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The relative volume growth of minimal submanifolds

By

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Abstract. The volume growth of certain well-defined subsets of minimal submanifolds in riemannian spaces are compared with the volume growth of balls and spheres in space forms of constant curvature.

1. Introduction. Suppose we intersect a minimal submanifold P^m with a metric $r-$ball in a given ambient space N^n. Suppose further that the center of the cutting ball is a point p on P and that $r \leqq \min\{i_N(p), \frac{\pi}{2\sqrt{b}}\}$, where b is the supremum of the sectional curvatures of N, and $i_N(p)$ is the injectivity radius of N from p. Then the connected component of the intersection which contains p is called an extrinsic $m-$dimensional minimal $r-$ball in N^n, and we denote it by D_r .

The quotient between the volume of D_r in N and the volume of any metric $r-$ball $B_r^{b,m}$ in the $m-$dimensional space form of constant curvature b is known to be a monotone non-decreasing function of r in case $b \leqq 0$. This monotonicity was first observed by Anderson in [1].

In the present note we apply the co-area formula (as previously considered and used in [11]) to present an alternative proof of this result.

We also use the same technique to obtain a similar monotonicity result in the case of ambient spaces with a positive upper bound on their sectional curvatures. In that case, the quotient considered is expressed as the *difference* in the volumes of D_r and $B_r^{b,m}$ divided by the volume of the $m-$dimensional sphere $S_r^{b,m}$ in the $(m+1)-$dimensional space form $\mathbb{K}^{m+1}(b)$ of constant curvature b.

In both cases, when $b \neq 0$, we also obtain a corresponding rigidity result to the effect that the derivative of any of the quotients considered can only vanish for a given value of the radius if it vanishes everywhere and P^m is a minimal radial cone in N^n.

Specifically we show the following theorem:

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Theorem 1. *Let P^m be a minimally immersed submanifold of N^n and let us suppose that the sectional curvatures K_N of N satisfy $K_N \leqq b$, ($b \in \mathbb{R}$), then:*

(i) *If $b \leqq 0$, the function $f(r) = \frac{\mathrm{vol}(D_r)}{\mathrm{vol}\left(B_r^{b,m}\right)}$ is monotone non-decreasing in r.*

(ii) *If $b > 0$, the function $g(r) = \frac{\mathrm{vol}(D_r) - \mathrm{vol}\left(B_r^{b,m}\right)}{\mathrm{vol}\left(S_r^{b,m}\right)}$ is monotone non-decreasing in r.*

When $b \neq 0$ we get the following associated rigidity result:

If there exists an $r_0 > 0$ such that $f'(r_0) = 0$, (respectively, $g'(r_0) = 0$), then the extrinsic ball D_{r_0} is a minimal cone in N^n. Thus, if furthermore $N^n = \mathbb{K}^n(b)$, the space form of constant curvature b, then P^m is a totally geodesic submanifold of N^n.

The proof of this theorem is based on the co-area formula and on isoperimetric inequalities which have been established by the authors in [10] and [11].

As a consequence of Theorem 1, we have the following

Corollary 2. *Let P^m be a minimally immersed submanifold of N^n and suppose that the sectional curvatures K_N of N satisfy $K_N \leqq b$, ($b \geqq 0$). Then, for every extrinsic ball in N^n we have*

$$\mathrm{vol}(D_r) \geqq \frac{\Gamma\left(\frac{m+1}{2}\right)}{m\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)} \mathrm{vol}\left(S_r^{b,m}\right).$$

If the equality is attained for some extrinsic ball D_{r_0}, then D_{r_0} is a minimal cone in N^n and $K_N \leqq 0$. In fact, therefore, if equality is attained and if N^n is assumed to be a simply connected manifold with sectional curvatures satisfying $0 \leqq K_N \leqq b$ then N^n is the $n-$dimensional Euclidean space $\mathbb{R}^n$, and P^m is an $m-$dimensional plane in $\mathbb{R}^n$.

The outline of the paper is as follows: we shall prove Theorem 1 and Corollary 2 in Sections 5 and 6. Sections 2, 3 and 4 are devoted to describe some previous results, to show an application of co-area formula and to show some relevant inequalities relating the volumes of balls and spheres in space forms of constant curvature.

