Theoretical and Mathematical Physics, Vol. 113, No. 2, 1997

# HIROTA'S DIFFERENCE EQUATIONS[1]

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*A review of selected topics for Hirota's bilinear difference equation (HBDE) is given. This famous three-dimensional difference equation is known to provide a canonical integrable discretization for most of the important types of soliton equations. Similar to continuous theory, HBDE is a member of an infinite hierarchy. The central point of our paper is a discrete version of the zero curvature condition explicitly written in the form of discrete Zakharov–Shabat equations for M-operators realized as difference or pseudo-difference operators. A unified approach to various types of M-operators and zero curvature representations is suggested. Different reductions of HBDE to two-dimensional equations are considered, with discrete counterparts of the KdV, sine-Gordon, Toda chain, relativistic Toda chain, and other examples.*

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 113, No. 2, pp. 179–230, November, 1997. Original article submitted June 20, 1997.

## 1. Introduction

In 1981, Hirota published [1], which summarized his earlier studies of discretized nonlinear integrable equations [2–6]. The main result was a compact bilinear equation which can be viewed as an integrable discrete analogue of the two-dimensional Toda lattice. In Hirota's original notation, it has the form

$$\left[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)\right]\tau \cdot \tau = 0, \tag{1.1}$$

where $Z_i$ are arbitrary constants, $\tau = \tau(x_1, x_2, x_3)$ is a function of three variables, $D_i \equiv D_{x_i}$, and Hirota's $D$-operator is defined for the linear differential operator $F(\partial_x)$ by the formula

$$F(D_x)f(x) \cdot g(x) = F(\partial_y)f(x+y)g(x-y)\Big|_{y=0}. \tag{1.2}$$

In more explicit notation, Eq. (1.1) looks as follows:

$$Z_1\tau(x_1+1, x_2, x_3)\tau(x_1-1, x_2, x_3) + Z_2\tau(x_1, x_2+1, x_3)\tau(x_1, x_2-1, x_3) + \\ + Z_3\tau(x_1, x_2, x_3+1)\tau(x_1, x_2, x_3-1) = 0. \tag{1.3}$$

This equation is often called the Hirota bilinear difference equation (HBDE). Its simplicity is surprising and elusive at the same time: each detail is controlled by the integrability and hides meaningful mathematical structures, whereas some even simpler looking equations turn out to be intractable by analytical methods.

One of the most impressive outcomes of Hirota's work is that HBDE unifies many, if not all, known soliton equations. More precisely, it contains them in an encoded form. Performing a scaling continuum limit for appropriate combinations of parameters and variables, one is able to obtain the Korteweg–de Vries (KdV) equation, the Kadomtsev–Petviashvili (KP) equation, the modified KdV (MKdV) and the modified KP (MKP) equations, the two-dimensional Toda lattice (2DTL) equation, the sine-Gordon (SG) equation, the Benjamin–Ono equation, etc. Their discrete analogues are produced from HBDE by choosing suitable dependent and independent variables. Furthermore, Eq. (1.1) was shown to possess soliton solutions and Bäcklund transformations for arbitrary parameter values. These facts suggest that HBDE should be considered as a *fundamental* classical soliton equation, from which the typical examples can be obtained as particular cases.

Recently, bilinear equations of this form emerged [7, 8] in the context of *quantum* integrable systems as the model-independent functional relations [9, 10] for eigenvalues of quantum transfer matrices. This is what motivated us to revisit the classical nonlinear difference equations.

These notes aim at reviewing selected topics of HBDE and clarifying the basic elements of the theory. In our discussion, we deal solely with the equations themselves, saying almost nothing about their solutions.[3] Likewise, their continuous counterparts, the completely discretized nonlinear integrable equations, are known to possess soliton and finite-gap solutions. However, a systematic treatment of these and other particular classes of solutions would be a separate enterprise requiring much more space. We confine ourselves to elaborating the discrete versions of commutation representations and auxiliary linear problems on a formal algebraic level. At the same time, some important elements of our approach are motivated by the finite-gap theory.

The difference soliton equations are intimately connected with differential equations. We have already mentioned that the latter are obtained from the former by a scaling limit. It would be better to say that HBDE was designed to enjoy this property. The fact that such an equation exists is by no means trivial. A link in the opposite direction was established by Miwa [11], who noticed that discrete Hirota equations can be obtained from the continuous KP hierarchy by choosing the time flows as certain infinite combinations of standard flows of the hierarchy. This idea was further developed in [12, 13] as a method of producing

[3] Because of this, we do not draw any distinction between the discrete and difference equations. It is usually implied in the latter case that solutions are functions of a continuous variable with certain analytical properties.

discrete soliton equations from continuous ones. The inter-relation between the discrete and continuous integrable hierarchies looks like a kind of Fourier duality: they provide complementary descriptions of the same object, namely, of the infinite-dimensional Grassmannian [14–17].

In this survey, we do not give a systematic treatment of the connection between the discrete and continuous hierarchies. The problem of describing a limiting procedure that would be compatible with the entire hierarchy is technically involved. However, it is impossible to refrain from referring to continuous hierarchies. We agree to a compromise and restrict ourselves to a few typical examples.

It is assumed that the reader is familiar with the basic notions of continuous theory such as the Lax and Zakharov–Shabat equations, zero curvature conditions in scalar and matrix forms, commuting flows, infinite hierarchies, $\tau$-functions, etc.

Let us outline the contents of the paper. (Detailed descriptions are given in the short introductions to each section.)

Section 2 can be considered as a part of the Introduction. Here we tried to collect the different forms of three-dimensional HBDE known in the literature. All of them are equivalent and simple transformations between them are listed.

Sections 3, 4, and 5 form the main body of the paper. As we need a number of definitions and axioms to discover the key principles underlying the variety of integrable difference equations, these are given in Sec. 3. All of the notions explained in Sec. 3 are used in what follows. In Sec. 4, the discrete version of the zero-curvature representation is presented. Filling in some gaps in the existing literature, we give explicit forms of the $M$-operators (realized as difference operators) for discrete flows. Section 5 is devoted to various types of redundant auxiliary linear problems. They provide a "linearization" of the original nonlinear equation. The related notion of the Bäcklund transformation is also discussed and the Baker–Akhiezer functions are introduced as special formal solutions to the linear problems.

Sections 6 and 7 are more technical and might be more interesting to experts. They can be skipped without loss of understanding. In Sec. 6, we explain how to extend the $M$-operator approach to the arbitrary discrete flows defined in Sec. 3. In the general case, the $M$-operators contain negative powers of first-order difference operators. In Sec. 7, we dwell on hierarchies of bilinear discrete equations and suggest the notion of "higher" discrete flows with the corresponding zero curvature representation. We postulate that all "higher" ($N$-term) discrete Hirota equations known in the literature are consequences of the 3-term equations. This assertion is proved for the first nontrivial example of a 4-term equation in four variables.

Section 8 deals with two-dimensional reductions of HBDE, the corresponding $(L-M)$-pairs, and auxiliary linear problems. The list of reductions includes discrete analogues of the KdV equation, the 1D Toda chain, the AKNS system, the relativistic Toda chain, the sine-Gordon equation, the Liouville equation, and others.

In the Appendix, we present the main elements of a different approach to HBDE based on Miwa's transformation. This method was suggested in [12] for generating discrete soliton equations. We show how it works for simple examples and comment on the continuum limit which is, in a sense, an "inverse" Miwa transformation.

## 2. Equivalent forms of the bilinear equation

Hirota's difference equation exists in several forms. Historically, they emerged as integrable discretizations of particular continuous hierarchies (e.g., KP, 2DTL). In this section, we give a list of most popular forms of Hirota's difference equation and explicitly demonstrate that they are equivalent. However, it is useful to bear all of them in mind since one or another may be more convenient in a particular problem.

**A.** *Hirota's original form:*

$$Z_1\tau(x_1+1)\tau(x_1-1)+Z_2\tau(x_2+1)\tau(x_2-1)+Z_3\tau(x_3+1)\tau(x_3-1)=0 \tag{2.1}$$

(hereafter, we often skip the variables that do not undergo shifts). Note that the three variables enter in a symmetric fashion and the equation is invariant under their permutations and a simultaneous permutation of the parameters $Z_i$. The equation is also invariant under a sign change for any of the variables and under the transformation

$$\tau(x_1,x_2,x_3)\to\chi_0(x_1+x_2+x_3)\chi_1(x_2+x_3-x_1)\chi_2(x_1+x_3-x_2)\chi_3(x_1+x_2-x_3)\tau(x_1,x_2,x_3). \quad (2.2)$$

where $\chi_i$ are arbitrary functions.

**A$'$.** *Canonical form*:

$$\tau(x_1+1)\tau(x_1-1)+\tau(x_2+1)\tau(x_2-1)+\tau(x_3+1)\tau(x_3-1)=0 \quad (2.3)$$

does not contain free parameters and is obtained from Eq. (1.3) by the transformation

$$\tau(x_1,x_2,x_3)\to Z_1^{-x_1^2/2}Z_2^{-x_2^2/2}Z_3^{-x_3^2/2}\tau(x_1,x_2,x_3). \quad (2.4)$$

**A$''$.** *"Gauge invariant" form*:

$$Y(x_1,x_2+1,x_3)Y(x_1,x_2-1,x_3)=\frac{\big(1+Y(x_1,x_2,x_3+1)\big)\big(1+Y(x_1,x_2,x_3-1)\big)}{\big(1+Y^{-1}(x_1+1,x_2,x_3)\big)\big(1+Y^{-1}(x_1-1,x_2,x_3)\big)}, \quad (2.5)$$

where the new unknown function

$$Y(x_1,x_2,x_3)\equiv\frac{\tau(x_1,x_2,x_3+1)\tau(x_1,x_2,x_3-1)}{\tau(x_1+1,x_2,x_3)\tau(x_1-1,x_2,x_3)} \quad (2.6)$$

is "gauge invariant" w.r.t. the "gauge" transformation (2.2). This form is a discrete counterpart of nonlinear integrable equations written in terms of potentials and fields rather than $\tau$-functions. Some particular cases of this equation emerge naturally in the thermodynamic Bethe ansatz [18, 19].

**B.** *KP-like form*:

$$(z_2-z_3)\tau^{p_1+1,p_2,p_3}\tau^{p_1,p_2+1,p_3+1}+(z_3-z_1)\tau^{p_1,p_2+1,p_3}\tau^{p_1+1,p_2,p_3+1}+$$
$$+(z_1-z_2)\tau^{p_1,p_2,p_3+1}\tau^{p_1+1,p_2+1,p_3}=0. \quad (2.7)$$

Here $\tau^{p_1,p_2,p_3}$ is a function of three variables $p_i$ and $z_i$ are arbitrary constants. This equation is invariant under cyclic permutations of the variables and simultaneous permutations of $z_i$. Changing signs of all of the variables also leaves it invariant. Invariance of Eq. (2.7) under transformation (2.2) is equivalent to the fact that if $\tau^{p_1,p_2,p_3}$ is a solution of Eq. (2.7), then

$$\chi_0(2p_1+2p_2+2p_3)\chi_1(2p_1)\chi_2(2p_2)\chi_3(2p_3)\tau^{p_1,p_2,p_3} \quad (2.8)$$

is also a solution of Eq. (2.7). Moreover, the coefficients in (2.7) can be set equal to unity by means of the transformation

$$\tau^{p_1,p_2,p_3}\to\left(\frac{z_1-z_3}{z_1-z_2}\right)^{p_1p_2}\left(\frac{z_1-z_3}{z_2-z_3}\right)^{p_2p_3}\tau^{p_1,p_2,p_3}, \quad (2.9)$$

bringing Eq. (2.7) to the canonical form.

**B$'$.** *MKP-like form*:

$$(z_0-z_1)(z_2-z_3)\tau_{p_0}^{p_1+1,p_2,p_3}\tau_{p_0+1}^{p_1,p_2+1,p_3+1}+$$
$$+(z_0-z_2)(z_3-z_1)\tau_{p_0}^{p_1,p_2+1,p_3}\tau_{p_0+1}^{p_1+1,p_2,p_3+1}+$$
$$+(z_0-z_3)(z_1-z_2)\tau_{p_0}^{p_1,p_2,p_3+1}\tau_{p_0+1}^{p_1+1,p_2+1,p_3}=0. \quad (2.10)$$

Note that the combination of the arguments $p_1 + p_2 + p_3 - p_0$ is the same for all $\tau$-functions in this equation. In other words, the hyperplane $p_1 + p_2 + p_3 - p_0 = \text{const}$ is invariant. Therefore, this equation actually depends on three variables rather than four. Choose them to be $p_1$, $p_2$, and $p_3$. Since, as in Eq. (2.7), the sum of the coefficients in Eq. (2.10) is zero, these equations differ only by a reparametrization of the quantities $z_i$.

