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Jacobi Polynomial Estimates and Fourier–Laplace Convergence

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1. Introduction

Wang has shown in [5] how to reduce convergence problems for Fourier–Laplace series on the sphere to the consideration of a certain equiconvergent operator defined by convolution with certain Jacobi polynomials. Here we derive new estimates for Jacobi polynomials with complex indices and use Stein's interpolation theorem to obtain L^p estimates for the corresponding maximal equiconvergent operator. This leads, in turn, to new results for almost everywhere convergence of Fourier–Laplace series.

The Jacobi polynomials, $P_k^{(\alpha,\beta)}(x)$ with $\alpha > -1, \beta > -1$, can be defined by (see [1, p. 62], formula (4.21.2))

$$P_k^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(k+1)} \sum_{j=0}^{k} C_k^j \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+k+j+1)}{\Gamma(\alpha+j+1)\Gamma(\alpha+\beta+k+1)} \left(\frac{x-1}{2}\right)^j .$$

This formula obviously also gives the definition for Jacobi polynomials with complex indices α, β such that $\Re\alpha > -1, \Re\beta > -1$. These are the polynomials we must estimate but our purposes require us to consider only certain types of complex index. In fact, we will prove the following.

Theorem 1.

Let $\alpha \in [0, 2n], \beta \in [0, n], n \in \mathbb{N}, \mu = \frac{1}{2} + i\tau, \tau \in \mathbb{R}$. *Then for* $k \in \mathbb{N}$

$$|P_k^{(\alpha+\mu,\beta)}(\cos\theta)| \le B_n e^{3|\tau|} k^{\alpha+\frac{1}{2}}, \qquad 0 < \theta < 2k^{-1}; \tag{1.1}$$

$$|P_k^{(\alpha+\mu,\beta)}(\cos\theta)| \le B_n e^{3|\tau|} k^{-\frac{1}{2}} \theta^{-\alpha-1} (\pi-\theta)^{-\beta-1}, \qquad 2k^{-1} < \theta < \pi - k^{-1}; \tag{1.2}$$

$$|P_k^{(\alpha+\mu,\beta)}(\cos\theta)| \le B_n e^{3|\tau|} k^{\beta+\frac{1}{2}}, \qquad \pi - k^{-1} < \theta < \pi . \tag{1.3}$$

Using this theorem and applying Stein's interpolation theorem (see [2]) we will prove the following.

Theorem 2.

Let $n \ge 3$, $\Omega_n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$. *If* $f \in L\log^2 L(\Omega_n)$, *i.e.,*

$$\int_{\Omega_n} |f(x)|(1 + \log_+^2 |f(x)|)dx < \infty$$

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Then

$$\lim_{N\to\infty} S_N(f)(x) = f(x), \qquad a.e. \tag{1.4}$$

where $S_N(f)$ denotes the Cesàro means at critical index of the Fourier–Laplace series of f.

We will prove Theorem 1 in Section 2. To prove Theorem 2 we will make use of the equicovergent operator of Cesàro means which was dicussed in detail in [5]. This operator is convenient for convergence problems. We will investigate the corresponding maximal operators in Section 3. Then, by the results in Section 3 we complete the proof of Theorem 2 in Section 4.

2. The Proof of Theorem 1

The following formula is obtained by analytic extension from the formula (3.9) of [3, p. 20].

$$\begin{aligned} P_k^{(\alpha+\mu,\beta)}(x) &= \frac{2^\mu\Gamma(\alpha+\mu+1)(1+x)^{\alpha+k+1}P_k^{(\alpha+\mu,\beta)}(1)}{\Gamma(\alpha+1)\Gamma(\mu)(1-x)^{\alpha+\mu}P_k^{(\alpha,\beta)}(1)} \\ &\int_x^1 \frac{(1-y)^\alpha}{(1+y)^{\alpha+\mu+1+k}} P_k^{(\alpha,\beta)}(y)(y-x)^{\mu-1}dy\,. \end{aligned} \tag{2.1}$$

For simplicity, we write

$$f(\theta,t) = \left(\frac{1}{\sin\frac{\theta}{2}\cos\frac{t}{2}}\right)^{i2\tau}\left(\frac{\sin\frac{t}{2}}{\sin\frac{\theta}{2}}\right)^{2\alpha+1}\left(\frac{\cos\frac{\theta}{2}}{\cos\frac{t}{2}}\right)^{2\alpha+2k+2}. \tag{2.2}$$