2. Preliminaries. Given an immersed submanifold P^m of a complete riemannian manifold N^n, the distance function on the ambient space N^n will be denoted by d, so, if $p \in P$, we define $r(q) := d(p, q)$ for every $q \in N$. We also denote by r the restriction $r|_P : P \longrightarrow \mathbb{R}$. This restriction is called the extrinsic distance to p in P.

The extrinsic ball of radius r and center $p \in P$, $D_r(p) \subseteq P$, can be viewed as the connected component of the restriction $B_r^n(p) \cap P = \{q \in P / r_p(q) \leqq r\}$ which contains p. It is a compact domain in P with boundary $\partial D_r(p)$. When we consider the totally geodesic submanifold $\mathbb{K}^m(b) \subseteq \mathbb{K}^n(b)$, then the corresponding extrinsic r-ball centered at $\tilde{p} \in \mathbb{K}^m(b)$, $D_r^b(\tilde{p})$ will be the geodesic r-ball $B_r^{b,m}$ centered at $\tilde{p}$ in this submanifold, and its boundary will be the geodesic sphere $S_r^{b,m-1}$. We shall refer to this setting as a *standard setting*.

We also recall that when we take the normal to the geodesic sphere in $\mathbb{K}^n(b)$, pointing inward, the constant mean curvature of any geodesic sphere of radius r in a space form of constant curvature b is given by the function

$$h_b(r) = \begin{cases} \sqrt{b}\cot\sqrt{b}r, & \text{if } b > 0 \\ \frac{1}{r}, & \text{if } b = 0 \\ \sqrt{-b}\coth\sqrt{-b}r, & \text{if } b < 0. \end{cases}$$

Further we shall denote by $\operatorname{grad}^N r$ and $\operatorname{grad}^P r$ the corresponding gradients of r in N and P respectively. Note that $\operatorname{grad}^P r(q)$ is just the tangential component in P of $\operatorname{grad}^N r(q)$, for all $q \in P$. Then we have the following basic relation on $\partial D_R(p)$, for all R: (see [6, eq. (2.1)])

$$\operatorname{grad}^N r = \operatorname{grad}^P r + (\operatorname{grad}^P r)^\perp$$

where $(\operatorname{grad}^P r)^\perp(q)$ is perpendicular to $T_q P$ for all $q \in \partial D_R(p)$.

As mentioned in the introduction, the proof of Theorem 1 is based on the following isoperimetric inequalities:

Theorem A ([10]). *Let P^m be a minimally immersed submanifold of N^n and let $D_r(p)$ be an extrinsic r-ball in P^m.*

If the sectional curvatures K_N of N satisfy $K_N \leqq b$, $(b > 0)$, then:

$$\frac{\operatorname{vol}(\partial D_r) - \operatorname{vol}\left(S_r^{b,m-1}\right)}{\operatorname{vol}(D_r) - \operatorname{vol}\left(B_r^{b,m}\right)} \geqq m h_b(r) \ \ \forall r. \tag{2.1}$$

Equality in (2.1) *(for some r_0), implies that D_{r_0} is a minimal radial cone in N^n. In particular, if $N^n = S^n(b)$, the sphere of constant curvature $b > 0$, then P^m is a totally geodesic submanifold of N^n.*

Theorem B ([11]). *Let P^m be a minimally immersed submanifold of N^n and let $D_r(p)$ be an extrinsic r-ball in P^m.*

If the sectional curvatures K_N of N satisfy $K_N \leqq b \leqq 0$, then

$$\frac{\operatorname{vol}(\partial D_r)}{\operatorname{vol}(D_r)} \geqq \frac{\operatorname{vol}\left(S_r^{b,m-1}\right)}{\operatorname{vol}\left(B_r^{b,m}\right)}. \tag{2.2}$$

When $b < 0$, equality in (2.2)*, (for some r_0), implies that D_{r_0} is a minimal radial cone in N^n. In particular, if $N^n = \mathbb{H}^n(b)$, the hyperbolic space of constant curvature $b < 0$, then P^m is a totally geodesic submanifold of N^n.*

An important tool for the proof of Theorem 1 is the co-area formula, see e.g. [2, p. 85].

