-1)^2\Delta\mu^*(k)\\ &= \sum_{k=2}^{M}(2k-1)\Delta\mu^*(k) + \Delta\mu^*(1) - M^2\Delta\mu^*(M+1).\end{aligned}$$

The value $\Delta\mu^*(1)$ is bounded; the last summand was previously estimated. Further,

$$\begin{aligned}\sum_{k=2}^{M}(2k-1)\Delta\mu^*(k) &= \sum_{k=2}^{M}(2k-1)\mu^*(k) - \sum_{k=2}^{M}(2k-1)\mu^*(k+1)\\ &= \sum_{k=2}^{M}(2k-1)\mu^*(k) - \sum_{k=3}^{M+1}(2k-3)\mu^*(k)\\ &= 2M\sum_{k=3}^{M}\mu^*(k) + 3\mu^*(2) - (2M-1)\mu^*(M+1).\end{aligned}$$

The last two values give the estimate needed. We have $\sum_{k=3}^{M}\mu^*(k) \le \sum_{k=3}^{M}\omega(k^{-2})$. Since ω is concave, we have by virtue of Jensen's inequality that

$$\left(\frac{1}{M}\right)M\sum_{k=3}^{M}\omega(k^{-2}) \le C\omega\left(\frac{1}{M}\right) \le C\omega(h).$$

Since the integral $\int_0^1 h^{-1}\omega(h)\,dh$ diverges, the integral $\int_0^1 t^{-1}\omega(t^2)\,dt$ diverges as well, and therefore the series $\sum_{k=1}^{\infty}\omega(2^{-2k})$ diverges. Consider the function f with the Fourier series

$$f(x) \sim \sum_{k=1}^{\infty} k^{-1}\mu^*(k)\sin kx.$$

We have $f'(x) = \varphi(x)$ and $\omega(f';t)_1 \le C\omega(t)$, and

$$\begin{aligned}\|f\|_{A^*(\mathbf{T})} &= \sum_{k=1}^{\infty}\sup_{k\le m<\infty}|\hat{f}(m)| = \sum_{k=1}^{\infty}\sup_{k\le m<\infty} m^{-1}\mu^*(m)\\ &\ge \sum_{k=1}^{\infty}2^k\sup_{2^{k+1}\le m<2^{k+2}} m^{-1}\mu^*(m) \ge \frac{1}{4}\sum_{k=1}^{\infty}\sup_{2^{k+1}\le m<2^{k+2}}\mu^*(m)\\ &\ge \frac{1}{8}\sum_{k=1}^{\infty}\omega(2^{-2k-4}) = \infty.\end{aligned}$$

The proof is complete. □

3.4.

Let us give some other necessary and (separately) sufficient conditions for belonging to $\mathbf{A}^*$.

Theorem 3.

i. *For each positive integer N*

$$\left|\sum_{k=0}^{N-1}\int_0^\infty u^{-1}\left(f\left(x+u+\frac{2k\pi}{N}\right)-f\left(x-u+\frac{2k\pi}{N}\right)\right)du\right|\le C\|f\|_{\mathbf{A}^*(\mathbf{T})}.$$

ii. *If $f'\in\mathbf{A}(\mathbf{T})$, then $f\in\mathbf{A}^*(\mathbf{T})$. If the Fourier series of f is lacunary in the Hadamard sense, then the converse statement holds.*

Proof. **i.** Assume

$$g_N(x)=\sum_{k=0}^{N-1}f\left(\frac{x}{N}-\frac{2k\pi}{N}\right)-N\hat f(0)=\sum_{k\neq 0}N\hat f(kN)e^{ikx}.$$

