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DESCRIPTION OF PERIODIC EXTREME GIBBS MEASURES OF SOME LATTICE MODELS ON THE CAYLEY TREE

N. N. Ganikhodzhaev[1] and U. A. Rozikov[1]

The uniqueness of the translation-invariant extreme Gibbs measure for the antiferromagnetic Potts model with an external field and the existence of an uncountable number of extreme Gibbs measures for the Ising model with an external field on the Cayley tree are proved. The classes of normal subgroups of finite index of the Cayley tree group representation are constructed. The periodic extreme Gibbs measures, which are invariant with respect to subgroups of index 2, are constructed for the Ising model with zero external field. From these measures, the existence of an uncountable number of nonperiodic extreme Gibbs measures for the antiferromagn. .ic Ising model follows.

Introduction

Let the Cayley tree $\mathcal{T}^k$ (or, in other terms, a Bethe lattice, see [1]) of order $k \geq 1$ be an infinite tree graph, i.e., a graph with no cycles and with exactly $k+1$ edges incident to each vertex of the graph. The absence of closed contours in the Cayley tree allows one to use the Markov random field theory and recurrent equations of this theory. This permits some lattice models [2–9] to be solved exactly.

In the present paper, we consider models on the Cayley tree for the purposes of

(1) proving the uniqueness of translation-invariant Gibbs measures in the antiferromagnetic Potts model with an external field;
(2) proving the existence of an uncountable number of extreme Gibbs measures in the Ising model with an external field;
(3) describing normal subgroups of finite index in the group representation of the Cayley tree; this enables one to consider periodic Gibbs measures for models determined in the Cayley tree;
(4) describing periodic Gibbs measures in the Ising model;
(5) describing a new class of extreme Gibbs measures in the antiferromagnetic Ising model with zero external field.

1. The construction of extreme Gibbs measures on the Cayley tree

Let $\mathcal{T}^k = (V, L)$ be a Bethe lattice of order $k \geq 1$. The distance $d(x,y)$, $x, y \in V$ in $\mathcal{T}^k$, is as follows:

$$d(x,y) = \min\{d : x = x_0, x_1, \dots, x_{d-1}, x_d = y\},$$

where the pairs $\langle x_0, x_1\rangle, \dots, \langle x_{d-1}, x_d\rangle$ are closest neighbors. The sequence

$$\pi = \{x = x_0, x_1, \dots, x_{d-1}, x_d = y \in V\}$$

that corresponds to this shortest distance is called the path from x to y.

For an arbitrary point $x^0 \in V$, we set

$$W_n = \{x \in V : d(x, x^0) = n\},$$

$$V_n = \bigcup_{m=0}^{n} W_m = \{x \in V : d(x, x^0) \leq n\},$$

$$L_n = \{l = \langle x, y\rangle \in L : x, y \in V_n\}.$$

[1] Mathematical Institute, Academy of Sciences of Uzbekistan.

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The partial ordering relation $x < y$ means that the path from x^0 to y passes through the vertex x. The vertex y is called a "direct descendant" of the vertex x if $y > x$ and x, y are closest neighbors. Let $S(x)$ be the set of all "direct descendants" of vertex x. Then, for any vertex $x \in V$, which is not equal to x^0, we have $|S(x)| = k$, while $|S(x^0)| = k + 1$ [6].

In the Cayley tree, the Hamiltonian of the Ising model with an external field reads

$$H_{V_n}(\sigma) = -\mathcal{J} \sum_{(x,y) \in L_n} \sigma(x)\sigma(y) - \alpha \sum_{x \in V_n} \sigma(x), \tag{1.1}$$

where $\alpha, \mathcal{J} \in R$, $\sigma(x) \in \{-1, 1\}$, and $x \in V$.