Theorem C (co-area formula). *Let M^q be a q-dimensional riemannian manifold. Let Ω be a connected domain in M, with smooth boundary $\partial\Omega$ and compact closure $\bar{\Omega}$. Let $h : \Omega \longrightarrow \mathbb{R}$ be a function such that $h \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$, and $h|_{\partial\Omega} = 0$.*

We now let:

$$\Omega(t) = \{p \in M / |h(p)| > t\},$$
$$V(t) = \operatorname{vol}(\Omega(t)),$$
$$\Sigma(t) = \{p \in M / |h(p)| = t\}.$$

When t is a regular value of $|h|$, we let $d\sigma_t$ denote the riemannian $(q-1)$-density on $\Sigma(t)$.

Then the function $V(t)$ is smooth on the set of regular values of $|h|$, and its derivative is given by

$$V'(t) = -\int_{\Sigma(t)} \|\mathrm{grad}^M h\|^{-1} d\sigma_t. \tag{2.3}$$

3. An application of the co-area formula. In this section we are going to relate the first derivative of the volume of the extrinsic balls with the volume of its boundary, using the co-area formula.

Proposition 3.1. *Let P^m be an immersed submanifold in a riemannian manifold N^n, and let $D_r(p)$ be an extrinsic ball in P^m. Then*

$$\frac{d}{dr}\mathrm{vol}(D_r) \geqq \mathrm{vol}(\partial D_r) \quad \forall r > 0$$

Proof. Given any fixed radius $r_0 > 0$, let us consider the extrinsic r_0-ball $D_{r_0}(p)$ as the domain Ω in Theorem C. Then, defining $h : \bar{D}_{r_0} \longrightarrow \mathbb{R}$ as

$$h(q) := {r_0}^2 - r^2(q)$$

where r denotes the extrinsic distance to p in P, we have that $h \in C^0(\bar{D}_{r_0}) \cap C^\infty(D_{r_0})$.

It is easy to check that, with the notation in Theorem C,

$$\Omega(t) = D_{\sqrt{{r_0}^2-t}}(p),$$

$$V(t) = \mathrm{vol}\left(D_{\sqrt{{r_0}^2-t}}(p)\right)$$

$$\text{and} \quad \Sigma(t) = \partial D_{\sqrt{{r_0}^2-t}}(p) \ \ \forall t \in \left]0, {r_0}^2\right].$$

Then, applying the co-area formula, we have that

$$V'(t) = -\int_{\partial D_{\sqrt{{r_0}^2-t}}(p)} \|\mathrm{grad}^P h\|^{-1} d\sigma_t \tag{3.1}$$

where $d\sigma_t$ is the $(m-1)$-density of $\partial D_{\sqrt{{r_0}^2-t}}(p)$ in P.

A straightforward computation gives that

$$\mathrm{grad}^P h = -2r \mathrm{grad}^P r \quad \text{on} \quad D_{r_0} \tag{3.2}$$

so we have, as $\|\mathrm{grad}^P r\| \leqq 1$,

$$-\|\mathrm{grad}^P h\|^{-1} \leqq -\frac{1}{2r} \quad \text{on} \quad D_{r_0} \tag{3.3}$$

and, therefore,

$$V'(t) \leqq -\frac{1}{2\sqrt{{r_0}^2 - t}}\mathrm{vol}\left(\partial D_{\sqrt{{r_0}^2-t}}\right) \quad \forall t \in \left[0, {r_0}^2\right]. \tag{3.4}$$

Now let us define $W(r) := \mathrm{vol}(D_r)$, $r \in [0, r_0]$ and

$$V(t) = \mathrm{vol}\left(D_{\sqrt{{r_0}^2-t}}\right) = W \circ \psi(t)$$

where $\psi : [0, r_0{}^2] \longrightarrow [0, r_0]$ is defined as $\psi(t) := \sqrt{r_0{}^2 - t}$. We can then write (3.4) as

$$(3.5)\qquad (W \circ \psi)'(t) \leqq -\frac{1}{2\sqrt{r_0{}^2 - t}} \operatorname{vol}\left(\partial D_{\sqrt{r_0{}^2 - t}}\right) \quad \forall t \in \left[0, r_0{}^2\right]$$

and, hence,

$$(3.6)\qquad W'(\psi(t)) \geqq \operatorname{vol}\left(\partial D_{\sqrt{r_0{}^2 - t}}\right) \quad \forall t \in \left[0, r_0{}^2\right].$$

