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ON THE INTEGRABILITY OF HYPERBOLIC SYSTEMS OF RICCATI-TYPE EQUATIONS

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We consider equations of the form $U_{xy} = U * U_x$, where $U(x,y)$ is a function taking values in an arbitrary finite-dimensional algebra T over the field $\mathbb{C}$. We show that every such equation can be naturally associated with two characteristic Lie algebras, L_x and L_y. We define the notion of a $\mathbb{Z}$-graded Lie algebra $\mathfrak{G}$ corresponding to a given equation. We prove that for every equation under consideration, the corresponding algebra $\mathfrak{G}$ can be taken as a direct sum of the vector spaces L_x and L_y if we define the commutators of the elements from L_x and L_y by means of the zero-curvature relations. Assuming that the algebra T has no left ideals, we classify the equations of the specified type associated with the finite-dimensional characteristic Lie algebras L_x and L_y. All of these equations are Darboux-integrable.

Introduction

In [1], the following nonlinear hyperbolic equation was considered:

$$U_{xy} = \big[[U,\, A],\, U_x\big], \tag{1}$$

where U is a function taking values in an arbitrary finite-dimensional Lie algebra $\mathfrak{A}$ and A is a constant element of $\mathfrak{A}$. Equation (1) is the compatibility condition of the over-determined linear system

$$L(\Psi) = \Psi_x - \frac{1}{\lambda} U_x \Psi = 0, \qquad A(\Psi) = \Psi_y - \Big(\lambda A + [U,\, A]\Big)\Psi = 0. \tag{2}$$

With the help of the (L, A)-pair (2), Eq. (1) can be integrated by the inverse scattering method. In [2–4], the case $\mathfrak{A} = sl(2)$ was considered in detail.

Equation (1) is a system of the form

$$u^i_{xy} = C^i_{jk} u^j u^k_x, \qquad i = 1, \dots, N, \tag{3}$$

where $U = u^i e_i$, $e_1, \dots, e_N$ is the basis of $\mathfrak{A}$ and summation over repeated indices has to be performed. The constants C^i_{jk} are defined by the coordinates of the element A and by the structure constants of the Lie algebra $\mathfrak{A}$.

In this paper, we consider a class of systems of form (3) with arbitrary coefficients C^i_{jk}. Let us note that, as a subclass, this class contains a system of ordinary differential equations of the form

$$u^i_y = \alpha^i_{jk} u^j u^k + f^i(y), \qquad i = 1, \dots, N. \tag{4}$$

Namely, if $C^i_{jk} = C^i_{kj}$, then Eq. (3) can be integrated with respect to x and the result is a system of form (4). Therefore, systems of form (3) are, in a certain sense, a two-dimensional generalization of the Riccati equation.

Let us note that the class of systems (3) contains many integrable systems.

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As the first example, we consider the equation

$$U_{xy} = [U, U_x], \tag{5}$$

where U is a function taking values in an arbitrary finite-dimensional Lie algebra $\mathfrak{A}$. It was shown in [5] that this equation is equivalent to the ordinary differential equation

$$Y_x = yY\beta(x), \qquad Y(0, y) = \alpha(y),$$

where $\beta(x)$ and $\alpha(y)$ are arbitrary functions taking values in the Lie algebra $\mathfrak{A}$ and in its Lie group, respectively. The general solution of Eq. (5) can be expressed through Y by the formula

$$U(x, y) = Y_y Y^{-1}.$$

Class (3) also contains systems of the form

$$u^i_{xy} = a^i_j u^j u^i_x, \tag{6}$$

where a^i_j are elements of the Cartan matrix of a simple Lie algebra or of a Kač–Moody algebra of rank N. Any system (6) is related to the corresponding two-dimensional Toda chain (see, for example, [6]) by means of a differential change of variables $v^i = \log(u^i_x)$ and, therefore, is exactly integrable. Both (5) and (6) belong to the class of systems related to $\mathbb{Z}$-graded Lie algebras.

Recall that a Lie algebra $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$ is called $\mathbb{Z}$-graded if $\mathfrak{G}_i$ are finite-dimensional vector subspaces for which $[\mathfrak{G}_k, \mathfrak{G}_j] \subseteq \mathfrak{G}_{k+j}$. Obviously, $\mathfrak{G}_0$ is a subalgebra of $\mathfrak{G}$.

Let $U(x, y)$ be a function taking values in $\mathfrak{G}_{-1}$ and $A \in \mathfrak{G}_1$ be a nonzero element. Let us consider Eq. (1). It is obvious that both the left and right-hand sides of (1) are in $\mathfrak{G}_{-1}$. Representing U as $U = u^i t_i$, where $t_1, \ldots, t_N$ is a basis in $\mathfrak{G}_{-1}$, we arrive at a system of form (3). We say, in this case, that the algebra $\mathfrak{G}$ gives rise to system (3). If, on the other hand, algebra $\mathfrak{G}$ and element A are such that the Lie algebra $\mathfrak{G}_0$ is generated by $[A, \mathfrak{G}_{-1}]$, we say that the pair $(\mathfrak{G}, A)$ corresponds to system (3). Clearly, the system from [1] mentioned at the beginning of the paper is generated by the Lie algebra $\mathfrak{G}$ consisting of Laurent polynomials with coefficients from $\mathfrak{A}$ with the grading $\mathfrak{G}_i = \mathfrak{A}\lambda^i$. Further, if one chooses $\mathfrak{G}$ to be polynomials in non-negative powers of λ, with coefficients from $\mathfrak{A}$ and the grading $\mathfrak{G}_{-i} = \mathfrak{A}\lambda^i$, $i = 0, 1, 2, \ldots$, and if one also takes $\mathfrak{G}_1 = \{A\}$, $A = \partial/\partial\lambda$, and $\mathfrak{G}_i = 0$, $i \geq 2$, one obtains Eq. (5).

Interpreting the constants C^i_{jk} as the structure constants of a finite-dimensional (in general, noncommutative and nonassociative) algebra T with the multiplication $*$ and, further, assuming $U = v^i e_i$ (where e_i is a basis in T) to be an element of this algebra, one can rewrite system (3) in a compact form,

$$U_{xy} = U * U_x. \tag{7}$$

Formal manipulations pertaining to system (3) are often conveniently performed in terms of the algebra T.

In the case where system (3) is generated by a $\mathbb{Z}$-graded Lie algebra $\mathfrak{G}$, the formula $X * Y = [[X, A], Y]$ endows the space $\mathfrak{G}_{-1}$ with the structure of an algebra T.

It is easy to verify that for any $\mathbb{Z}$-graded algebra $\mathfrak{G}$, Eq. (1) admits a zero-curvature representation $L_y - A_x = [L, A]$, where $L = \partial/\partial x - U_x$ and $A = \partial/\partial y - \big(A + [U, A]\big)$. However, this equation appears to be exactly integrable only in the case where $\mathfrak{G}$ has a finite growth [7].

One of the main results of this work is the statement that every system of form (3) is generated by the corresponding $\mathbb{Z}$-graded Lie algebra.

