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# Old Friends Revisited: the Multifractal Nature of Some Classical Functions

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ABSTRACT. *We study some explicit functions introduced by Riemann, Jordan, Lévy, Kahane... These functions share the property of having a dense set of discontinuities. We prove that they are examples of multifractal functions.*

## 1. Introduction

In his famous *Habilitationsshrift* of 1854 on trigonometric series [15], Bernard Riemann introduced the integral that now bears his name. To show that this new integral did in fact extend the Cauchy integral, Riemann defined several functions that were too irregular to be Cauchy integrable but nevertheless "Riemann integrable." One of these examples is

$$R(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}, \tag{1}$$

where $(x) = x - m$ if $|x - m| < 1/2$ and $(x) = 0$ if $x = m + 1/2, m \in \mathbf{Z}$.

Riemann remarked that $R$ is continuous except at the rationals $p/2q$ ($p$ and $2q$ having no common factor, which we denote by $p \wedge 2q = 1$), with the following right and left limits at these points

$$R\left(\frac{p}{2q}^-\right) = R\left(\frac{p}{2q}\right) + \frac{\pi^2}{16q^2} \qquad R\left(\frac{p}{2q}^+\right) = R\left(\frac{p}{2q}\right) - \frac{\pi^2}{16q^2}. \tag{2}$$

Thus $R$ is discontinuous on a dense set; but, in contrast with the characteristic function of rationals, the following property holds: For all $\epsilon > 0$, the set of points where $R$ has a discontinuity of amplitude larger than $\epsilon$ is finite. Thus $R$ is Riemann integrable, and for this reason we will call $R$ the *Integrable Riemann function*. It is possible, however, to analyze the regularity of $R$ more precisely.

Consider, for example, the neighborhood of a rational $\frac{p}{2q+1}$ (with an odd denominator). The function $(nx)$ has its discontinuities at the points $\frac{2k+1}{2n}$ ($k \in \mathbf{Z}$), so the distance between $\frac{p}{2q+1}$ and a discontinuity of $(nx)$ is at least $1/2n(2q+1)$; thus $(nx)$ is linear on the interval $[\frac{p}{2q+1} - h, \frac{p}{2q+1} + h]$ provided $h < \frac{1}{2n(2q+1)}$. Fix $h$ and take $N + 1 = [\frac{1}{2(2q+1)h}]$. Then $h < \frac{1}{2n(2q+1)}$ for $n = 1, \dots, N$ and

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$$\left(R\left(\frac{p}{2q+1}+h\right)-R\left(\frac{p}{2q+1}\right)\right)=\sum_{n=1}^{N}\frac{h}{n}+\sum_{n=N+1}^{\infty}\frac{\left(n\left(\frac{p}{2q+1}+h\right)\right)-\left(n\frac{p}{2q+1}\right)}{n^2}$$
$$=h\log\left(\frac{1}{2qh}\right)+O(h).$$

Thus $R$ is quite smooth at rationals with an odd denominator, and its modulus of continuity is $h\log(1/h)$ at these points. So we see that the regularity of $R$ can change completely from one point to another; this is in sharp contrast, for instance, with the Weierstrass functions

$$W_{a,b}(x)=\sum a^n\sin(b^n x)$$

that have a very regular irregularity. We make this point more precise.

The regularity of a function $f$ at a point $x_0$ is measured by the $C^\alpha(x_0)$ conditions, that is, $f$ is $C^\alpha(x_0)$ ($\alpha\geq 0$) if there exists a polynomial $P$ of degree at most $[\alpha]$ and a constant $C>0$ such that

$$|f(x)-P(x-x_0)|\leq C|x-x_0|^\alpha.$$

In practice one often uses the *Hölder exponent* of $f$ at $x_0$ defined by

$$h_f(x_0)=\sup\{\alpha : f \quad\text{is}\quad C^\alpha(x_0)\}$$

to measure the regularity of $f$ at $x_0$.

The Hölder exponent of $W_{a,b}$ is constant (and equal to $-\log a/\log b$), whereas we saw that $R$'s can change from point to point. Such behavior of the Hölder exponent is characteristic of *multifractal functions*, whose Hölder exponent jumps from one point to another in such an erratic way that the set of points $E_\alpha$ where the function has a given exponent $\alpha$ is a fractal set. The relevant parameters that one tries to determine are contained in the *spectrum of singularities*

$$d(\alpha)=\dim\{x_0 : h(x_0)=\alpha\}, \tag{3}$$

where dim stands for the Hausdorff dimension (and by convention $\dim(\emptyset)=-\infty$).

We will show that $R$ is a multifractal function. More precisely, let $\frac{p_n}{q_n}$ be the sequence of convergents of the continued fraction expansion of $x_0$. We define $\tau_n$ (as in [4]) by

$$\left|x_0-\frac{p_n}{q_n}\right|=\frac{1}{q_n^{\tau_n}}.$$

It is an elementary result from continued fractions that $\tau_n\geq 2$. Let

$$\tau(x_0)=\limsup_{n\in A}\tau_n, \tag{4}$$

where $A$ is the set of integers $n$ such that $q_n$ is even (if this set contains only a finite number of elements, we set $\tau(x_0)=2$). We will prove the following theorem in §2.

***Theorem 1.***

*The Hölder exponent of the integrable Riemann function $R$ at $x_0$ is*

$$h_R(x_0)=\frac{2}{\tau(x_0)},$$

*and its spectrum of singularities is given by*

$$d(\alpha)=\begin{cases}\alpha & \textit{for}\quad \alpha\in[0,1],\\ -\infty & \textit{elsewhere.}\end{cases}$$

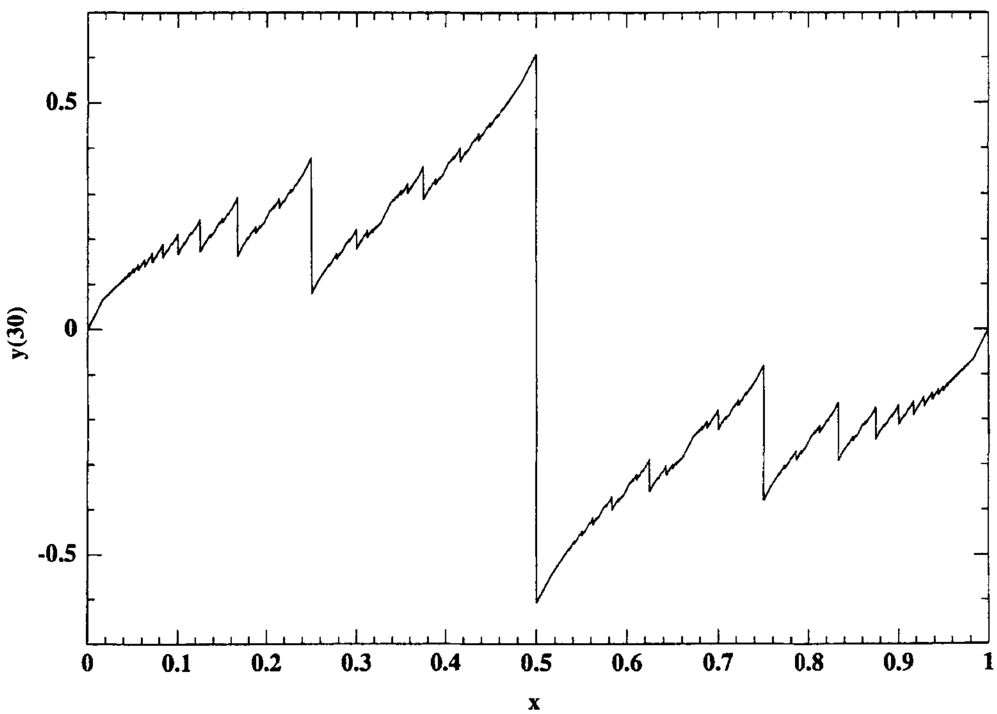


FIGURE 1. Riemann's integrable function $R$.

Note that this Hölder exponent is strikingly similar to the Hölder exponent of another function attributed to Riemann:

$$\mathcal{R}(x) = \sum \frac{1}{n^2} \sin(\pi n^2 x).$$

Indeed, let

$$\tau'(x_0) = \limsup_{n \in B} \tau_n,$$

where $B$ is the set of integers $n$ such that $p_n$ and $q_n$ are not both odd. The Hölder exponent of $\mathcal{R}$ is

$$h_{\mathcal{R}}(x_0) = \frac{1}{2} + \frac{1}{2\tau'(x_0)}$$

(see [4]). $\mathcal{R}$ is also a multifractal function with spectrum of singularities

$$d_{\mathcal{R}}(\alpha) = \begin{cases} 4\alpha - 2 & \text{if} \quad \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 0 & \text{if} \quad \alpha = \frac{3}{2}, \\ -\infty & \text{elsewhere.} \end{cases} \tag{5}$$

It would certainly be interesting to determine if there is a deeper relationship between these two functions.