Then $\|g_N\|_{\mathbf{A}^*(\mathbf{T})}\le 2\|f\|_{\mathbf{A}^*(\mathbf{T})}$. Indeed,

$$\begin{aligned}\|g_N\|_{\mathbf{A}^*(\mathbf{T})}&=N\sup_{1\le|k|<\infty}|\hat f(kN)|+\sum_{m=1}^{\infty}N\sup_{m\le|k|<\infty}|\hat f(kN)|\\&\le 2\sum_{m=1}^{\infty}N\sup_{m\le|k|<\infty}|\hat f(kN)|\\&\le\sum_{m=1}^{\infty}\sum_{p=(m-1)N+1}^{mN}\sup_{p\le|k|<\infty}|\hat f(kN)|=2\|f\|_{\mathbf{A}^*(\mathbf{T})}.\end{aligned}$$

But analogously $\|g_N(Nx)\|_{\mathbf{A}(\mathbf{T})}=\|g_N\|_{\mathbf{A}(\mathbf{T})}\le 2\|g_N\|_{\mathbf{A}^*(\mathbf{T})}\le 4\|f\|_{\mathbf{A}^*(\mathbf{T})}$, and it is enough to apply the necessary condition for belonging to $\mathbf{A}$ (see, e.g., [K, Chapter II, §10]).

ii. Indeed,

$$\sum_{m=0}^{\infty}\sup_{m\le k<\infty}|\hat f(k)|\le\sum_{m=0}^{\infty}\sum_{m\le|k|}|\hat f(k)|\le\sum_{k=-\infty}^{\infty}|\hat f(k)|(|k|+1).$$

Conversely, if $|m_{(|p|+1)\operatorname{sign}p}|\ge q|m_p|$, where $m_0=0$ and $q>1$, and $\hat f(k)=0$ for $k\neq m_p$, then

$$\begin{aligned}\sum_{m=0}^{\infty}\sup_{m\le k<\infty}|\hat f(k)|&\ge\frac12\sum_{p=-\infty}^{\infty}(|m_{(|p|+1)\operatorname{sign}p}|-|m_p|)|\hat f(m_p)|\\&\ge\frac{q-1}{2}\sum_{p=-\infty}^{\infty}|m_p||\hat f(m_p)|,\end{aligned}$$

and the boundedness of the right-hand sides of both inequalities is equivalent to the fact that $f'\in\mathbf{A}(\mathbf{T})$. □

4. On Some Properties of $\mathbf{A}^*(\mathbf{R})$

4.1.

As in the case of $\mathbf{A}$ there are many properties which hold both for $\mathbf{A}^*(\mathbf{R})$ and $\mathbf{A}^*(\mathbf{T})$. This is so because the two spaces coincide locally.

Proposition 6.

A function belongs locally to $\mathbf{A}^*(\mathbf{T})$ *if and only if it belongs locally to* $\mathbf{A}^*(\mathbf{R})$.

Proof. Because the norm in $\mathbf{A}^*$ is invariant with respect to shift, we can restrict to a neighborhood of the origin. If $\operatorname{supp} f \subset \mathbf{T}$ and $f \in \mathbf{A}^*(\mathbf{R})$, then

$$\|f\|_{\mathbf{A}^*(\mathbf{T})} = \sum_{m=0}^{\infty} \sup_{m \le |k| < \infty} |\hat{f}(k)| \le \sum_{m=0}^{\infty} \sup_{m \le |x| < \infty} |\hat{f}(x)|$$

$$\le \sup_x |\hat{f}(x)| + \int_0^\infty \sup_{u \le |x| < \infty} |\hat{f}(x)|\, du \le 2\|f\|_{\mathbf{A}^*(\mathbf{R})}.$$

If now $f \in \mathbf{A}^*(\mathbf{T})$ and $\operatorname{supp} f \subset [-\pi + \varepsilon, \pi - \varepsilon]$, $\varepsilon \in (0, \pi)$, then $f \in \mathbf{A}^*(\mathbf{R})$. This statement, similar to Wiener's well-known result for $\mathbf{A}$ (see [W, Chapter II]) may be proved analogously. In this case, $\hat{f}$ is an entire function of exponential type less than or equal to $\pi - \varepsilon$. We have (see, e.g., [W, Chapter II, §11])

$$\hat{f}(x) = 2 \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{\sin(\pi - \varepsilon/2)(x-k) \sin(\varepsilon(x-k)/2)}{(x-k)^2 \varepsilon/2}.$$