Following Kač, we call a $\mathbb{Z}$-graded Lie algebra $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$ transitive if

for $x \in \mathfrak{G}_i$, $i \geq 0$, the equation $[x, \mathfrak{G}_{-1}] = 0$ implies $x = 0$ or,

for $x \in \mathfrak{G}_i$, $i \leq 0$, the equation $[x, \mathfrak{G}_1] = 0$ implies $x = 0$.

Theorem 1. *For any system* (3), *there exist a corresponding* $\mathbb{Z}$*-graded Lie algebra* $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$ *and an element* $A \in \mathfrak{G}_1$. *This Lie algebra is transitive if* T *is not a Lie algebra and does not contain left ideals.*

In [8, 9], the notion of a characteristic Lie algebra was defined for systems like the two-dimensional Toda chain. It was shown in [10] that hyperbolic-type systems possess two characteristic Lie algebras (one for each characteristic). It was also hypothesized in [10] that the $\mathbb{Z}$-graded algebra corresponding to systems of form (3) and more general ones can be obtained by "gluing" two characteristic Lie algebras with the help of a zero-curvature representation. In such a case, the algebra is called complete.

Characteristic Lie algebras of vector fields are defined for the infinitely extended system (3). Let us introduce the notation $p_i = D_y^i U$ and $q_i = D_x^{i+1} U$, $i = 0, 1, 2, \ldots$, where D_x and D_y are operators of the full derivatives taken in accordance with Eq. (7). The variables p_i and q_i are elements of the algebra T and, in the chosen basis, have some coordinates $p_i = (p_i^1, \ldots, p_i^N)$ and $q_i = (q_i^1, \ldots, q_i^N)$.

Rewritten in form (7), system (3) can be naturally extended to the variables $p_i, q_i, i = 0, 1, 2, \ldots$ as

$$\begin{aligned} &(p_0)_x = q_0, \quad (p_1)_x = D_y q_0 = p_0 * q_0, \\ &(p_2)_x = D_y^2 q_0 = p_1 * q_0 + p_0 * (p_0 * q_0), \ldots; \end{aligned} \tag{8}$$

$$\begin{aligned} &(q_0)_y = p_0 * q_0, \quad (q_1)_y = D_x(p_0 * q_0) = q_0 * q_0 + p_0 * q_1, \\ &(q_2)_y = D_x^2(p_0 * q_0) = q_1 * q_0 + 2 q_0 * q_1 + p_0 * q_2, \ldots . \end{aligned} \tag{9}$$

Let $A[p]$ and $A[q]$ be the algebras of smooth functions in a finite number of variables p_i^j and q_i^j, respectively, $i = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, N$. It follows from (8) and (9) that there are elements $X_j \in \operatorname{Der} A[p]$ such that for any function $v \in A[p]$,

$$D_x(v) = \sum_{j=1}^{N} q_0^j X_j(v). \tag{10}$$

Similarly, we have

$$D_y(w) = Y_0(w) + \sum_{j=1}^{N} p^j Y_j(w), \quad Y_j \in \operatorname{Der} A[q], \tag{11}$$

for any function $w \in A[q]$. Therefore, the relation $D_x(v) = 0$ is equivalent to the over-determined system $X_j(v) = 0$, $j = 1, \ldots, N$, while the relation $D_y(w) = 0$ is equivalent to the system $Y_j(w) = 0$, $j = 0, \ldots, N$.

The subalgebra $L_x \subset \operatorname{Der} A[p]$, generated as a Lie algebra by X_j, $j = 1, \ldots, N$, is called the x-characteristic Lie algebra of system (3). Similarly, the y-characteristic algebra $L_y \subset \operatorname{Der} A[q]$ is generated by the vector fields Y_j, $j = 0, \ldots, N$.

The following theorem shows that the above-mentioned hypothesis is valid for system (3).

Theorem 2. *Let* L_x *and* L_y *be the characteristic algebras of system* (3). *Then there exist a* $\mathbb{Z}$*-graded Lie algebra* $\mathfrak{G}$, *an element* A *corresponding to system* (3), *and the isomorphisms* $\varphi\colon \bigoplus_{i=1}^{+\infty} \mathfrak{G}_{-i} \to L_x$ *and* $\psi\colon \bigoplus_{i=0}^{+\infty} \mathfrak{G}_i \to L_y$.

An important consequence of this theorem is the Darboux-integrability of systems corresponding to the finite-dimensional Lie algebras $\mathfrak{G}$.

Let us consider the case where the Lie algebra $\mathfrak{G}$ is finite-dimensional. This case comprises, for example, systems (6) associated with the Cartan matrix of a simple Lie algebra $\mathfrak{G}$. Such systems correspond to choosing the canonical $\mathbb{Z}$-grading in $\mathfrak{G}$ [11]. In addition to the canonical one, simple Lie algebras also admit other $\mathbb{Z}$-gradings, which give rise, in general, to different systems. In what follows, we consider the case of standard gradings [7].

System (3) is called Darboux-integrable if there are functions $v_i(p_0, \dots, p_{n_i})$ and $w_i(q_0, \dots, q_{k_i})$, $i = 1, \dots, N$, that satisfy the essential independence condition

$$\det\left[\frac{\partial v_1}{\partial p_{n_1}}, \dots, \frac{\partial v_N}{\partial p_{n_N}}\right] \neq 0, \qquad \det\left[\frac{\partial w_1}{\partial q_{k_1}}, \dots, \frac{\partial w_N}{\partial q_{k_N}}\right] \neq 0$$

and such that

$$D_x(v_i) = 0, \quad D_y(w_i) = 0, \quad i = 1, \dots, N.$$

Theorem 3. *Let the characteristic algebras L_x and L_y of system (3) be finite-dimensional. Then system (3) is Darboux-integrable.*

Proof. This is similar to the proof of the corresponding statement for the two-dimensional Toda chain [9] and is therefore omitted.

It is known that finding the general solution of a Darboux-integrable system amounts to integrating a system of ordinary differential equations possessing a large number of Lie symmetries.

Let us call the $\mathbb{Z}$-graded Lie algebra $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$ irreducible if the representation of $\mathfrak{G}_0$ in $\mathfrak{G}_{-1}$ is irreducible.

In this paper, we restrict ourselves to those systems of form (3) that do not contain subsystems. It is easy to see that the absence of subsystems in (3) is equivalent to the absence of left ideals in the algebra T with the operation $*$. If one imposes the more restricting condition that two-sided ideals be absent in T, then, for the pair $(\mathfrak{G}, A)$ corresponding to system (3), this condition can be easily seen to be equivalent to the irreducibility of the representation of $\mathfrak{G}_0$ in $\mathfrak{G}_{-1}$. In that case, Theorem 1 allows us to use the results of [7] in the classification of transitive $\mathbb{Z}$-graded irreducible Lie algebras.