Let $A \subset V$ be a finite subset and let $\Omega_A = \{-1, 1\}^A$ denote the space of configurations on the set A. Let $h_x \in R$ be a real-valued function of $x \in V$. For any n, consider the measure μ_n on the space Ω_{V_n}, which is determined as follows:

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\left\{ T^{-1} H_{V_n}(\sigma) + \sum_{x \in V_n} h_x \sigma(x) \right\}. \tag{1.2}$$

Here $T > 0$, $\sigma_n = \{\sigma(x) : x \in V_n\} \in \Omega_{V_n}$, and Z_n is the normalizing factor. Consistency conditions for the measures $\mu_n(\sigma_n)$, $n \geq 1$, are

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \tag{1.3}$$

where $\sigma^{(n)} = \{\sigma(x) : x \in W_n\}$. In this case, the Gibbs measure μ exists in the space Ω_V. This measure is Markovian and it is called the *Markov chain* with interactions $\mathcal{J}$ and α.

Theorem 1.1 [7, 8]. *Measures* (1.2) *satisfy the consistency condition* (1.3) *iff* $\forall x \in V$,

$$h_x = \alpha/T + \sum_{y \in S(x)} \operatorname{arctanh}(\theta \tanh h_y), \tag{1.4}$$

where $\theta = \tanh(\mathcal{J}/T)$.

In the Potts model with an external field in the Cayley tree, spin variables $\sigma(x)$, $x \in V$, take the values $\sigma_1, \sigma_2, \ldots, \sigma_q$ and the Hamiltonian reads

$$\tilde{H}_{V_n}(\sigma) = -\mathcal{J} \sum_{(x,y) \in L_n} \delta_{\sigma(x)\sigma(y)} - \alpha \sum_{x \in V_n} \delta_{\sigma(x)\sigma_1}, \tag{1.5}$$

where σ is the Kronecker symbol. The antiferromagnetic Potts model is determined by Hamiltonian (1.5) for $\mathcal{J} < 0$.

Assume that $\sigma_1, \sigma_2, \ldots, \sigma_q \in R^{q-1}$ are such that

$$\sigma_i \sigma_j = \begin{cases} 1, & \text{if } \; i = j, \\ -1/q - 1, & \text{if } \; i \neq j. \end{cases} \tag{1.6}$$

Then, for any $x, y \in V$, we obtain that

$$\frac{q-1}{q} \left(\sigma(x)\sigma(y) + \frac{1}{q-1} \right) = \delta_{\sigma(x)\sigma(y)},$$

whence $\tilde{H}_{V_n}(\sigma)$ has the following form:

$$\tilde{H}_{V_n}(\sigma) = -\mathcal{J} \sum_{(x,y) \in L_n} \sigma(x)\sigma(y) - \alpha \sum_{x \in V_n} \sigma(x)\sigma_1. \tag{1.7}$$

Let $A \subset V$ be a finite subset. Denote by $\Omega_A = \{\sigma_1, \ldots, \sigma_q\}^A$ the space of configurations on the set A. A finite-dimensional distribution of a measure μ (μ is a Gibbs measure) in a volume V_n is determined as follows:

$$\mu_n(\sigma_n) = \tilde{Z}_n^{-1} \exp\left\{ T^{-1} \tilde{H}_{V_n}(\sigma) + \sum_{x \in W_n} \tilde{h}_x \sigma(x) \right\}, \tag{1.8}$$

where $\tilde{Z}_n$ is a normalizing factor and $\tilde{h}_x \in R^{q-1}$.

Theorem 1.2 [3, 4]. *Measures (1.8) satisfy Eq. (1.3) iff* $\forall x \in V$,

$$\tilde{h}_x = \sum_{y \in S(x)} F(\tilde{h}_y, \theta_1, \alpha). \tag{1.9}$$

Here F is the transformation that maps R^{q-1} *onto itself. It reads*

$$h_i' = \frac{\alpha}{T}\delta_{1i} + \log\left[\frac{(\theta_1 - 1)\exp h_i + \sum_{j=1}^{q-1}\exp h_j + 1}{\sum_{j=1}^{q-1}\exp h_j + \theta_1}\right], \quad i = \overline{1, q-1},$$

where $\theta_1 = \exp(\mathcal{J}/T)$ *and* $h = (h_1, h_2, \ldots, h_{q-1}) \in R^{q-1}$. *A unique Gibbs measure* μ *on* Ω_V *corresponds to each set of vectors* $\{h_x, x \in V\}$ *satisfying* (1.9).