**C.** *2DTL-like form*:

$$\nu\tau_n^{l,\bar{l}+1}\tau_n^{l+1,\bar{l}} + (\mu-\nu)\tau_n^{l,\bar{l}}\tau_n^{l+1,\bar{l}+1} = \mu\tau_{n+1}^{l,\bar{l}+1}\tau_{n-1}^{l+1,\bar{l}}, \tag{2.11}$$

where $\tau_n^{l,\bar{l}}$ is a function of three variables and $\mu$, $\nu$ are arbitrary constants. The variables $l, \bar{l}$ are called light-cone coordinates. Note that, in this form, the permutation symmetry is lost. However, an analogue of Eq. (2.2) holds: if $\tau_n^{l,\bar{l}}$ solves Eq. (2.11), then $\chi_0(2n+2l)\chi_1(2l)\chi_2(2\bar{l})\chi_3(2n-2\bar{l})\tau_n^{l,\bar{l}}$ is a solution as well. The transformation

$$\tau_n^{l,\bar{l}} \to \left(\frac{\mu}{\nu}-1\right)^{-l\bar{l}}\left(-\frac{\mu}{\nu}\right)^{-n^2/2}\tau_n^{l,\bar{l}} \tag{2.12}$$

allows one to hide the coefficients in (2.11),

$$\tau_n^{l,\bar{l}+1}\tau_n^{l+1,\bar{l}} + \tau_n^{l,\bar{l}}\tau_n^{l+1,\bar{l}+1} + \tau_{n+1}^{l,\bar{l}+1}\tau_{n-1}^{l+1,\bar{l}} = 0. \tag{2.13}$$

By analogy with the previous cases, we call this form canonical.

For the reader's convenience, we present below the linear substitutions that make the canonical forms of equations **A**, **B**, and **C** equivalent.

**A** $\leftrightarrow$ **B**: $\tau(x_1, x_2, x_3) = \tau^{p_1,p_2,p_3}$,

$$p_1 = \frac{1}{2}(-x_1 + x_2 + x_3), \qquad p_2 = \frac{1}{2}(x_1 - x_2 + x_3), \qquad p_3 = \frac{1}{2}(x_1 + x_2 - x_3), \tag{2.14}$$

$$x_1 = p_2 + p_3, \qquad x_2 = p_1 + p_3, \qquad x_3 = p_1 + p_2, \tag{2.15}$$

**B** $\leftrightarrow$ **C**: $\tau^{p_1,p_2,p_3} = \tau_n^{l,\bar{l}}$,

$$n = p_2 + p_3, \qquad l = p_1, \qquad \bar{l} = p_2, \tag{2.16}$$

$$p_1 = l, \qquad p_2 = n - \bar{l}, \qquad p_3 = \bar{l}, \tag{2.17}$$

**A** $\leftrightarrow$ **C**: $\tau(x_1, x_2, x_3) = \tau_x^{l,\bar{l}}$,

$$n = x_1, \qquad l = \frac{1}{2}(-x_1 + x_2 + x_3), \qquad \bar{l} = \frac{1}{2}(x_1 - x_2 + x_3), \tag{2.18}$$

$$x_1 = n, \qquad x_2 = n + l - \bar{l}, \qquad x_3 = l + \bar{l}. \tag{2.19}$$

Clearly, these linear substitutions are not unique. All other possibilities can be obtained from the ones given by applying a transformation of the form $(x_1, x_2, x_3) \to (\pm x_{P(1)}, \pm x_{P(2)}, \pm x_{P(3)})$, where $P$ is a permutation. Using formulas (2.14)–(2.19), one can easily obtain gauge invariant forms of **B** and **C**.

## 3. Definitions. The nomenclature of flows

Here we introduce a practical set of definitions and axioms which helped us develop a systematic viewpoint of the zoo of nonlinear integrable equations and their commutation representations. This viewpoint is, in fact, more general than we need for the HBDE itself. Differential and "mixed" differential-difference nonlinear equations also fit the scheme. Our approach is motivated by the algebro-geometric solutions [20] to soliton equations expressed through Riemannian $\theta$-functions. However, since the goal is to clarify formal algebraic structures, we do not refer to the solutions explicitly.

## 3.1. Variables and kinematic constraints

The "unknown function" entering bilinear equations is always denoted by $\tau$. This function depends on an infinite set of independent variables, which are called *flows* or *times*. The last two words are used as synonyms. For each particular equation, only a finite number of the time variables take nonzero values.

The flows are labeled by points of the complex plane $\mathbf{C}$. Call the points $\lambda \in \mathbf{C}$ *labels*. We make a distinction between *discrete* and *continuous* flows.[4]

*Discrete flows*: A discrete flow $l = l_{\lambda\mu}$ is associated with each ordered pair of points $\lambda, \mu \in \mathbf{C}$, $\lambda \neq \mu$. To say it differently, the flows are attached to vectors $\overrightarrow{\lambda\mu}$, i.e., each discrete flow has *two* labels.

*Continuous flows*: An infinite sequence of times $\{t_1, t_2, t_3, \dots\}^{(\lambda)}$ is associated with each point $\lambda \in \mathbf{C}$. All of the (continuous) variables $t_j$ have the common label $\lambda$.

In each particular equation we consider, only a finite number of labels are involved. Therefore, we assume that for all but a finite number of labels $\lambda \in \mathbf{C}$ and for all but a finite number of ordered pairs of labels, the corresponding variables are zero. This condition makes the definition very close to the adelic ideology from algebraic number theory. A definition in such an abstract form may seem to be overcomplicated and too general. However, this standpoint is useful since it provides an adequate formalization of the simple fact that the number of independent variables in the equations of an integrable hierarchy can be arbitrary but also finite.

Sometimes it is convenient to say that those variables which are nonzero are *switched on* while all others are *switched off*. According to the above definition, the set of labels corresponding to the switched on variables is always finite.

Having this in mind, it is worthwhile to reformulate the definition, making it a little more concrete.[5]

Let $\{\lambda_\alpha\}$ , $\alpha \in I$, be a finite set of marked points in $\mathbf{C}$. Here $I$ is just the finite set of labels corresponding to the variables that are switched on. By $l_{\alpha\beta}$ ($\alpha \neq \beta$) denote the "discrete" variable associated with $\overrightarrow{\lambda_\alpha\lambda_\beta}$. By $t_j^{(\alpha)}$, $j = 1, 2, \dots$ denote the "continuous" times associated with $\lambda_\alpha$. The $\tau$-function is a function of these variables,

$$\tau = \tau\big(\{l_{\alpha\beta}\}; \{t_j^{(\alpha)}\}\big).$$

Let $\mathcal{G}$ be the graph whose vertices are the marked points (labels) $\lambda_\alpha$, $\alpha \in I$, and whose edges are the vectors $\overrightarrow{\lambda_\alpha\lambda_\beta}$. The edges have an orientation that is indicated by an arrow looking from $\alpha$ to $\beta$. This graph is referred to as the *graph of flows*. It encodes the kinematic structure of the equation.

We stress that the only essential elements of the graph are the vertices and their ordered pairs. All other graphic elements are introduced for convenience of the visualization. In particular, the vectors may intersect on the complex plane, but the intersection points should not be considered as belonging to the graph. It is also worth emphasizing that the vectors are just convenient names of flows. They should not be confused with the "directions" of the flows in any sense of this word.

The introduced variables are not independent. There are certain "kinematic" constraints imposed on them.

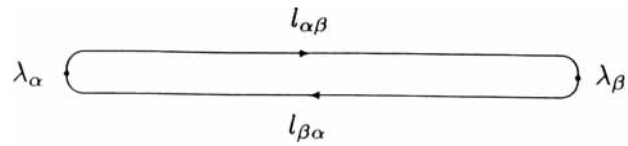
The first group of constraints involves the discrete variables only. The constraints arise when the graph of flows $\mathcal{G}$ has cycles. It is sufficient to fix the constraints for the following two cases:

(i) The 2-cycle:

$$\tau\big(l_{\alpha\beta} + 1, l_{\beta\alpha} + 1\big) = \tau\big(l_{\alpha\beta}, l_{\beta\alpha}\big), \tag{3.1}$$

[4] These are not more than conventional names. In general, both time variables may take complex values.

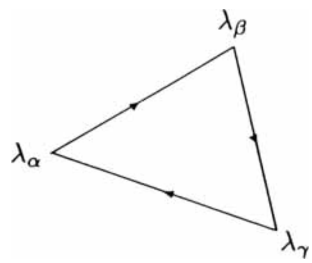
[5] For each concrete example, this unified notation is still not very convenient to work with and will be simplified. However, for the sake of clarity and definiteness, it is better to introduce general notions and definitions using the unified notation.




Informally, this means that $l_{\beta\alpha}$ is identified with $-l_{\alpha\beta}$.

(ii) The 3-cycle:

$$\tau\left(l_{\alpha\beta}+1, l_{\beta\gamma}+1, l_{\gamma\alpha}+1\right) = \tau\left(l_{\alpha\beta}, l_{\beta\gamma}, l_{\gamma\alpha}\right), \tag{3.2}$$



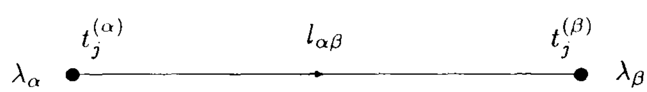

The corresponding rules for longer cycles follow from these two. According to these rules, one can subsequently remove all of the cycles and reduce the graph to a tree. The tree graphs correspond to kinematically independent flows. Formally, it is sufficient to consider graphs without cycles. However, the introduction of cycles sometimes makes the set of variables more symmetrical, though nonminimal.

The last constraint describes the inter-relation between a discrete flow $\overrightarrow{\lambda_\alpha\lambda_\beta}$ and the "adjacent" continuous flows (i.e., the flows corresponding to the endpoints $\lambda_\alpha$ and $\lambda_\beta$).

(iii) Miwa's rules [11, 12]:

$$\tau\left(l_{\alpha\beta}+1; t^{(\alpha)}, t^{(\beta)}\right) = \tau\left(l_{\alpha\beta}; t^{(\alpha)} - [\lambda_\beta - \lambda_\alpha], t^{(\beta)}\right), \tag{3.3}$$

$$\tau\left(l_{\alpha\beta}-1; t^{(\alpha)}, t^{(\beta)}\right) = \tau\left(l_{\alpha\beta}; t^{(\alpha)}, t^{(\beta)} - [\lambda_\alpha - \lambda_\beta]\right), \tag{3.4}$$




Here $\tau(t) \equiv \tau(\{t_j\})$ and the short-hand notation

$$f(t \pm [z]) \equiv f\left(t_1 \pm z, t_2 \pm \frac{1}{2}z^2, t_3 \pm \frac{1}{3}z^3, \dots\right) \tag{3.5}$$

is used for the function $f$ of the infinite sequence of variables $t = \{t_1, t_2, \dots\}$. Relation (3.4) follows from rule (3.1) and relation (3.3).

Relations (3.3), (3.4) should be understood as formal rules which allow one to translate the infinite sequence of continuous time shifts into the shift of a single discrete variable and vice versa. We are not concerned about the convergence of infinite substitutions, i.e., the $\tau$-function is considered to be a formal series in $\lambda$ (on the left-hand sides of equalities (3.3), (3.4), $\lambda_\alpha$ and $\lambda_\beta$ are implicitly present in the definition of the discrete flows). In known examples of algebro-geometric solutions, the $\tau$-function is a *true function*, not merely a formal series. In this case, there are some additional restrictions on the domains of all variables and labels. These ensure the convergence of the infinite substitutions. Meanwhile, for algebro-geometric solutions, the vertices and edges of the graph $\mathcal{G}$ can be presented as punctures and cuts on a Riemann surface. Furthermore, the discrete time variables describe discontinuities of the Baker–Akhiezer function on the cuts.

We refer to the discrete flows $\overrightarrow{\lambda_\alpha\lambda_\beta}$ as *elementary discrete flows*. One may introduce more complicated flows, which can be thought of as "superpositions" of the elementary ones. Specifically, fix several elementary flows, say, $l_1, l_2, \dots, l_M$ (here $l_i \equiv l_{\alpha_i\beta_i}$ for some $\alpha_i, \beta_i$) and consider the $\tau$-function as the function of a new variable $y$ as follows: $\tau[y] \equiv \tau(l_1 + y, l_2 + y, \dots, l_M + y)$. In the time evolution with respect to the new variable $y$, the "elementary" variables $l_i$ are simultaneously shifted by $y$ while the others are constants. Let us call flows of this type *composite discrete flows*.

To say it differently, let $\partial_i$ be the vector field corresponding to the flow $l_i$. Then the vector field corresponding to the composite flow $y$ is

$$\partial_y := \sum_{i=1}^{M} \partial_i,$$

therefore,

$$\exp(\partial_y)\tau\big(\{l_{\alpha_i\beta_i}\}\big) = \tau\big(\{l_{\alpha_i\beta_i} + 1\}\big)\exp(\partial_y).$$

However, one should be careful because this procedure does not necessarily generate a composite flow. For example, due to (3.2), the simultaneous shift of $l_{\alpha\beta}$ and $l_{\beta\gamma}$ is equivalent to an elementary flow.