Then, by (2.1) we have for $0<\theta<\pi$ and $\mu=\frac{1}{2}+i\tau(\tau\in\mathbb{R})$

$$P_k^{(\alpha+\mu,\beta)}(\cos\theta) = \frac{\Gamma(\alpha+\mu+k+1)}{2^\mu\Gamma(\mu)\Gamma(\alpha+k+1)}\int_0^\theta \frac{f(\theta,t)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}}P_k^{(\alpha,\beta)}(\cos t)dt\,. \tag{2.3}$$

Now we apply the formulae 1.3.(2) and 1.3.(3) of [4], which state

$$\frac{\Gamma(u+v)}{\Gamma(u)} = e^{-\gamma v}\prod_{j=0}^\infty \frac{1}{1+\frac{v}{u+j}}e^{\frac{v}{j+1}}, \qquad u>0, v>0 \tag{2.4}$$

$$\frac{1}{|\Gamma(u+iv)|} = \frac{1}{\Gamma(u)}\prod_{j=0}^\infty\sqrt{1+\frac{v^2}{(j+u)^2}}, \qquad u>0, v\in\mathbb{R}\,. \tag{2.5}$$

By (2.4) we have

$$\log\frac{\Gamma(u+v)}{\Gamma(u)} = -\gamma v + \sum_{j=0}^\infty\left(\frac{v}{j+1}+\log\frac{j+u}{j+u+v}\right).$$

Write

$$f(x) = \frac{v}{x+1}+\log\frac{x+u}{x+u+v}, \quad x\ge 0\,.$$

Noticing $u\ge 1$, $v>0$ we have

$$\frac{d}{dx}f(x) = -\frac{v}{(x+1)^2}+\frac{v}{(x+u)(x+u+v)}<0\,.$$

Hence, we get

$$\int_0^\infty f(x)dx < \sum_{j=0}^\infty f(j) < f(0)+\int_0^\infty f(x)dx\,.$$

Integrating by parts we find

$$\int_0^\infty f(x)dx = -\int_0^\infty x\frac{d}{dx}f(x)dx = -v + u\log\frac{u+v}{u} + v\log(u+v)\,.$$

From these we derive that

$$e^{-1-\gamma v}(u+v)^v \le \frac{\Gamma(u+v)}{\Gamma(u)} \le e^{(1-\gamma)v}(u+v)^v, \qquad u \ge 1,\ v > 0\,. \tag{2.6}$$

Similarly, by (2.5) we have

$$\log\frac{1}{|\Gamma(u+iv)|} = \log\frac{1}{\Gamma(u)} + \sum_{j=0}^\infty \log\sqrt{1+\frac{v^2}{(j+u)^2}}\,.$$

Since

$$\begin{aligned}\sum_{j=0}^\infty \log\sqrt{1+\frac{v^2}{(j+u)^2}} &< \log\sqrt{1+\frac{v^2}{u^2}} + \frac{1}{2}\int_0^\infty \log\left(1+\frac{v^2}{(x+u)^2}\right)dx \\ &< \log\sqrt{1+\frac{v^2}{u^2}} + \int_0^\infty \frac{v^2}{(x+u)^2+v^2}\,dx\end{aligned}$$

we get

$$\sqrt{1+\frac{v^2}{u^2}}\frac{1}{\Gamma(u)} \le \frac{1}{|\Gamma(u+iv)|} \le \sqrt{1+\frac{v^2}{u^2}}\,\frac{1}{\Gamma(u)}e^{\frac{\pi}{2}|v|}, \qquad u > 0, v \in \mathbb{R}\,. \tag{2.7}$$

From (2.3), (2.6), and (2.7), we derive

$$|P_k^{(\alpha+\mu,\beta)}(\cos\theta)| \le B_n k^{\frac{1}{2}} e^{2|\tau|}\left|\int_0^\theta \frac{f(\theta,t)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}}P_k^{(\alpha,\beta)}(\cos t)dt\right|\,. \tag{2.8}$$

Then from the well-known estimate (see [1, p. 197])

$$|P_k^{(\alpha,\beta)}(\cos t)| \le B_n k^\alpha, \qquad 0 < t < \frac{2}{k}$$

we get (1.1).