Therefore,

$$\int_0^\infty \sup_{u \le |x| < \infty} |\hat{f}(x)|\, du \le \sum_{m=0}^{\infty} \sup_{m \le |x| < \infty} |\hat{f}(x)|$$

$$\le \sum_{m=0}^{\infty} \sup_{m \le |x| < \infty} \sum_{k=-\infty}^{\infty} |\hat{f}(k)| \left| \frac{\sin(\pi - \varepsilon/2)(x-k) \sin(\varepsilon(x-k)/2)}{(x-k)^2 \varepsilon/2} \right|.$$

But for $x \in \mathbf{R}$ the inequality $|\sin \alpha x/(\alpha x)| \le (4 + [\alpha])/(|\alpha|(|[x]| + 1))$ holds, and inequality (5) completes the proof. □

An integral analogue of Proposition 2 (in slightly weaker form) may be found in [Bo].

4.2.

Let us give one more result. Consider for $\alpha > 0$ the following integral operators on $\mathbf{A}(\mathbf{R})$:

$$H_\alpha f = H_\alpha(f; x) = |x|^{-\alpha} \operatorname{sign} x \int_0^x |x - t|^{\alpha-1} f(t)\, dt = \int_0^1 (1-t)^{\alpha-1} f(xt)\, dt,$$

integral means of fractional order α, and

$$A_\alpha f = A_\alpha(f; x) = x^{-\alpha} f(x), \quad |x|^{-\alpha} f(x) \qquad \left(\text{or } |x|^{-\alpha} f(x)/\operatorname{sign} x\right),$$

for α integer or noninteger, respectively, the operator of division by the power function.

If $f = \hat{g}$, then we consider the fractional derivative of f

$$f^{(\alpha)}(x) = \int_{\mathbf{R}} (-iu)^\alpha g(u) e^{-iux}\, du.$$

Theorem 4.

i. *H_α is a linear bounded operator taking $\mathbf{A}(\mathbf{R})$ into $\mathbf{A}(\mathbf{R})$ for $\alpha > 0$, and $\mathbf{A}(\mathbf{R})$ into $\mathbf{A}^*(\mathbf{R})$ for $\alpha \ge 1$.*

ii. *If f and $f^{(\alpha)} \in \mathbf{A}(\mathbf{R})$, and $f^{(\nu)}(0) = 0$ for integer $\nu \in [0, \alpha)$, then $A_\alpha f \in \mathbf{A}(\mathbf{R})$. If $\alpha \geq 1$ then $A_\alpha f \in \mathbf{A}^*(\mathbf{R})$.*

Proof. Let us introduce a multiplicative convolution for functions from $L^1[0, \infty)$

$$(f * g)(x) = \int_0^\infty f\left(\frac{x}{t}\right) g(t) t^{-1}\, dt = \int_0^\infty g\left(\frac{x}{t}\right) f(t) t^{-1}\, dt$$

and assume $L^*[0, \infty) = \{f : \|f\|_{L^*} = \int_0^\infty \operatorname{ess\,sup}_{t \leq x < \infty} |f(x)|\, dt < \infty\}$.

Lemma 2.

i. *If $f, g \in L^1[0, \infty)$, then $f * g \in L^1[0, \infty)$ and $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$.*

ii. *If $f \in L^1$ and $g \in L^*$, then $f * g \in L^*$ and $\|f * g\|_{L^*} \leq \|f\|_{L^1} \|g\|_{L^*}$.*