It was stated in [7], in particular, that if a finite-dimensional transitive $\mathbb{Z}$-graded algebra is irreducible, it coincides with one of the finite-dimensional simple Lie algebras taken in one of the standard gradings. A posteriori, it turns out that with the exception of systems related to the algebras G_2 and F_4 in the standard gradings, either the remaining systems have subsystems or the corresponding Lie algebra in the standard grading has a height 1, i.e., $\mathfrak{G}_i = 0$ for $|i| > 1$. In the latter case, the algebra T is a Jordan algebra [12], while the corresponding system is given by the x-derivative of the system of ordinary differential equations (4).

In the standard grading, the algebra G_2 corresponds to the system

$$\begin{cases} u^1_{xy} = 2u^1u^1_x - 3u^3u^2_x - u^4u^1_x, \\ u^2_{xy} = u^1u^2_x + u^2u^1_x - 4u^3u^3_x, \\ u^3_{xy} = u^2u^2_x + u^3u^4_x + u^4u^3_x, \\ u^4_{xy} = 2u^4u^4_x - 3u^2u^3_x - u^1u^4_x, \end{cases} \tag{12}$$

with no subsystems. Interestingly, a different choice of element A in the same grading gives rise to a linearly inequivalent system (see Sec. 4). For other standard gradings of G_2, the system consists of two equations and possesses a subsystem.

We plan to consider the systems corresponding to Lie algebras of finite growth in another paper.

1. Constructing $\mathbb{Z}$-graded Lie algebras associated with the system

In this section, we prove Theorem 1, but, first, we need the definition of a local Lie algebra and a number of related statements from [7].

Let $\tilde{\mathfrak{G}} = \mathfrak{G}_{-1} \oplus \mathfrak{G}_0 \oplus \mathfrak{G}_1$ be the direct sum of the finite-dimensional vector spaces. Let us assume that whenever $|i + j| \leq 1$, there is an anti-commutative bilinear operation $\mathfrak{G}_i \times \mathfrak{G}_j \to \mathfrak{G}_{i+j}$ $((x, y) \to [xy])$ such that the Jacobi identity is fulfilled for every triple of vectors as soon as all of the commutators involved in that identity are defined. Then, $\tilde{\mathfrak{G}}$ is called a local Lie algebra. Transitivity, irreducibility, and the homomorphisms of local algebras are defined in the same way as for graded Lie algebras.

To the graded algebra $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$, there corresponds the local Lie algebra $\mathfrak{G}_{-1}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_1$, which we call the local component of the Lie algebra $\mathfrak{G}$.

A graded Lie algebra $\mathfrak{G} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{G}_i$ is called minimal if, for any other graded algebra G', every isomorphism of the local components of G and G' can be extended to a homomorphism of G' onto G.

Proposition 1. *Let $\tilde{\mathfrak{G}} = \mathfrak{G}_{-1}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_1$ be a local Lie algebra. Then there exists a minimal graded algebra $\mathfrak{G}$ whose local component is isomorphic to $\tilde{\mathfrak{G}}$.*

Proposition 2. *The minimal graded Lie algebra with a transitive local component is transitive.*

Let us proceed to the proof of Theorem 1.

Consider the following elements of the Lie algebra $\operatorname{Der} A[q]$:

$$B_i^k = \sum_{l=0}^{+\infty} q_l^k \frac{\partial}{\partial q_l^i}, \qquad B_i^{jk} = \sum_{l=1}^{+\infty}\sum_{m=1}^{l} C_l^m q_{m-1}^j q_{l-m}^k \frac{\partial}{\partial q_l^i},$$

where C_l^m are binomial coefficients, and let $H_0 = \{B_i^k\}$ and $H_1 = \{B_i^{jk}\}$ be the linear spaces spanned by the corresponding sets. Obviously, all of the derivations B_i^k and B_i^{jk} are linearly independent. Let $t_1,\dots,t_N$ be the basis of some linear space H_{-1}. Consider the local Lie algebra $H = H_{-1}\oplus H_0\oplus H_1$ in which the commutator operation is defined in the following way:

$$[B_i^k, t_j] = -\delta_j^k t_i, \qquad [B_i^{jk}, t_p] = \delta_p^j B_i^k, \qquad [B_i^k, B_m^p] = \delta_i^p B_m^k - \delta_m^k B_i^p,$$
$$[B_i^{jk}, B_m^p] = \delta_i^p B_m^{jk} - \delta_m^j B_i^{pk} - \delta_m^k B_i^{jp}$$

(the last two commutators are defined in the same way as in $\operatorname{Der} A[q]$).

Let us verify that this definition of the local Lie algebra H is correct. In the case where the commutators do not contain elements from H_{-1}, the Jacobi identity is satisfied, since $\operatorname{Der} A[q]$ is a Lie algebra. If the commutator contains more that one element from H_{-1}, then, obviously, the Jacobi identity is not defined. Let us consider the remaining cases:

(1) $[[B_i^k, B_m^p], t_j] = \delta_i^p[B_m^k, t_j] - \delta_m^k[B_i^p, t_j] = -\delta_i^p\delta_j^k t_m + \delta_m^k\delta_j^p t_i$,
$[[B_i^k, t_j], B_m^p] + [B_i^k, [B_m^p, t_j]] = -\delta_j^k[t_i, B_m^p] - \delta_j^p[B_i^k, t_m] = -\delta_i^p\delta_j^k t_m + \delta_m^k\delta_j^p t_i$;

(2) $[[B_i^{jk}, B_m^p], t_l] = \delta_i^p[B_m^{jk}, t_l] - \delta_m^j[B_i^{pk}, t_l] - \delta_m^k[B_i^{jp}, t_l] = \delta_i^p\delta_l^j B_m^k - \delta_m^j\delta_l^p B_i^k - \delta_m^k\delta_l^j B_i^p$,
$[B_i^{jk}, [B_m^p, t_l]] + [[B_i^{jk}, t_l], B_m^p] = -\delta_l^p[B_i^{jk}, t_m] + \delta_l^j[B_i^k, B_m^p] = -\delta_l^p\delta_m^j B_i^k + \delta_l^j\delta_i^p B_m^k - \delta_l^j\delta_m^k B_i^p$.

Thus, H is indeed a local Lie algebra.

Consider the local subalgebra $\tilde{\mathfrak{G}} \subset H$ generated by the elements $t_1,\dots,t_N$ and $A = C_{jk}^i B_i^{jk}$. Let $\mathfrak{G}_i = \tilde{\mathfrak{G}}\cap H_i$, $i = 0,\pm1$, then $\tilde{\mathfrak{G}} = \mathfrak{G}_{-1}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_1$.

Denoting by $\mathfrak{G}_A$ the Lie span of the set $[\mathfrak{G}_{-1}, A]$, we have

$$[\mathfrak{G}_{-1}, [\mathfrak{G}_A, A]] \subset [[\mathfrak{G}_{-1}, \mathfrak{G}_A], A] + [\mathfrak{G}_A, \mathfrak{G}_A] \subset \mathfrak{G}_A$$

and, thus, $\mathfrak{G}_A = \mathfrak{G}_0$. Consider the minimal Lie algebra $\mathfrak{G}$ whose local part coincides with $\tilde{\mathfrak{G}}$. Introducing the notation $s_j = [A, t_j] = C_{jk}^i B_i^k$, we have $[t_k, s_j] = C_{jk}^i t_i$. Therefore, the algebra $\mathfrak{G}$ corresponds to system (3). The first assertion of the theorem is proved.