When $2k^{-1} < \theta < \pi$, we break the integral on the right side of (2.8) into two parts, $\int_0^{k^{-1}}$ and $\int_{k^{-1}}^\theta$, and write them as I_k^1 and I_k^2, respectively.

For the first part we note that

$$\cos t - \cos\theta > \cos\frac{\theta}{2} - \cos\theta > B\theta^2, \quad 0 < t < 2k^{-1} < \theta < \pi\,.$$

Then we have

$$\begin{aligned}|I_k^1| &= \left|\int_0^{k^{-1}} \frac{f(\theta,t)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}}P_k^{(\alpha,\beta)}(\cos t)dt\right| \\ &\le B_n k^\alpha \int_0^{k^{-1}} \frac{t^{2\alpha+1}}{\theta^{2\alpha+\frac{3}{2}}}\frac{dt}{(\theta-t)^{\frac{1}{2}}} \le B_n\frac{1}{k\theta^{\alpha+1}}\,.\end{aligned} \tag{2.9}$$

In order to estimate I_k^2, we apply the following asymptotic estimate for Jacobi polynomial with real indices, which is the formula (8.21.18) in [1, p. 192].

$$P_k^{(\alpha,\beta)}(\cos\theta) = \sqrt{\frac{1}{\pi k}}\,\frac{\cos(N\theta-\gamma) + r(\alpha,\beta,k,\theta)(k\sin\theta)^{-1}}{\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}}, \quad \frac{1}{k} < \theta < \pi - \frac{1}{k}, \tag{2.10}$$

where $N = k + \frac{\alpha+\beta+1}{2}$, $\gamma = (\alpha+\frac{1}{2})\frac{\pi}{2}$ and $r(\alpha,\beta,k,\theta)$ satisfies

$$|r(\alpha,\beta,k,\theta)| \le B_n, \qquad \alpha,\beta \in [0,2n],\ k^{-1} < \theta < \pi - k^{-1}. \tag{2.11}$$

We see that

$$I_k^2 = (\pi k)^{-\frac{1}{2}}\int_{k^{-1}}^{\theta}\frac{f(\theta,t)\cos(Nt-\gamma)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}\left(\sin\frac{t}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{t}{2}\right)^{\beta+\frac{1}{2}}}dt + R(\alpha,\beta,k,\theta) \tag{2.12}$$

where

$$R(\alpha,\beta,k,\theta) = (\pi k)^{-\frac{1}{2}}\int_{k^{-1}}^{\theta}\frac{f(\theta,t)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}}\frac{r(\alpha,\beta,k,t)}{k\sin t\left(\sin\frac{t}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{t}{2}\right)^{\beta+\frac{1}{2}}}dt\,. \tag{2.13}$$

By (2.2) and (2.11), we get for $2k^{-1} < \theta < \pi - 2k^{-1}$

$$|R(\alpha,\beta,k,\theta)| \le \frac{B_n}{k\left(\sin\frac{\theta}{2}\right)^{\alpha+1}\left(\cos\frac{\theta}{2}\right)^{\beta}}\int_0^{\theta}\frac{dt}{\cos\frac{t}{2}(\cos t-\cos\theta)^{\frac{1}{2}}}\,. \tag{2.14}$$

By an easy calculation we know

$$\int_0^{\theta}\frac{dt}{\cos\frac{t}{2}(\cos t-\cos\theta)^{\frac{1}{2}}} = \frac{\pi}{\sqrt{2}\cos\frac{\theta}{2}}\,.$$

Hence, we obtain for $2k^{-1} < \theta < \pi - k^{-1}$

$$|R(\alpha,\beta,k,\theta)| \le \frac{B_n}{k\left(\sin\frac{\theta}{2}\right)^{\alpha+1}\left(\cos\frac{\theta}{2}\right)^{\beta+1}}\,. \tag{2.15}$$

Write

$$h(t) = \left(\sin\frac{t}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{t}{2}\right)^{-(2\alpha+\beta+2k+\frac{5}{2}+i2\tau)}$$

and write the integral on the right side of (2.12) as F_k. Then

$$F_k = \left(\cos\frac{\theta}{2}\right)^{2\alpha+2k+2}\left(\sin\frac{\theta}{2}\right)^{-(2\alpha+1+i2\tau)}\int_{k^{-1}}^{\theta}\frac{h(t)\cos(Nt-\gamma)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}}dt\,.$$