We quote from [9]: the function $R$ "is exactly what Paul Lévy called a compensated jump function: all jumps are negative and their sum is infinite but the continuous parts of $(nx)$ provide a shift such that the series converges. Paul Lévy considerd the simpler function

$$\mathcal{L}(x) = \sum_{n=0}^{\infty} \frac{(2^n x)}{2^n}$$

as an illustration of what occurs frequently in the theory of stochastic processes with independant increments."

We will study the pointwise regularity of Lévy's function $\mathcal{L}$ and prove in §5 that it is another example of a multifractal function. Can we infer from Lévy's intuition that there are natural examples of stochastic processes with independant increments that are multifractal? It is actually the case for Lévy processes; see [6]. Note also that in [5] we prove that the simplest model of (random) lacunary wavelet series yields almost surely multifractal functions.

In a note to Compte Rendu [8] published in 1881, Jordan introduced the notion of bounded variation and proposed the following example of a function of bounded variation, which is nonetheless discontinuous on a dense set.

Let $\psi(m,n) > 0$ $(m, n \in \mathbf{Z})$ such that $\sum \psi(m,n) < \infty$; Jordan's example is

$$J(x) = \sum_{\frac{m}{n} < x} \psi(m,n).$$

In [9], Kahane remarks that Jordan could have used the following (slight) modification of $R$ as well

$$K(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^3}.$$

Indeed, this example of Kahane fulfills Jordan's purpose as well, for the following reason. Let us compute the amplitude of the jump of $J$ at $(2k+1)/2n$ $(2k+1$ and $2n$ having no common factor). All the functions $(mx)$ that have a jump at $(2k+1)/2n$ satisfy

$$\exists l \in \mathbf{Z} \qquad \frac{l + \frac{1}{2}}{m} = \frac{2k+1}{2n},$$

so $m = An$ and $2l + 1 = A(2k+1)$. The possible values of $A$ are exactly the odd integers, so the total jump at $(2k+1)/2n$ is

$$\sum_{A \text{ odd}} \frac{1}{(An)^3} = \frac{1}{n^3} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^3} = \frac{1}{n^3} \frac{7\zeta(3)}{8}.$$

$K$ can almost be considered as a particular case of the example proposed by Jordan using

$$\psi(m,n) = \begin{cases} \dfrac{1}{m^3} & \text{if } m \text{ is odd ,} \\ 0 & \text{otherwise.} \end{cases}$$

$K$ is the sum of this function composed of *pure jumps* and a linear compensation term.

The study of Kahane's example is quite similar to that of Riemann, and $K$ is another example of a multifractal function. We will prove the following theorem in §3.

***Theorem 2.***

*The Hölder exponent of Kahane's example $K$ at $x_0$ is*

$$h_K(x_0) = \frac{3}{\tau(x_0)}$$

*except when $x_0$ is a rational with an odd denominator, in which case*

$$h_K(x_0) = 3.$$

*The spectrum of singularities of $K$ is given by*

$$d_K(\alpha) = \begin{cases} \dfrac{2\alpha}{3} & \text{if} \quad \alpha \in \left[0, \dfrac{3}{2}\right], \\ 0 & \text{if} \quad \alpha = 3, \\ -\infty & \text{elsewhere.} \end{cases} \tag{6}$$

The only qualitative difference from $R$ is the regularity at rationals with an odd denominator. (Note that the resemblance to $\mathcal{R}$ is even more striking here.)

One can see Kahane's example as a modification of Riemann's function $R$. But an important property that is not shared with Riemann's function is that it is the primitive of a singular measure (up to a linear term). Indeed, the derivative of Kahane's example is

$$\frac{\pi^2}{6}x - \sum_{2k+1 \wedge 2n = 1} \frac{7\zeta(3)}{8n^3} \delta_{\frac{2k+1}{2n}}.$$

We can thus reinterpret Theorem 2 as follows. The measure

$$\mu_K = \sum_{2k+1 \wedge 2n = 1} \frac{1}{n^3} \delta_{\frac{2k+1}{2n}} \tag{7}$$

is a multifractal positive measure whose spectrum is given by (6). We will explain this assertion.

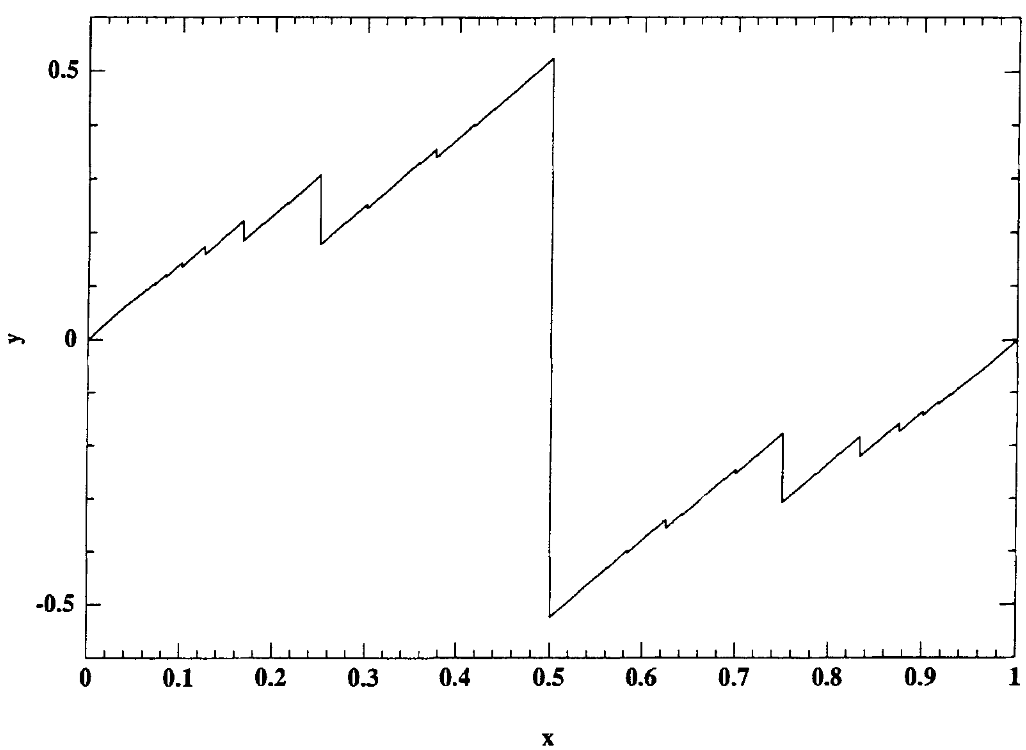


FIGURE 2. Kahane's example $K$.

The Hölder exponent of a measure $\mu$ at $x_0$ is defined by

$$h_\mu(x_0) = \sup\{\alpha : \exists C > 0;\ \ \forall\epsilon < 1\ \ \mu([x_0 - \epsilon, x_0 + \epsilon]) \le C\epsilon^\alpha\}. \tag{8}$$

The spectrum of singularities of $\mu$ is then defined as in (3). If $f$ is the primitive of the measure $\mu$,

$$\mu([x_0 - \epsilon, x_0 + \epsilon]) = f(x_0 + \epsilon) - f(x_0 - \epsilon)$$

and the exponents of $\mu$ and $f$ at $x_0$ are the same as long as one can choose $P(x - x_0) = f(x_0)$ in (2).This is always the case if $0 \le h(x_0) \le 1$. If $1 < h(x_0) \le 2$, the two exponents will coincide for an important class of functions: *purely singular increasing functions* (as called by Lebesgue and later by Salem). By definition, these functions are differentiable almost everywhere with a vanishing derivative at every point of differentiablility. (Actually, these authors require $f$ to be continuous; we do not make this asumption here.) The Devil's staircase is an example of a purely singular increasing function. We will see that $\frac{\pi^2 x}{6} - J(x)$ is another example.

Properties of such functions, or equivalently of their derivatives, have been extensively studied by many authors, starting with Jordan and Lebesgue and including Denjoy, Rajchman, and Salem. These authors often considered examples of specific functions, many of which we can now interpret as multifractal functions. We will examine some of them.