Let us show the transitivity of $\tilde{\mathfrak{G}}$.

Let $s\in\mathfrak{G}_1$, with $s = a_{jk}^i B_i^{jk}$ being the decomposition with respect to the basis, and let $[s, \mathfrak{G}_{-1}] = 0$. Then $0 = a_{jk}^i[B_i^{jk}, t_p] = a_{pk}^i B_i^k$. Further, since B_i^k are linearly independent and the number p is arbitrary, we can see that all a_{jk}^i vanish and that $s = 0$.

Similarly, if $s \in \mathfrak{G}_0$ and $[s, \mathfrak{G}_{-1}] = 0$, then $s = a^i_k B^k_i$, $0 = a^i_k [B^k_i, t_p] = a^i_p t_i$; therefore, all a^i_p vanish and $s = 0$.

Let $K = \{t \in \mathfrak{G}_{-1} | [t, \mathfrak{G}_1] = 0\}$. Assume that $K \neq \{0\}$. If $s \in \mathfrak{G}_0$ and $t \in K$, then $[[t, s], \mathfrak{G}_1] = [t, [s, \mathfrak{G}_1]] + [[t, \mathfrak{G}_1], s] = 0$ and $[K, \mathfrak{G}_0] \subseteq K$. Since the absence of left ideals in T is equivalent to the absence in $\mathfrak{G}_{-1}$ of subspaces invariant under the action of $\mathfrak{G}_0$, we have $K = \mathfrak{G}_{-1}$. Further, since $A \in \mathfrak{G}_1$, we see from the definition of K that $[A, \mathfrak{G}_{-1}] = 0$. As proved above, the last equality implies $A = 0$, which is impossible. Therefore, $K = \{0\}$.

Let $K = \{s \in \mathfrak{G}_0 | [s, \mathfrak{G}_1] = 0\}$. Assume that $K \neq \{0\}$. If $s \in K$ and $s' \in \mathfrak{G}_0$, then $[[s, s'], \mathfrak{G}_1] = [s, [s', \mathfrak{G}_1]] + [[s, \mathfrak{G}_1], s'] = 0$ and $[K, \mathfrak{G}_0] \subseteq K$. Setting $P = [K, \mathfrak{G}_{-1}]$, we can see from the Jacobi identity and the invariance of K under $\mathfrak{G}_0$ that

$$[\mathfrak{G}_0, P] = [\mathfrak{G}_0, [K, \mathfrak{G}_{-1}]] = [[\mathfrak{G}_0, K], \mathfrak{G}_{-1}] + [K, \mathfrak{G}_{-1}] \subseteq P.$$

Therefore, subspace P is also invariant under the action of $\mathfrak{G}_0$. Since $K \neq \{0\}$, we also have $P \neq \{0\}$ in accordance with the results proved above and, therefore, $P = \mathfrak{G}_{-1}$. Using the definition of the set K and its invariance under $\mathfrak{G}_0$, we find that

$$[\mathfrak{G}_{-1}, \mathfrak{G}_1] = [P, \mathfrak{G}_1] = [[K, \mathfrak{G}_{-1}], \mathfrak{G}_1] \subseteq [K, [\mathfrak{G}_{-1}, \mathfrak{G}_1]] \subseteq K.$$

However, $[\mathfrak{G}_{-1}, \mathfrak{G}_1]$ generates $\mathfrak{G}_0$. Therefore, $K = \mathfrak{G}_0$ and (by the definition of K) $[\mathfrak{G}_0, \mathfrak{G}_1] = 0$. Given this, we have $\mathfrak{G}_1 = \{A\}$, $[s_j, A] = 0$ and, then, $0 = [t_k, [s_j, A]] = -C^i_{jk} s_i - [s_j, s_k]$. We also have $[s_j, s_k] = -C^i_{jk} s_i$ and, similarly, $[s_k, s_j] = -C^i_{kj} s_i$, whence $(C^i_{jk} + C^i_{kj}) s_i = 0$.

It has already been proved that $[t, \mathfrak{G}_1] = 0$ implies $t = 0$, i.e., $[t, A] = 0$ implies $t = 0$. Therefore, $0 = (C^i_{jk} + C^i_{kj}) s_i = [A, (C^i_{jk} + C^i_{kj}) t_i]$ implies $(C^i_{jk} + C^i_{kj}) t_i = 0$. Since $t_1, \dots, t_N$ is a basis, the last equality implies that $C^i_{jk} = -C^i_{kj}$ for all $i, j, k = 1, \dots, N$. Thus, the $*$ operation is anti-commutative.

Taking $U, V, W \in \mathfrak{G}_{-1}$, we can see from the definition of $*$ that

$$\begin{aligned} U * (V * W) &= [[U, A], [[V, A], W]] = [[[U, A], [V, A]], W] + [[V, A], [[U, A], W]] = \\ &= [[[[U, A], V], A], W] + [[V, [[U, A], A]], W] + V * (U * W) = \\ &= (U * V) * W + V * (U * W) + [[V, [[U, A], A]], W]. \end{aligned}$$

Since $[\mathfrak{G}_0, \mathfrak{G}_1] = 0$, the last commutator vanishes and the operation $*$ satisfies the Jacobi identity. In view of the condition of the theorem, T is not a Lie algebra. Therefore, our assumption is wrong and, thus, $K = \{0\}$ and the local algebra $\widetilde{\mathfrak{G}}$ is transitive. As follows from Proposition 2, the algebra $\mathfrak{G}$ is also transitive. The theorem is proved.

Let us note that the derivations B^k_i and B^{jk}_i were introduced in such a way that the generating elements of the characteristic algebra L_y are expressed through B^k_i and B^{jk}_i by the formulas

$$Y_j = C^i_{jk} B^k_i \equiv s_j, \qquad Y_0 = C^i_{jk} B^{jk}_i \equiv A. \tag{13}$$

2. Characteristic algebras

In the study of characteristic Lie algebras, it is important to know how their elements commute with the operators of the derivatives D_x and D_y. In the corresponding formulas, one uses the "derivations" f_0, f_j, and g_j, $j = 1, \dots, N$, of the characteristic Lie algebras L_x and L_y, respectively. The term "derivations" is put in quotation marks since these mappings do not act on the characteristic Lie algebras themselves, but, rather, on some formal objects. Let us give the precise definitions of these mappings.