If we write $a = \alpha + \frac{1}{2}$, $b = 2\alpha+\beta+2k+\frac{5}{2}$, then we have

$$\frac{d}{dt}h(2t) = a\frac{(\sin t)^{a-1}}{(\cos t)^{b-1+i2\tau}} + (b+i2\tau)\frac{(\sin t)^{a+1}}{(\cos t)^{b+1+i2\tau}}\,.$$

Because $b > 2$, we have $|b+i2\tau| < be^{|\tau|}$ and hence

$$\left|\frac{d}{dt}h(t)\right| \le e^{|\tau|}\frac{d}{dt}|h(t)|\,. \tag{2.16}$$

Write

$$u_k(\theta) = \int_{k^{-1}}^{\theta-k^{-1}} \frac{h(t)\cos(Nt-\gamma)}{(\cos t - \cos\theta)^{\frac{1}{2}-i\tau}}dt, \quad v_k(\theta) = \int_{\theta-k^{-1}}^{\theta} \frac{h(t)\cos(Nt-\gamma)}{(\cos t - \cos\theta)^{\frac{1}{2}-i\tau}}dt .$$

Integrating by parts we have

$$u_k(\theta) = N^{-1}\frac{h(t)\sin(Nt-\gamma)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}}\Big|_{k^{-1}}^{\theta-k^{-1}}$$
$$+ N^{-1}\int_{k^{-1}}^{\theta-k^{-1}}\sin(Nt-\gamma)\left(\frac{h'(t)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}} + \frac{(\frac{1}{2}-i\tau)h(t)\sin t}{(\cos t-\cos\theta)^{\frac{3}{2}-i\tau}}\right)dt . \tag{2.17}$$

By (2.16) we have

$$\left|\int_{k^{-1}}^{\theta-k^{-1}}\sin(Nt-\gamma)\left(\frac{h'(t)}{(\cos t-\cos\theta)^{\frac{1}{2}-i\tau}} + \frac{(\frac{1}{2}-i\tau)h(t)\sin t}{(\cos t-\cos\theta)^{\frac{3}{2}-i\tau}}\right)dt\right| \le$$
$$\int_{k^{-1}}^{\theta-k^{-1}} 2e^{|\tau|}\frac{d}{dt}\frac{|h(t)|}{(\cos t-\cos\theta)^{\frac{1}{2}}}dt = 2e^{|\tau|}\frac{|h(t)|}{(\cos t-\cos\theta)^{\frac{1}{2}}}\Bigg|_{k^{-1}}^{\theta-k^{-1}} . \tag{2.18}$$

Meanwhile, we have

$$|v_k(\theta)| \le \int_{\theta-k^{-1}}^{\theta}\frac{|h(t)|}{(\cos t-\cos\theta)^{\frac{1}{2}}}dt \le B\frac{|h(\theta)|}{\sqrt{k\sin\theta}} . \tag{2.19}$$

Combining (2.17), (2.18), and (2.19) we get

$$|F_k| \le B_n e^{|\tau|}\left(\sqrt{k\sin\theta}\left(\sin\frac{\theta}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{\beta+\frac{1}{2}}\right)^{-1} \tag{2.20}$$

Substituting (2.15) and (2.20) into (2.12), we get for $2k^{-1} < \theta < \pi - k^{-1}$

$$|I_k^2| \le B_n e^{|\tau|}\frac{1}{k\left(\sin\frac{\theta}{2}\right)^{\alpha+1}\left(\cos\frac{\theta}{2}\right)^{\beta+1}} . \tag{2.21}$$

A combination of (2.8), (2.9), and (2.21) yields (1.2).

Next we assume $\pi - k^{-1} < \theta < \pi$. By (2.8) we have

$$|P_k^{(\alpha+\mu,\beta)}(\cos\theta)| \le B_n k^{\frac{1}{2}} e^{2|\tau|}\int_0^{\theta}\frac{|f(\theta,t)|}{\sqrt{\cos t-\cos\theta}}|P_k^{(\alpha,\beta)}(\cos t)|dt .$$

We have for $\pi - k^{-1} < \theta < \pi$

$$\frac{|f(\theta,t)|}{\sqrt{\cos t-\cos\theta}}\left|P_k^{(\alpha,\beta)}(\cos t)\right| \le B_n \begin{cases} t^{\alpha+1}, & \text{if } \ 0 < t < k^{-1}, \\ \frac{1}{\sqrt{k(\cos t-\cos\theta)}(\pi-t)^{\beta+\frac{1}{2}}}, & \text{if } \ k^{-1} < t < \pi - k^{-1}, \\ \frac{k^{\beta}}{\sqrt{\cos t-\cos\theta}}, & \text{if } \ \pi - k^{-1} < t < \theta . \end{cases}$$

From this we derive (1.3).