The precise definition is as follows:

*Composite discrete flows* are labeled by finite sets of vectors $\{\overrightarrow{\lambda_{\alpha_i}\lambda_{\beta_i}}\}$, $i = 1, 2, \dots, M$, such that $\beta_i \neq \alpha_j$ for any $i, j$. Let $y$ be the corresponding time variable. In this case, evolution along the direction of the composite flow is defined by the formula

$$\tau[y] = \tau\big(\{l_{\alpha_i\beta_i} + y\}\big),$$

where $l_{\alpha_i\beta_i}$ and other elementary variables are assumed to be constants.

The distinction between elementary and composite flows can be extended to continuous flows as well. For reasons which are clarified later, it is natural to consider the continuous times $t_1^{(\alpha)}$ as elementary flows. At this stage, we defend this definition by the fact that, due to Miwa's rule (3.3), they can be obtained as a result of the scaling limit from discrete elementary flows. Similarly, higher continuous times $t_j^{(\alpha)}$ with $j \geq 2$ are limits of composite discrete flows. Therefore, we call them composite times.

To summarize, we have introduced several notions and definitions which are extensively used throughout the paper. First of all, a partial classification of flows and time variables in soliton equations has been suggested. We have defined discrete and continuous flows, and distinguished between elementary and composite flows. One may assign a graph to any particular equation, which explicitly shows the kinematic structure of the equation and possible constraints imposed on the flows. In order to make this clear, we give some examples below.

## 3.2. Examples

To make the graphs of flows more informative, let us add a new graphic element: fat dots mean that the corresponding continuous times are nonzero.[6]

In the KP hierarchy, the graph of flows consists of a single "fat" point with corresponding continuous "times" $\{t_j\}$ (Fig. 1). The set of discrete flows is empty and the $\tau$-function is $\tau(t) \equiv \tau(t_1, t_2, \dots)$.

In the 2DTL hierarchy, the graph of flows consists of two "fat" points with corresponding times $\{t_j\}$ and $\{\bar{t}_j\}$ (Fig. 2). The discrete flow associated with the vector connecting the two points is the discrete "time" $n$ of the 2DTL $\tau$-function $\tau_n(t; \bar{t})$.

The graphs of flows for the discrete KP and the discrete 2DTL equations are as in Fig. 3 and Fig. 4, respectively.

All continuous times are switched off. In both cases, only three independent discrete flows are switched on. This agrees with the continuous case, where the first nontrivial equations of the KP and 2DTL hierarchies

[6] It would be more precise to say that the continuous times cannot be set equal to zero by a transformation of form (3.3), (3.4).

**Fig. 1.** Graph of flows for the KP hierarchy.

**Fig. 2.** Graph of flows for the 2DTL hierarchy.

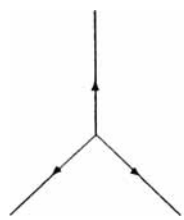


**Fig. 3.** The discrete KP equation.

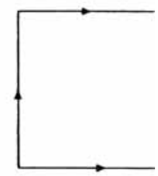


**Fig. 4.** The discrete 2DTL equation.

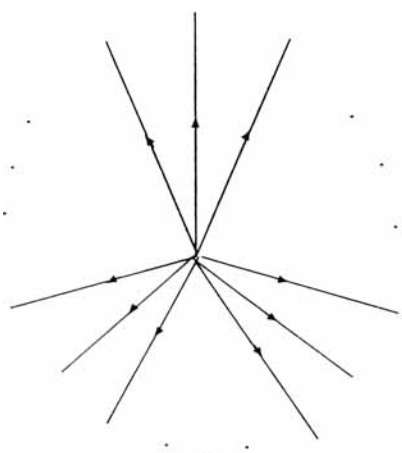


**Fig. 5.** The discrete KP hierarchy.

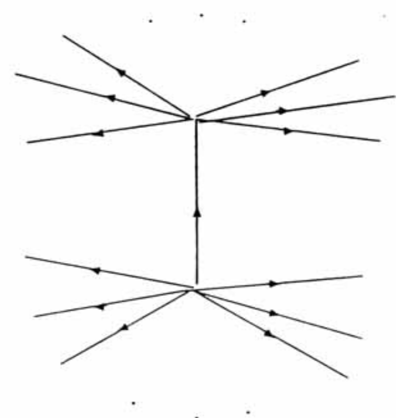


**Fig. 6.** The discrete 2DTL hierarchy.

have three independent variables, $t_1, t_2, t_3$ and $t_1$, $\bar{t}_1$, $n$, respectively. In the continuum limit, all of the lines except the vertical one in the 2DTL figures shrink to fat points.

Figures 5 and 6 represent the discretized KP and 2DTL hierarchies, respectively.
Higher equations of the hierarchies involve more than three elementary flows. The labels of these flows are analogous to the number of the higher flow in the continuous hierarchies; these labels are complex numbers. This scheme looks like a kind of Fourier duality between a parameter that numbers the equations of the hierarchy and the time variable corresponding to a particular flow: continuous flows are marked by a discrete "label," whereas discrete flows are marked by a continuous label.

## 4. Discrete Zakharov–Shabat representation of the Hirota equation

The reformulation of classical nonlinear integrable equations as flatness conditions for a two-dimensional connection is the basic constituent of the theory. Flatness means that subsequent shifts along any pair of time flows commute. These conditions are known as the *Zakharov–Shabat equations* or the *zero curvature*

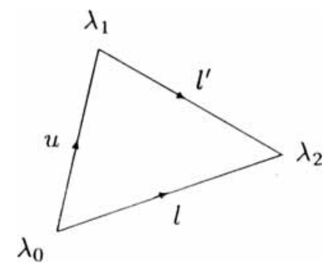



**Fig. 7**

*representation.* In [1], Hirota gave an example of the discretized zero curvature representation for Eq. (1.1). In physical language, the discrete connection is a lattice gauge field. The approach emphasizing the relationship with gauge field theories on a lattice was developed by Saito and Saitoh [21]. We present these results in a modified form, which makes the theory parallel to the 2DTL theory [22].

The discrete zero curvature condition is equivalent to the commutativity of certain multivariable difference operators. The existence of such a "commutation representation" is a hall-mark of integrability. At the same time, if a commutation representation exists, it is not unique. In particular, there are different (in fact, infinitely many) ways of representing the HBDE as a zero curvature condition.

The general scheme is as follows. Choose *any* time flow as the "reference" one, i.e., the one in which all of the $M$-operators act as differential or difference operators. Commutativity of the flows means that any pair of such $M$-operators obeys a compatibility condition, which turns out to be one of the Zakharov–Shabat equations. This fact allows one to relate the different hierarchies to each other. In general, $M$-operators are pseudo-difference operators[7] or difference operators with matrix coefficients. Here, we consider only the case of difference operators. Examples that are more general are given later in Sec. 6.

When the reference flow is taken as an elementary one, the coefficients are scalar functions. Section 4.3 contains an example of the zero curvature condition for HBDE realized by $2 \times 2$ matrix difference operators.

## 4.1. Basic $M$-operators

Thus far, all elementary discrete flows have been treated on equal footing. None was any better than another. Now, we are going to break this equal treatment and distinguish the reference flow. This may be any flow, including composite and continuous flows. For simplicity, we start with the case where the reference flow is discrete and elementary. Other cases are discussed later.

The idea is to assign difference operators to all of the flows. These operators act on functions of the reference flow variable and we call them *$M$-operators.* In this section, we consider the simplest $M$-operators, which are the basic blocks of more general operators.

Let us specify the notation and take the reference flow to be $\overrightarrow{\lambda_0\lambda_1}$. However, the double index notation is inconvenient for practical purposes. When dealing with a limited number of flows, it is worthwhile to give them simpler though less systematic names. Unless otherwise stated, the letter $u$ is reserved for the reference variable corresponding to an elementary discrete flow. Therefore, we set

$$u = l_{01}.$$

Let $\lambda_2$ be any label different from $\lambda_0, \lambda_1$. In this situation, where the three elementary flows with these labels are switched on while all of the other flows are switched off, the graph of the flows is the triangle depicted in Fig. 7. Its sides $\overrightarrow{\lambda_0\lambda_2}$ and $\overrightarrow{\lambda_1\lambda_2}$ define flows that we call adjacent to the reference flow $u$ in the obvious sense. In general, a flow $\overrightarrow{\lambda_\alpha\lambda_\gamma}$ (respectively, $\overrightarrow{\lambda_\beta\lambda_{\gamma'}}$) is said to be *left adjacent* (respectively, *right adjacent*) to the flow $\overrightarrow{\lambda_\alpha\lambda_\beta}$. Coming back to the triangle graph of the flows, we set $l_{02} = l$, $l_{12} = l'$.

[7] We use this shorter name for what is usually called a "quantum pseudo-differential operator."

The $\tau$-function is denoted by $\tau_u^{l,l'}$. There are only two independent flows. According to (ii), we have the connection

$$\tau_{u+1}^{l,l'+1} = \tau_u^{l+1,l'}. \tag{4.1}$$

The coefficients of the $M$-operators are expressed via $\tau$.

Let us take $u, l$ as independent variables. By definition, the $M$-operator assigned to the left adjacent flow $l$ is

$$M_u^l = e^{\partial_u} - \lambda_2^{01} \frac{\tau_u^l \tau_{u+1}^{l+1}}{\tau_u^{l+1} \tau_{u+1}^l}, \tag{4.2}$$

where the coefficient $\lambda_2^{01}$ is expressed through the three labels $\lambda_0$, $\lambda_1$, $\lambda_2$ as follows:

$$\lambda_2^{01} = \frac{1}{\lambda_2 - \lambda_0} - \frac{1}{\lambda_1 - \lambda_0}, \qquad \lambda_1^{02} = -\lambda_2^{01}. \tag{4.3}$$

The shift operator $e^{\partial_u}$ has standard commutation relations with functions of $u$: $e^{\pm\partial_u} f(u) = f(u \pm 1) e^{\pm\partial_u}$. Note that $M_u^u = e^{\partial_u}$.

In (4.2), it is implied that the $\tau$-function depends on all of the other variables, which are switched off in this particular case. When they are switched on, they enter Eq. (4.2) as parameters. Their values are the same for each of the four $\tau$-functions in the ratio. As a rule, we do not indicate them explicitly where there is no confusion.

Once the $M$-operator for the left adjacent flow is written, it can be translated into the $M$-operator for the right adjacent flow $l'$ by passing to the independent variables $u, l'$. On the right-hand side of Eq. (4.2), the (implicit) argument $l'$ is the same in each $\tau$-function. Using (4.1), we rewrite the right-hand side in such a way that the argument $l$ is the same and implicit. Rule (ii) tells us that the shift $l' \to l' + 1$ is equivalent to the simultaneous shifts $u \to u + 1$ and $l \to l + 1$. Considering $M$-operators as the generating shifts of discrete variables by unity, it is natural to *define* the $M$-operator for the right adjacent flow $l'$ as follows:

$$\overline{M}_u^{l'} = e^{-\partial_u} M_u^l,$$

or, more explicitly,

$$\overline{M}_u^{l'} = 1 - \lambda_2^{01} \frac{\tau_{u+1}^{l'+1} \tau_{u-1}^{l'}}{\tau_u^{l'+1} \tau_u^{l'}} e^{-\partial_u}. \tag{4.4}$$

It is also useful to introduce the operators

$$\mathcal{M}_u^l = e^{-\partial_l} M_u^l, \qquad \overline{\mathcal{M}}_u^{l'} = e^{-\partial_{l'}} \overline{M}_u^{l'}, \tag{4.5}$$

which are difference operators in two variables. It follows from the construction above that

$$[\mathcal{M}_u^l, \overline{\mathcal{M}}_u^{l'}] = 0. \tag{4.6}$$

We have defined the $M$-operators for elementary discrete flows adjacent to the reference one. In this case, these operators have a simple form: they are first-order difference operators in $u$. The $M$-operators corresponding to composite and nonadjacent flows have a more complicated structure, which is discussed in the sections that follow.

Let us comment on continuous reference flows. According to (3.3), in the continuum limit in $u$, we have $e^{\partial_u} \to 1 - \lambda^{-1}\partial_{t_1} + O(\lambda^{-2})$, where $t_1$ is the first continuous time with the label $\lambda_0$, while $\lambda = (\lambda_1 - \lambda_0)^{-1}$. The limiting form of the $M$-operator (4.2) as $\lambda \to \infty$ is

$$M^{(l)} = \partial_{t_1} - \partial_{t_1} \log \frac{\tau^{l+1}}{\tau^l} - (\lambda_2 - \lambda_0)^{-1}.$$

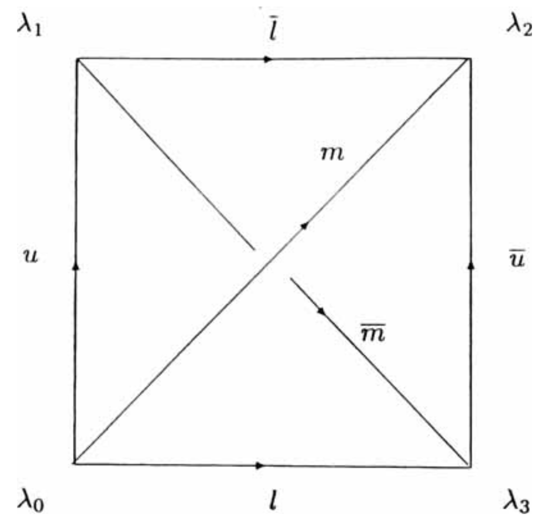



**Fig. 8**

This is a first-order differential operator in the reference continuous time variable $t_1$. It generates shifts in the discrete variable $l$. Since it is an operator of the *first* order, we call the continuous flow $t_1$ *elementary* (see the end of Sec. 3.1).