Let us try to differentiate Riemann's function $R$. If we are not careful, we obtain the difference between an infinite linear function $(\sum \frac{1}{n})x$ and an infinite measure

$$\sum_{2k+1 \wedge 2n=1} \frac{1}{n^2}\delta_{\frac{2k+1}{2n}}. \tag{9}$$

Of course, this calculation should be given credibility by differentiating $R$ in the sense of distributions. Thus, if $\psi$ is a $C^\infty$, one-periodic function, we obtain

$$\langle R'|\psi\rangle = \lim_{N\to\infty} \int \left(\sum_{n=1}^{N} \frac{1}{n}\right) \psi(x)\,dx - \left\langle \left(\sum_{n=1}^{N} \frac{1}{n^2} \sum_{2k+1\wedge 2n=1} \delta_{\frac{2k+1}{2n}}\right) \middle| \psi \right\rangle. \tag{10}$$

The limits of the two terms usually do not exist independently, except if $\int \psi = 0$. Thus (9) makes sense when integrated against functions with a vanishing integral and we can interpret (10) as the correct way to renormalize the infinite measure (9) by substracting the correct floating constant.

We will study measures (finite or infinite) similar to (9) in §4. We can actually slightly simplify the model given by (9) or by $\mu_K$ without changing the specific properties of these measures as follows. Consider the expressions

$$\mu_\beta = \sum_{m\wedge n=1} \frac{1}{n^\beta}\delta_{\frac{m}{n}}; \tag{11}$$

these are real measures if $\beta > 2$, but they need to be renormalized if $\beta \le 2$. We will explain how one can define a Hölder exponent for the measure $\mu_\beta$ when $1 < \beta \le 2$. When using this generalization, all measures $\mu_\beta$ will be multifractal ($1 < \beta < \infty$).

***Theorem 3.***

*If $\beta \ge 2$, the spectrum of singularities of* (11) *is*

$$d_\beta(\alpha) = \begin{cases} \dfrac{2\alpha}{\beta} & \textit{for}\ \ \alpha \in \left[0, \dfrac{\beta}{2}\right], \\ -\infty & \textit{elsewhere.} \end{cases}$$

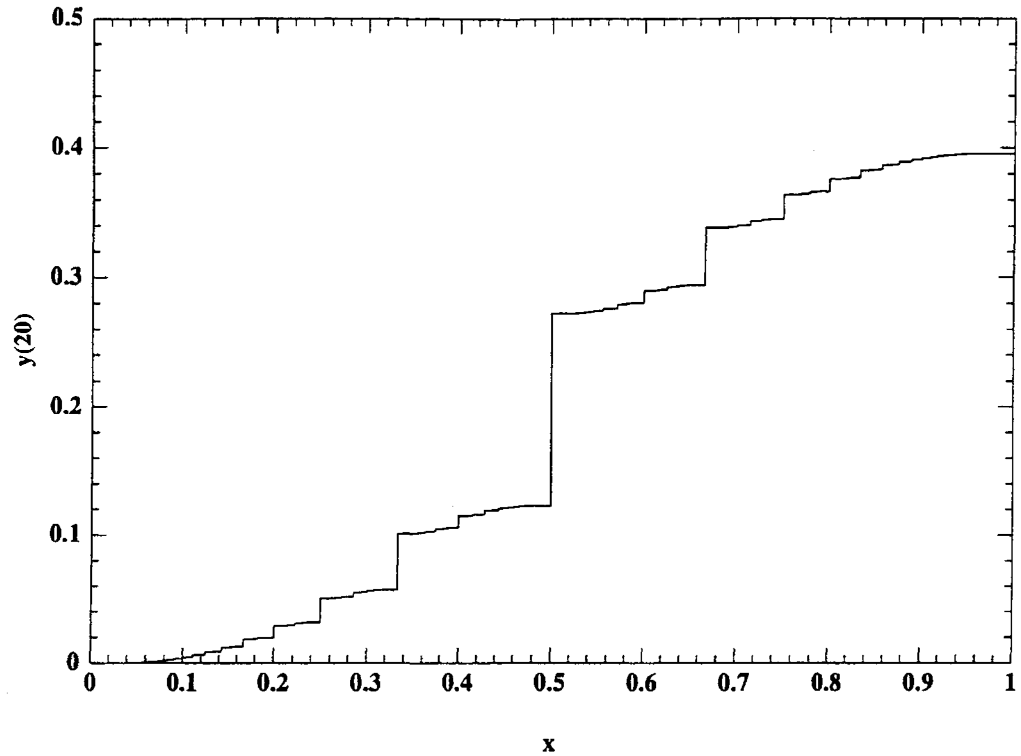


FIGURE 3. Rajchman's function.

*If* $1 < \beta < 2$, *the spectrum of singularities of* (11) *satisfies*

$$d_\beta(\alpha) = \begin{cases} \dfrac{2\alpha}{\beta} & \mathit{for} \quad \alpha \in [0, \beta - 1), \\ -\infty & \mathit{for} \quad \alpha > \dfrac{\beta}{2}. \end{cases}$$

Note that, in the case $\beta = 3$, (11) is the derivative of

$$R_a(x) = \sum_{n=1}^{\infty} \frac{E(nx)}{n^3}$$

(up to the numerical factor $\zeta(3)$). This function appears in a paper of Rajchman [13], where he studied purely singular increasing functions and proved that a convergent series of such functions is still purely singular. As an example he considers $R_a$ and thus obtains directly that it is almost everywhere differentiable. The proof of Theorem 3 will actually yield a more precise result: Rajchman's function $R_a$ is differentiable except on a set of Hausdorff dimension $2/3$.

Other functions that have been introduced in the past as examples or counterexamples of functions with unexpected properties turn out to be multifractal, for example, Polya's "triangle-filling" function, studied by B. Mandelbrot and the author in [7] and de Rham's function, studied by Y. Meyer in [12].

The motivation to consider multifractal functions came from physical problems [2, 11]. Within mathematics, it leads one to reconsider the historical examples we have mentioned with a new eye;

and by extending our understanding of these functions, we are able to perceive similarities and recurrent structures in what was before a collection of unrelated curiosities.

## 2. The Integrable Riemann Function

### 2.1. Upper Bounds of the Hölder Exponent

We start with an easy lemma that yields an upper bound for the Hölder exponent of any function having a dense set of singularities.

***Lemma 1.***

*Let $f$ be a function discontinuous on a dense set of points; $x_0 \in \mathbb{R}$; and $r_n$ be a sequence converging to $x_0$ such that, at each point $r_n$, $f$ has right and left limits; denote by $s_n$ the jump of $f$ at $r_n$. Then*

$$h_f(x_0) \leq \liminf \frac{\log s_n}{\log |r_n - x_0|}.$$

**Proof.** Let $P$ be a polynomial; since

$$|(f(r_n^+) - P(r_n - x_0)) - (f(r_n^-) - P(r_n - x_0))| = s_n,$$

there exists $r_n'$ arbitrarily close to $r_n$ such that

$$|f(r_n') - P(r_n' - x_0)| \geq \frac{s_n}{2}$$

and $|r_n' - x_0| \geq \frac{1}{2}|r_n - x_0|$. We choose $h = |r_n' - x_0|$ and deduce Lemma 1. □

We will now apply this lemma to Riemann's function $R$. Since $R$ has discontinuities at the rationals $p/2q$, we expect $R$ to be irregular at points well-approximated by these rationals and to be smooth at points badly approximated. (Actually we used this property of bad approximation to prove regularity at rationals with an odd denominator.) The Hölder regularity of $F$ at a point $x_0$ will thus depend on properties of diophantine approximation of $x_0$.