The proof of Theorem 1 is complete. □

Remark. If we take $\mu = \varepsilon + i\tau$, $\varepsilon \in (0, 1)$ in Theorem 1, then the same argument will yield the following estimate:

$$\left|P_k^{(\alpha+\varepsilon+i\tau,\beta)}(\cos\theta)\right|$$
$$\leq B_n e^{3|\tau|} \begin{cases} k^{\alpha+\varepsilon} & \text{if } 0 < \theta < k^{-1}, \\ \varepsilon^{-1} k^{-\frac{1}{2}} \theta^{-\alpha-\varepsilon-\frac{1}{2}} (\pi - \theta)^{-\beta-\varepsilon-\frac{1}{2}} & \text{if } k^{-1} < \theta < \pi - k^{-1}, \\ \varepsilon^{-1} k^{\beta+\varepsilon} & \text{if } \pi - k^{-1} < \theta < \pi. \end{cases}$$

This estimate is better than Theorem 1, which is good enough for our use but is definitely not the best possible. We guess that the factor ε^{-1} in the above estimate is removable and hence it should yield an estimate for $\varepsilon = 0$.

3. Equiconvergent Operators

Denote by σ_N^δ the Cesàro means of the Fourier–Laplace series on the unit sphere Ω_n. We will apply the main result of [5]:

Proposition 1.

Assume $\delta > -1$, D is a non-empty subset of Ω_n, $f \in L(\Omega_n)$ and $\sup\{|f(x)| : x \in D\} < \infty$. For uniform convergence on D, the following two relations are equivalent:

$$\lim_{N\to\infty} S_N^\delta(g_x)(x) = 0, \tag{3.1}$$

$$\lim_{N\to\infty} \sigma_N^\delta(g_x)(x) = 0 \tag{3.2}$$

where the function g_x is defined for each $x \in D$ by $g_x(y) = f(y) - f(x)$ and the operator S_N^δ is defined by

$$S_N^\delta(f)(x) = a_N^\delta \left(f * P_N^{(\frac{n-1}{2}+\delta, \frac{n-3}{2})}\right)(x), \qquad f \in L(\Omega_n), x \in \Omega_n, \delta > -1 \tag{3.3}$$

where

$$a_N^\delta = |\Omega_{n-1}|^{-1} \frac{\Gamma(N+\delta+n-1)\Gamma(2N+\delta+n)}{2^{n-2}\Gamma(\frac{n-1}{2}) A_N^\delta \Gamma(N+\frac{n-1}{2})\Gamma(2N+2\delta+n)}, \tag{3.4}$$

$$A_k^\delta = \frac{\Gamma(\delta+k+1)}{\Gamma(\delta+1)\Gamma(k+1)} \tag{3.5}$$

Remark. The expression (3.4) is a correction of the formula (4) in [5] by inserting a factor

$$2\,|\Omega_{n-1}|^{-1} = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi^{\frac{n-1}{2}}}.$$

Now we normalize the operator S_N^δ to obtain the following.

Definition 1.

We call the following operator

$$E_N^\delta(f) = S_N^\delta(f)\left(S_N^\delta(\mathbf{1})\right)^{-1}$$

where $\mathbf{1}$ denotes the constant function of value 1, the equiconvergent operator of σ_N^δ.