Changing the variable from $t \in [0, r_0{}^2]$ to $r = \psi(t) \in [0, r_0]$, we obtain

$$(3.7)\qquad W'(r) \geqq \operatorname{vol}(\partial D_r) \quad \forall r \in [0, r_0] \quad \forall r_0 > 0. \qquad \square$$

4. The volume growth of $B^{b,m}$ versus $S^{b,m}$. We show that the derivative of the function $\frac{\operatorname{vol}(B_r^{b,m})}{\operatorname{vol}(S_r^{b,m})}$, depends on b in the following way:

Proposition 4.1. *Let $B_r^{b,m}$ and $S_r^{b,m-1}$ be any geodesic r-ball and geodesic r-sphere respectively in the real space form $\mathbb{K}^m(b)$, and correspondingly, let $S_r^{b,m}$ denote any geodesic r-sphere in $\mathbb{K}^{m+1}(b)$. Then, for all r, ($r \in]0, \pi/2\sqrt{b}[$ if $b > 0$),*

$$(4.1)\qquad \frac{d}{dr}\left\{\frac{\operatorname{vol}\left(B_r^{b,m}\right)}{\operatorname{vol}\left(S_r^{b,m}\right)}\right\} \begin{cases} > 0, & \text{if } b > 0 \\ = 0, & \text{if } b = 0 \\ < 0, & \text{if } b < 0. \end{cases}$$

And, equivalently we have:

$$(4.2)\qquad \text{If } \ b > 0 \quad \operatorname{vol}\left(S_r^{b,m-1}\right) > m h_b(r) \operatorname{vol}\left(B_r^{b,m}\right).$$

$$(4.3)\qquad \text{If } \ b = 0 \quad \operatorname{vol}\left(S_r^{0,m-1}\right) = m h_0(r) \operatorname{vol}\left(B_r^{0,m}\right).$$

$$(4.4)\qquad \text{If } \ b < 0 \quad \operatorname{vol}\left(S_r^{b,m-1}\right) < m h_b(r) \operatorname{vol}\left(B_r^{b,m}\right).$$

Proof. We first recall the following well known volume formulae for geodesic balls and spheres in space forms of constant curvature b, (see e.g. [5]).

$$(4.5)\qquad \frac{d}{dr} \operatorname{vol}\left(B_r^{b,m}\right) = \operatorname{vol}\left(S_r^{b,m-1}\right)$$

$$(4.6)\qquad \operatorname{vol}\left(B_r^{b,m}\right) = \operatorname{vol}\left(S_1^{0,m-1}\right) \int_0^r (Q_b(t))^{m-1} dt.$$

In (4.6), the function $Q_b(r)$ denotes the unique solution to the differential equation

$$(4.7)\qquad Q_b'(r) = h_b(r) Q_b(r)\,, \ Q_b(0) = 0$$

such that e.g. for $b > 0$ we have $Q_b(r) = \sin\sqrt{b}r/\sqrt{b}$

Now for short-hand we let $\gtreqless_b$ denote the inequality $>$ if $b > 0$, the inequality $<$ if $b < 0$, and equality if $b = 0$.

It is straightforward to check that, using (4.7), the inequalities in (4.1) are equivalent to

$$(4.8)\qquad (Q_b(r))^m \gtreqless_b m Q_b'(r) \int_0^r (Q_b(t))^{m-1} dt,$$

(In the limiting case of $r = 0$ we get equality for all $b \in \mathbb{R}$). To check (4.8) for $r > 0$ a division on both sides by $m Q_b'(r)$ followed by a differentiation gives the following inequality, (which is easily verified using (4.7) and the fact that $Q'_b(r) \geqq 0$ for all $r \in]0, \pi/2\sqrt{b}[$ and thus proves (4.8))

$$-m(Q_b(r))^m Q''_b(r) \gtreqless_b 0.$$

Using the volume formulae (4.5) and (4.6) it is easy to show that the inequalities (4.2), (4.3) and (4.4) are also equivalent to inequalities in (4.1), and the Proposition is proved. □

5. Proof of Theorem 1. We show the assertions (i) and (ii) separately.

To prove (i), let $G(r)$ be the function defined as

$$G(r) = \ln(f(r)) = \ln\left(\frac{\mathrm{vol}(D_r)}{\mathrm{vol}\left(B_r^{b,m}\right)}\right).$$