Proof of Lemma 2. Applying Fubini's theorem and then changing variable we obtain

$$\int_0^\infty \left| \int_0^\infty g\left(\frac{x}{t}\right) f(t) t^{-1}\, dt \right| dx \leq \int_0^\infty dx \int_0^\infty t^{-1} |f(t)| \left| g\left(\frac{x}{t}\right) \right| dt$$
$$= \int_0^\infty |f(t)|\, dt \int_0^\infty \left| g\left(\frac{x}{t}\right) \right| t^{-1}\, dx = \|f\|_{L^1} \|g\|_{L^1}$$

which proves i, and

$$\int_0^\infty \operatorname*{ess\,sup}_{u \leq x < \infty} \left| \int_0^\infty g\left(\frac{x}{t}\right) f(t) t^{-1}\, dt \right| du \leq \int_0^\infty du \int_0^\infty t^{-1} |f(t)| \operatorname*{ess\,sup}_{u \leq x < \infty} \left| g\left(\frac{x}{t}\right) \right| dt$$
$$= \int_0^\infty |f(t)|\, dt \int_0^\infty \operatorname*{ess\,sup}_{u/t \leq v < \infty} |g(v)| t^{-1}\, du$$
$$= \int_0^\infty |f(t)|\, dt \int_0^\infty \operatorname*{ess\,sup}_{x \leq v < \infty} |g(v)|\, dx = \|f\|_{L^1} \|g\|_{L^*},$$

which proves ii. $\square$

Let us prove the first part of Theorem 4. If $f = \hat{g}$ and $g \in L^1(\mathbf{R})$, we obtain by the substitution of v for ut that

$$H_\alpha f = \int_0^1 (1 - t)^{\alpha - 1} dt \int_{\mathbf{R}} g(u) e^{-iuxt} du$$
$$= \int_0^1 (1 - t)^{\alpha - 1} dt \int_{\mathbf{R}} g\left(\frac{v}{t}\right) e^{-ixv} t^{-1}\, dv$$
$$= \int_{\mathbf{R}} e^{-ixv} dv = \int_0^1 (1 - t)^{\alpha - 1} g\left(\frac{v}{t}\right) t^{-1}\, dt.$$

The right-hand side is the Fourier transform of the convolution of functions $g \in L^1$ and $\varphi_\alpha(t) = (1-t)^{\alpha-1}$ for $t \in [0,1]$ and $\varphi_\alpha(t) = 0$ for $t > 1$. It is clear that $\varphi_\alpha \in L^1$ for each $\alpha > 0$ and that $\varphi_\alpha \in L^*$ for $\alpha \geq 1$ only. Nothing remains as to apply Lemma 2.

Let us prove the second part of the theorem. Since $f^{(\alpha)} \in \mathbf{A}$, we have $|u|^{(\alpha)} g(u) \in L^1(\mathbf{R})$, and $f^{(\nu)}(0) = 0$ $(0 \leq \nu < \alpha)$. Consider the function

$$\varphi_\alpha(x) = |x|^{-\alpha}\delta(x)\sum_{k\geq\alpha}\frac{(ix)^k}{k!},$$

where $\alpha > 0$, $\delta(x) = \delta_\alpha(x) = \operatorname{sign} x$ when α is odd, $\delta(x) = 1$ when α is even, and $\delta(x) = \operatorname{sign} x$ or 1 when α is not integer. If $\varphi_\alpha = \hat{g_\alpha}$ with $g_\alpha \in L^1(\mathbf{R})$, and

$$h_\alpha(t) = \int_{\mathbf{R}} |u|^{\alpha-1}\delta(u)g(u)g_\alpha\left(\frac{-t}{u}\right)du,$$

then

$$\begin{aligned}\hat{h_\alpha}(x) &= \int_{\mathbf{R}} h_\alpha(t)e^{-itx}\,dt = \int_{\mathbf{R}} |u|^\alpha\delta(u)g(u)\,du\int_{\mathbf{R}} g_\alpha\left(\frac{-t}{u}\right)e^{-itx}|u|^{-1}\,dt \\ &= \int_{\mathbf{R}} |u|^\alpha\delta(u)g(u)\,du\int_{\mathbf{R}} g_\alpha(y)e^{iuyx}\,dy = \int_{\mathbf{R}} |u|^\alpha\delta(u)g(u)\,du\varphi_\alpha(-ux)\,du.\end{aligned}$$

