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From the Original Framer to Present-Day Time-Frequency and Time-Scale Frames

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In 1952, the paper "A Class of Nonharmonic Fourier Series" by R.J. Duffin and A.C. Schaeffer was published in the Transactions of the American Mathematical Society. The nonharmonic Fourier series referred to in the title are series of the type

$$\sum_n c_n e^{i\lambda_n x}, \tag{1}$$

where the λ_n need not be equally spaced. (That is what makes the series nonharmonic.) The paper proves, for instance, that for appropriate sequences $(\lambda_n)_{n\in\mathbb{Z}}$ ("of uniform density 1"), and for square summable $(c_n)_{n\in\mathbb{Z}}$, the series (1) converges in the mean to a function $g \in L^2(-\pi, \pi)$; moreover the series (1) converges pointwise in $(-\pi, \pi)$ at every point where the Fourier series of g converges. (It converges therefore almost everywhere, but that was not known then.) But before that, the authors prove many other interesting results on these nonharmonic $e^{i\lambda_n x}$. If $(\lambda_n)_{n\in\mathbb{Z}}$ is of uniform density $d > 1$, then they show that there exist $A > 0$, $B < \infty$ so that, for every $g \in L^2(-\pi, \pi)$,

$$A \int_{-\pi}^{\pi} |g(t)|^2 \, dt \le \frac{1}{2\pi} \sum_n \left| \int_{-\pi}^{\pi} g(t) e^{i\lambda_n t} dt \right|^2 \le B \int |g(t)|^2 \, dt, \tag{2}$$

and they name this property: if (2) is satisfied then the sequence of functions $\{\exp(i\lambda_n t)\}$ is a **frame** for $L^2(-\pi, \pi)$. Thus was born the notion that would turn out to be the right framework for a different family of questions later. Section 3 of Duffin and Schaeffer's paper handles the general concept of frame in a Hilbert space; they discuss many of the basic properties, including the reconstruction of $v \in \mathcal{H}$ from the sequence $(\langle v, \phi_n \rangle)_n$ (where the $(\phi_n)_n$ constitute a frame in $\mathcal{H}$) via an algorithm with exponential convergence; the notion of an exact frame (a frame where every ϕ_n "counts": if any of them is deleted, the remaining sequence no longer constitutes a frame); and the observation that an exact frame constitutes a Riesz basis.

For many years, the role of frames was confined to their use as the tool for studying nonharmonic Fourier series constructed in Duffin and Schaeffer's paper, but a new life was waiting for them. In the early 80's, A. Grossmann, J. Morlet, and T. Paul had given a firm mathematical footing to earlier numerical results by J. Morlet, by showing that any function in $L^2(\mathbb{R})$ can be reconstructed via an integral transform from its wavelet coefficients:

$$f = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, \psi^{a,b} \rangle \psi^{a,b} \frac{da\,db}{a^2}, \tag{3}$$

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where $\psi^{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right)$, and ψ is such that $C_\psi = 2\pi \int_{-\infty}^{\infty} |\xi|^{-1} \left|\hat{\psi}(\xi)\right|^2 d\xi < \infty$. (This was in fact a rediscovery of a decomposition formula due to A. Calderón, as was realized a few years later, leading to a fraternization of harmonic analysts and researchers interested in signal analysis that is still having reverberations.) Although (3) provided some mathematical justification for Morlet's results, more was needed: in practice, one also obtained very good reconstructions by replacing the integrals in (3) by sums, discretizing $a = a_0^j$, $b = a_0^j b_0^k$, with $j, k \in \mathbb{Z}$. Everything would go reasonably well for a whole range of choices a_0, b_0, and then crash spectacularly after some thresholds were crossed. Rather than trying to view these expansions as discrete approximations of the integral (3), Grossmann was looking for a formulation of the problem that would be discrete from the start, and he knew he had found it when he came across a description of Duffin and Schaeffer's frames in Young's book [1]. Thus did frames start their second career, now in windowed Fourier expansions and in wavelet series. The first published use of the term "frames" in this context was in [2]; further elaborations were presented in [4]. (Note that it later turned out that some of the constructions in [2] were very similar to the work of Frazier and Jawerth [3], of which the authors of [2] were not aware at the time.) In [2] a new term in connection with frames was coined: if $A = B$ in (2), then the frame is **tight**.

Many more papers using frames, tight or not, in the time-frequency and time-scale analysis have followed since then, to the delighted amusement of frame father R.J. Duffin, to whom this issue is dedicated. There are still many aspects to be explored, as illustrated by the many different articles in this issue. Please join us in the world of frames and framers.

References

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