## 4.2. Discrete Zakharov–Shabat equations

Two independent flows are not enough to derive bilinear equations, as nontrivial bilinear equations for $\tau$ arise from three independent discrete flows. In this case, the graph of the flows should contain at least four vertices. Therefore, let us fix four labels $\lambda_\alpha$, $\alpha = 0,1,2,3$, and consider a general graph with four vertices (Fig. 8). The simplified notation for the flows is as in Fig. 8. As in (4.3), we set

$$\lambda_\gamma^{\alpha\beta} = \frac{1}{\lambda_\gamma - \lambda_\alpha} - \frac{1}{\lambda_\beta - \lambda_\alpha}, \qquad \lambda_\beta^{\alpha\gamma} = -\lambda_\gamma^{\alpha\beta}, \tag{4.7}$$

for all possible values of the pairwise distinct indices.

Let $\overrightarrow{\lambda_0\lambda_1}$ be the reference flow as above. The left (respectively, right) adjacent flows are $\overrightarrow{\lambda_0\lambda_2}$, $\overrightarrow{\lambda_0\lambda_3}$ (respectively, $\overrightarrow{\lambda_1\lambda_2}$, $\overrightarrow{\lambda_1\lambda_3}$). Each has its own $M$-operator of form (4.2) (respectively, (4.4)).

The key point is to extend the trivial commutation properties (4.6) to *all* of the flows in the graph adjacent to $u$,

$$[\mathcal{M}_u^l, \mathcal{M}_u^m] = [\mathcal{M}_u^l, \overline{\mathcal{M}}_u^{\bar{l}}] = [\overline{\mathcal{M}}_u^{\bar{l}}, \overline{\mathcal{M}}_u^{\overline{m}}] = 0. \tag{4.8}$$

In contrast to Eq. (4.6), these are nontrivial requirements which give rise to the bilinear equations for $\tau$. Written in terms of $M$-operators, commutation relations (4.8) are discrete Zakharov–Shabat equations.

In the following proposition, we use the notation such as $M_u^l = M_u^l(u, l, \bar{l}, \ldots)$ to indicate that the $M$-operators are variable dependent.

**Proposition 4.1.** *The discrete Zakharov–Shabat equations*

$$M_u^m(m, l+1)M_u^l(m, l) = M_u^l(m+1, l)M_u^m(m, l), \tag{4.9}$$

$$\overline{M}_u^{\overline{m}}(\overline{m}, \bar{l}+1)\overline{M}_u^{\bar{l}}(\overline{m}, \bar{l}) = \overline{M}_u^{\bar{l}}(\overline{m}+1, \bar{l})\overline{M}_u^{\overline{m}}(\overline{m}, \bar{l}), \tag{4.10}$$

$$\overline{M}_u^{\bar{l}}(l+1, \bar{l})M_u^l(l, \bar{l}) = M_u^l(l, \bar{l}+1)\overline{M}_u^{\bar{l}}(l, \bar{l}) \tag{4.11}$$

are equivalent, respectively, to the following bilinear relations for $\tau$:

$$\lambda_2^{01}\tau_u^{l+1,m}\tau_{u+1}^{l,m+1} - \lambda_3^{01}\tau_u^{l,m+1}\tau_{u+1}^{l+1,m} + H_1(l,m;u)\tau_u^{l+1,m+1}\tau_{u+1}^{l,m} = 0, \tag{4.12}$$

$$\lambda_3^{01}\tau_u^{\bar{l}+1,\overline{m}}\tau_{u+1}^{\bar{l},\overline{m}+1} - \lambda_2^{01}\tau_u^{\bar{l},\overline{m}+1}\tau_{u+1}^{\bar{l}+1,\overline{m}} + H_2(\bar{l},\overline{m};u)\tau_u^{\bar{l},\overline{m}}\tau_{u+1}^{\bar{l}+1,\overline{m}+1} = 0, \tag{4.13}$$

$$\lambda_3^{01}\tau_u^{l,l+1}\tau_u^{l+1,l} - \lambda_2^{01}\tau_{u+1}^{l,l+1}\tau_{u-1}^{l+1,l} + H_3(l,l;u)\tau_u^{l,l}\tau_u^{l+1,l+1} = 0, \tag{4.14}$$

where $H_i$ are arbitrary functions such that $H_i(l,m;u+1) = H_i(l,m;u)$.

**Proof.** The proof consists of the straightforward commutation of the $M$-operators. The $M$-operators read as

$$M_u^l = e^{\partial_u} - \lambda_3^{01}\frac{\tau_u^l\tau_{u+1}^{l+1}}{\tau_u^{l+1}\tau_{u+1}^l}, \tag{4.15}$$

$$\overline{M}_u^{\overline{m}} = 1 - \lambda_3^{01}\frac{\tau_{u-1}^{\overline{m}}\tau_{u+1}^{\overline{m}+1}}{\tau_u^{\overline{m}+1}\tau_u^{\overline{m}}}e^{-\partial_u}. \tag{4.16}$$

Operator $M_u^m$ is given by (4.15) with the changes $l \to m$ and $3 \to 2$, and $\overline{M}_u^{\bar{l}}$ is given by (4.16) with the changes $\overline{m} \to \bar{l}$ and $3 \to 2$. For the case of Eq. (4.9), the equation is

$$\left(e^{\partial_u} - \lambda_2^{01}\frac{\tau_u^{m,l+1}\tau_{u+1}^{m+1,l+1}}{\tau_u^{m+1,l+1}\tau_{u+1}^{m,l+1}}\right)\left(e^{\partial_u} - \lambda_3^{01}\frac{\tau_u^{m,l}\tau_{u+1}^{m,l+1}}{\tau_u^{m,l+1}\tau_{u+1}^{m,l}}\right) =$$

$$= \left(e^{\partial_u} - \lambda_3^{01}\frac{\tau_u^{m+1,l}\tau_{u+1}^{m+1,l+1}}{\tau_u^{m+1,l+1}\tau_{u+1}^{m+1,l}}\right)\left(e^{\partial_u} - \lambda_2^{01}\frac{\tau_u^{m,l}\tau_{u+1}^{m+1,l}}{\tau_u^{m+1,l}\tau_{u+1}^{m,l}}\right). \tag{4.17}$$

The terms $e^{2\partial_u}$ and those that do not contain the shift operator are automatically canceled. A comparison of the coefficients in front of $e^{\partial_u}$ yields

$$\frac{\tau_{u+1}^{m,l}}{\tau_{u+2}^{m,l}}\left(\lambda_3^{01}\frac{\tau_{u+2}^{m,l+1}}{\tau_{u+1}^{m,l+1}} - \lambda_2^{01}\frac{\tau_{u+2}^{m+1,l}}{\tau_{u+1}^{m+1,l}}\right) = \frac{\tau_{u+1}^{m+1,l+1}}{\tau_u^{m+1,l+1}}\left(\lambda_3^{01}\frac{\tau_u^{m+1,l}}{\tau_{u+1}^{m+1,l}} - \lambda_2^{01}\frac{\tau_u^{m,l+1}}{\tau_{u+1}^{m,l+1}}\right), \tag{4.18}$$

or

$$\frac{\lambda_3^{01}\tau_u^{m+1,l}\tau_{u+1}^{m,l+1} - \lambda_2^{01}\tau_u^{m,l+1}\tau_{u+1}^{m+1,l}}{\lambda_3^{01}\tau_{u+1}^{m+1,l}\tau_{u+2}^{m,l+1} - \lambda_2^{01}\tau_{u+1}^{m,l+1}\tau_{u+2}^{m+1,l}} = \frac{\tau_{u+1}^{m,l}\tau_u^{m+1,l+1}}{\tau_{u+2}^{m,l}\tau_{u+1}^{m+1,l+1}}. \tag{4.19}$$

The denominators on both sides differ from the numerators by the shift $u \to u+1$. Therefore, their ratio is a "quasi-constant" in $u$ with the unit period and, therefore, the equation is equivalent to Eq. (4.12). This completes the proof.

Now we restore the "equality of treatment" for the elementary flows by imposing the requirement that the Zakharov–Shabat equations hold for any choice of reference flow. For example, let $l$ be the reference flow. Construct $\mathcal{M}$-operators for the flows $u, \overline{u}$ adjacent to $l$ (see Fig. 8). Then, we require that the operators $\mathcal{M}_l^u$, $\mathcal{M}_l^m$, $\overline{\mathcal{M}}_l^{\overline{u}}$, and $\overline{\mathcal{M}}_l^{-\overline{m}}$ commute with each other (of course, some of them commute automatically due to (4.6)). Note, however, that the $\mathcal{M}$-operators constructed with respect to different reference flows are not required to be commuting, e.g., $[\overline{\mathcal{M}}_u^{\overline{m}}, \mathcal{M}_l^{\overline{m}}] \neq 0$.

**Theorem 4.1.** *Let $x$ be any of the elementary flows shown in Fig. 8 and let $v, \overline{v}$ be the corresponding left and right adjacent flows such that $x$, $v$, and $\overline{v}$ are independent. Then the commutativity conditions*

$$[\mathcal{M}_x^v, \mathcal{M}_x^{\overline{v}}] = 0$$

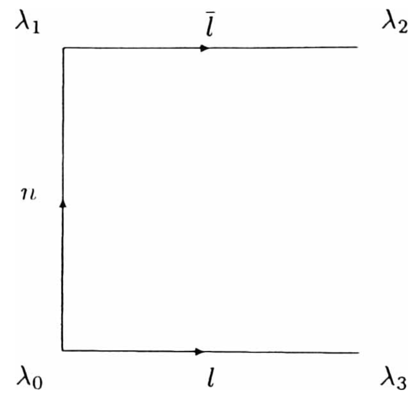



**Fig. 9**

*imposed simultaneously on three arbitrary independent reference flows $x$ are equivalent to*

$$\lambda_1^{03}\tau_u^{l,m+1}\tau_{u+1}^{l+1,m}+\lambda_2^{01}\tau_u^{l+1,m}\tau_{u+1}^{l,m+1}+\lambda_3^{02}\tau_u^{l+1,m+1}\tau_{u+1}^{l,m}=0, \tag{4.20}$$

$$\lambda_1^{02}\tau_u^{\bar{l},\overline{m}+1}\tau_{u+1}^{\bar{l}+1,\overline{m}}+\lambda_2^{03}\tau_u^{\bar{l},\overline{m}}\tau_{u+1}^{\bar{l}+1,\overline{m}+1}+\lambda_3^{01}\tau_u^{\bar{l}+1,\overline{m}}\tau_{u+1}^{\bar{l},\overline{m}+1}=0, \tag{4.21}$$

$$\lambda_3^{01}\tau_u^{l,\bar{l}+1}\tau_u^{l+1,\bar{l}}-\lambda_3^{02}\tau_u^{l,\bar{l}}\tau_u^{l+1,\bar{l}+1}=\lambda_2^{01}\tau_{u+1}^{l,\bar{l}+1}\tau_{u-1}^{l+1,\bar{l}}. \tag{4.22}$$

**Sketch of the proof.** By virtue of Proposition 4.1, it is sufficient to show that the functions $H_\iota$ are constants, $H_1=-H_2=-H_3=\lambda_3^{02}$. This can be done straightforwardly by writing the bilinear equations arising from the Zakharov–Shabat equations for the $M$-operators corresponding to each choice of the reference flow and requiring that they be consistent with each other.

We can see that Eqs. (4.20) and (4.22) coincide with the KP- and Toda-like forms of HBDE, (2.7) and (2.11), respectively. Equation (4.21) coincides with the KP-like form after the change $u\to -u$. The three equations differ by the choice of only those independent variables that agree with substitutions (2.14)–(2.19). The transition from one triad of independent variables to another should be done according to rules (i) and (ii) (see (3.1) and (3.2)). Using these rules, it is easy to see that the three equations (4.20)–(4.22) are equivalent to each other.

The four-variable MKP-like form (2.10) of the Hirota equation follows from (4.20) by applying rule (3.2). Namely, fix an extra label $\mu_0$ and consider the flows $\overrightarrow{\mu_0\lambda_\alpha}$, $\alpha=0,\dots 3$, with time variables $p_\alpha$. From (3.2) we have, for instance, $\tau_{u,p_0+1}^{m+1,p_2}=\tau_{u,p_0}^{m,p_2+1}$, etc. This change of variables converts Eq. (4.20) into Eq. (2.10).

## 4.3. Matrix realization of the zero curvature condition

We restrict ourselves to giving an example that illustrates the general scheme outlined in the introduction to this section.