***Proposition 1.***

*Let $x_0$ be an irrational number; then*

$$h_R(x_0) \leq \frac{2}{\tau(x_0)}$$

*(in particular, for any $x$, $0 \leq h(x) \leq 1$).*

**Proof.** First we consider the case where $A$ is infinite. Let $n \in A$;

$$R\left(\frac{p_n}{q_n}^+\right) - R\left(\frac{p_n}{q_n}^-\right) = \frac{\pi^2}{8q_n^2}.$$

Since $|x_0 - \frac{p_n}{q_n}| = 1/q_n^{\tau_n}$, Lemma 1 implies that

$$h(x_0) \leq \liminf \frac{2}{\tau_n} = \frac{2}{\tau(x_0)}.$$

Suppose now that $q_n$ is odd for all $n \geq N$. In that case we consider

$$r_n = \frac{p_n + p_{n+1}}{q_n + q_{n+1}}.$$

Since $p_n q_{n+1} - q_n p_{n+1} = (-1)^{n+1}$, this fraction is under irreducible form; thus it has an even denominator. The jump of $R$ at $r_n$ is

$$\frac{\pi^2}{8(q_n + q_{n+1})^2} \geq \frac{1}{4q_{n+1}^2}. \tag{12}$$

On the other hand,

$$|x_0 - r_n| = \frac{1}{q_n + q_{n+1}} |(p_n - x_0 q_n) + (p_{n+1} - x_0 q_{n+1})|;$$

but $|p_n - x_0 q_n| \leq 1/q_{n+1}$, so

$$|x_0 - r_n| \leq \frac{1}{q_{n+1}} \left( \frac{1}{q_{n+1}} + \frac{1}{q_{n+2}} \right) \leq \frac{2}{q_{n+1}^2}. \tag{13}$$

Using Lemma 1, we obtain $h_R(x_0) \leq 1$, hence Proposition 1 in this case. □

## 2.2. An Estimate of the Hölder Exponent

In this section, we show how the determination of the Hölder exponent at irrationals can be reduced to a problem of Diophantine approximation that we will solve in the following section. Since $h_R \leq 1$, we have to estimate $R(x_0 + h) - R(x_0)$. Suppose that $h > 0$ and let $N = [1/h]$;

$$\sum_{n=N+1}^{\infty} \frac{(nx)}{n^2} \leq \frac{C}{N} \leq Ch,$$

so

$$R(x_0 + h) - R(x_0) = \sum_{n=1}^{N} \frac{(n(x_0 + h)) - (nx_0)}{n^2} + O(h)$$

(and the term $O(h)$ can be neglected since the Hölder exponent of $R$ is at most 1).

Let $E(x_0, h)$ be the set of rationals $r = p/q$ such that

$$\begin{aligned} &q \quad \text{is even,} \\ &r \in [x_0, x_0 + h], \\ &q \leq N \qquad \left( = \left[ \frac{1}{h} \right] \right). \end{aligned} \tag{14}$$

Each function $(nx)$ is linear on $[x_0, x_0 + h]$, with perhaps one jump (at most) of amplitude $\pi^2/8q^2$ if $r \in E(x_0, h)$. Thus

$$R(x_0 + h) - R(x_0) = \sum_{n=1}^{N} \frac{nh}{n^2} - \frac{\pi^2}{8} \sum_{r \in E(x_0,h)} \frac{1}{q^2} + O(h) = -\frac{\pi^2}{8} \sum_{r \in E(x_0,h)} \frac{1}{q^2} + O(h \log h). \tag{15}$$

The determination of the Hölder exponent of $R$ at $x_0$ is thus reduced to the estimation of

$$\sum_{r \in E(x_0,h)} \frac{1}{q^2}. \tag{16}$$

We separate two contributions in this sum. The first one comes from the rationals that are convergents of $x_0$, and the second one from the other rationals.

Let us first compute the contribution of the convergents. Since the $q_n$ grow at least geometrically, the order of magnitude of the sum is given by its first term; so, if $h = |x_0 - \frac{p_n}{q_n}|$,

$$h^{2/\tau_n} = \frac{1}{q_n^2} \leq \sum_{r \in E(x_0,h)} \frac{1}{q^2} \leq \frac{C}{q_n^2} = Ch^{2/\tau_n} \tag{17}$$

(where the sum in the middle is restricted to convergents). And the estimate

$$\sum_{r\in E(x_0,h)} \frac{1}{q^2} \leq Ch^{2/\tau_n}$$

a fortiori holds if

$$\left|x_0 - \frac{p_n}{q_n}\right| \leq h < \left|x_0 - \frac{p_{n-1}}{q_{n-1}}\right|.$$

We denote by $E'(x_0, h)$ the subset of $E(x_0, h)$ composed of rationals that are not convergents. If $p/q$ is not a convergent,

$$\left|\frac{p}{q} - x_0\right| \geq \frac{1}{2q^2}.$$

Furthermore, if $p/q \in E'(x_0, h)$, then $q \geq 1/\sqrt{2h}$. Thus the denominators of the rationals of $E'(x_0, h)$ satisfy

$$\frac{1}{\sqrt{h}} \leq q \leq \frac{1}{h}.$$

In order to estimate $\sum_{r\in E'(x_0,h)} \frac{1}{q^2}$, we take a large integer $m$ and split the interval $[1/2, 1]$ of the exponents of $h$ into $m$ subintervals

$$I_k = [\gamma_k, \gamma_{k+1}) = \left[\frac{1}{2} + \frac{k}{2^m}, \frac{1}{2} + \frac{k+1}{2^m}\right).$$

The following proposition will be proved in the following section.

***Proposition 2.***

*Denote by $E(m, k)$ the set of rationals $r \in E'(x_0, h)$ such that*

$$\frac{1}{h^{\gamma_k}} \leq q \leq \frac{1}{h^{\gamma_{k+1}}}. \tag{18}$$

*The number $N(h, k)$ of elements of $E(m, k)$ is bounded by*

$$\frac{1}{2 - \dfrac{1}{\gamma_{k+1}}} h^{1-2\gamma_{k+1}}.$$

Using Proposition 2, $\sum_{r\in E(m,k)} \frac{1}{q^2}$ is bounded by

$$\frac{2}{2 - \dfrac{1}{\gamma_{k+1}}} h^{2\gamma_k+1-2\gamma_{k+1}} \leq \frac{2}{2 - \dfrac{1}{\gamma_{k+1}}} h^{1-1/m}$$

and the following bound holds:

$$\sum_{r\in E(m,k)} \frac{1}{q^2} \leq C(m)h^{1-\frac{1}{m}}.$$

Using this bound, we deduce that

$$\sum_{r\in E'(x_0,h)} \frac{1}{q^2} \leq C(m)h^{1-\frac{1}{m}}$$

for all integers $m$, so

$$\sum_{r \in E'(x_0,h)} \frac{1}{q^2} = O(h^{1-\epsilon}) \quad \forall \epsilon > 0.$$

This estimate, together with (18), proves the first part of Theorem 1 (the proof is exactly the same if $h < 0$). In order to determine the number of rationals satisfying (18), we need to make an excursion into diophantine approximation.

## 2.3. Some Diophantine Approximation

The results we use in this section can all be found in Serge Lang's book [10]. Let $x_0$ be an irrational number and $g$ an increasing function, larger than 1.

**Definition 1.** The number $x_0$ is said to be of type less than $g$ if for any $B$ large enough, there exists a solution of the system

$$\left| x_0 - \frac{p}{q} \right| < \frac{1}{q^2} \tag{19}$$

$$\frac{B}{g(B)} \le q < B \tag{20}$$

($p$ and $q$ having no common factor). □

First note that the convergents $\frac{p_n}{q_n}$ satisfy (19). If $B = q_{n+1}$, then (20) holds if $\frac{B}{g(B)} \le q_n$ and, hence, if $g(q_{n+1}) \ge \frac{q_{n+1}}{q_n}$. But

$$\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| \le 2 \left| x_0 - \frac{p_n}{q_n} \right|,$$

which can also be written $\frac{1}{q_n q_{n+1}} \le 2/q_n^{\tau_n}$ or $q_{n+1} \ge \frac{1}{2} q_n^{\tau_n - 1}$; thus

$$g(q_{n+1}) \ge 2(q_{n+1})^{(\tau_n - 2)/(\tau_n - 1)}. \tag{21}$$

If we choose $g$ increasing and satisfying (21), a fortiori,

$$g(B) \ge 2B^{(\tau_n - 2)/(\tau_n - 1)}$$

if $B \in (q_n, q_{n+1}]$. Thus, the following corollary holds.

***Corollary 1.***

*Let $\tau' > \tau(x_0)$; the number $x_0$ is of type less than $t^{\frac{\tau'-2}{\tau'-1}}$.*

Let $\psi(t)$ be a positive decreasing function such that

$$\sum_{q=1}^{\infty} \psi(q) = +\infty.$$

Let

$$\theta(N) = \int_1^{\infty} \psi(t)\, dt,$$

and $\lambda(N)$ be the number of solutions of the inequalities

$$0 < x_0 - \frac{p}{q} < \frac{\psi(q)}{q}, \qquad 1 \le q < N.$$

Finally, let $\omega(t) = t\psi(t)$. The following result holds [10, Theorem 8].