2. Translation-invariant Gibbs measures in the antiferromagnetic Potts model with an external field

In this section, we construct Gibbs measures that are invariant w.r.t. all spatial shifts of the lattice $\mathcal{T}^k$. In [3, 4], for the ferromagnetic Potts model with a zero external field, the existence of q translation-invariant and of uncountably many translation-noninvariant Gibbs measures was proved. The constructive description of these measures was also presented there.

Assume that $\tilde{h}_x = h = (h_1, h_2, \ldots, h_{q-1})$ for $\forall x \in V$. Then, from Theorem 1.2, making the substitution $z_i = \exp h_i$, we obtain

$$z_i = \exp\left(\frac{\alpha}{T}\delta_{1i}\right)\left[\frac{(\theta_1 - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\sum_{j=1}^{q-1} z_j + \theta_1}\right]^k, \quad i = \overline{1, q-1}. \tag{2.1}$$

Note that for $0 < \theta_1 < 1$ $(\mathcal{J} < 0)$, system (2.1) has a unique solution, $z^* = (z_*, 1, 1, \ldots, 1)$. Denote by μ_* the Gibbs measure corresponding to the set of vectors $\{h_x = (\log z_*, 0, 0, \ldots, 0)\ \forall x \in V\}$.

Therefore, we have just proved the following theorem.

Theorem 2.2. *For* $q > 1$, $k \geq 1$, $\mathcal{J} < 0$, *and* $\alpha \in R$, *a translation-invariant measure in the Potts model is unique and equal to* μ_*.

3. Extreme Gibbs measures in the Ising model with an external field

In model (1.1), the necessary condition for a measure to be translation-invariant is $h_x = h \in R\ \forall x \in V$. Then, from (1.4) we obtain

$$h = \frac{\alpha}{T} + k\,\mathrm{arctanh}(\theta \tanh h). \tag{3.1}$$

From the properties of the function $\alpha/T + k\,\mathrm{arctanh}(\theta\tanh h)$, it follows that for $\alpha \in (-(k-1)\mathcal{J}, (k-1)\mathcal{J})$, $\mathcal{J} > 0$, Eq. (3.1) has two stable solutions $h_*^{(1)} < h_*^{(2)}$ and one unstable solution $h_*^{(3)}$, which lies between $h_*^{(1)}$ and $h_*^{(2)}$.

Consider the sets $\{h_x = h_*^{(1)}\ \forall x \in V\}$ and $\{h_x = h_*^{(2)}\ \forall x \in V\}$ and, denote by $\mu^{(1)}$ and $\mu^{(2)}$ the corresponding Gibbs measures.

Theorem 3.1 [5, 9]. *The measures* $\mu^{(1)}$ *and* $\mu^{(2)}$ *are extreme Gibbs measures.*

To any path π, which is finite or infinite, we can put into correspondence, in a standard way, the number $t \in [0, 1]$ (see [2 4]). One can easily show that for any $t \in [0, 1]$, the set of quantities $h^{\pi(t)} = \left\{h_x^{\pi(t)} = h_x^{\pi(t)}(h_*^{(1)}, h_*^{(2)}), x \in V\right\}$ satisfying Eq. (1.4) is uniquely determined.

Denote by μ^t the Gibbs measure corresponding to $h^{\pi(t)}$.

Theorem 3.2. *For any* $t \in [0, 1]$, *the Gibbs measure* μ^t *is extreme.*

Proof. The proof is analogous to the proof of Theorem 3.2 of [2].