Taking into account that $\delta(xy) = \delta(x)\delta(y)$, we obtain for $\alpha = n + \varepsilon$ $(\varepsilon \in (0,1))$

$$\begin{aligned}\hat{h_\alpha}(x) &= \int_{\mathbf{R}} |x|^{-\alpha}g(u)\left[e^{-iux} - 1 - \cdots - \frac{(-iux)^n}{n!}\right]\delta(-x)\,du \\ &= |x|^{-\alpha}\delta(-x)\int_{\mathbf{R}} g(u)e^{-iux}\,du = |x|^{-\alpha}\delta(-x)f(x).\end{aligned}$$

Thus we see from the definition of h_α and Lemma 2 that the question is whether φ_α belongs either to $\mathbf{A}$ or to $\mathbf{A}^*$. In fact, $\varphi_\alpha \in \mathbf{A}^*(\mathbf{R})$ for $\alpha \geq 1$ only. For $\alpha = 1$ it may be verified immediately since the Fourier transform of φ_1 is bounded. When $\alpha = n + \varepsilon$, the function

$$\varphi_\alpha(x) = \left\{e^{ix} - \sum_{k=0}^{n}\frac{(ix)^k}{k!}\right\}|x|^{-\alpha}\delta(x)$$

is absolutely continuous, and for each r the rth derivative is estimated as $\varphi_\alpha^{(r)} = O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$. Now we can apply Theorem 3 (see ii, $\alpha = 1$). When $\alpha \in (0,1)$ we have $\varphi_\alpha(x) = |x|^{-\alpha}(e^{ix} - 1)\delta(x) \in \mathbf{A}(\mathbf{R})$ (it may be shown by various methods, for example, by applying Boman's result [Bn]), but $\varphi_\alpha \notin \mathbf{A}^*$, because $\lim_{x\to 1}|\hat{\varphi_\alpha}(x)| = \infty$. Indeed, for $x \in (0,1)$

$$\begin{aligned}\int_{\mathbf{R}} |t|^{-\alpha}|t|^{-\alpha}e^{-itx}\,dt &= 2\int_0^\infty t^{-\alpha}[\cos t(1-x) - \cos tx]\,dt \\ &= [(1-x)^{\alpha-1} - x^{\alpha-1}]\,2\int_0^\infty u^{-\alpha}\cos u\,du.\end{aligned}$$

Theorem 4 is proved. □

Let us note that H_α is sometimes called the Hardy operator, and that ii of Theorem 4 is an analog of well-known Hadamard's lemma on division by the power function in the C-space.

5. Summability at Lebesgue Points

5.1.

The following Lebesgue theorem is very well-known.

If f is integrable, then

$$\int_0^h |f(x \pm t) - f(x)|\, dt = o(h) \quad as \quad h \to 0 \tag{14}$$

for almost all x. Every point x for which (14) holds is called a Lebesgue point.

Let $f \in L^1(\mathbf{T})$ be a 2π-periodic function with the Fourier series

$$\sum_k \hat{f}(k) e^{ikx}. \tag{15}$$

Let λ be a continuous function on $\mathbf{R}$ representable as

$$\lambda(x) = \int_{\mathbf{R}} e^{-ixt}\, d\mu(t),$$

where μ is a finite Borel measure. Assume that

$$\int_{\mathbf{R}} d\mu(t) = 1.$$

Consider the means

$$(f * d\mu)_N \stackrel{\text{def}}{=} \lim_{h \to 0} \int_{\mathbf{R}} f_h\left(\frac{x-u}{N}\right) d\mu(u),$$

where

$$f_h(x) = h^{-1} \int_{\mathbf{R}} \varphi\left(\frac{t}{h}\right) f(x-t)\, dt.$$

Here $\varphi(t)$ is infinitely differentiable, equal to 1 for $|t| < 1$, vanishing for $|t| > 2$, and such that

$$\int_{\mathbf{R}} \varphi(t)\, dt = 1.$$

For f sufficiently smooth it is possible to take the limit inside the integral, yielding

$$(f * d\mu)_N(x) = \int_{\mathbf{R}} f\left(\frac{x-u}{N}\right) d\mu(u) = \sum_k \lambda\left(\frac{k}{N}\right) \hat{f}(k) e^{ikx},$$

so the linear means of the series (15) are considered. These means are generated by the function λ.