***Theorem 4.***

*Let $x_0$ be an irrational of type less than $g$. If $\omega$ satisfies the three following properties:*

- $g = o(\omega)$
- *$\omega$ is increasing and tends to $+\infty$*
- *$\sqrt{\omega(t)g(t)}/t$ is decreasing for $t$ large enough.*

*Then*

$$\lambda(N) = \theta(N) + o(\theta(N)).$$

Recall that we want to estimate the number $N(h,k)$ of rationals $r = p/q$ satisfying

$$r \in [x_0, x_0 + h] \quad \text{and} \quad \frac{1}{h^{\gamma_k}} \leq q < \frac{1}{h^{\gamma_{k+1}}}.$$

These two conditions imply that

$$0 < x_0 - \frac{p}{q} \leq \left(\frac{1}{q}\right)^{\frac{1}{\gamma_{k+1}}}, \tag{22}$$

so $N(h,k)$ is bounded by

$$\lambda_{k+1}\left(\frac{1}{h^{\gamma_{k+1}}}\right), \tag{23}$$

where $\lambda_{k+1}(N)$ is the counting function associated with

$$\psi_{k+1}(q) = q\left(\frac{1}{q}\right)^{\frac{1}{\gamma_{k+1}}}.$$

First, when $\tau(x_0) > 2$, we will estimate $\lambda_{k+1}(N)$ using a function $g$ of the form $g(t) = t^{\beta}$.

Let us check the hypotheses of Theorem 4. First $\gamma_{k+1} \in (1/2, 1]$, so $\psi$ is positive decreasing and satisfies $\sum \psi(q) = +\infty$. Since $\omega(t) = t^{2-\frac{1}{\gamma_{k+1}}}$, the hypotheses of Theorem 4 will be satisfied if

$$\begin{aligned} &\beta \geq 0, \qquad 2 - \frac{1}{\gamma_{k+1}} > \beta, \qquad 2 - \frac{1}{\gamma_{k+1}} > 0, \\ &\frac{1}{2}\left(2 - \frac{1}{\gamma_{k+1}} + \beta\right) - 1 \leq 0, \qquad \beta < \frac{\tau(x_0) - 2}{\tau(x_0) - 1}. \end{aligned} \tag{24}$$

Since $1/2 < \gamma_{k+1} \leq 1$, we can choose any $\beta$ satisfying

$$0 < \beta < \inf\left(\frac{\tau(x_0) - 2}{\tau(x_0) - 1}, 2 - \frac{1}{\gamma_{k+1}}\right)$$

(as long as $\tau(x_0) > 2$).

Thus $\lambda_{k+1}(N) \sim \frac{1}{2-1/(\gamma_{k+1})} N^{2-\frac{1}{\gamma_{k+1}}}$; so, using Theorem 4, $\lambda_{k+1}(\frac{1}{h^{\gamma_{k+1}}})$ is bounded by

$$\frac{2}{2 - \dfrac{1}{\gamma_{k+1}}}\left(\frac{1}{h^{\gamma_{k+1}}}\right)^{2-\frac{1}{\gamma_{k+1}}} = \frac{2}{2 - \dfrac{1}{\gamma_{k+1}}} h^{1-2\gamma_{k+1}}.$$

Hence, Proposition 2 follows in this case.

If $\tau(x_0) = 2$, we take for $g$ an increasing function satisfying

$$g(q_{n+1}) \geq \frac{q_{n+1}}{q_n};$$

$g$ increases slower than any positive power of $t$, so the hypotheses of Theorem 4 still hold and Proposition 2 holds in this case.

## 2.4. The Spectrum of Singularities of R

Let $H_\tau$ be the set of all real numbers $\rho$ such that

$$\exists C > 0, \qquad \left|\rho - \frac{p_n}{q_n}\right| \leq \frac{C}{q_n^\tau}$$

for an infinity of values of $n$ such that $q_n$ is even (and if there is only a finite number of values of $n$ such that $q_n$ is even, we decide that $\rho \in H_2$). Let us prove that the $\mathcal{H}^{2/\tau}$ Hausdorff measure of $H_\tau$ is positive.

First, if $\tau = 2$, then $H_\tau = \mathbb{R}$ and the result holds. If $\tau > 2$, one uses the following classical lemma.

***Lemma 2.***

*If $p$ and $q$ have no common factor and if*

$$|qx_0 - p| < \frac{1}{2q},$$

*then $p/q$ is a convergent of $x_0$.*

Let $F_\tau$ be the set of all real numbers $x_0$ such that

$$\exists C > 0, \qquad \left|x_0 - \frac{p_n}{q_n}\right| \leq \frac{C}{q_n^\tau}$$

for an infinity of values of $n$ such that $p_n$ and $q_n$ are both odd.

The $\mathcal{H}^{2/\tau}$ Hausdorff measure of $F_\tau$ satisfies (see [4])

$$\mathcal{H}^{2/\tau}(F_\tau) > 0.$$

But, if $x_0 \in F_\tau$, $x_0/2 \in H_\tau$, then $p_n/2q_n$ is an irreducible fraction,

$$\left|\frac{x_0}{2} - \frac{p_n}{2q_n}\right| \leq \frac{C}{2p_n^\tau},$$

and Lemma 2 implies that $p_n/q_n$ is a convergent of $x_0/2$; thus $\mathcal{H}^{2/\tau}(H_\tau) > 0$.

Consider the set

$$H_\tau - \bigcup_{\tau'>\tau} H_{\tau'}.$$

The $\mathcal{H}^{2/\tau}$ measure of $\bigcup_{\tau'>\tau} H_{\tau'}$ vanishes; since $H_\tau$ has a $\mathcal{H}^{2/\tau}$ measure positive, $H_\tau - \bigcup_{\tau'>\tau} H_{\tau'}$ has dimension $2/\tau$.

If $x_0 \in H_\tau - \bigcup_{\tau'>\tau} H_{\tau'}$, since $x_0 \in F_\tau$, the first part of Theorem 1 implies that R is not smoother than $\frac{2}{\tau}$ at $x_0$ and, since $x_0 \notin \bigcup_{\tau'>\tau} H_{\tau'}$, $\varphi$ is $C^{\frac{2}{\tau}-\epsilon}(x_0)$ $\forall\epsilon > 0$; thus $h_R(x_0) = \frac{2}{\tau}$ and the dimension of $\{x_0 : \ h_R(x_0) = \frac{2}{\tau}\}$ is at least $2/\tau$.

Suppose that $h_R(x_0) = \frac{2}{\tau}$. Then $R$ is $C^{\frac{2}{\tau}-\epsilon}(x_0)$ $\forall\epsilon > 0$ and thus $x_0 \in H_{\tau'}$ $\forall\tau' < \tau$; thus

$$\left\{x_0 : \ h_R(x_0) = \frac{2}{\tau}\right\} \subset \bigcup_{\tau'>\tau} H_{\tau'}$$

and the dimension of $\{x_0 : \ h_R(x_0) = \frac{2}{\tau}\}$ is bounded by $2/\tau$, hence the second part of Theorem 1.

## 3. Jordan's Function

The study of Jordan's function differs from the study of Riemann's integrable function only at rationals with an odd denominator, and we will detail that point.

Recall that

$$J(x) = \sum \frac{(nx)}{n^3}$$

is continuous except at rationals $p/2q$, where $J$ has right and left limits, the jump of $J$ at such a point being $7\zeta(3)/8q^3$. Let $r = p/(2q+1)$ be a rational with an odd denominator. In order to bound the regularity of $J$ at $r$, we use Lemma 1; since $J$ has a discontinuity at a distance $h = C/n$ of $r$, the jump being of $C'/n^3$, we obtain that the Hölder exponent of $J$ at $r$ is at most 3. In contrast with Riemann's function $R$, this upper bound will turn out to be the right exponent at these rationals. Indeed, using the same notation as in §2.2,

$$J\left(\frac{p}{2q+1}+h\right) - J\left(\frac{p}{2q+1}\right) = \sum_{n=1}^{N-1} \frac{h}{n^2} + \sum_{n=N}^{\infty} \frac{\left(n\left(\frac{p}{2q+1}+h\right)\right) - \left(n\frac{p}{2q+1}\right)}{n^3} \tag{25}$$

We split each term $(n(\frac{p}{2q+1}+h)) - (n\frac{p}{2q+1}))$ as a sum of a linear term $nh$ and a certain number $\xi(n)$ of jumps; thus (25) is the sum of two terms, the first one being

$$\sum_{n=1}^{N-1} \frac{h}{n^2} + \sum_{n=N}^{\infty} \frac{h}{n^2} = \frac{h\pi^2}{6}$$

and the second one being

$$\sum_{n=N}^{\infty} \frac{\xi(n)}{n^3}. \tag{26}$$

Since $(x)$ has discontinuities at $(2k+1)/2$ $(k \in \mathbf{Z})$, $t$ is a point where $(n(\frac{p}{2q+1}+t))$ jumps if there exists $k$ such that

$$\left(n\left(\frac{p}{2q+1}+t\right)\right) = \frac{2k+1}{2},$$

hence

$$t = \frac{q - np + \frac{1}{2} + k(2q+1)}{n(2q+1)}.$$

Suppose that $h > 0$. The numerator is always larger than $1/2$; thus if

$$n < \frac{1}{2(2q+1)h} \quad (= A),$$

the function $(n(\frac{p}{2q+1}+t))$ has no jump on $[0, h]$.