4. Group representation and automorphisms of the Cayley tree

Let G_{k+1} be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \dots, a_{k+1}$, respectively.

Proposition 4.1 [10]. *There exists a one-to-one correspondence between the set of vertices V of the Cayley tree $\mathcal{T}_k$ and the group G_{k+1}.*

In the group G_{k+1}, let us consider the left (right) shift transformations defined as follows. For $g_0 \in G_{k+1}$, let us set

$$T_{g_0}(h) = g_0 h \qquad (T_{g_0}(h) = hg_0) \quad \forall h \in G_{k+1}. \tag{4.1}$$

The set of all left (right) shifts in G_{k+1} is isomorphic to the group G_{k+1}. By virtue of Proposition 4.1, any transformation S of the group G_{k+1} induces the transformation $\widehat{S}$ of the set of vertices V of the Cayley tree. The following theorem obviously holds.

Theorem 4.1. *The group of left (right) shifts on the right (left) representation of the Cayley tree is the group of translations of the Cayley tree.*

5. Normal subgroups of finite index for the group representation of the Cayley tree

Any element $x \in G_{k+1}$ has the following form:

$$x = a_{i_1} a_{i_2} \dots a_{i_n}, \quad \text{where} \quad 1 \le i_m \le k+1, \quad m = \overline{1, n}.$$

The number n is called the length of the word x and is denoted by $l(x)$. The number of letters a_i, $i = \overline{1, k+1}$, that enter the noncontractible representation of the word x is denoted by $\omega_x(a_i)$.

Proposition 5.1 (see [11, 12]). *Let φ be a homomorphism of the group G_{k+1} with the kernel H. Then H is a normal subgroup of the group G_{k+1} and $\varphi(G_{k+1}) \simeq G_{k+1} : H$ ($G_{k+1} : H$ is a factor-group), i.e., the index $|G_{k+1} : H|$ coincides with the order $|\varphi(G_{k+1})|$ of the group $\varphi(G_{k+1})$.*

By virtue of Proposition 5.1, in order to construct a normal subgroup of a finite index of the group G_{k+1}, one should construct a homomorphism of the group G_{k+1} into some finite group.

Definition 5.1. Let $M_1, M_2, \dots, M_m$ be some sets and $M_i \neq M_j$ for $i \neq j$ $(i, j = \overline{1, m})$. We call the intersection $\bigcap_{i=1}^m M_i$ *contractible* if there exists i_0 $(1 \le i_0 \le m)$ such that

$$\bigcap_{i=1}^m M_i = \Big(\bigcap_{i=1}^{i_0-1} M_i\Big) \bigcap \Big(\bigcap_{i=i_0+1}^{m} M_i\Big).$$

Let $N_k = \{1, 2, \dots, k+1\}$.

Theorem 5.1. *For any $\varnothing \neq A \subseteq N_k$, there exists a subgroup $H_A \subset G_{k+1}$ with the following properties:*

(a) *H_A is a normal subgroup and $|G_{k+1} : H_A| = 2$;*
(b) *$H_A \neq H_B$ for $\forall A \neq B \subseteq N_k$;*
(c) *$|H_A \cap H_B| = \infty$ and $H_A \cap H_B \subseteq H_{A \triangle B}$, for $A, B \subseteq N_k$;*
(d) *If $A_1, A_2, \dots, A_m \subseteq N_k$ and $A_i \cap A_j = \varnothing$ for any $i \neq j = \overline{1, m}$, then*

$$\bigcap_{i=1}^m H_{A_i} \subset H_{\bigcup_{i=1}^m A_i};$$

(e) *Let* $A_1, A_2, \ldots, A_m \subseteq N_k$. *If* $\bigcap_{i=1}^{m} H_{A_i}$ *is a noncontractible intersection, then it is a normal subgroup of index* 2^m;