Switching off the unnecessary variables, we consider the graph of flows (Fig. 9), which is a reduced version of the graph from Sec. 4.2.

The simplified ad hoc notation is clear from Fig. 9. This choice of independent variables corresponds to the discretized 2DTL equation.

Our goal here is to write the zero curvature condition with another choice of the reference flow. Specifically, let it be the composite flow labeled by the pair of vectors $\overrightarrow{\lambda_0\lambda_3}$, $\overrightarrow{\lambda_1\lambda_2}$ and let $y$ be the corresponding "composite" time variable. According to the definition given in Sec. 3.1, the $\tau$-function depends on $y$ as follows:

$$\tau_u^{l,\bar{l}}[y]:=\tau_u^{l+y,\bar{l}+y}.$$

In other words, we set, by definition, $\partial_y := \partial_l + \partial_{\bar{l}}$. Then the shift operator $e^{\partial_y}$ acts on the $\tau$-function by shifting $l, \bar{l}$ simultaneously,

$$e^{\partial_y}\tau_u^{l,\bar{l}} = \tau_u^{l+1,\bar{l}+1}e^{\partial_y}.$$

Introduce the following difference operators with $2 \times 2$ matrix coefficients:

$$\begin{aligned}
L_n(l,\bar{l}) &= \begin{pmatrix} e^{\partial_y} + \nu \dfrac{\tau_n^{l,\bar{l}}\tau_{n+1}^{l+1,\bar{l}}}{\tau_n^{l+1,\bar{l}}\tau_{n+1}^{l,\bar{l}}} + \mu \dfrac{\tau_{n+1}^{l+1,\bar{l}+1}\tau_{n-1}^{l+1,\bar{l}}}{\tau_n^{l+1,\bar{l}+1}\tau_n^{l+1,\bar{l}}} & -\mu\nu \dfrac{\tau_{n+1}^{l+1,\bar{l}+1}}{\tau_n^{l+1,\bar{l}+1}} \\ \dfrac{\tau_n^{l,\bar{l}}}{\tau_{n+1}^{l,\bar{l}}} & 0 \end{pmatrix}, \\
M_n(l,\bar{l}) &= \begin{pmatrix} \nu & -\mu\nu \dfrac{\tau_{n+1}^{l,\bar{l}+1}}{\tau_n^{l,\bar{l}+1}} \\ \dfrac{\tau_{n-1}^{l+1,\bar{l}+1}}{\tau_n^{l+1,\bar{l}+1}} & -e^{\partial_y} - \mu \dfrac{\tau_{n+1}^{l,\bar{l}+1}\tau_{n-1}^{l+1,\bar{l}+1}}{\tau_n^{l,\bar{l}+1}\tau_n^{l+1,\bar{l}+1}} \end{pmatrix},
\end{aligned} \tag{4.23}$$

where we set $\mu \equiv \lambda_2^{01}$, $\nu \equiv \lambda_3^{01}$ for the sake of brevity.

**Proposition 4.2.** *The matrix discrete Zakharov-Shabat equation*

$$L_n(l,\bar{l}+1)M_n(l,\bar{l}) = M_{n+1}(l,\bar{l})L_n(l,\bar{l}) \tag{4.24}$$

*is equivalent to the bilinear relation*

$$\tau_n^{l,\bar{l}+1}\tau_n^{l+1,\bar{l}} - H_n(l,\bar{l})\tau_n^{l,\bar{l}}\tau_n^{l+1,\bar{l}+1} = (\mu/\nu)\tau_{n+1}^{l,\bar{l}+1}\tau_{n-1}^{l+1,\bar{l}}, \tag{4.25}$$

*where $H_n(l,\bar{l})$ is periodic in $n$ with period 1: $H_{n+1}(l,\bar{l}) = H_n(l,\bar{l})$.*

This bilinear equation coincides with (4.14). We omit the proof since it is absolutely straightforward after the $(L-M)$-pair is given. A means of deriving matrix $M$-operators from the scalar ones is discussed in Sec. 5.

As in Theorem 4.1, the validity of the zero curvature condition for $M$-operators constructed with respect to all possible independent reference flows implies a bilinear equation with a fixed constant function $H$. It has form (2.11).

**Remark 4.1.** In the 2DTL interpretation, the operator $M_n$ generates the evolution in the chiral discrete "space-time" variable $\bar{l}$, whereas $L_n$ generates shifts along the $n$-lattice. In our scheme, both $M_n$ and $L_n$ are "$M$-operators" rather than "$L$-operators." We write $L_n$ according to tradition, which goes back to the case where an additional reduction of the 2DTL is implied.

It is instructive to look at the continuous version of this zero curvature condition. It provides the zero curvature representation of the 2DTL with the composite continuous reference flow defined by the vector field $\partial_y := \partial_{t_1} + \partial_{\bar{t}_1}$ in the space of the times (see Sec. 3). This representation naturally arises when one embeds the 2DTL into the 2-component KP hierarchy. The Zakharov–Shabat equation

$$\partial_{\bar{t}_1}L_n = M_{n+1}L_n - L_nM_n, \tag{4.26}$$

where

$$\begin{aligned}
L_n &= \begin{pmatrix} \partial_y - \partial_{t_1}\left(\log \frac{\tau_{n+1}}{\tau_n}\right) & -\frac{\tau_{n+1}}{\tau_n} \\ \frac{\tau_n}{\tau_{n+1}} & 0 \end{pmatrix}, \\
M_n &= \begin{pmatrix} 0 & \frac{\tau_{n+1}}{\tau_n} \\ -\frac{\tau_{n-1}}{\tau_n} & \partial_y \end{pmatrix},
\end{aligned} \tag{4.27}$$

is equivalent to the equation

$$\partial_{t_1}\partial_{\bar{t}_1}\log\frac{\tau_{n+1}}{\tau_n}=\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}-\frac{\tau_{n+2}\tau_n}{\tau_{n+1}^2},\tag{4.28}$$

which is the 2DTL equation in bilinear form.

**Remark 4.2.** The one-dimensional Toda chain (1DTC) is a reduction of the 2DTL such that $\tau_n$ does not depend on $t_1+\bar{t}_1$, i.e., $\partial_y$ commutes with $\tau_n$. Therefore, in this case, $\partial_y$ can be considered as a $c$-number. Identifying it with the spectral parameter, one recognizes Eqs. (4.27) as the standard $(L-M)$-pair for the 1DTC realized by $2\times 2$ matrices depending on the spectral parameter (see, e.g., [23]).

## 5. Linearizing the Hirota equations

The zero curvature conditions studied in the previous section are equivalent to the compatibility of an overdetermined system of linear difference equations for a "wave function" $\psi$. These linear equations are called *auxiliary linear problems* (ALP) and they play a very important role in the theory. Common solutions to the ALP contain complete information about solutions to nonlinear equations. All of the properties of the latter can be translated into the language of the ALP. This is what we mean by linearization of the HBDE.

In accordance with the diversity of zero curvature representations, there are many types of ALP. This section deals with the most important examples.

We begin with the scalar linear problems associated with the $M$-operators (4.15), (4.16) for elementary discrete flows adjacent to the reference flow. They are simple first-order linear difference equations with coefficients expressed through the $\tau$-function. The formal solution in a special form is called (the formal) Baker–Akhiezer function and it depends on the spectral parameter. Baker–Akhiezer functions are formal analogues of Bloch solutions. The formula for the Baker–Akhiezer functions in terms of the $\tau$-function was suggested for the first time in [24]. General solutions to the ALP are linear combinations of Baker–Akhiezer functions with different spectral parameters. In a similar way, one may define the dual Baker–Akhiezer functions as formal solutions to the linear problems for adjoint operators.

Given a solution to the ALP, one may consider the Bäcklund transformations. Furthermore, the "duality" between the coefficient functions and solutions of the ALP allows one to define a chain of successive Bäcklund transformations described by the Bäcklund flow. We consider two types of Bäcklund flows. It is shown that in the particular case where the solutions to the ALP are Baker–Akhiezer functions, the Bäcklund flows can be identified with elementary discrete flows adjacent to the reference flow.

There is a "gauge freedom" in the ALP which can be fixed by certain normalizations of $\psi$, but we usually use the gauge that leads to the simplest possible form of the linear equations. Another choice—the $z_0$-gauge—is briefly discussed in Sec. 5.4. This gauge makes the equations more symmetric at the price of introducing an auxiliary point $z_0\in\mathbf{C}$ and complicating the coefficient functions.

The ALP associated with the matrix $M$-operators are also discussed. In fact, matrix $M$-operators can be most conveniently derived using the scalar ALP, as the matrix linear problems can be obtained by combining the scalar problems. More precisely, in order to rearrange the scalar ALP in such a way that the reference flow is taken to be composite, one has to pass to difference operators with matrix coefficients.

### 5.1. Scalar linear problems

The commutativity of the $M$-operators (4.8) implies that they have a common set of eigenfunctions. Equivalently, the discrete Zakharov–Shabat equations (4.9)–(4.11) for $M$-operators imply the compatibility of the linear problems

$$M_u^l\psi^{l,\bar{l}}(u)=\psi^{l+1,\bar{l}}(u),\tag{5.1}$$

$$\overline{M}^{\bar{l}}\psi^{l,\bar{l}}(u)=\psi^{l,\bar{l}+1}(u),\tag{5.2}$$

for arbitrary elementary discrete flows $l, \bar{l}$ adjacent to $u$. Note that the "eigenvalues," which are set equal to unity on the r.h.s., can be made arbitrary by changing the normalization of $\psi$. Our choice in (5.1), (5.2) is most convenient in the purely discrete case, though it does not lead to a smooth continuum limit.

More explicitly, Eqs. (5.1), (5.2) read (see (4.15), (4.16))

$$\psi^{l,\bar{l}}(u+1) - \lambda_3^{01} V^{l,\bar{l}}(u)\psi^{l,\bar{l}}(u) = \psi^{l+1,\bar{l}}(u), \tag{5.3}$$

$$\psi^{l,\bar{l}}(u) - \lambda_2^{01} C^{l,\bar{l}}(u)\psi^{l,\bar{l}}(u-1) = \psi^{l,\bar{l}+1}(u), \tag{5.4}$$

where

$$V^{l,\bar{l}}(u) := \frac{\tau_u^{l,\bar{l}}\tau_{u+1}^{l+1,\bar{l}}}{\tau_u^{l+1,\bar{l}}\tau_{u+1}^{l,\bar{l}}}, \tag{5.5}$$

$$C^{l,\bar{l}}(u) := \frac{\tau_{u+1}^{l,\bar{l}+1}\tau_{u-1}^{l,\bar{l}}}{\tau_u^{l,\bar{l}+1}\tau_u^{l,\bar{l}}}. \tag{5.6}$$

These formulas become more symmetric in terms of the "unnormalized" wave function

$$\rho_u^{l,\bar{l}} = \psi^{l,\bar{l}}(u)\tau_u^{l,\bar{l}}. \tag{5.7}$$

Substituting (5.7) into (5.3), (5.4), we obtain

$$\tau_u^{l+1,\bar{l}}\rho_{u+1}^{l,\bar{l}} - \lambda_3^{01}\tau_{u+1}^{l+1,\bar{l}}\rho_u^{l,\bar{l}} = \tau_{u+1}^{l,\bar{l}}\rho_u^{l+1,\bar{l}}, \tag{5.8}$$

$$\tau_u^{l,\bar{l}+1}\rho_u^{l,\bar{l}} - \lambda_2^{01}\tau_{u+1}^{l,\bar{l}+1}\rho_{u-1}^{l,\bar{l}} = \tau_u^{l,\bar{l}}\rho_u^{l,\bar{l}+1}. \tag{5.9}$$

Let us show that a properly performed continuum limit of these equations yields the familiar linear problems for the 2DTL. Let us set

$$\lambda_3 = \lambda_0 + \epsilon, \qquad \lambda_2 = \lambda_1 + \bar{\epsilon}, \qquad \epsilon, \bar{\epsilon} \to 0,$$

such that

$$e^{\partial_l} \to 1 - \epsilon\partial_{t_1}, \qquad e^{\partial_{\bar{l}}} \to 1 - \bar{\epsilon}\partial_{\bar{t}_1},$$

by virtue of Miwa's rule (3.3) applied to the discrete flows $\overrightarrow{\lambda_0\lambda_3}$, $\overrightarrow{\lambda_1\lambda_2}$, respectively. Let us change the normalization of the $\psi$-function by introducing the $\varphi$-function as follows:

$$\varphi^{l,\bar{l}}(u) = (\lambda_0 - \lambda_3)^l \psi^{l,\bar{l}}(u). \tag{5.10}$$