If $A \le n < 3A$, the only contributions to (26) come from the values of $n$ satisfying

$$q - np \equiv 0 \mod 2q+1. \tag{27}$$

There exists a unique solution of (27) between $A$ and $A + 2q$; the other solutions form an arithmetic sequence of reason $2q+1$. The contribution of this whole sequence to (26) is thus between

$$\sum_{m=0}^{\infty} \frac{1}{(A + m(2q+1))^3} \quad \text{and} \quad \sum_{m=0}^{\infty} \frac{1}{(A + 2q + m(2q+1))^3},$$

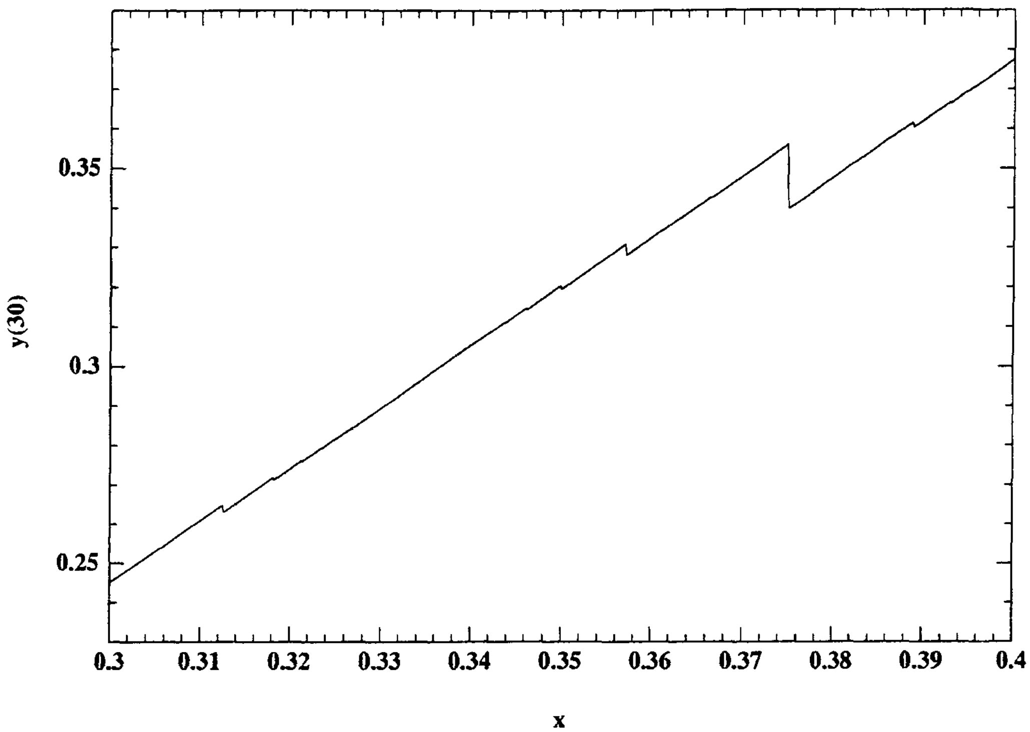


FIGURE 4. Jordan's function near $1/3$.

and the value of these two sums is

$$\frac{1}{2(2q+1)A^2} + O\left(\frac{1}{A^3}\right).$$

If $3a \leq n < 5A$, the values of $n$ satisfying

$$q - np \equiv 1 \mod 2q+1 \tag{28}$$

also contribute to (26). As above, this contribution amounts to

$$\frac{1}{2(2q+1)(3A)^2} + O\left(\frac{1}{(3A)^3}\right).$$

The same argument works for all possible values of $q - np + 1/2$, and finally (26) is equal to

$$\sum_{l=0}^{\infty} \frac{1}{2(2q+1)(2l+1)^2A^2} + O\left(\frac{1}{(2l+1)^3A^3}\right) = \frac{\pi^2(2q+1)}{4}h^2 + O(h^3).$$

We just proved the following proposition.

***Proposition 3.***

*The Hölder exponent of $J$ at $\frac{p}{2q+1}$ is 3, and the following expansion holds:*

$$J\left(\frac{p}{2q+1} + h\right) = J\left(\frac{p}{2q+1}\right) + \frac{\pi^2}{6}h - \frac{\pi^2(2q+1)}{4}h^2 + O(h^3).$$

Suppose now that $x_0$ is an irrational number. Proposition 1 becomes

$$h(x_0) \leq \frac{3}{\tau(x_0)} \tag{29}$$

The Hölder exponents at irrationals are thus between 0 and $3/2$.

Since

$$\sum_{N+1}^{\infty} \frac{(nx)}{n^3} \leq \frac{C}{N^2} \leq Ch^2,$$

it follows that

$$J(x_0+h) - J(x_0) = \sum_{n=1}^{N} \frac{(n(x_0+h)) - (nx_0)}{n^3} + O(h^2)$$

and (15) becomes

$$J(x_0+h) - J(x_0) = \sum_{n=1}^{N} \frac{nh}{n^3} - \frac{7\zeta(3)}{8} \sum_{r\in E(x_0,h)} \frac{1}{q^3} + O(h^2) = \frac{\pi^2}{6}h - \frac{7\zeta(3)}{8} \sum_{r\in E(x_0,h)} \frac{1}{q^3} + O(h^2).$$

The contributions of the convergents to

$$\sum_{r\in E(x_0,h)} \frac{1}{q^3} \tag{30}$$

are bounded by $Ch^{3/\tau_n}$; using the same method as in §2.2, the contribution to (30) of the rationals satisfying (22) is bounded by

$$Ch^{3\gamma_k} \left(\frac{1}{h^{\gamma_{k+1}}}\right)^{2-\frac{1}{\gamma_{k+1}}} \leq Ch^{\gamma_k+1-\frac{1}{m}}.$$

Since $\gamma_k \geq 1/2$,

$$\sum_{r\in E(m,h)} \frac{1}{q^3} \leq Ch^{\frac{3}{2}-\frac{1}{m}}.$$

We deduce Theorem 2 as in the case of Riemann's function.

Note that the derivative of $J(x) - \frac{\pi^2}{6}x$ vanishes at the points where it exists, so it is an example of increasing singular function. However, the term in $h^2$ in the Taylor expansion of $J$ at rationals with an odd denominator does not vanish, so the the spectrum of the singular measure $\mu_K$ differs slightly from the spectrum of $K$: they are the same except that the value $d(3) = 0$ in the spectrum of $K$ is replaced for $\mu_K$ by $d(2) = 0$.

## 4. From Rajchman Function to Renormalized Measures

We now consider the Rajchman function $R_a$, its derivative, the measure $\zeta(3)\sum_{p\wedge q=1} \frac{1}{q^3}\delta_{p/q}$, and more generally the distributions

$$\mu_\beta = \sum_{p\wedge q=1} \frac{1}{q^\beta}\delta_{p/q}.$$

The study of the pointwise regularity of $R_a$ is very similar to the study of Jordan's function. The exception is that, here, we do not have to make a specific study at rationals. The reader will easily check that Theorem 2 must be reformulated as follows.

If $x_0$ is an irrational number, let

$$\tau''(x_0) = \limsup_{n\to\infty} \tau_n;$$

the Hölder exponent of Rajchman's function $R_a$ at $x_0$ is

$$h_{R_a}(x_0) = \frac{3}{\tau''(x_0)}.$$

More generally, if $\beta > 2$, the same analysis as above yields the exponent

$$h_{\mu_\beta}(x_0) = \frac{\beta}{\tau''(x_0)}. \tag{31}$$

Using Lemma 1, (31) is also an upper bound for the Hölder exponent of the primitive $f_\beta$ of $\mu_\beta$, so the Hölder exponents of $f_\beta$ and $\mu_\beta$ coincide everywhere.