(f) *For any* $m = \overline{1, 2k}$ *(where* k *is the order of the lattice), there exist noncontractible intersections*

$$H_m = \bigcap_{i=1}^{m} H_{A_i}.$$

Proof. Let $\varnothing \neq A \subseteq N_k$. Define the mapping as follows:

$$f_A(x) = \begin{cases} 1, & \text{if } \sum_{i \in A} \omega_x(a_i) \text{ is even,} \\ -1, & \text{if } \sum_{i \in A} \omega_x(a_i) \text{ is odd.} \end{cases}$$

One can easily find that f_A is a homomorphism, i.e., for any $x, y \in G_{k+1}$, the equality $f_A(xy) = f_A(x) f_A(y)$ holds. By virtue of Proposition 5.1, $H_A = \left\{ x \in G_{k+1} : \sum_{i \in A} \omega_x(a_i) \text{ is even} \right\}$ is a normal subgroup of index 2. One can easily check that H_A satisfies all assertions of Theorem 5.1. The theorem is proved.

Let $x = a_{i_1} a_{i_2} \ldots a_{i_n}$, where $1 \leq i_m \leq k+1$, $m = \overline{1, n}$. For $x \in \{y \in G_{k+1} : l(y) = n\}$, we set $\nu_x(a_j) = \{m \in N_{n-1} : i_m = j\}$. For instance, if $x = a_1 a_4 a_8 a_4 a_1 a_2 a_4$, then $\nu_x(a_4) = \{2, 4, 7\}$. Let $\alpha(x)$ denote the number of constituents of the element x.

Theorem 5.2. *Let* $e \neq x_0 \in G_{k+1}$. *Then there exists a normal finite-index subgroup* H_{x_0}, *which does not contain the element* x_0 *and for which the inequality*

$$\alpha(x_0) + 1 \leq |G_{k+1} : H_{x_0}| \leq (l(x_0) + 1)!$$

holds.

Proof. Let $\alpha(x_0) = m$. Denote by $a'_1, a'_2, \ldots, a'_m$, $1 \leq m \leq k+1$, the constituents a_i, $i = \overline{1, k+1}$, in the noncontractible form of the element x. Then, x_0 reads

$$x_0 = a'_{i_1} a'_{i_2} \ldots a'_{i_s}, \quad \text{where} \quad 1 \leq i_s \leq k+1, \quad s = \overline{1, n}.$$

In the symmetric group S_{n+1} that acts on the symbols $1, 2, \ldots, n+1$, we choose the substitutions π_i, $i = \overline{1, m}$ as follows: if $\nu_{x_0}(a_j) = \{j_1, j_2, \ldots, j_{m_1}\}$, where $j_{k_1} \in N_{n-1}$ and $k_1 = \overline{1, m_1}$, then

$$\pi_j = \begin{pmatrix} 1 \ldots j_1 & j_1 + 1 \ldots j_2 & j_2 + 1 \ldots j_{m_1} & j_{m_1} + 1 \ldots n+1 \\ 1 \ldots j_1 + 1 & j_1 \ldots j_2 + 1 & j_2 \ldots j_{m_1} + 1 & j_{m_1} \ldots n+1 \end{pmatrix}, \qquad j = \overline{1, m}.$$

Obviously, $\pi_j^2 = \pi_0$, $j = \overline{1, m}$, where π_0 is the identical permutation. Let us define the mappings $u \colon \{a_1, \ldots, a_{k+1}\} \to \{\pi_1, \ldots, \pi_m\}$ and $f_{x_0} \colon G_{k+1} \to S_{n+1}$ as follows:

$$u(x) = \begin{cases} \pi_0, & \text{if } x \neq a'_j, \\ \pi_j, & \text{if } x = a'_j, \end{cases} \qquad j = \overline{1, m},$$

$$f_{x_0}(x) = f_{x_0}(a_{i_1} a_{i_2} \ldots a_{i_s}) = u(a_{i_1}) u(a_{i_2}) \ldots u(a_{i_s}).$$