On the set of formal commutators of the elements X_k, $k = 1, 2, \dots, N$, and their linear combinations, we introduce the linear mappings f_j, $j = 0, 1, \dots, N$ (where f_0 pertains to the set of commutators of length not less than 2) according to the following rules:

(1) for $j > 0$, we have

$$f_j(X_k) = -C^n_{jk} X_n,$$
$$f_j([X', X'']) = [f_j(X'), X''] + [X', f_j(X'')],$$

where X' and X'' are some commutators;

(2) for $j = 0$,

$$f_0[X_i, X_j] = f_i(X_j) - f_j(X_i),$$
$$f_0[X_i, X'] = f_i(X') + [X_i, f_0(X')],$$
$$f_0([X', X'']) = [f_0(X'), X''] + [X', f_0(X'')],$$

where X' and X'' are some commutators of length not less than 2.

Consider the set of formal commutators consisting of the elements Y_j, $j = 0, 1, \dots, N$, and of their linear combinations. Let Y and Y' be elements from this set. We introduce on this set the linear mappings g_k, $k = 1, \dots, N$, according to the rules

$$g_k(Y_0) = -Y_k,$$
$$g_k([Y, Y']) = [g_k(Y), Y'] + [Y, g_k(Y')],$$
$$g_k([Y_j, Y]) = C^i_{jk} g_i(Y) + [Y_j, g_k(Y)], \quad j \neq 0.$$

If an element of the set under consideration is of the form $[Y'', Y]$, where Y'' is a commutator of length greater than 1 and such that it does not contain Y_0, then $[Y'', Y] = [[\widetilde{Y}, \widehat{Y}], Y] = [\widetilde{Y}, [\widehat{Y}, Y]] - [\widehat{Y}, [\widetilde{Y}, Y]]$, where $\widetilde{Y}$ and $\widehat{Y}$ are commutators whose length is less than the length of Y''. Continuing with this process, we can transform the commutator $[Y'', Y]$ into a unique linear combination of commutators of the form $[Y_j, Y']$. We set $g_k([Y'', Y])$ equal to the image of this sum under the mapping g_k.

In some of the lemmas in this section, we prove the desired relations only for individual commutators, each time assuming that the statement is true for their linear combinations in view of the linearity of the mappings f_j and g_k, and of the commutation operation.

Lemma 1. *There is a relation* $[D_y, X] = f_0(X) + p^j_0 f_j(X)$, *where* X *is the commutator of elements* X_j *of length not less than 2 and the right-hand side is viewed as an element of* Der $A[p]$.

Proof. Let us first show that $[D_y, X_k] = -C^i_{jk} p^j_0 X_i$. Using (10), we obtain the following relations in which we assume that the operators act on functions from $A[p]$,

$$0 = [D_y, D_x] = [D_y, q^i_0 X_i] = D_y(q^i_0) X_i + q^i_0 [D_y, X_i] =$$
$$= C^i_{jk} p^j_0 q^k_0 X_i + q^i_0 [D_y, X_i] = q^k_0 \left(C^i_{jk} p^j_0 X_i + [D_y, X_k]\right).$$

It follows that $[D_y, X_k] = -C^i_{jk} p^j_0 X_i$.

We now prove the desired relation by induction on the length d of the commutator X.

1. Case $d = 2$. Inserting p^j_0 into (10), we can see that $X_k(p^j_0) = \delta^j_k$. Let us note in passing that these equalities imply the linear independence of X_k. Further, we have

$$[D_y, [X_k, X_l]] = -[C^i_{jk} p^j_0 X_i, X_l] - [X_k, C^i_{jl} p^j_0 X_i] =$$
$$= -C^i_{jk} p^j_0 [X_i, X_l] + C^i_{lk} X_i - C^i_{jl} p^j_0 [X_k, X_i] - C^i_{kl} X_i.$$

The right-hand side of the relation we have to prove can be represented as

$$f_0[X_k, X_l] + p_0^j f_j[X_k, X_l] = f_k(X_l) - f_l(X_k) + p_0^j([f_j(X_k), X_l] + [X_k, f_j(X_l)]) = \\ = -C_{kl}^i X_i + C_{lk}^i X_i + p_0^j(-C_{jk}^i[X_i, X_l] - C_{jl}^i[X_k, X_i]).$$

By comparing the right-hand sides of the formulas obtained, we verify that the relation is fulfilled.

2. Let the statement be true for $d \le m$. Let us prove it for $d = m+1$. If X is a commutator of length d, then $X = [X', X'']$, where X' and X'' are commutators of lengths l and k, respectively, $l, k \le m$.

First, take $k, l > 1$. Then $X'(p_0^j) = X''(p_0^j) = 0$, whence we can see that

$$[D_y, [X', X'']] = [[D_y, X'], X''] + [X', [D_y, X'']] = \\ = [f_0(X') + p_0^j f_j(X'), X''] + [X', f_0(X'') + p_0^j f_j(X'')] = \\ = [f_0(X'), X''] + [X', f_0(X'')] + p_0^j([f_j(X'), X''] + [X', f_j(X'')]) = \\ = f_0([X', X'']) + p_0^j f_j([X', X'']).$$

Now, let one of the numbers k, l, for instance k, be equal to 1. Then $X'' = X_l$ for some $l = 1, \dots, N$, which allows us to write

$$[D_y, [X', X'']] = [[D_y, X'], X_l] + [X', [D_y, X_l]] = \\ = [f_0(X') + p_0^j f_j(X'), X_l] - C_{jl}^i [X', p_0^j X_i] = \\ = [f_0(X'), X_l] + [p_0^j f_j(X'), X_l] - C_{jl}^i [X', p_0^j X_i] = \\ = [f_0(X'), X_l] + p_0^j [f_j(X'), X_l] - f_l(X') - C_{jl}^i p_0^j [X', X_i],$$

and then we transform the right-hand side of the relation to the same form,

$$f_0([X', X'']) + p_0^j f_j([X', X'']) = \\ = [f_0(X'), X_l] - f_l(X') + p_0^j [f_j(X'), X_l] + p_0^j [X', f_j(X_l)] = \\ = [f_0(X'), X_l] - f_l(X') + p_0^j [f_j(X'), X_l] - C_{jl}^i p_0^j [X', X_i].$$

The lemma is proved.

Lemma 2. *There is the relation $[D_x, Y] = q_0^j g_j(Y)$, where Y is a commutator of elements Y_j involving Y_0 and the right-hand side is viewed as an element of* Der $A[q]$.

Proof. Let us prove the desired relation by induction on the length d of the commutator Y.

1. Case $d = 1$. Using (11), we obtain the following relations, where we assume the operators act on functions from $A[q]$:

$$0 = [D_x, D_y] = [D_x, Y_0 + p_0^i Y_i] = [D_x, Y_0] + p_0^i [D_x, Y_i] + D_x(p_0^i) Y_i = [D_x, Y_0] + q_0^i Y_i + p_0^i [D_x, Y_i].$$

Since $[D_x, Y_0] + q_0^i Y_i, [D_x, Y_i] \in$ Der $A[q]$, the last equality allows us to obtain $[D_x, Y_0] + q_0^i Y_i = 0$ and $[D_x, Y_i] = 0$. Thus, $[D_x, Y_0] = -q_0^i Y_i = q_0^i g_i(Y_0)$.