Then the linear problems read

$$\begin{aligned} &(\lambda_0 - \lambda_3)\varphi^{l,\bar{l}}(u+1) - \lambda(\lambda_3 - \lambda_1)V^{l,\bar{l}}(u)\varphi^{l,\bar{l}}(u) = \varphi^{l+1,\bar{l}}(u), \\ &(\lambda_0 - \lambda_2)\varphi^{l,\bar{l}}(u) - \lambda(\lambda_2 - \lambda_1)C^{l,\bar{l}}(u)\varphi^{l,\bar{l}}(u-1) = (\lambda_0 - \lambda_2)\varphi^{l,\bar{l}+1}(u), \end{aligned} \tag{5.11}$$

where $\lambda \equiv (\lambda_1 - \lambda_0)^{-1}$. For $\epsilon, \bar{\epsilon} \to 0$, we have

$$\begin{cases} \partial_{t_1}\varphi(u) = \varphi(u+1) + \left(\lambda + \partial_{t_1}\log\dfrac{\tau_{u+1}}{\tau_u}\right)\varphi(u), \\ \partial_{\bar{t}_1}\varphi(u) = \lambda^2 \dfrac{\tau_{u+1}\tau_{u-1}}{\tau_u^2}\varphi(u-1). \end{cases} \tag{5.12}$$

The transformation

$$\tau_u \to \lambda^{-u^2} e^{-\lambda u t_1} \tau_u$$

eliminates the constant $\lambda$, reducing the linear problems to the familiar form,

$$\begin{cases} \partial_{t_1}\varphi(u) = \varphi(u+1) + v(u)\varphi(u), \\ \partial_{\bar t_1}\varphi(u) = c(u)\varphi(u-1). \end{cases} \tag{5.13}$$

Here

$$v(u) = \partial_{t_1} \log \frac{\tau_{u+1}}{\tau_u}, \qquad c(u) = \frac{\tau_{u+1}\tau_{u-1}}{\tau_u^2}$$

(for the sake of simplicity of the notation, we use the same letter for the transformed function). The Zakharov–Shabat equation

$$\left[\partial_{t_1} - e^{\partial_u} - v(u), \partial_{\bar t_1} - c(u)e^{-\partial_u}\right] = 0$$

yields the first 2DTL equation in the form

$$\partial_{t_1}\partial_{\bar t_1} \log \frac{\tau_{u+1}}{\tau_u} = \frac{\tau_{u+1}\tau_{u-1}}{\tau_u^2} - \frac{\tau_{u+2}\tau_u}{\tau_{u+1}^2}. \tag{5.14}$$

These equations are continuous analogues of Eqs. (4.11) and (4.19), respectively.

## 5.2. Bäcklund transformations

The ALP in form (5.8), (5.9) have a remarkable property (see [21]). These equations are almost symmetrical w.r.t. the interchange of $\tau$ and $\rho$. Furthermore, one may treat them as linear problems for the function $\tau$, the compatibility condition being a bilinear equation for $\rho$. This equation is, again, the HBDE of the same form. In [21], this fact was referred to as the *duality* between the "potentials" $\tau$ and the "wave functions" $\rho$. This "duality" emerges most transparently in the fully discretized case.

More precisely, rewriting Eqs. (5.3), (5.4) as linear equations for

$$\tilde\psi^{l,\bar l}(u) = \frac{\tau_{u+1}^{l+1,\bar l+1}}{\rho_{u+1}^{l+1,\bar l+1}} = \left(\psi^{l+1,\bar l+1}(u+1)\right)^{-1}$$

(see (5.7)), we obtain

$$(e^{-\partial_u} - \lambda_3^{01}\tilde V^{l,\bar l}(u))\tilde\psi^{l,\bar l}(u) = \tilde\psi^{l-1,\bar l}(u), \tag{5.15}$$

$$(1 - \lambda_2^{01}\tilde C^{l,\bar l}(u+1)e^{\partial_u})\tilde\psi^{l,\bar l}(u) = \tilde\psi^{l,\bar l-1}(u), \tag{5.16}$$

where $\tilde V$ and $\tilde C$ are given by the same formulas (5.5), (5.6) with $\rho$ instead of $\tau$. The difference operators on the l.h.s. are adjoint to the operators (4.15), (4.16) with $\tau \to \rho$. The formal adjoint operator is defined by the rule $(f(u)e^{k\partial_u})^\dagger = e^{-k\partial_u}f(u)$. It then follows that the compatibility conditions are described by Theorem 4.1 with $\rho$ substituted for $\tau$.

Therefore, passing from a given solution $\tau$ to $\rho$, we obtain a new solution to HBDE. This is a Bäcklund-type transformation that is also known as the "Darboux" or "Bäcklund–Darboux" transformation. For a comprehensive discussion of transformations of this kind, see [25]. The bilinear form of the Bäcklund transformations was suggested by Hirota [5].

One may repeat the procedure once again, starting from $\rho$ and, moreover, consider a chain of successive transformations of this type. It is natural to introduce an additional discrete variable $b$ to mark the steps of the "flow" along this chain and let $\tau_{u,b}^{l,\bar l}$, $\rho_{u,b}^{l,\bar l}$ be $\tau$ and $\rho$ at the $b$th step, respectively. The first Bäcklund flow can be defined by identifying

$$\tau_{u,b+1}^{l,\bar l} = \rho_{u,b}^{l,\bar l}. \tag{5.17}$$

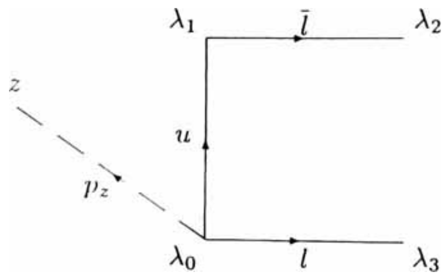



**Fig. 10**

This means that $\tau$ at the next step of the "Bäcklund time" $b$ is set equal to a solution $\rho$ of the linear equations (5.8), (5.9). In this case, these linear problems themselves become *bilinear equations* for $\tau_b$,

$$\tau^{l+1}_{u,b}\tau^{l}_{u+1,b+1} - \lambda_3^{01}\tau^{l+1}_{u+1,b}\tau^{l}_{u,b+1} - \tau^{l}_{u+1,b}\tau^{l+1}_{u,b+1} = 0, \tag{5.18}$$

$$\tau^{\bar{l}+1}_{u,b}\tau^{\bar{l}}_{u,b+1}(u) - \tau^{\bar{l}}_{u,b}(u)\tau^{\bar{l}+1}_{u,b+1} = \lambda_2^{01}\tau^{\bar{l}+1}_{u+1,b}\tau^{\bar{l}}_{u-1,b+1}. \tag{5.19}$$

Here $\bar{l}$ (respectively, $l$) in Eq. (5.18) (respectively, (5.19)) is skipped since it is the same for all of the terms.

Analogously, one may define the second Bäcklund flow (the Bäcklund "time" is now denoted by $\bar{b}$),

$$\tau^{l,\bar{l}}_{u,\bar{b}+1} = \rho^{l,\bar{l}}_{u-1,\bar{b}}. \tag{5.20}$$

From (5.8), (5.9), we have

$$\lambda_3^{01}\tau^{l+1}_{u,\bar{b}}\tau^{l}_{u,\bar{b}+1} + \tau^{l}_{u,\bar{b}}\tau^{l+1}_{u,\bar{b}+1} = \tau^{l}_{u+1,\bar{b}+1}\tau^{l+1}_{u-1,\bar{b}}, \tag{5.21}$$

$$\tau^{\bar{l}}_{u,\bar{b}}\tau^{\bar{l}+1}_{u+1,\bar{b}+1} - \tau^{\bar{l}+1}_{u,\bar{b}}\tau^{\bar{l}}_{u+1,\bar{b}+1} + \lambda_2^{01}\tau^{\bar{l}+1}_{u+1,\bar{b}}\tau^{\bar{l}}_{u,\bar{b}+1} = 0. \tag{5.22}$$

In these equations, one immediately recognizes different forms of the HBDE. A time discretization of the Toda chain by means of Darboux transformations was considered in [26].

The Bäcklund flows can also be defined by a zero curvature condition. Given any solution $\psi$ to the ALP (5.3), (5.4), introduce the operator

$$\mathcal{B}^b_u = e^{-\partial_b}\left(e^{\partial_u} - \frac{\psi(u+1)}{\psi(u)}\right). \tag{5.23}$$

Then Eq. (5.18) is represented as the commutativity condition $[\mathcal{B}^b_u, \mathcal{M}^l_u] = 0$. A similar $\mathcal{B}$-operator exists for the second Bäcklund flow.

## 5.3. Baker–Akhiezer functions

Each of the ALP (5.3), (5.4) is a first-order linear difference equation in two variables. Assuming HBDE (4.20)–(4.22) hold, we construct a parameter family of their common solutions in a special form. These solutions $\psi(u) = \psi(u; z)$ are called *Baker–Akhiezer functions* and they depend on the *spectral parameter* $z \in \mathbf{C}$.

Let us switch on the extra elementary flow shown by the dotted line in Fig. 10; the corresponding time variable is $p_z$. Let

$$\lambda_z^{\alpha\beta} = \frac{1}{z - \lambda_\alpha} - \frac{1}{\lambda_\beta - \lambda_\alpha}. \tag{5.24}$$

Note the useful identity satisfied by $\lambda_z^{\alpha\beta}$,

$$\lambda_z^{\alpha\beta}\lambda_z^{\beta\gamma} = \lambda_\alpha^{\beta\gamma}\lambda_z^{\alpha\gamma}. \tag{5.25}$$

Then, assuming the 3-term Hirota equations hold for the triads $(u, l, p_z)$ and $(u, \bar{l}, p_z)$ of independent variables, we find that the function

$$\psi^{l,\bar{l}}(u;z) = (\lambda_z^{01})^u (\lambda_z^{03})^l \left(\frac{\lambda_z^{02}}{\lambda_z^{01}}\right)^{\bar{l}} \left.\frac{\tau_{u,p_z+1}^{l,\bar{l}}}{\tau_{u,p_z}^{l,\bar{l}}}\right|_{p_z=0} \tag{5.26}$$

is a formal common solution to Eqs. (5.3), (5.4) for any $z$. Indeed, substituting (5.26) into Eq. (5.3), the latter becomes Eq. (4.20) for the triad $(u, l, p_z)$, while Eq. (5.4) becomes Eq. (4.22) for the triad $(u, \bar{l}, p_z)$. Therefore, the new label $z$ is identified with the spectral parameter. Formula (5.26) for the $\psi$-function coincides with the formula of the Kyoto school [15, 24] because, due to (3.3), we have

$$\left.\frac{\tau_{u,p_z+1}^{l,\bar{l}}}{\tau_{u,p_z}^{l,\bar{l}}}\right|_{p_z=0} = \frac{\tau_u^{l,\bar{l}}(-[z-\lambda_0])}{\tau_u^{l,\bar{l}}(0)}.$$

The general solution to the ALP can be represented in the form

$$\psi(u) = \int d^2z\, \mu(z)\psi(u;z) \tag{5.27}$$

with an arbitrary measure $\mu(z)$ on the complex plane. In other words, this is a linear combination of Baker–Akhiezer functions with different spectral parameters.

Note that the $\mathcal{B}$-operator (5.23), in which the function $\psi$ is taken as the Baker–Akhiezer function, coincides with an $\mathcal{M}$-operator. Indeed, we have the following formula for the $M$-operator (4.15) in terms of the Baker–Akhiezer function:

$$M_u^l = \lim_{z\to\lambda_3}\left(e^{\partial_u} - \frac{\psi(u+1;z)}{\psi(u;z)}\right). \tag{5.28}$$

The *dual Baker–Akhiezer function* $\psi^*$ is defined by the formula

$$\psi^{*l,\bar{l}}(u;z) = (\lambda_z^{01})^{-u}(\lambda_z^{03})^{-l}\left(\frac{\lambda_z^{02}}{\lambda_z^{01}}\right)^{-\bar{l}} \left.\frac{\tau_{u,p_z-1}^{l,\bar{l}}}{\tau_{u,p_z}^{l,\bar{l}}}\right|_{p_z=0}. \tag{5.29}$$

This function satisfies the equations

$$\begin{aligned}
\big(M_u^l(u-1,l-1)\big)^\dagger \psi^{*l}(u;z) &= \psi^{*l-1}(u;z),\\
\big(\overline{M}_u^{\bar{l}}(u-1,\bar{l}-1)\big)^\dagger \psi^{*\bar{l}}(u;z) &= \psi^{*\bar{l}-1}(u;z),
\end{aligned} \tag{5.30}$$

where the difference operators on the l.h.s. are formally adjoint to the operators (4.15), (4.16).

## 5.4. The $z_0$-gauge

Equations (5.3), (5.4) imply a specific choice for the normalization of the $\psi$-function. Indeed, by multiplying $\psi$ by any function, one can change the form of the equations. This is a kind of "gauge freedom" as there is no canonical way to fix the gauge. The gauge that we systematically use throughout this paper

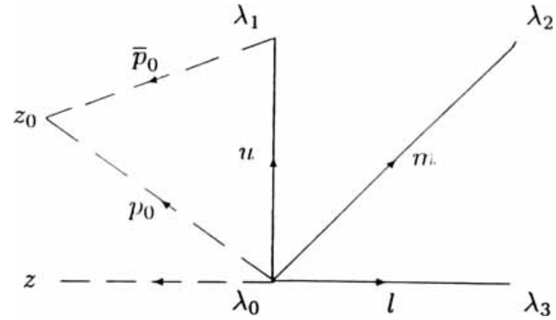



**Fig. 11**

is the most economical one in the sense that the ALP for discrete flows have the simplest possible form. Below, however, we are going to discuss another choice, which has its own advantages.