Next we give a multiplier representation for this operator. Denote by Y_k the projection operator on the subspace $\mathcal{H}_k^n$ of spherical harmonics of degree k. Write $P_N^{(\frac{n-1}{2}+\delta,\frac{n-3}{2})}$ as g_N, $c_{n,k}P_k^n$ as h_k temporarily for simplicity. Then we have

$$\begin{aligned} Y_k\left(S_N^\delta(f)\right)(x) &= a_N^\delta\left((f*g_N)*h_k\right) \\ &= a_N^\delta\int_{\Omega_n}\left(\int_{\Omega_n} f(u)g_N(uv)du\right)h_k(xv)dv \\ &= a_N^\delta\int_{\Omega_n} f(u)\left(\int_{\Omega_n} g_N(uv))h_k(xv)dv\right)du\,. \end{aligned}$$

By the Funk-Hecke formula (see [6]) we have

$$\int_{\Omega_n} g_N(uv)h_k(xv)dv = \alpha_{N,k}^\delta h_k(xu)\,,$$

where

$$\alpha_{N,k}^\delta = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{-1}^1 g_N(t)P_k^n(t)(1-t^2)^{\frac{n-3}{2}}dt\,. \tag{3.6}$$

Hence, we get

$$Y_k\left(S_N^\delta(f)\right)(x) = a_N^\delta\alpha_{N,k}^\delta\int_{\Omega_n} f(u)h_k(xu)du = a_N^\delta\alpha_{N,k}^\delta Y_k(f)(x)\,. \tag{3.7}$$

To calculate the value of $\alpha_{N,k}^\delta$, we apply Rodrigues' formula (see [6]) and get

$$\alpha_{N,k}^\delta = \frac{2\pi^{\frac{n-1}{2}}}{2^k\Gamma(k+\frac{n-1}{2})}\int_{-1}^1(1-t^2)^{k+\frac{n-3}{2}}\left(\frac{d}{dt}\right)^k g_N(t)dt\,.$$

We see that $\alpha_{N,k}^\delta = 0$, if $k > N$. When $0 \le k \le N$ we make use of the formula (see [1, p. 63], (4.21.7))

$$\left(\frac{d}{dt}\right)^k g_N(t) = 2^{-k}\frac{\Gamma(N+\delta+n+k-1)}{\Gamma(N+\delta+n-1)}P_{N-k}^{(\frac{n-1}{2}+\delta+k,\frac{n-3}{2}+k)}(t)$$

and get by partial integration that

$$\begin{aligned} \alpha_{N,k}^\delta &= \frac{2\pi^{\frac{n-1}{2}}\Gamma(N+\delta+n+k-1)}{4^k\Gamma(k+\frac{n-1}{2})\Gamma(N+\delta+n-1)}\int_{-1}^1 P_{N-k}^{(\frac{n-1}{2}+\delta+k,\frac{n-3}{2}+k)}(t)(1-t^2)^{k+\frac{n-3}{2}}dt \\ &= 2^{n-1}\pi^{\frac{n-1}{2}}A_{N-k}^\delta\frac{\Gamma(N+\delta+n+k-1)\Gamma(N+\frac{n-1}{2})}{\Gamma(N+\delta+n-1)\Gamma(N+k+n-1)}\,. \end{aligned} \tag{3.8}$$

Then, by (3.7) we get

$$S_N^\delta(\mathbf{I}) = Y_0(S_N^\delta(\mathbf{I})) = a_N^\delta\alpha_{N,0}^\delta\,.$$

We write

$$\gamma_N^\delta = (\alpha_{N,0}^\delta)^{-1} = \frac{\Gamma(\alpha+1)\Gamma(N+1)\Gamma(N+n-1)}{(4\pi)^{\frac{n-1}{2}}\Gamma(N+\delta+1)\Gamma(N+\frac{n-1}{2})}\,. \tag{3.9}$$

$$b_{N,k}^\delta = \frac{\alpha_{N,k}^\delta}{\alpha_{N,0}^\delta} = \frac{A_{N-k}^\delta}{A_N^\delta}\frac{\Gamma(N+\delta+n+k-1)\Gamma(N+n-1)}{\Gamma(N+\delta+n-1)\Gamma(N+n+k-1)}\,. \tag{3.10}$$

Then, by an analytic extension for the indexes we get the following.

Proposition 2.

Let δ be a complex number with $\Re\delta > -1$ and $N \in \mathbb{Z}_+$. Then

$$\begin{aligned} E_N^\delta(f)(x) &= \gamma_N^\delta \left(f * P_N^{(\frac{n-1}{2}+\delta, \frac{n-3}{2})}\right)(x) = \sum_{k=0}^{N} b_{N,k}^\delta Y_k(f)(x), \quad f \in L(\Omega_n), \quad x \in \Omega_n\,, \\ E_N^\delta(\mathbf{1})(x) &= 1, \qquad \textit{for constant function} \quad \mathbf{1}(x) = 1, x \in \Omega_n\,. \end{aligned}$$

Definition 2.