2. Let the statement be true for $d \le m$ and let us prove it for $d = m+1$. Inserting q_0^i into (11), we have $Y_0(q_0^i) = 0$ and $Y_j(q_0^i) = C_{jk}^i q_0^k$, $i, j = 1, 2, \dots, N$. Obviously, if a commutator Y of the elements Y_j contains Y_0, then $Y(q_0^i) = 0$. If Y is a commutator of length d, then $Y = [Y', Y'']$, where Y' and Y'' are commutators of lengths l and k, respectively, $l, k \le m$.

First, let both elements Y' and Y'' contain Y_0. Then $Y'(q_0^j) = Y''(q_0^j) = 0$, in which case we have

$$[D_x, [Y', Y'']] = [[D_x, Y'], Y''] + [Y', [D_x, Y'']] =$$
$$= [q_0^j g_j(Y'), Y''] + [Y', q_0^j g_j(Y'')] =$$
$$= q_0^j [g_j(Y'), Y''] + q_0^j [Y', g_j(Y'')] = q_0^j g_j([Y', Y'']).$$

Now, let one of the commutators, for example Y'', be equal to Y_l for some $l = 1, 2, \ldots, N$. Then

$$[D_x, [Y', Y'']] = [[D_x, Y'], Y_l] = [q_0^j g_j(Y'), Y_l] =$$
$$= q_0^j [g_j(Y'), Y_l] - C_{lk}^j q_0^k g_j(Y') = -q_0^j (C_{lj}^i g_i(Y') + [Y_l, g_j(Y')]) =$$
$$= -q_0^j g_j([Y_l, Y']) = q_0^j g_j([Y', Y_l]).$$

If Y'' does not contain Y_0 and is of a length greater than 1, we should consider the representation of Y as a sum of commutators of the form $[Y_j, \widetilde{Y}]$. It is clear that whenever the relation holds for each of these commutators, it is also true for their sum and, thus, for Y as well. The lemma is proved.

We say that elements $T \in \bigoplus_{i=1}^{+\infty} \mathfrak{G}_{-i}$, $X \in L_x$ ($s \in \bigoplus_{i=0}^{+\infty} \mathfrak{G}_i$ and $Y \in L_y$) are of the same form if there are representations of these elements in terms of the linear combinations of commutators that can be obtained from each other by replacing all of the t_p symbols with X_p symbols (A with Y_0 and s_j with Y_j, $j = 1, 2, \ldots, N$) and vice versa.

Lemma 3. *If $T \in \mathfrak{G}_{-m}$ and $X \in L_x$ are of the same form, then $[s_j, T]$ and $f_j(X)$ are of the same form for $m > 0$. If, in addition, $m > 1$, then $[A, T]$ and $f_0(X)$ are also of the same form.*

Proof.

1. Let $T = t_k$ and $X = X_k$. Let us consider $[s_j, T]$ and $f_j(X)$:

$$[s_j, T] = [s_j, t_k] = -C_{jk}^i t_i,$$
$$f_j(X) = f_j(X_k) = -C_{jk}^i X_i.$$

As can be seen, the assertion holds for $m = 1$.

2. Let the assertion hold for $m < p$ and let us prove it for $m = p$. Let T and X be commutators of length p of the same form. Then $T = [T', T'']$ and $X = [X', X'']$, where T' and X' have length k, while T'' and X'' have length l, $k, l < p$, where, in addition, T' and X', as well as T'' and X'', are of the same form:

$$[s_j, T] = [s_j, [T', T'']] = [[s_j, T'], T''] + [T', [s_j, T'']],$$
$$f_j(X) = f_j([X', X'']) = [f_j(X'), X''] + [X', f_j(X'')].$$

By the induction hypothesis, the right-hand sides of these equations are of the same form, whence we deduce the first statement of the lemma.

The statement regarding $[A, T]$ and $f_0(X)$ is proved in a similar way.

Lemma 4. *If $s \in \bigoplus_{i=1}^{+\infty} \mathfrak{G}_i$ and $Y \in \bigoplus_{i=1}^{+\infty} L_y^i$ are of the same form, then $[t_k, s]$ and $g_k(Y)$ also are of the same form.*

Proof. This follows the scheme of the proof of Lemma 3.

Introducing the corresponding weights of the variables x, y, and u^i, it is not difficult to check that the characteristic algebras L_x and L_y admit the natural gradings $L_x = \bigoplus_{i=1}^{+\infty} L_x^i$ and $L_y = \bigoplus_{i=0}^{+\infty} L_y^i$. The spaces L_x^i are spanned by commutators of length i constructed from elements X_j, $j = 1, \ldots, N$. The spaces L_y^i are linear spans of the commutators of Y_j, $j = 0, \ldots, N$, that contain Y_0 precisely i times.

The next statement provides a sufficient condition for a set of commutators in the characteristic algebra L_x to vanish.

Lemma 5. *Let $\alpha_1, \dots, \alpha_k \in L_x^i$, $i \geq 2$, be a set of elements such that $[D_y, \alpha_l] = f^{lj}(p)\alpha_j$ for every l. Then $\alpha_l = 0$, $l = 1, \dots, k$.*

Proof. Since $X_\nu(p_0^r) = \delta_\nu^r$, it follows from $X \in L_x^i$, $i \geq 2$, that $X(p_0^r) = 0$. Let us prove by induction on s that, under the conditions of the lemma, $\alpha_l(p_s^r) = 0$ for any $l = \overline{1,k}$, $r = \overline{1,n}$, and $s \in \mathbb{N}$. The case of $s = 0$ has already been considered. Let the statement be true for $s = m$, then

$$\alpha_l(p_{m+1}^r) = \alpha_l(D_y(p_m^r)) = D_y(\alpha_l(p_m^r)) - f^{lj}\alpha_j(p_m^r) = 0.$$

The following lemma can be proved similarly.

Lemma 6. *Let $\beta_1, \dots, \beta_k \in L_y^i$, $i \geq 1$, be a set of elements such that $[D_x, \beta_l] = g^{lj}(q)\beta_j$ for any l. Then $\beta_l = 0$, $l = 1, \dots, k$.*

3. Proof of the hypothesis regarding the existence of a complete algebra.

In this section, we prove Theorem 2.

Let us consider the algebra $\mathfrak{G}$ constructed in the proof of Theorem 1. We first show that there are epimorphisms φ and ψ, and then show that $\operatorname{Ker}\varphi = \operatorname{Ker}\psi = 0$.

1. We specify a linear mapping $\varphi\colon \bigoplus_{i=1}^{+\infty} \mathfrak{G}_{-i} \to L_x$ in the following way: to every commutator t consisting of elements t_j, $j = 1, \dots, N$, we associate the commutator of elements X_j that has the same form. It is clear that this defines a surjective mapping.