The spectrum of singularities of $\mu_\beta$ is calculated using the following remark. Let $E_\tau$ be the set of all real numbers $x_0$ such that

$$\exists C > 0 \qquad \left| x_0 - \frac{p_n}{q_n} \right| \le \frac{C}{q_n^\tau}$$

for an infinity of values of $n$ . The $\mathcal{H}^{2/\tau}$ Hausdorff measure of $H_\tau$ satisfies

$$\mathcal{H}^{2/\tau}(H_\tau) > 0$$

(see [1] or [4]). The spectrum of singularities of $\mu_\beta$ is thus obtained exactly as in the case of Riemann's function $R$:

$$d_{\mu_\beta}(\alpha) = \begin{cases} \dfrac{2\alpha}{\beta} & \text{for } \alpha \in \left[0, \dfrac{\beta}{2}\right], \\ -\infty & \text{elsewhere.} \end{cases}$$

We now consider the renormalized measures

$$\sum_{p\wedge q=1} \frac{1}{q^\beta} \delta_{p/q} \quad \text{when} \quad 1 < \beta \le 2.$$

Recall that this must be understood as the distribution

$$\lim_{N\to\infty} \sum_{q=1}^{N} \left( \frac{1}{q^\beta} \left\langle \sum_p \delta_{p/q} | \psi \right\rangle - \frac{1}{q^{\beta-1}} \int \psi \right) \qquad (= \lim \langle S_N | \psi \rangle) \tag{32}$$

(if $\psi$ is $C^\infty$ and 1-periodic). In order to be able to define an Hölder exponent of this distribution, we must check if we can take for $\psi$ the characteristic function of an interval. If $\psi = 1_{[a,b]}$,

$$\frac{1}{q^\beta} \left\langle \sum_q \delta_{p/q} | \psi \right\rangle - \frac{1}{q^{\beta-1}} \int \psi = \frac{|b-a|q + r}{q^\beta} - \frac{|b-a|}{q^{\beta-1}} = \frac{r}{q^\beta} \quad \text{with} \quad r \in \{-1, 0, 1\} \tag{33}$$

and the limit in (32) exists. We can now try to determine the order of magnitude of (32) when $\psi = 1_{[x_0,x_0+h]}$.

Thus, we denote by $A(x_0, h)$ the limit of (32) when $\psi = 1_{[x_0,x_0+h]}$ and define the Hölder exponent of the measure $\sum_{p\wedge q=1} \frac{1}{q^\beta}\delta_{p/q}$ by

$$h_\beta(x_0) = \liminf_{|h|\to 0} \frac{\log |A(x_0, h)|}{\log |h|}. \tag{34}$$

Of course, this is the same as computing the Hölder exponent of the function

$$A(x) = \lim_{N\to\infty} \langle S_N | 1_{[0,x]} \rangle,$$

which is the "renormalized primitive" of $\mu_\beta$. Furthermore, it coincides with the usual definition of the Hölder exponent of a measure when $\beta > 2$. Let us first show that $A$ is continuous at irrationals and estimate its Hölder exponent.

As usual, $\frac{p_n}{q_n}$ denotes the convergents of $x_0$, and we consider an increment

$$h = \left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{\tau_n}}.$$

Note first from (33) that we deduce

$$\sum_{q \geq 1/h} \left| \frac{1}{q^\beta} \left\langle \sum_p \delta_{p/q} | 1_{[x_0, x_0+h]} \right\rangle - \frac{h}{q^{\beta-1}} \right| \leq \sum_{q \geq 1/h} \frac{1}{q^\beta} \leq \frac{h^{\beta-1}}{\beta - 1} + O(1). \tag{35}$$

Of course,

$$\sum_{q < 1/h} \frac{h}{q^{\beta-1}} = \frac{h}{2-\beta} \left( \left( \frac{1}{h} \right)^{2-\beta} + O(1) \right) = \frac{h^{\beta-1}}{2-\beta} + O(1). \tag{36}$$

We must still estimate

$$\sum_{q<1/h} \frac{1}{q^\beta} \left\langle \sum_p \delta_{p/q} | 1_{[x_0, x_0+h]} \right\rangle.$$

- If $q < q_n$, because of the best approximation properties of convergents, no Dirac mass $\delta_{p/q}$ is supported in $[x_0, x_0 + h]$.
- The contribution of $q = q_n$ is
$$\frac{1}{q_n^\beta} = h^{\beta/\tau_n}. \tag{37}$$
- If $q > q_n$, we have
$$\frac{1}{h^{1/2}} < q < \frac{1}{h}.$$
As usual, we split the intervals $[1/2, 1]$ of the exponents of $1/h$ into arbitrarily small subintervals $[\gamma_k, \gamma_{k+1}]$ and apply Proposition 2 to each of these subintervals. We obtain a contribution bounded by
$$C_k h^{1-2\gamma_{k+1}} h^{\beta\gamma_{k+1}} \leq C_k h^{\beta-1}.$$

We see that the contribution of the convergents dominates when

$$\frac{\beta}{\tau''(x_0)} < \beta - 1;$$

in this case, the Hölder exponent is $\beta/\tau$. (The estimation for values of $h$ different from $|x_0 - \frac{p_n}{q_n}| = 1/q_n^{\tau_n}$ is straightforward.) When $\frac{\beta}{\tau''(x_0)} \geq \beta - 1$, we can only say that the Hölder exponent is larger than $\beta - 1$ (and smaller than $\beta/\tau''(x_0)$), hence Theorem 3. Note that the method we use cannot yield the spectrum between $\beta - 1$ and $\beta/2$.

## 5. Lévy's Function

Paul Lévy introduced

$$\mathcal{L}(x) = \sum_{n=0}^{\infty} \frac{(2^n x)}{2^n}$$

as an illustration of the type of discontinuities that a stochastic process with independent increments can have. $\mathcal{L}$ is clearly continuous at nondyadic points and discontinuous with right and left limits at dyadic points; at $\lambda = \frac{2k+1}{2^{n+1}}$, the jump of $\mathcal{L}$ is $2^{-n}$.

The regularity of $\mathcal{L}$ at a nondyadic point $x_0$ will clearly depend on the quality of approximation of $x_0$ by dyadics. Let us introduce the notation

$$\Delta_n(x) = \operatorname{dist}(x, 2^{-n}\mathbf{Z}).$$

***Proposition 4.***

*The Hölder exponent of $\mathcal{L}$ at $x_0$ is*

$$h_{\mathcal{L}}(x_0) = \liminf \frac{n}{\log_2 \Delta_n(x)}, \tag{38}$$

*and the spectrum of singularities of $\mathcal{L}$ is*

$$d_{\mathcal{L}}(\alpha) = \alpha \quad \text{for} \quad \alpha \in [0, 1]. \tag{39}$$

**Proof.** Let $x_0$ be given. Since $\mathcal{L}$ has, at a distance of $h = \Delta_n(x)$ from $x_0$, a jump of size at least $2^{-n}$, Lemma 1 implies that

$$h_{\mathcal{L}}(x_0) \leq \liminf \frac{n}{\log_2 \Delta_n(x)};$$

in particular, $h_{\mathcal{L}}(x_0) \leq 1 \;\; \forall x_0$.

Note that for $h = \Delta_n(x)$,

$$|\mathcal{L}(x_0 + h) - \mathcal{L}(x_0)| \leq 2^{-n} + O(h) \tag{40}$$

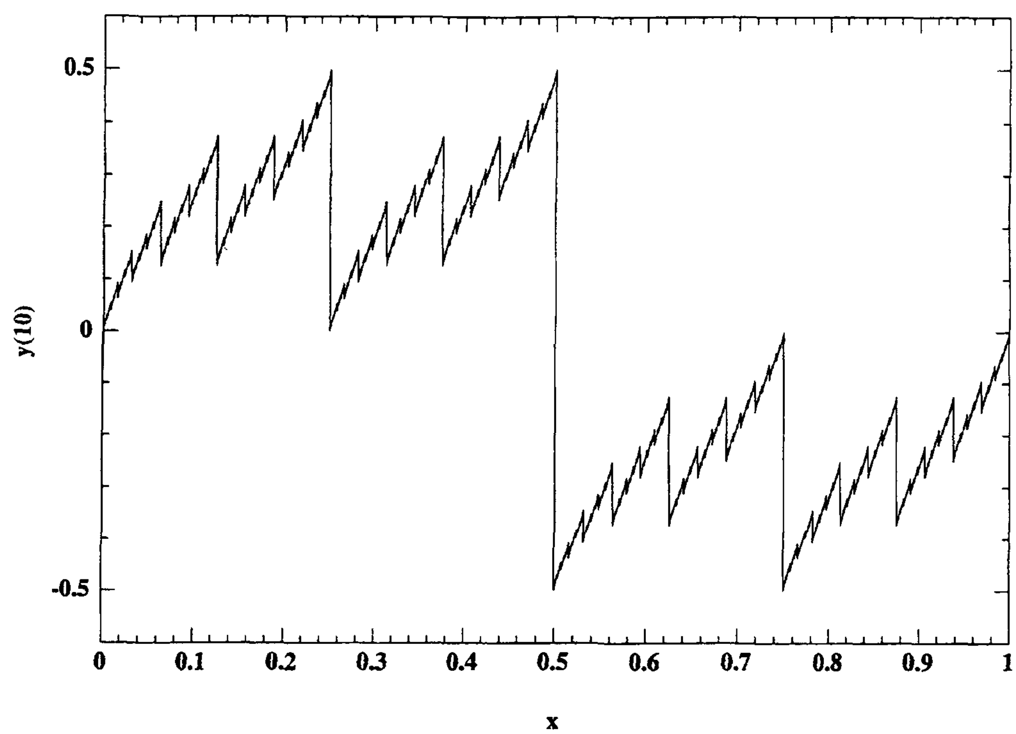


FIGURE 5. Lévy's function.