Since $\pi_j^2 = \pi_0$, the mapping f_{x_0} is a homomorphism. The kernel H_{x_0} of this homomorphism is a normal finite-index subgroup. Obviously, the element x does not belong to this normal subgroup. Further, by virtue of Proposition 5.1, $|G_{k+1} : H_{x_0}| = |f_{x_0}(G_{k+1})|$ and, from the procedure of constructing f_{x_0}, it is clear that

$$\alpha(x_0) + 1 \leq |G_{k+1} : H_{x_0}| \leq (l(x_0) + 1)!.$$

The theorem is proved.

Proposition 5.3. *The following relations hold:*

(1) $H_{a_i} = H_{a_i^{-1}}$, $i = \overline{1, k+1}$;

(2) *if* $\alpha(x) \geq 2$, *then* $|G_{k+1} : H_x| \geq 3$;

(3) $|G_{k+1} : H_{a_i a_j}| = 6$ *for* $\forall i \neq j \in N_k$.

Proof. The proof follows from Theorems 5.1 and 5.2.

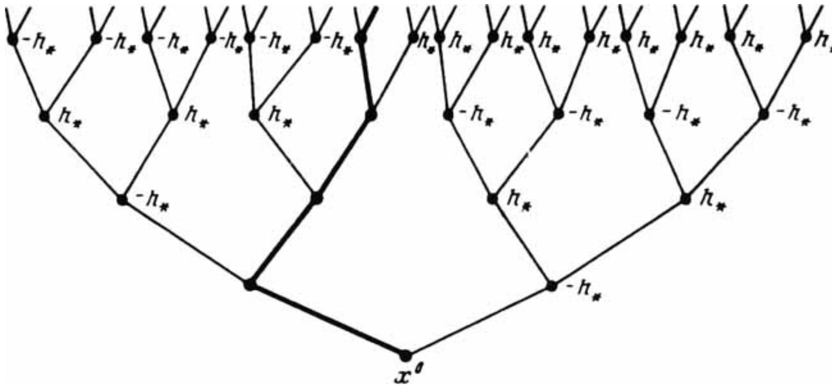

Fig. 1

6. Periodic Gibbs measures of the Ising model on the Cayley tree

The notions of the periodic Gibbs measure, the Hamiltonian, configurations, etc., are introduced in a standard way (see [13, 14]). Let us consider the Ising model with a zero external field, i.e., with $\alpha = 0$. Let $G_{k+1}/\widehat{G}_* = \{\widehat{G}_*, \widehat{G}_{*1}\}$ be a factor-group, where

$$\widetilde{G}_* = H_{N_k} = \{x \in G_{k+1} : l(x) \text{ is even}\}$$

(see Sec. 5). Let us construct $\widehat{G}_*$-periodic Gibbs measures. To obtain the periodic measures, we put

$$h_x = \begin{cases} h_1, & \text{if} \quad x \in \widehat{G}_*, \\ h_2, & \text{if} \quad x \in \widehat{G}_{*1} \end{cases}$$

into (1.4). Then we obtain

$$\begin{cases} h_1 = kf(h_2, \theta), \\ h_2 = kf(h_1, \theta), \end{cases} \tag{6.1}$$

where $\theta = \tanh(\mathcal{J}/T)$, $-1 < \theta < 1$, and $f(x, \theta) = \operatorname{arctanh}(\theta \tanh x)$.

One may easily note that this system of equations has a unique solution $h_* = (0, 0)$ for $-k^{-1} \le \theta \le k^{-1}$, three solutions $h_*^{(1)} = (-h_*, -h_*)$, $h_*^{(2)} = (0, 0)$, and $h_*^{(3)} = (h_*, h_*)$ $(h_* > 0)$ for $k^{-1} \le \theta \le 1$, and three solutions $h_*^{\mp} = (h_*, -h_*)$, $h_*^0 = (0, 0)$, and $h_*^{\pm} = (-h_*, h_*)$ for $-1 < \theta < -k^{-1}$.