Let δ be complex number such that $\Re\delta > -1$. The maximal equiconvergent operator E_*^δ of Cesàro mean is defined by

$$E_*^\delta(f)(x) = sup\{|E_N^\delta(f)(x)| : \ N \in \mathbb{N}\}\,.$$

Theorem 3.

Let $n \geq 3$, $\lambda = \frac{n-2}{2}$ $\delta = \lambda + \varepsilon + i\tau$, $\tau \in \mathbb{R}$. If $\varepsilon \in (0, n)$, then

$$E_*^\delta(f)(x) \leq B_n \varepsilon^{-1} e^{3|\tau|}(HL(f)(x) + HL(f)(-x))$$

where HL denotes the Hardy-Littlewood maximal operator on sphere.

Proof. $\delta = \lambda + \varepsilon + i\tau$, $\varepsilon \in (0, n)$, $\tau \in \mathbb{R}$. First we note that by (3.9)

$$\left|\gamma_N^\delta\right| \leq B_n e^{2|\tau|} N^{\frac{1}{2}-\varepsilon}, \qquad N \in \mathbb{N}\,.$$

Then

$$\left|E_N^\delta(f)(x)\right| \leq B_n e^{2|\tau|} N^{\frac{1}{2}-\varepsilon} \left|f * P_N^{(\alpha,\beta)}(x)\right|$$

where $\alpha = n - \frac{3}{2} + \varepsilon + i\tau$, $\beta = \frac{n-3}{2}$. write

$$F(x,t) = \int_{xy=\cos t} f(y) d\sigma_t(y) \quad (0 < t < \pi)$$

where $\sigma_t(y)$ denotes the measure element on the surface $\{y \in \Omega_n : \quad xy = \cos t\}$. We have

$$\int_0^\theta |F(x,t)| dt \leq \theta^{n-1} HL(f)(x), \quad \theta \in (0,\pi) \tag{3.11}$$

and

$$F(x,t) = F(-x, \pi - t), \quad t \in (0,\pi)\,. \tag{3.12}$$

Then we get

$$\left|E_N^\delta(f)(x)\right| \leq B_n e^{2|\tau|} N^{\frac{1}{2}-\varepsilon} \int_0^\pi \left|F(x,t) P_N^{(\alpha,\beta)}(\cos t)\right| dt\,. \tag{3.13}$$

Now we break the integral in (3.13) into four parts:

$$I_1 = N^{\frac{1}{2}-\varepsilon} \int_0^{N^{-1}}, I_2 = N^{\frac{1}{2}-\varepsilon} \int_{N^{-1}}^{\frac{\pi}{2}}, I_3 = N^{\frac{1}{2}-\varepsilon} \int_{\frac{\pi}{2}}^{\pi - N^{-1}}, I_4 = N^{\frac{1}{2}-\varepsilon} \int_{\pi-N^{-1}}^{\pi},$$

and apply Theorem 1 to estimate these integrals. Accounting (3.11) and (3.12) we get

$$|I_1| \leq B_n e^{3|\tau|} N^{n-1} \int_0^{N^{-1}} |F(x,t)|dt \leq B_n e^{3|\tau|} HL(f)(x)\,, \tag{3.14}$$

$$|I_2| \leq B_n e^{3|\tau|} \frac{1}{N^\varepsilon} \int_{N^{-1}}^{\frac{\pi}{2}} \frac{|F(x,t)|}{t^{n-1+\varepsilon}} dt \leq B_n e^{3|\tau|} \varepsilon^{-1} HL(f)(x)\,, \tag{3.15}$$

$$|I_3| \leq B_n e^{3|\tau|} \frac{1}{N^\varepsilon} \int_{N^{-1}}^{\frac{\pi}{2}} \frac{|F(-x,t)|}{t^{\frac{n-1}{2}}} dt \leq B_n e^{3|\tau|} N^{-\varepsilon} HL(f)(-x)\,, \tag{3.16}$$

$$|I_4| \leq B_n e^{3|\tau|} N^{\frac{n-1}{2}-\varepsilon} \int_0^{N^{-1}} |F(-x,t)|dt \leq B_n e^{3|\tau|} N^{-\frac{n-1}{2}-\varepsilon} HL(f)(-x)\,. \tag{3.17}$$