This more general gauge requires the fixing of an additional point $z_0 \in \mathbf{C}$, which is different from the vertices of the graph of flows. The gauge is defined by the following normalization condition for the Baker–Akhiezer function $\Psi(u; z)$:

$$\Psi(u; z_0) = 1 \tag{5.31}$$

and we call it the $z_0$-*gauge*. Given this condition, it is natural to represent $\Psi$ in the form

$$\Psi(u; z) = \frac{\psi(u; z)}{\psi(u; z_0)}$$

and rewrite the ALP (5.3), (5.4) for $\psi$ in terms of $\Psi$. We obtain

$$(\lambda_{z_0}^{01} U(u,l) e^{\partial_u} - \lambda_{z_0}^{03} W(u,l) e^{\partial_l}) \Psi(u; z) = \lambda_3^{01} \Psi(u; z), \tag{5.32}$$

where

$$U(u,l) = \frac{\tau^{l+1}_{u,p_0} \tau^{l}_{u+1,p_0+1}}{\tau^{l}_{u,p_0+1} \tau^{l+1}_{u+1,p_0}}, \qquad W(u,l) = \frac{\tau^{l+1}_{u,p_0+1} \tau^{l}_{u+1,p_0}}{\tau^{l}_{u,p_0+1} \tau^{l+1}_{u+1,p_0}}. \tag{5.33}$$

A general prescription for writing the equations should be clear from a comparison with Fig. 11. With this method in hand, a similar equation can be written for any pair of flows such that one of them is the reference flow and the other is left adjacent.

An attractive feature of the $z_0$-gauge is that there is no need be concerned about the equations for right adjacent flows. They are automatically produced by the same prescription if one changes the "orientation" of the reference flow (i.e., consider $\overrightarrow{\lambda_1 \lambda_0}$ as the reference flow). In order to express everything in the same variables, one should apply the rules (3.1) ($u \to -u$) and (3.2) ($\overline{p}_0 \to p_0$). We stress, however, that Eq. (5.4) cannot be obtained from Eq. (5.3) in this way. In this gauge, we need the two types of equations for left and right adjacent flows to be treated separately.

The Baker–Akhiezer function that solves all of these equations in the $z_0$-gauge has the form

$$\Psi(u; z) = \prod_{\alpha\beta} \left( \frac{\lambda_z^{\alpha\beta}}{\lambda_{z_0}^{\alpha\beta}} \right)^{l_{\alpha\beta}} \left. \frac{\tau_{p_0, p_z+1}}{\tau_{p_0+1, p_z}} \right|_{p_0 = p_z = 0}. \tag{5.34}$$

As usual, the $\tau$-functions depend on all of the skipped variables as parameters. Due to (5.25), the form of the prefactor is consistent with properties (3.1) and (3.2).

Our previous gauge is a limiting case of the $z_0$-gauge as $z_0 \to \lambda_0$. However, the limit is singular and, therefore, it needs a regularization. As a result, the symmetry of the $z_0$-gauge is broken.

## 5.5. Two-component formalism

Linear equations (5.3), (5.4) can be brought to another form in which they become first-order partial difference equations for a two-component vector function. Their compatibility yields the matrix Zakharov Shabat equations presented in Sec. 4.3.

Let us use the notation of Sec. 4.3 and, accordingly, denote $\psi_n \equiv \psi(n)$, $V_n^{l,\bar{l}} \equiv V^{l,\bar{l}}(n)$, and $C_n^{l,\bar{l}} = C^{l,\bar{l}}(n)$. Then Eqs. (5.3), (5.4) read

$$\psi_n^{l+1,\bar{l}} = \psi_{n+1}^{l,\bar{l}} - \nu V_n^{l,\bar{l}} \psi_n^{l,\bar{l}}, \tag{5.35}$$

$$\psi_n^{l,\bar{l}+1} = \psi_n^{l,\bar{l}} - \mu C_n^{l,\bar{l}} \psi_{n-1}^{l,\bar{l}}. \tag{5.36}$$

These equations allow us to find out how the vector

$$\begin{pmatrix} \psi_n^{l,\bar{l}} \\ \psi_{n-1}^{l,\bar{l}} \end{pmatrix} \tag{5.37}$$

transforms under shifts of $n$ and $\bar{l}$.

Combining Eqs. (5.35), (5.36) we have, for instance (here $\partial_y \equiv \partial_l + \partial_{\bar{l}}$),

$$\begin{aligned} \psi_{n+1}^{l,\bar{l}} &= \psi_n^{l+1,\bar{l}} + \nu V_n^{l,\bar{l}} \psi_n^{l,\bar{l}} = \psi_n^{l+1,\bar{l}+1} + \mu C_n^{l+1,\bar{l}} \psi_{n-1}^{l+1,\bar{l}} + \nu V_n^{l,\bar{l}} \psi_n^{l,\bar{l}} = \\ &= (e^{\partial_y} + \nu V_n^{l,\bar{l}}) \psi_n^{l,\bar{l}} + \mu C_n^{l+1,\bar{l}} (\psi_n^{l,\bar{l}} - \nu V_{n-1}^{l,\bar{l}} \psi_{n-1}^{l,\bar{l}}) = \\ &= (e^{\partial_y} + \nu V_n^{l,\bar{l}} + \mu C_n^{l+1,\bar{l}}) \psi_n^{l,\bar{l}} - \mu\nu C_n^{l+1,\bar{l}} V_{n-1}^{l,\bar{l}} \psi_{n-1}^{l,\bar{l}}. \end{aligned} \tag{5.38}$$

Proceeding in the same way, we obtain

$$\begin{aligned} \begin{pmatrix} \psi_{n+1}^{l,\bar{l}} \\ \psi_n^{l,\bar{l}} \end{pmatrix} &= \begin{pmatrix} e^{\partial_y} + \nu V_n^{l,\bar{l}} + \mu C_n^{l+1,\bar{l}} & -\mu\nu C_n^{l+1,\bar{l}} V_{n-1}^{l,\bar{l}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n^{l,\bar{l}} \\ \psi_{n-1}^{l,\bar{l}} \end{pmatrix}, \\ \begin{pmatrix} \psi_n^{l,\bar{l}+1} \\ \psi_{n-1}^{l,\bar{l}+1} \end{pmatrix} &= \begin{pmatrix} 1 & -\mu C_n^{l,\bar{l}} \\ (\nu V_{n-1}^{l,\bar{l}+1})^{-1} & -(\nu V_{n-1}^{l,\bar{l}+1})^{-1} (e^{\partial_y} + \mu C_n^{l,\bar{l}}) \end{pmatrix} \begin{pmatrix} \psi_n^{l,\bar{l}} \\ \psi_{n-1}^{l,\bar{l}} \end{pmatrix}. \end{aligned} \tag{5.39}$$

The operators on the right-hand sides of (5.39) provide a matrix $(L-M)$-pair for HBDE, which differs from (4.23) by a diagonal "gauge" transformation. Recall that the Baker–Akhiezer function (5.26) has $\tau_n$ in the denominator; thus, the two components of the vector (5.37) have different denominators. In the two-component formalism, it is natural to require the denominators be the same for both components. This condition partially fixes the gauge.

Therefore, introducing the vector $(\psi_n, \chi_n)$ with the second component

$$\chi_n^{l,\bar{l}} = \frac{\tau_{n-1}^{l,\bar{l}}}{\tau_n^{l,\bar{l}}} \psi_{n-1}^{l,\bar{l}}, \tag{5.40}$$

we can rewrite Eqs. (5.39) in the form

$$L_n(l,\bar{l}) \begin{pmatrix} \psi_n^{l,\bar{l}} \\ \chi_n^{l,\bar{l}} \end{pmatrix} = \begin{pmatrix} \psi_{n+1}^{l,\bar{l}} \\ \chi_{n+1}^{l,\bar{l}} \end{pmatrix}, \tag{5.41}$$

$$M_n(l,\bar{l}) \begin{pmatrix} \psi_n^{l,\bar{l}} \\ \chi_n^{l,\bar{l}} \end{pmatrix} = \nu \begin{pmatrix} \psi_n^{l,\bar{l}+1} \\ \chi_n^{l,\bar{l}+1} \end{pmatrix}, \tag{5.42}$$

where the $L$- and $M$-operators are given by Eqs. (4.23).

Equations (5.41), (5.42) imply some useful difference equations for $\psi_n^{l,\bar{l}}$. Excluding $\chi_n$ from (5.41), we obtain

$$\psi_n^{l+1,\bar{l}+1} = \psi_{n+1}^{l,\bar{l}} - \left( \nu \frac{\tau_n^{l,\bar{l}} \tau_{n+1}^{l+1,\bar{l}}}{\tau_n^{l+1,\bar{l}} \tau_{n+1}^{l,\bar{l}}} + \mu \frac{\tau_{n-1}^{l+1,\bar{l}} \tau_{n+1}^{l+1,\bar{l}+1}}{\tau_n^{l+1,\bar{l}+1} \tau_n^{l+1,\bar{l}}} \right) \psi_n^{l,\bar{l}} + \mu\nu \frac{\tau_{n-1}^{l,\bar{l}} \tau_{n+1}^{l+1,\bar{l}+1}}{\tau_n^{l,\bar{l}} \tau_n^{l+1,\bar{l}+1}} \psi_{n-1}^{l,\bar{l}}. \tag{5.43}$$

Similarly, excluding $\chi_n$ from Eq. (5.42) and using HBDE (2.11), we obtain a 4-term equation for $\psi_n$,

$$\psi_n^{l+1,\bar{l}+1} - \psi_n^{l+1,\bar{l}} = -\nu \frac{\tau_n^{l,\bar{l}+1} \tau_{n+1}^{l+1,\bar{l}+1}}{\tau_n^{l+1,\bar{l}+1} \tau_{n+1}^{l,\bar{l}+1}} \psi_n^{l,\bar{l}+1} + (\nu - \mu) \frac{\tau_n^{l,\bar{l}} \tau_{n+1}^{l+1,\bar{l}+1}}{\tau_n^{l+1,\bar{l}} \tau_{n+1}^{l,\bar{l}+1}} \psi_n^{l,\bar{l}}. \tag{5.44}$$

Here it is implied that $\tau_n^{l,\bar{l}}$ satisfies HBDE (4.22) in the corresponding notation. For more information on Eq. (5.44), see the next section.

The continuous analogue of Eq. (5.43) is

$$(\partial_{t_1} + \partial_{\bar{t}_1})\varphi_n = \varphi_{n+1} + \left(\partial_{t_1} \log \frac{\tau_{n+1}}{\tau_n}\right) \varphi_n + \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \varphi_{n-1}, \tag{5.45}$$

which is obvious from (5.12) (we write $\varphi_n$ instead of $\varphi(u)$ according to the present notation). It is a discrete nonstationary Schrödinger equation. Equation (5.44) in the continuum limit becomes

$$\partial_{t_1}\partial_{\bar{t}_1}\varphi_n - \left(\partial_{t_1} \log \frac{\tau_{n+1}}{\tau_n}\right) \partial_{\bar{t}_1}\varphi_n - \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \varphi_n = 0, \tag{5.46}$$

which is the continuous two-dimensional Schrödinger equation in a magnetic field. Its quasi-periodic solutions were studied in [27] by means of the algebro-geometric approach. The corresponding theory for discrete two-dimensional equations similar to (5.44) was proposed in [28].

## 6. Pseudo-difference $M$-operators

In this section, we study the general form of $M$-operators that satisfy the conditions of the zero curvature form, with $M$-operators for the elementary discrete flows adjacent to the reference flow.

Starting from the scalar ALP for a pair of left and right adjacent flows, it is not difficult to find the $M$-operators for *nonadjacent flows*. Indeed, it is possible to exclude the reference flow from the pair of linear equations. Then the right adjacent flow can be written in terms of the left adjacent flow. Considering the latter as a new reference flow, one can obtain a general $M$-operator for *any* elementary discrete flow in terms of *any* (elementary discrete) reference flow. In general, these are the pseudo-difference operators, i.e., they contain negative powers of the first-order difference operators.

This construction can be extended to more general operators that generate new flows commuting with the elementary discrete flows. We call these adjoint flows. The corresponding pseudo-difference operators are constructed in Sec. 6.2 with the help of two arbitrary independent solutions $\psi$, $\psi^*$ to the ALP and the adjoint ALP.

### 6.1. $M$-operators for arbitrary elementary discrete flows and corresponding linear problems

Transforming Eqs. (5.3), (5.4), it is possible to find the $M$-operators for arbitrary elementary discrete flows, in addition to those adjacent to the reference flow. The idea is to exclude shifts in $u$ and then consider $l$ as a new reference flow.