The following theorem investigates the behavior of $(f * d\mu)_N$, as $N \to \infty$, at Lebesgue points of integrable functions f. It has an exact form and, in certain sense, is the most general in this field.

Theorem 5.

The linear means $(f * d\mu)_N(x)$ converge to $f(x)$ as $N \to \infty$ at all the Lebesgue points of each $f \in L_1(\mathbf{T})$ if and only if the measure μ is absolutely continuous with respect to Lebesgue measure and $\mu' \in L^*(\mathbf{R})$.

Proof. Let us note that the sufficient part of this theorem was known earlier (see [SW, Chapter I]). Therefore, we will only give a proof of the necessity.

It will be convenient for us to assume $x = 0$ to be a Lebesgue point and $f(0) = 0$. It follows, from the existence of $(f_h * d\mu)_N(0)$ for every h and from the Banach–Steinhaus theorem, that $(f * d\mu)_N(0)$ is a bounded linear functional on

$$\mathbf{PM}^*(\mathbf{R}) = \left\{ g : \|g\|_{\mathbf{PM}^*(\mathbf{R})} = \sup_t t^{-1} \int_{|x|<t} |f(x)|\, dx < \infty \right\}$$

(see [Bo, BT2]). Using the condition of convergence of $(f * d\mu)_N(0)$ for every $f \in \mathbf{PM}^*$ and applying the Banach–Steinhaus theorem to this sequence, we obtain

$$\sup_N \sup_{\|f\|_{\mathbf{PM}^*}<1} |(f * d\mu)_N(0)| < \infty;$$

this value is not less than

$$\sup_N \sup_{\bar{f}_h : \|f\|_{\mathbf{PM}^*}<C} |(\bar{f}_h * d\mu)_N(0)|,$$

where $\bar{f}_h(x) = f_h(x) - f_h(0)$. Indeed, the equality $f_h(0) = 0$ is obvious, and it is easy to verify that $\|\bar{f}_h\|_{\mathbf{PM}^*} \leq (1/C)\|f\|_{\mathbf{PM}^*}$. Thus substituting $\bar{f}_h$ by the explicit representation we have

$$\sup_N \sup_{\|f\|_{\mathbf{PM}^*}<C} \left| \int_{\mathbf{R}} h^{-1} \int_{\mathbf{R}} \varphi\left(\frac{t}{h}\right) f\left(\frac{u}{N-t}\right) dt\, d\mu(u) \right| < \infty.$$

Expanding f_h into the Fourier series and using the inverse formula for the Fourier transform we obtain

$$\sup_N \sup_{\|f\|_{\mathbf{PM}^*}<C} \left| \int_{N\mathbf{T}} f\left(\frac{x}{N}\right) N^{-1} \sum_k \lambda\left(\frac{k}{N}\right) \hat{\varphi}(|k|h) e^{ikx/N}\, dx \right| < \infty.$$

Put $h = \varepsilon/N$. Then the left-hand side is not less than

$$\int_0^\infty \sup_{u\leq|x|<\infty} \left| \int_{\mathbf{R}} \hat{\varphi}(|t|\varepsilon)\lambda(t) e^{itx}\, dt \right| du$$

(see [B2]). Substituting the definition of λ and using the inverse formula, we obtain that

$$\int_0^\infty \sup_{u\leq|x|<\infty} \left| \varepsilon^{-1} \int_{\mathbf{R}} \varphi\left(\frac{t-x}{\varepsilon}\right) d\mu(t) \right| du < \infty \tag{16}$$

uniformly with respect to $\varepsilon > 0$. Let us show that μ is absolutely continuous everywhere except a neighborhood of the origin. Let $r > 0$. Then for each continuous function $g(x)$, vanishing for $|x| < r$, we have

$$\left| \int_{\mathbf{R}} g(x) d\mu(x) \right| = \left| \int_{\mathbf{R}} \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{\mathbf{R}} \varphi\left(\frac{t-x}{\varepsilon}\right) g(t)\, dt d\mu(x) \right|.$$