A combination of (3.13) through (3.17) yields the desired inequality. □

Corollary of Theorem 3. If $\delta = \lambda + \varepsilon + i\tau$, $\varepsilon \in (0, n)$, $\tau \in \mathbb{R}$ and $1 < p < \infty$, then

$$\|E_*^\delta(f)\|_p \leq B_n e^{3|\tau|} \frac{p}{\varepsilon(p-1)} \|f\|_p\,. \tag{3.18}$$

Remark. For the maximal Cesàro operator $\sigma_*^\delta(f) = \sup\{|\theta_N^\delta(f)| : N \in \mathbb{N}\}$ it is well known (see [7, p. 239]) that

$$\|\sigma_*^\delta(f)\|_p \leq B_{\varepsilon,p} e^{c\tau^2} \|f\|_p, \; 1 < p < \infty\,.$$

But no estimate for the bound $B_{\varepsilon,p}$ exists. By the method in [7], which is based on the estimates for the *Cesàro* coefficients with complex indices, we would be able to obtain only $B_{\varepsilon,p} \leq B_n \varepsilon^{-2} \frac{p}{p-1} (\varepsilon \in (0,n))$ where the power of ε is -2. Our theorem gives -1. This is important because this power decides the power of $\log_+$ appearing in the conclusion of Theorem 2. This is the reason that we first give an appropriate estimate for Jacobi polynomials with complex indices and then make use of the equiconvergent operator which is a convolution with Jacobi polynomial as kernel.

Now we turn to estimate the L^2 bound of E_*^δ.

Theorem 4.

Let $\delta = \varepsilon + i\tau$, $\tau \in \mathbb{R}$. *If* $\varepsilon \in (0, n)$, *then*

$$\|E_*^\delta(f)\|_2 \leq B_n e^{4|\tau|} \varepsilon^{-1} \|f\|_2\,. \tag{3.19}$$

Proof. It is also known from [7, p. 239] that

$$\|\sigma_*^\delta(f)\|_2 \leq B e^{c\tau^2} \|f\|_2\,. \tag{3.20}$$

In [7] the dependence of the constant B on the parameters n, ε was not discussed (and the factor $e^{c\tau^2}$ is not exact.) But by the same argument we may refine (3.20) and get

$$\|\sigma_*^\delta(f)\|_2 \leq B_n \varepsilon^{-1} e^{3|\tau|} \|f\|_2, \quad \varepsilon > 0\,. \tag{3.21}$$

Then we apply the formula (5) of [5]:

$$\sigma_N^\delta(f) = S_N^\delta(f) + |\Omega_{n-1}|^{-1} T_N^\delta(f)$$

where the factor $|\Omega_{n-1}|^{-1}$ in front of T_N^δ was missed in [5]. Then we get

$$T_N^\delta(f) = \sum_{k=1}^{\infty} b_k(N,\delta) \sigma_N^{\delta+k}(f)$$

with $|b_k(N,\delta)| \leq B_n e^{|\tau|} k^{-1-n-\varepsilon}$ (see [5], Lemma 1) and hence

$$E_N^\delta(f) = (a_N^\delta \alpha_{N,0}^\delta)^{-1} \{\sigma_N^\delta(f) - |\Omega_{n-1}|^{-1} T_N^\delta(f)\}$$

So, by (3.21) we derive (3.19). □

4. Proof of Theorem 2

It is a routine application of Stein's interpolation theorem (see [2] or [8] for more details) to derive the $L^p(1 < p < 2)$ bound from Theorem 3 and Theorem 4. We omit the details and state the result:

Theorem 5.

If $1 < p < 2$, *then*

$$\|E_*^\lambda(f)\|_p \le B_n(p-1)^{-2}\|f\|_p \,. \tag{4.1}$$

Also, it can be regarded as routine to derive the $L\log_+^r L$ bound ($r > 0$) from the known L^p bound $(p-1)^{-r}$ following the argument stated in [8] (see [8]: Lemma 2 of Section 9 and the proof of Theorem D^*). Then we have the following.

Theorem 6.

$$\|E_*^\lambda(f)\|_1 \le B_n\left(\int_{\Omega_n} |f(x)|(1+\log_+^2|f(x)|)dx\right) . \tag{4.2}$$

As a consequence of Theorem 6 we establish Theorem 2.

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