For every k, the mapping φ induces a mapping $\varphi_k\colon \mathfrak{G}_{-k} \to L_x^k$. The necessary and sufficient condition for φ to be a homomorphism of the linear spaces $\bigoplus_{i=1}^{+\infty} \mathfrak{G}_{-i}$ and L_x is that every mapping φ_k be a homomorphism of the corresponding linear spaces. This is equivalent to the following condition: if a linear combination of commutators of length k consisting of elements t_j vanishes, its image under φ_k is zero as well.

Let us show that φ_k is a homomorphism by induction on k:

(a) as noted above, the elements X_j, $j = 1, \dots, N$, form a basis of L_x^1, therefore, φ_1 is an isomorphism;

(b) let the mapping φ_m be a homomorphism.

Let t be a linear combination of commutators of length $m+1$ ($t \in \mathfrak{G}_{-m-1}$), $t = 0$, and $X = \varphi_{m+1}(t)$. Then $[A, t]$, $[A, [s_i, t]]$, and $[A[s_{i_1}[s_{i_2} \dots [s_{i_p}, t] \dots]]]$ are linear combinations from $\mathfrak{G}_{-m}$ which vanish for every $p \in \mathbb{N}$ and for every $i_k = \overline{1,N}$. By Lemma 3, they are taken by φ_m into $f_0(X)$, $f_0(f_i(X))$, and $f_0(f_{i_1}(f_{i_2} \dots (f_{i_p}(X)) \dots))$, respectively. These, therefore, also vanish by the induction hypothesis.

Let us consider the smallest set $K \subseteq L_x^{-m-1}$ defined by the conditions $K \subset \operatorname{Ker} f_0$, $f_i(K) \subseteq K$ for any $i = \overline{1,N}$ and $X \in K$. In K, we choose a basis $B^i, i = \overline{1,l}$. Since $f_0(B^i) = 0$, we can see from Lemma 1 that $[D_y, B^i] = u^j f_j(B^i)$, where $f_j(B^i) \in K$. According to Lemma 5, we have $K = \{0\}$, whence $X = 0$. Thus, φ_{m+1} is a homomorphism and, therefore, so is φ. This last statement, taken together with the definition of φ, implies that the commutator operation is preserved. Further, since the mapping $\varphi\colon \bigoplus_{i=1}^{+\infty} \mathfrak{G}_{-i} \to L_x$ is surjective, it is a Lie algebra epimorphism.

2. Let us consider the mapping $\psi\colon \bigoplus_{i=0}^{+\infty} \mathfrak{G}_i \to L_y$ defined like the mapping φ; it is surjective. For every $k \in \mathbb{N}_0$, the mapping ψ induces a mapping $\psi_k\colon \mathfrak{G}_k \to L_y^k$. In order that ψ be a homomorphism of the linear spaces $\bigoplus_{i=0}^{+\infty} \mathfrak{G}_i$ and L_y, it is necessary and sufficient that every mapping ψ_k be a homomorphism between $\mathfrak{G}_k$ and L_y^k. This is true if and only if every linear combination of commutators from $\mathfrak{G}_k$ that is equal to zero is mapped into zero.

Let us show that ψ_k is a homomorphism by induction on k:

(a) In view of Eqs. (13), the spaces $\mathfrak{G}_0$ and $\mathfrak{G}_1$ are, by definition, identical to L_y^0 and L_y^1, respectively. Therefore, ψ_0 and ψ_1 are isomorphisms;

(b) Let ψ_m be a homomorphism.

Let $s \in \mathfrak{G}_{m+1}$ be an arbitrary linear combination of commutators, $s = 0$, and $Y = \psi_{m+1}(s)$. Then $[t_i, s]$ are linear combinations of the commutators of elements s_j and A lying in $\mathfrak{G}_m$ and vanishing for any $i = \overline{1, N}$. Under ψ_m, according to Lemma 4, they are taken into $g_i(Y)$, which vanishes, since ψ_m is a homomorphism. From Lemma 2, we have $[D_x, Y] = p^i g_i(Y) = 0$. Since $Y \in \mathfrak{G}_{m+1}$, $m \geq 0$, we can see from Lemma 6 that $Y = 0$ and ψ_{m+1} is a homomorphism. Thus, ψ is a homomorphism of $\bigoplus_{i=0}^{+\infty} \mathfrak{G}_i$ onto L_y. This and the definition of ψ imply that the commutator operation is preserved, whence, by surjectivity, we obtain that ψ is a Lie algebra epimorphism.

We set $K = \operatorname{Ker}\varphi \oplus \operatorname{Ker}\psi$. Since $\operatorname{Ker}\psi = \bigoplus_{i=0}^{+\infty} \operatorname{Ker}\psi_i$ and $\operatorname{Ker}\varphi = \bigoplus_{i=1}^{+\infty} \operatorname{Ker}\varphi_i$, the factor algebra $\mathfrak{G}/K$ inherits the natural grading and there is a homomorphism $\chi\colon \mathfrak{G} \to \mathfrak{G}/K$. Since φ_1, ψ_0, and ψ_1 are isomorphisms, the local components of $\mathfrak{G}$ and $\mathfrak{G}/K$ are isomorphic. Further, since $\mathfrak{G}$ is a minimal algebra, it follows that χ is an isomorphism. Therefore, $\operatorname{Ker}\psi = \operatorname{Ker}\varphi = 0$. The theorem is proved.

4. Systems corresponding to simple finite-dimensional Lie algebras in the standard grading

As noted in the Introduction, a transitive irreducible $\mathbb{Z}$-graded algebra that is also finite-dimensional coincides with one of simple Lie algebras taken in one of the standard gradings. At the same time, the requirement that the algebra generating system (3) be irreducible is too restrictive: systems of form (6) correspond to simple Lie algebras with canonical $\mathbb{Z}$-grading that are reducible, whereas system (6) itself does not contain subsystems. In this paper, however, we restrict ourselves to considering only the irreducible case.

A straightforward analysis of root systems of type A_n, D_n, E_6, E_7, and E_8 shows that any standard grading of the corresponding simple Lie algebras has a height 1. Then the corresponding algebra T is a Jordan algebra [12] and, therefore, its structure constants are symmetric, $C^i_{jk} = C^i_{kj}$. Thus, as was noted in the Introduction, system (3) reduces after the integration to a system of ordinary differential equations of form (4). In the case of the Lie algebras B_2, C_2, B_3, and C_3, some of the standard gradings may have a height 2. In that case, however, inspection shows that subsystems are present in the corresponding systems of form (3) as well. Moreover, the Lie algebras corresponding to the subsystems are of height 1. From our point of view, such systems of equations are of little interest and this appears to be the case for all of the Lie algebras B_n and C_n, $n \geq 2$.