Using step by step the Lebesgue theorem and Fubini's theorem we obtain

$$\left|\lim_{\varepsilon\to 0}\varepsilon^{-1}\int_{\mathbf{R}} g(t)\left[\int_{\mathbf{R}}\varphi\left(\frac{t-x}{\varepsilon}\right)d\mu(x)\right]dt\right|$$

$$\le \int_{\mathbf{R}} |g(t)|\sup_{\varepsilon}\left|\varepsilon^{-1}\int_{\mathbf{R}}\varphi\left(\frac{t-x}{\varepsilon}\right)d\mu(x)\right|dt$$

$$\le \sup_{\varepsilon}\sup_{|t|>r}\left|\varepsilon^{-1}\int_{\mathbf{R}}\varphi\left(\frac{t-x}{\varepsilon}\right)d\mu(x)\right|\int_{|t|>r}|g(t)|\,dt.$$

Because of (16) this value is less than or equal to $C_r\int_{|t|>r}|g(t)|\,dt$. Hence, the functional $\int_{|x|>r}g(x)\,d\mu(x)$ may be extended by continuity to the space of all the functions integrable in the domain $\{x : |x| > r\}$. Taking into account a general form of the linear functional in this space (see [DS, Chapter IV, Theorem 5]), we obtain that there exists a function Ψ, with $\operatorname{ess\,sup}_{|x|>r}|\Psi(x)| < \infty$, such that

$$\int_{|x|>r} g(x)\,d\mu(x) = \int_{|x|>r} g(x)\Psi(x)\,dx.$$

This means that the measure μ is absolutely continuous. Now we complete the proof by substitution of $\Psi(x)\,dx$ for $d\mu(x)$ in (16), letting $\varepsilon \to 0$. □

5.2.

In the assumption that μ is absolutely continuous, Theorem 5 was obtained earlier by E. Belinskii [B1, B2] and P.-A. Boo [Boo]. Observe that Theorem 1 from the paper of Oberlin [O] is contained in these papers (and, of course, in Theorem 5). We must say a few words about Theorem 2 in [O], which establishes necessary and sufficient conditions of summability at L^p-Lebesgue points. This result is contained in the paper od Dyachkov [Dy] among many other interesting results (for example, necessary and sufficient conditions of summability at the points at which a function is the derivative of the indefinite integral of the function).

5.3.

The following statement establishes certain relations between $\mathbf{A}^*(\mathbf{T})$ and $\mathbf{A}^*(\mathbf{R})$ for functions of compact support.

Proposition 7.

If $\operatorname{supp}\lambda \subset \mathbf{T}$, *then the condition* $\lambda \in \mathbf{A}^*(\mathbf{R})$ *is equivalent to the following two conditions: after periodic continuation* $\lambda(x) \in \mathbf{A}^*(\mathbf{T})$ *and* $x\lambda(x) \in \mathbf{A}^*(\mathbf{T})$.

Proof. Let $\lambda \in \mathbf{A}^*(\mathbf{R})$. Then $\lambda(x + 2\pi) \in \mathbf{A}^*(\mathbf{R})$ yields $\lambda(x) + \lambda(x + 2\pi) \in \mathbf{A}^*(\mathbf{R})$, and hence $\mathbf{A}^*(\mathbf{R})_{\text{loc}}$. Therefore, we can continue λ periodically such that it is in $\mathbf{A}^*(\mathbf{T})_{\text{loc}}$, and consequently $\lambda \in \mathbf{A}^*(\mathbf{T})$ in virtue of Proposition 1 i. Since λ is boundedly supported, $x\lambda(x) \in \mathbf{A}^*(\mathbf{R})$, and this yields $x\lambda(x) \in \mathbf{A}^*(\mathbf{T})$ as above.