Using Proposition 3.1, equations (4.6), and Theorem B, we have that

$$G'(r) = \frac{\frac{d}{dr}\mathrm{vol}(D_r)}{\mathrm{vol}(D_r)} - \frac{\mathrm{vol}\left(S_r^{b,m-1}\right)}{\mathrm{vol}\left(B_r^{b,m}\right)} \geqq \frac{\mathrm{vol}(\partial D_r)}{\mathrm{vol}(D_r)} - \frac{\mathrm{vol}\left(S_r^{b,m-1}\right)}{\mathrm{vol}\left(B_r^{b,m}\right)} \geqq 0 \tag{5.1}$$

for all $r \in]0, r_0]$, and hence also $f'(r) \geqq 0$ for all $r \in]0, r_0]$

If there exists an $r_0 > 0$ such that $f'(r_0) = 0$, then $G'(r_0) = 0$, so inequalities in (5.1) become equalities, and hence the equality assertion in Theorem B applies.

Now for the proof of (ii):

Let $H(r)$ be the function defined as

$$H(r) = \ln(g(r)) = \ln\left(\frac{\mathrm{vol}(D_r) - \mathrm{vol}\left(B_r^{b,m}\right)}{\mathrm{vol}\left(S_r^{b,m}\right)}\right).$$

With the same arguments as before, using equations (4.5), (4.6), Proposition 3.1 and Theorem A, we have that

$$\begin{aligned} H'(r) &= \frac{\frac{d}{dr}\left(\mathrm{vol}(D_r) - \mathrm{vol}\left(B_r^{b,m}\right)\right)}{\mathrm{vol}(D_r) - \mathrm{vol}\left(B_r^{b,m}\right)} - mh_b(r) \\ &\geqq \frac{\mathrm{vol}(\partial D_r) - \mathrm{vol}\left(S_r^{b,m-1}\right)}{\mathrm{vol}(D_r) - \mathrm{vol}\left(B_r^{b,m}\right)} - mh_b(r) \geqq 0 \end{aligned} \tag{5.2}$$

for all r, and hence, applying Proposition 4.1,

$$\frac{d}{dr}\left(\frac{\mathrm{vol}(D_r)}{\mathrm{vol}\left(S_r^{b,m}\right)}\right) \geqq \frac{d}{dr}\left(\frac{\mathrm{vol}\left(B_r^{b,m}\right)}{\mathrm{vol}\left(S_r^{b,m}\right)}\right) > 0 \ \ \forall r. \tag{5.3}$$

If there exists r_0 such that $g'(r_0) = 0$, then the first inequality in (5.3) becomes an equality at $r = r_0$, so $H'(r_0) = 0$ and both the inequalities in (5.2) become equalities and therefore finally the equality assertion in Theorem A applies and proves the theorem. □

Remark. We observe that when $b \geqq 0$, the function $\frac{\mathrm{vol}(D_r)}{\mathrm{vol}(S_r^{b,m})}$ is non-decreasing function of r. Indeed, if $b > 0$, this follows directly from equation (5.3), and if $b = 0$, it follows from assertion (i) in Theorem 1 when taking into account that $\frac{\mathrm{vol}(B_r^{b,m})}{\mathrm{vol}(S_r^{b,m})}$ is constant for $b = 0$.

In contrast, when $b < 0$, it is not in general true, that $\frac{\mathrm{vol}(D_r)}{\mathrm{vol}(S_r^{b,m})}$ is non-decreasing. In fact, every geodesic ball $B_r^{b,m}$ considered as an extrinsic ball in the submanifold $\mathbb{K}^m(b) \subseteqq \mathbb{K}^n(b)$, gives a sharp counterexample (by virtue of Proposition 4.1) for every $b < 0$:

$$\frac{d}{dr}\left\{\frac{\mathrm{vol}\left(B_r^{b,m}\right)}{\mathrm{vol}\left(S_r^{b,m}\right)}\right\} < 0.$$

6. Proof of Corollary 2. When $b \geqq 0$, the functions $\frac{\mathrm{vol}(D_r)}{\mathrm{vol}(S_r^{b,m})}$ and $\frac{\mathrm{vol}(B_r^{b,m})}{\mathrm{vol}(S_r^{b,m})}$ are both non-decreasing as we have just remarked above.