The simple Lie algebra of type G_2 has two standard gradings of height 3 and 2, respectively. In the first case, we obtain a system consisting of two equations that have subsystems. In the second case, we have the following grading:

$$
\begin{aligned}
\mathfrak{G}_{-2} &= \{e_5\},\\
\mathfrak{G}_{-1} &= \{e_1, e_3, e_4, e_6\},\\
\mathfrak{G}_0 &= \{h_1, h_2, e_2, f_2\},\\
\mathfrak{G}_1 &= \{f_1, f_3, f_4, f_6\},\\
\mathfrak{G}_2 &= \{f_5\}.
\end{aligned}
$$

Choosing the element $A = f_1 + f_6/12$ and performing the rescaling $v^4 = 3u^4$, we arrive at the system of equations (12). It is interesting that for $A = f_3$ we have another system,

$$
\begin{cases}
u^1_{xy} = 3u^1u^2_x + 3u^2u^1_x,\\
u^2_{xy} = 4u^3u^1_x + 4u^1u^3_x + 2u^2u^2_x,\\
u^3_{xy} = 4u^3u^2_x + u^2u^3_x + 3u^1u^4_x,\\
u^4_{xy} = u^3u^3_x.
\end{cases}
$$

Here, the set $[\mathfrak{G}_{-1}, A]$ is three-dimensional, i.e., the left multiplication operators in T constitute a three-dimensional space, unlike the first case, where this space is four-dimensional. Therefore, these two systems are linearly inequivalent. With a different choice of element A, either the system is linearly equivalent to one we have considered or the set $[\mathfrak{G}_{-1}, A]$ does not generate $\mathfrak{G}_0$.

Let us prove that neither system contains subsystems. Since $[\mathfrak{G}_{-1}, A]$ generates $\mathfrak{G}_0$ in either case, it suffices to prove, as we have already noted, that the representation of $\mathfrak{G}_0$ on $\mathfrak{G}_{-1}$ is irreducible. The commutation table of the algebra G_2 looks as follows.

	e_1	e_2	e_3	e_4	e_5	e_6	f_1	f_2	f_3	f_4	f_5	f_6	h_1	h_2
e_1	0	$-e_3$	0	0	0	$-e_5$	$-h_1$	0	$-f_2$	0	$-f_6$	0	$2e_1$	$-3e_1$
e_2	e_3	0	e_4	e_6	0	0	0	$-h_2$	$3f_1$	$4f_3$	0	$3f_4$	$-e_2$	$2e_2$
e_3	0	$-e_4$	0	e_5	0	0	e_2	$-3e_1$	α	$4f_2$	$-3f_4$	0	e_3	$-e_3$
e_4	0	$-e_6$	$-e_5$	0	0	0	0	$-4e_3$	$-4e_2$	$-\beta$	$-12f_3$	$-12f_2$	0	e_4
e_5	0	0	0	0	0	0	e_6	0	$3e_4$	$12e_3$	$-\delta$	$36e_1$	e_5	0
e_6	e_5	0	0	0	0	0	0	$-3e_4$	0	$12e_2$	$-36f_1$	γ	$-e_6$	$3e_6$
f_1	h_1	0	$-e_2$	0	$-e_6$	0	0	$-f_3$	0	0	0	$-f_5$	$-2f_1$	$3f_1$
f_2	0	h_2	$3e_1$	$4e_3$	0	$3e_4$	$-f_3$	0	f_4	f_6	0	0	f_2	$-2f_2$
f_3	f_2	$-3f_1$	$-\alpha$	$4e_2$	$-3e_4$	0	0	$-f_4$	0	f_5	0	0	$-f_3$	f_3
f_4	0	$-4f_3$	$-4f_2$	β	$-12e_3$	$-12e_2$	0	$-f_6$	$-f_5$	0	0	0	0	$-f_4$
f_5	f_6	0	$3f_4$	$12f_3$	δ	$36f_1$	0	0	0	0	0	0	$-f_5$	0
f_6	0	$-3f_4$	0	$12f_2$	$-36e_1$	$-\gamma$	f_5	0	0	0	0	0	f_6	$-3f_6$
h_1	$-2e_1$	e_2	$-e_3$	0	$-e_5$	e_6	$2f_1$	$-f_2$	f_3	0	f_5	$-f_6$	0	0
h_2	$3e_1$	$-2e_2$	e_3	$-e_4$	0	$-3e_6$	$-3f_1$	$2f_2$	$-f_3$	f_4	0	$3f_6$	0	0

Here, $\alpha = 3h_1 + h_2$, $\beta = 12h_1 + 8h_2$, $\gamma = 36(h_1 + h_2)$, and $\delta = 72h_1 + 36h_2$. Let us consider an arbitrary element $s = \alpha e_1 + \beta e_3 + \gamma e_4 + \delta e_6 \in \mathfrak{G}_{-1}$. Then

$$[e_2, s] = [e_2, \alpha e_1 + \beta e_3 + \gamma e_4 + \delta e_6] = -\alpha e_3 - \beta e_4 - \gamma e_6,$$
$$[f_2, s] = [f_2, \alpha e_1 + \beta e_3 + \gamma e_4 + \delta e_6] = -3\beta e_1 - 4\gamma e_3 - 3\delta e_4.$$

Assume that $\mathfrak{G}_{-1}$ contains a subspace H that is closed with respect to commutation with $\mathfrak{G}_0$. Let us show that either $H = \{0\}$ or $H = G_{-1}$.

Lemma 7. *If $e_6 \in H$, then $H = G_{-1}$.*

Proof. Assume that $e_6 \in H$. Then $[f_2, e_6] = -3e_4 \in H$ and, thus, $e_4 \in H$. Also, $[f_2, e_4] = -4e_3 \in H$, $e_3 \in H$, and $[f_2, e_3] = -3e_1 \in H$, $e_1 \in H$; thus, $H = G_{-1}$.

The lemma is proved.

Assume that H contains a nonzero element $s = \alpha e_1 + \beta e_3 + \gamma e_4 + \delta e_6$. If $\alpha \neq 0$, then $[e_2, s] = -\alpha e_3 - \beta e_4 - \gamma e_6 = s_1 \in H$, $[e_2, s_1] = \alpha e_4 + \beta e_6 = s_2 \in H$, and $[e_2, s_2] = -\alpha e_6 \in H$, whence $e_6 \in H$ and $H = G_{-1}$. Similarly, $H = G_{-1}$ whenever $\alpha = 0$, $\beta \neq 0$ or $\alpha = \beta = 0$, $\gamma \neq 0$. If $\alpha = \beta = \gamma = 0$, then $\delta \neq 0$, $e_6 \in H$, and $H = G_{-1}$. Thus, $H \neq \{0\}$ implies $H = G_{-1}$. Therefore, the representation of $\mathfrak{G}_0$ on $\mathfrak{G}_{-1}$ is irreducible and (12) does not contain any subsystems.

The algebra F_4 admits three standard gradings of height 2 and one grading of height 4. In each of these, the representation of $\mathfrak{G}_0$ on $\mathfrak{G}_{-1}$ is irreducible and there exists an element $A \in \mathfrak{G}_1$ such that $[\mathfrak{G}_{-1}, A]$ generates $\mathfrak{G}_0$. Thus, the corresponding systems do not contain subsystems. They consist of fourteen, ten, eight, and six equations, respectively, which will be given elsewhere.

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