Now let $\lambda(x), x\lambda(x) \in \mathbf{A}^*(\mathbf{T})$. Set

$$\ell(x) = \begin{cases} 2x, & |x| \le \dfrac{\pi}{2}, \\ \pi\operatorname{sign}x, & \dfrac{\pi}{2} < |x| \le \pi, \end{cases}$$

and continue this function 2π-periodically. Then

$$\ell(x)\lambda(x) = (\ell(x) - x)\lambda(x) + x\lambda(x) \in \mathbf{A}^*(\mathbf{T})$$

since $\ell(x) - x$ is piecewise linear and equals 0 at the points $-\pi$ and π.

$$l_1(x) = (2\pi)^{-1}(\pi + \ell(x))\lambda(x),\ l_2(x) = (2\pi)^{-1}(\pi - \ell(x))\lambda(x) \in \mathbf{A}^*(\mathbf{T}) \text{ and } \mathbf{A}^*(\mathbf{T})_{\mathrm{loc}}.$$

If we consider a function $\lambda(x)$ on $\mathbf{T}$, as vanishing outside $\mathbf{T}$, then within $\mathbf{T}$ we have, by virtue of Proposition 6, that $\lambda \in \mathbf{A}^*(\mathbf{T})_{\mathrm{loc}}$ is equivalent to $\lambda \in \mathbf{A}^*(\mathbf{R})_{\mathrm{loc}}$. In neighborhoods of the points $-\pi$ and π, zero-continuation of λ coincides with l_1 and l_2, respectively. Membership of λ in $\mathbf{A}^*(\mathbf{R})_{\mathrm{loc}}$ follows from Proposition 6. Since λ has compact support, $\lambda \in \mathbf{A}^*(\mathbf{R})$. □

6. Concluding Remarks

We finished our survey of the main results around $\mathbf{A}^*$, both published and unpublished. We tried to give only such results that have some difference with $\mathbf{A}$ (either by their formulations or by their proofs). Of course, various results may be formulated and proved completely analogously as their $\mathbf{A}$-prototypes. In particular, this may be done for many results from [K] or, for example, [Ru, Chapter 7].

Let us introduce clarity into authorship, make some additional remarks, and point out several open problems.

Propositions 1, 2, and 7 and Theorem 3 obtained by RMT were the first results by which a systematic investigation of $\mathbf{A}^*$ [T1, T3] was begun. The short proof of Proposition 7 may be applied to obtaining the A-prototype (see [T2]) as well. To prove Proposition 2 we follow the method proposed in [T3, BT1]. Note that exact constants are what is significant in this proof. A very short proof without exact constants, given in [BT1], uses the equivalent norm (3).

Proposition 4 was obtained first by RMT [T1, T3], and Proposition 3 is obtained by ERL. Both proofs are slightly modified in order to give quantitative estimates (9) and (10). It will be very interesting to find some applications of these estimates.

We must note that the central problem of spectral synthesis, that is, existence of sets that are not of synthesis as well as existence of functions not admitting synthesis, is still open for $\mathbf{A}^*$.

Another question is connected with Beurling's result [Be]. It would be interesting to know whether the following statement, converse to that in §1.2, is true or not:

If for every $f \in \mathbf{A}^*$ we have $F \circ f \in \mathbf{A}$, then $F \in \mathrm{Lip}\,1$.

In Proposition 5 certain alterations in comparison with the proof known earlier, and, in particular, the proof of ii, which gives another method of proving Katznelson's corresponding result for $\mathbf{A}$, were made by ESB. Theorem 1 as well as the construction in Theorem 2 were found by ESB. It is very probable that ii in Theorem 1 is sharp for $p > 1$ as well as for $p = 1$.

Theorem 4 was obtained by RMT [T3, T5]. Theorem 5 was obtained jointly by ESB and RMT [BT2].

We can add that the asymptotic behavior of Fourier transforms of functions with a derivative from $\mathbf{A}^*(\mathbf{R})$ was studied by different methods in [T6, L2]

Some of the results presented have already been generalized to the multidimensional case, but they are not of special interest and do not demonstrate an essential difference between the one-dimensional and the multidimensional cases. So, we do not formulate them and can refer a reader to the papers [B1, BT2, T4, T6, L1–L3].

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Received January 24, 1994

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