Theorem 6.1. *For the ferromagnetic* $(\mathcal{J} > 0)$ *(respectively, antiferromagnetic* $(\mathcal{J} < 0)$*) Ising model, there exist three (respectively, two) extreme periodic Gibbs measures* $\hat{\mu}^{(1)}$, $\hat{\mu}^{(2)}$, *and* $\hat{\mu}_*$ *(respectively,* $\hat{\mu}^{\mp}$ *and* $\hat{\mu}^{\pm}$*).*

Proof. The proof is analogous to the proof of Theorem 3.2 of [4].

Using the measures $\mu^{\mp}$ and $\mu^{\pm}$, we can construct an uncountable set of extreme Gibbs measures. One may prove (cf. [2, 4]) that for $-1 < \theta < -k^{-1}$, the following sets of quantities satisfy Eq. (1.4):

$$h^{\pi(t)} = \left\{ h_x^{\pi(t)} = \begin{cases} \begin{cases} -h_*, & \text{if} \quad x \prec x_n, \quad x, x_n \in W_{n=2l}, \\ h_*, & \text{if} \quad x \prec x_n, \quad x, x_n \in W_{n=2l+1}, \end{cases} \\ \begin{cases} h_*, & \text{if} \quad x_n \prec x, \quad x, x_n \in W_{n=2l}, \\ -h_*, & \text{if} \quad x_n \prec x, \quad x, x_n \in W_{n=2l+1}, \end{cases} \quad x \in V, \\ h_0 \in [-h_*, h_*], \quad \text{if} \quad x = x_n, \end{cases} \right\},$$

where $n = 0, 1, 2, \ldots$ (see Fig. 1). These sets of quantities are different for different $t \in [0, 1]$.

Let us denote by μ^t the Gibbs measure that corresponds to the set $h^{\pi(t)}$. The following statement holds.

Theorem 6.2. *For any $t \in [0, 1]$, the Gibbs measure μ^t is extreme.*

REFERENCES

1. R. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London–New York–Paris (1982).
2. P. M. Blekher and N. N. Ganikhodzhaev, *Teor. Veroyatn. Primenen.*, **35**, 920 (1990).
3. N. N. Ganikhodzhaev, *Dokl. Akad. Nauk Resp. Uzbekistan*, Nos. 6–7, 4 (1992).
4. N. N. Ganikhodzhaev, *Theor. Math. Phys.*, **85**, 1125 (1990).
5. C. J. Preston, *Gibbs States on Countable Sets*, Cambridge Univ., Oxford (1974).
6. P. M. Bleher, *Commun. Math. Phys.*, **128**, 411 (1990).
7. H. O. Georgii, *Gibbs Measures and Phase Transitions*, Walter de Gruyter, Berlin–New York (1988).
8. F. Spitzer, *Ann. Prob.*, **3**, 387 (1975).
9. Y. Higuchi, *Publ. RIMS Kyoto Univ.*, **3**, 335 (1977).
10. N. N. Ganikhodzhaev, *Dokl. Akad. Nauk Resp. Uzbekistan*, No. 5, 3 (1994).
11. A. G. Kurosh, *Gruppentheorie*, Akademic Verlag, Berlin (1953).
12. M. I. Kargapolov and Yu. I. Merzlyakov, *Foundations of Group Theory* [in Russian], Nauka, Moscow (1982).
13. S. A. Pirogov and Ya. G. Sinai, *Theor. Math. Phys.*, **25**, 1185 (1975); **26**, 61 (1976).
14. Ya. G. Sinai, *Theory of Phase Transitions: Rigorous Results*, Pergamon, Oxford–New York–London–Paris (1982).