(if $n$ is the first index such that $h = \Delta_n(x)$) and (40) is a fortiori satisfied if $h$ lies between two values $\Delta_n$.

We still have to calculate the dimension of the set of points where $\mathcal{L}$ has a given Hölder exponent.

If $\alpha \geq 1$, let

$$E_\alpha = \limsup_{n\to\infty} \bigcup_k [k2^{-n} - 2^{-n\alpha}, k2^{-n} + 2^{-n\alpha}].$$

Clearly, dim $E_\alpha \leq 1/\alpha$; the converse inequality is almost as easy. We pick a very lacunary sequence $n_m$ ($n_m = 2^{n_{m-1}}$, for instance), and we construct a probability measure $\mu$ supported by

$$F_\alpha = \bigcap_m \left( \bigcup_k [k2^{-n_m} - 2^{-n_m\alpha}, k2^{-n_m} + 2^{-n_m\alpha}] \right).$$

If $m = 0$, we put on each interval $[k2^{-n_0} - 2^{-n_0\alpha}, k2^{-n_0} - 2^{-n_0\alpha}]$ the same mass $2^{-n_0}$. Each of these intervals contains $A(k, n_0) = 2.2^{-n_0\alpha}(2^{n_1} + O(1))$ intervals $[l2^{-n_1} - 2^{-n_1\alpha}, l2^{-n_1} + 2^{-n_1\alpha}]$; on each of these intervals, we put the measure $2^{-n_0}/A(k, n_0)$. We iterate this construction and thus obtain at the limit a probability measure $\mu$ supported by $F_\alpha$. One easily checks that $\forall x \in F_\alpha$

$$\mu([x - h, x + h]) \leq ch^{1/\alpha}.$$

We use Proposition 4.9 of [1], which implies that

$$\mathcal{H}_{1/\alpha}(F_\alpha) > 0. \tag{41}$$

Since $F_\alpha \subset E_\alpha$,

$$\dim (E_\alpha) = \frac{1}{\alpha}. \tag{42}$$

Using (38), $h_{\mathcal{L}}(x_0) = \beta$ if and only if

$$x_0 \in \bigcap_{\gamma>\beta} E_{1/\gamma} - \bigcup_{\gamma<\beta} E_{1/\gamma}.$$

From (41) and (42) we deduce that the dimension of the set of points where $h_{\mathcal{L}}(x_0) = \beta$ is $\beta$; hence Proposition 4 is proven. □

## 6. Concluding Remarks: Direct Methods vs. Wavelets

Lévy's function can be seen as a modification of the Weierstrass function

$$\sum 2^{-n} \sin 2^n x,$$

where the sine function is replaced by $(x)$. Let us compare the determination of the pointwise regularity of these two functions. As regards Lévy's function, Lemma 1 immediately yields the right upper bound for the Hölder exponent. As regards the Weierstrass function, the oscillations of the sine functions make it difficult to obtain upper bounds for the Hölder exponent. Actually, Weierstrass, using only 'by hand' methods did not get optimal results; Hardy in 1916 had the idea of estimating a convolution product of the analyzed function with a well-localized function having one vanishing moment (the derivative of the Poisson kernel). This idea, which announces wavelet methods, yields optimal results (see [3, 4]). Up to now, wavelet methods were used to study multifractal functions (see [4, 5, 7] and references therein). However, wavelet methods cannot be applied to the functions we study in this paper, since these functions have a dense set of discontinuities; and all existing criteria on the wavelet transform that imply a pointwise Hölder condition also imply that the function is continuous in a small neighborhod of the point that is considered. Thus no existing wavelet criterium could possibly give the pointwise regularity of functions that have a dense set of discontinuities.

Roughly speaking, if $f$ is a series of piecewise linear functions, direct methods for estimating the modulus of continuity usually yield optimal results; and if $f$ has a minimal regularity, one should probably use wavelet methods.

Of course, particularly simple cases are the functions that belong to both of these categories. An example is the Takagi function

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{|(2^n x)|}{2^n}. \tag{43}$$

(Note that $|(x)|$ is the "hat function," which is the first function of the Schauder basis.) This function was introduced by Takagi in [15] as a particularly simple example of a continuous nowhere differentiable function, and it was rediscoverd by de Rham later [14]. To study this (monofractal) function, we can either compute directly the increments of the function or notice that (43) yields immediately the expansion of $T(x)$ in the Schauder basis and use a wavelet criterion. Both methods give $h_T(x) = 1$ everywhere.

More interesting is the case of functions that belong to none of the categories we mentioned. Consider, for instance,

$$\mathcal{F}(x) = \sum_{n=1}^{\infty} \frac{\phi(2^n x)}{2^n}, \tag{44}$$

where $\phi$ is one-periodic, discontinuous, piecewise smooth, but not piecewise linear; and suppose, for instance, that it is discontinuous at $1/2$. None of the methods we considered applies. The situation is not desperate however, we can write $\phi$ as a linear combination of the function $(x)$ and a Lipschitz

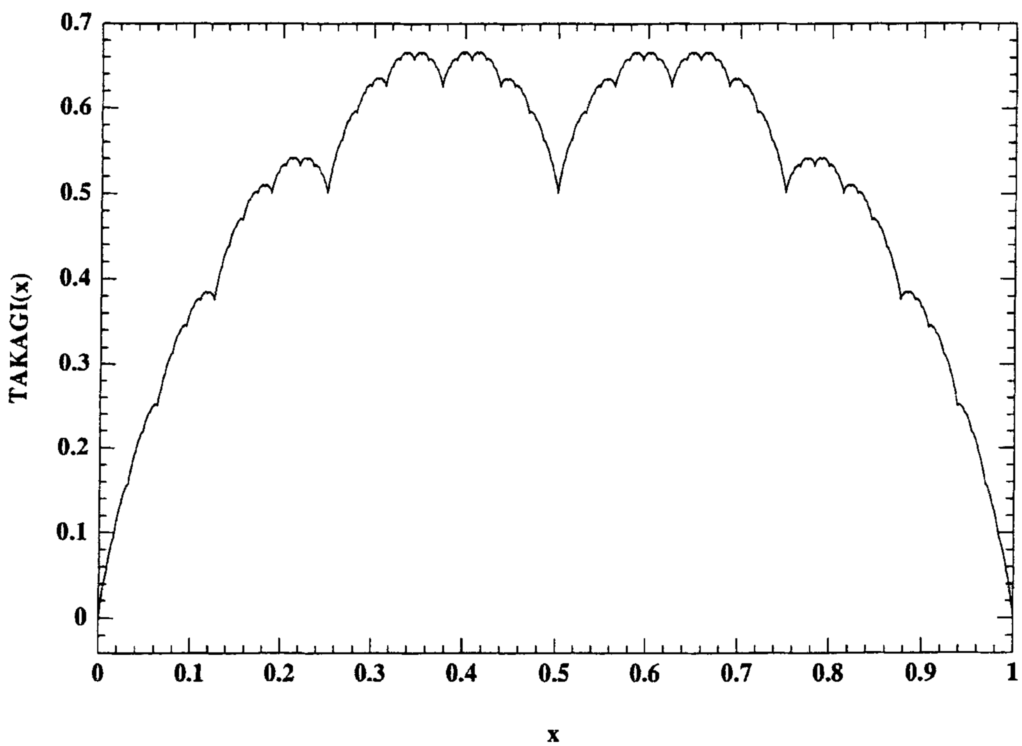


FIGURE 6. Takagi's function.

function. The Hölder exponent is thus the same as for Lévy's function. That is, the only problem might be at the points where both functions have the same exponent; but, in this case it is equal to 1, so that the exponent of $\mathcal{F}$ is larger than 1; and it is actually equal to 1 because of Lemma 1. We leave to the reader the amusing cases where the discontinuity of $\phi$ is not at $1/2$.

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