On the other hand, it was proved in [9] that (under the assumptions in Theorem 1) the volume of any extrinsic minimal ball satisfies the inequality

$$\mathrm{vol}(D_r) \geqq \mathrm{vol}\left(B_r^{b,m}\right). \tag{6.1}$$

Hence, using equations (4.5), (4.6) and (4.7), we have for $b \geqq 0$:

$$\frac{\mathrm{vol}(D_r)}{\mathrm{vol}\left(S_r^{b,m}\right)} \geqq \frac{\mathrm{vol}\left(B_r^{b,m}\right)}{\mathrm{vol}\left(S_r^{b,m}\right)} \geqq \lim_{r\to 0}\frac{\mathrm{vol}\left(B_r^{b,m}\right)}{\mathrm{vol}\left(S_r^{b,m}\right)} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{m\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}$$

and therefore

$$\mathrm{vol}(D_r) \geqq \mathrm{vol}\left(B_r^{b,m}\right) \geqq \frac{\Gamma\left(\frac{m+1}{2}\right)}{m\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}\mathrm{vol}\left(S_r^{b,m}\right). \tag{6.2}$$

In Corollary 2 we assume the equality

$$\mathrm{vol}\left(D_{r_0}\right) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{m\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}\mathrm{vol}\left(S_{r_0}^{b,m}\right)$$

for some fixed extrinsic ball D_{r_0} in P with $r_0 > 0$.

In view of (6.2) it is natural to expect strong consequences from this assumption. The inequalities in (6.2) both become equalities:

$$\mathrm{vol}(D_{r_0}) = \mathrm{vol}\left(B_{r_0}^{b,m}\right) \tag{6.3}$$

$$\frac{\mathrm{vol}\left(B_{r_0}^{b,m}\right)}{\mathrm{vol}\left(S_{r_0}^{b,m}\right)} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{m\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)}. \tag{6.4}$$

Since $\frac{\mathrm{vol}(B_r^{b,m})}{\mathrm{vol}(S_r^{b,m})}$ is a strictly increasing function of r if b is strictly positive (cf. Proposition 4.1), then this quotient cannot be equal to the limit at $r = 0$ as in (6.4) unless $b = 0$.

On the other hand, the equality (6.3) already has as a consequence that D_{r_0} is a minimal radial cone in N. This follows directly from [10], Theorem 2. For the sake of completeness we include below an alternative proof of this fact based on the proof of the co-area formula:

Consider the function $G(r) = \ln(f(r))$ which we defined in the proof of Theorem 1. Then $G(r)$ has a continuous extension to $r = 0$, namely $G(0) = 0$. (This follows from the asymptotic expansion of the volume of an extrinsic ball in a submanifold of an arbitrary riemannian manifold as considered e.g. in [7].)

Applying Theorem B, equations (4.5) and (4.6) and Proposition 3.1, we then get

$$G'(r) = \frac{\frac{d}{dr}\mathrm{vol}\,(D_r)}{\mathrm{vol}(D_r)} - \frac{\mathrm{vol}\left(S_r^{b,m-1}\right)}{\mathrm{vol}\left(B_r^{b,m}\right)} \geqq \frac{\mathrm{vol}(\partial D_r)}{\mathrm{vol}(D_r)} - \frac{\mathrm{vol}\left(S_r^{b,m-1}\right)}{\mathrm{vol}\left(B_r^{b,m}\right)} \geqq 0. \tag{6.5}$$

From equality (6.3), we have $G(0) = G(r_0) = 0$. Since $G(r)$ is non decreasing, we thus have $G(r) = 0$ for all $r \in [0, r_0]$. Then the inequalities in (6.5) all become equalities. In particular $\frac{d}{dr}\mathrm{vol}(D_r) = \mathrm{vol}(\partial D_r)$ for all r, and therefore $\|\mathrm{grad}^P r\| = 1$ on ∂D_r for all $r \in [0, r_0]$ (cf. the proof of the co-area formula). Hence D_{r_0} is a minimal radial cone in N^n.

Moreover, if we suppose that the sectional curvatures of N^n satisfies $0 \leqq K_N \leqq b$, then we have that $K_N = 0$ and N^n is the euclidean space $\mathbb{R}^n$.

In this case, it follows by analytic prolongation from $D_{r_0} = B^{0,m}_{r_0}$, that all of P^m is a totally geodesic submanifold of N, and this proves the corollary. □

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