From (5.3), we have

$$\psi^{l,l}(u+1) = \psi^{l+1,\bar{l}}(u) + \lambda_3^{01} V^{l,\bar{l}}(u)\psi^{l,l}(u).$$

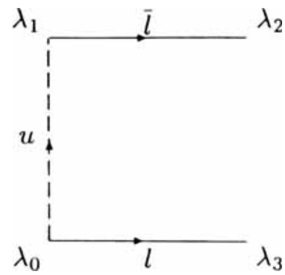


**Fig. 12**

Substituting this expression instead of $\psi^{l,\bar{l}}(u+1)$ and $\psi^{l,\bar{l}+1}(u+1)$ into Eq. (5.4), written in the form

$$\psi^{l,\bar{l}}(u+1) - \psi^{l,\bar{l}+1}(u+1) = \lambda_2^{01} C^{l,\bar{l}}(u+1)\psi^{l,\bar{l}}(u),$$

we obtain

$$\psi^{l+1,\bar{l}+1}(u) + \lambda_3^{01} V^{l,\bar{l}+1}(u)\psi^{l,\bar{l}+1}(u) = \psi^{l+1,\bar{l}}(u) + \big(\lambda_3^{01} V^{l,\bar{l}}(u) - \lambda_2^{01} C^{l,\bar{l}}(u+1)\big)\psi^{l,\bar{l}}(u).$$

Using (4.22), we find

$$\lambda_3^{01} V^{l,\bar{l}}(u) - \lambda_2^{01} C^{l,\bar{l}}(u+1) = \lambda_3^{01} V^{l,\bar{l}+1}(u) - \lambda_2^{01} C^{l+1,\bar{l}}(u) = \lambda_3^{01} \frac{\tau_u^{l,\bar{l}}\tau_{u+1}^{l+1,\bar{l}+1}}{\tau_u^{l+1,\bar{l}}\tau_{u+1}^{l,\bar{l}+1}}. \tag{6.1}$$

Note that the first equality follows from Eq. (4.11), which is a weaker condition than (4.22).

Therefore, $\psi$ obeys the following 4-term linear equation:

$$\psi^{l+1,\bar{l}+1}(u) - \psi^{l+1,\bar{l}}(u) = \lambda_1^{03} \frac{\tau_u^{l,\bar{l}+1}\tau_{u+1}^{l+1,\bar{l}+1}}{\tau_u^{l+1,\bar{l}+1}\tau_{u+1}^{l,\bar{l}+1}} \psi^{l,\bar{l}+1}(u) + \lambda_3^{02} \frac{\tau_u^{l,\bar{l}}\tau_{u+1}^{l+1,\bar{l}+1}}{\tau_u^{l+1,\bar{l}}\tau_{u+1}^{l,\bar{l}+1}} \psi^{l,\bar{l}}(u), \tag{6.2}$$

in which we recognize Eq. (5.43) from Sec. 5.5.

Relation (6.1) allows us to rewrite (6.2) in the form

$$\frac{\tau_u^{l+1,\bar{l}}}{\tau_{u-1}^{l+1,\bar{l}}} \tilde{\Delta}_l \frac{\tau_u^{l,\bar{l}+1}}{\tau_{u+1}^{l,\bar{l}+1}} \overline{\Delta}_{\bar{l}} \psi^{l,\bar{l}}(u) + \lambda_2^{01}\psi^{l,\bar{l}}(u) = 0, \tag{6.3}$$

where

$$\tilde{\Delta}_l \equiv e^{\partial_l} + \lambda_3^{01}, \qquad \overline{\Delta}_{\bar{l}} \equiv e^{\partial_{\bar{l}}} - 1. \tag{6.4}$$

This equation looks like a discrete two-dimensional Laplace equation in curved space. It can be formally rewritten as

$$\left( \overline{\Delta}_{\bar{l}} + \lambda_2^{01} \frac{\tau_{u+1}^{l,\bar{l}+1}}{\tau_u^{l,\bar{l}+1}} \tilde{\Delta}_l^{-1} \frac{\tau_{u-1}^{l+1,\bar{l}}}{\tau_u^{l+1,\bar{l}}} \right) \psi^{l,\bar{l}}(u) = 0, \tag{6.5}$$

or, finally,

$$\psi^{l,\bar{l}+1}(u) = \left( 1 - \frac{\tau_{u+1}^{l,\bar{l}+1}}{\tau_u^{l,\bar{l}+1}} \frac{\lambda_2^{01}}{e^{\partial_l} + \lambda_3^{01}} \frac{\tau_{u-1}^{l+1,\bar{l}}}{\tau_u^{l+1,\bar{l}}} \right) \psi^{l,\bar{l}}(u). \tag{6.6}$$

To avoid a misunderstanding, we stress that the pseudo-difference operator inside the parentheses acts on the variable $l$, whereas $u$ enters as a parameter. This operator, being a *pseudo-difference operator in* $l$, should be identified with the $M$-operator generating the flow $\bar{l}$ (Fig. 12). In other words, letting $l$ be the reference flow, we obtain an $M$-operator for the flow $\bar{l}$, which is not adjacent to $l$.

In the limit where the points $\lambda_0$, $\lambda_1$ merge, the flow $\bar{l}$ becomes left adjacent to $l$. Let us demonstrate how the corresponding difference $M$-operator is reproduced from the r.h.s. of Eq. (6.6). Let $\lambda_1 - \lambda_0 = \epsilon$, $\epsilon \to 0$. For $\epsilon \to 0$, we have

$$\frac{\lambda_2^{01}}{e^{\partial_l} + \lambda_3^{01}} \to 1 + \epsilon(e^{\partial_l} + \lambda_3^{02}) + O(\epsilon^2),$$

$$\frac{\tau_{u\pm 1}^{l,\bar{l}}}{\tau_u^{l,\bar{l}}} \to 1 + O(\epsilon),$$

$$\frac{\tau_{u+1}^{l,\bar{l}+1}\tau_{u-1}^{l+1,\bar{l}}}{\tau_u^{l+1,\bar{l}}\tau_u^{l,\bar{l}+1}} \to 1 - \epsilon\lambda_3^{02}\left(1 - \frac{\tau^{l,\bar{l}}\tau^{l+1,\bar{l}+1}}{\tau^{l+1,\bar{l}}\tau^{l,\bar{l}+1}}\right) + O(\epsilon^2)$$

(in the last line, Eq. (4.22) was used). Therefore, the naive limit of the r.h.s. of (6.6) is zero. However, we should take into account the change in the normalization of the $\psi$-function, which is implied when the former flow $\bar{l}$ becomes a left adjacent flow to $l$. This is achieved by replacing $\psi^{\bar{l}} \to (-\epsilon)^{\bar{l}}\psi^{\bar{l}}$. Thus, in the limiting case, the correct $M$-operator,

$$M_l^{\bar{l}} = e^{\partial_l} - \lambda_2^{03}\frac{\tau^{l,\bar{l}}\tau^{l+1,\bar{l}+1}}{\tau^{l+1,\bar{l}}\tau^{l,\bar{l}+1}},$$

is reproduced.

For purposes of illustration, let us give continuous analogues of the above formulas. Rather than perform the limit directly, it is much easier to use the continuous version (5.13) of the linear problems from the very beginning. Making use of Eq. (5.14), we find the analogue of Eq. (6.2),

$$\partial_{t_1}\partial_{\bar{t}_1}\varphi(u) - v(u)\partial_{\bar{t}_1}\varphi(u) - c(u)\varphi(u) = 0, \tag{6.7}$$

and, respectively, the analogues of Eqs. (6.3) and (6.6),

$$\frac{\tau_u}{\tau_{u-1}}\partial_{t_1}\frac{\tau_u}{\tau_{u+1}}\partial_{\bar{t}_1}\varphi(u) = \varphi(u), \tag{6.8}$$

$$\left(\partial_{\bar{t}_1} - \frac{\tau_{u+1}}{\tau_u}\partial_{t_1}^{-1}\frac{\tau_{u-1}}{\tau_u}\right)\varphi(u) = 0. \tag{6.9}$$

## 6.2. Adjoint flows

Finally, we extend the above scheme to incorporate the more general flows that we call *adjoint*. Let

$$A_l^a = 1 + w\Delta_l^{-1}w^*, \qquad \Delta_l = e^{\partial_l} - 1, \tag{6.10}$$

be a pseudo-difference operator, where $w$ and $w^*$ are still undefined functions of all of the time variables. We denote the time variable corresponding to the adjoint flow we are going to define by $a$. In this section, the reference flow is $l$. The $M$-operator for an elementary discrete flow $p$ (see Fig. 13) has the standard form

$$M_l^p = e^{\partial_l} - \lambda_p\frac{\tau_l^p\tau_{l+1}^{p+1}}{\tau_{l+1}^p\tau_l^{p+1}}, \qquad \lambda_p \equiv \lambda_4^{03}. \tag{6.11}$$

**Proposition 6.1.** *The commutativity condition*

$$[e^{-\partial_a}A_l^a, e^{-\partial_p}M_l^p] = 0 \tag{6.12}$$

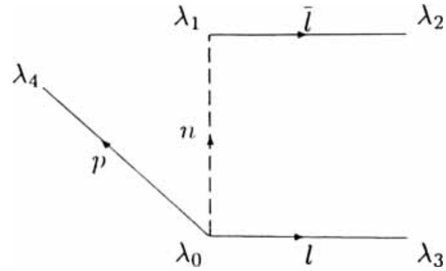



**Fig. 13**

*holds only if $w$ and $w^*$ satisfy the linear equations*

$$\begin{cases} (e^{\partial_l} + \lambda_p V_l^{p,a+1}) w_l^p = \omega w_l^{p+1}, \\ (e^{-\partial_l} + \lambda_p V_l^{p,a}) w_l^{*p+1} = \omega w_l^{*p}, \end{cases} \tag{6.13}$$

*where*

$$V_l^{p,a} \equiv \frac{\tau_l^{p,a} \tau_{l+1}^{p+1,a}}{\tau_{l+1}^{p,a} \tau_l^{p+1,a}}$$

*and $\omega$ is an arbitrary constant.*

**Proof.** The proof is by straightforward computation. Equations (6.13) are necessary conditions for the vanishing of the pseudo-difference part of the commutator. Here are the main steps of the proof. Equation (6.12), i.e.,

$$(e^{\partial_l} - \lambda_p V_l^{p,a+1})(1 + w_l^p \Delta_l^{-1} w_l^{*p}) = (1 + w_l^{p+1} \Delta_l^{-1} w_l^{*p+1})(e^{\partial_l} - \lambda_p V_l^{p,a}),$$

after opening the brackets, can be rewritten as

$$\begin{aligned} \lambda_p \left( V_l^{p,a+1} - V_l^{p,a} \right) + w_l^{p+1} w_{l-1}^{*p+1} - w_{l+1}^p w_l^{*p} = \\ = (w_{l+1}^p - \lambda_p V_l^{p,a+1} w_l^p - \omega w_l^{p+1}) \Delta_l^{-1} w_l^{*p} - \\ - w_l^{p+1} \Delta_l^{-1} (w_{l-1}^{*p+1} - \lambda_p V_l^{p,a} w_l^{*p+1} - \omega w_l^{*p}). \end{aligned} \tag{6.14}$$

Since the l.h.s. does not contain negative powers of $\Delta_l$, the r.h.s. should be zero. This condition implies Eqs. (6.13).

The ALP for the $p$ left adjacent to the reference flow $l$ and its adjoint reads as follows (cf. (5.3)):

$$\begin{cases} \psi^p(l+1) - \lambda_p V_l^p \psi^p(l) = \psi^{p+1}(l), \\ \psi^{*p}(l-1) - \lambda_p V_{l-1}^{p-1} \psi^{*p}(l) = \psi^{*p-1}(l). \end{cases} \tag{6.15}$$

Comparing (6.15) with (6.13), we identify

$$w_l^{p,a} = \psi^{p,a+1}(l), \qquad w_l^{*p,a} = \psi^{*p,a}(l+1),$$

where $\psi$ and $\psi^*$ are arbitrary solutions to the linear problems (6.15). Then, operator (6.10) acquires the form

$$A_l^a = 1 + \psi^{a+1}(l) \Delta_l^{-1} \psi^{*a}(l+1). \tag{6.16}$$

Commutativity condition (6.12) is equivalent to a nonlinear equation for $\tau_l^{p,a}$. The adjoint flow $a$ is defined by two *arbitrary* solutions $\psi$, $\psi^*$ to the linear problems (6.15). For continuous hierarchies, pseudo-differential analogues of operators (6.16) and corresponding adjoint flows were studied in [29].

As an example, let us show that by taking $\psi$, $\psi^*$ as the Baker–Akhiezer function $\psi(l;z)$ and its dual, one reproduces Eq. (6.6). According to the method of Sec. 5.3, the Baker–Akhiezer function and its dual read

$$\psi^a(l;z) = \zeta^a(\lambda_z^{03})^l \left.\frac{\tau_{p_z+1}^{l,a}}{\tau_{p_z}^{l,a}}\right|_{p_z=0}, \qquad \psi^{*a}(l;z) = \zeta^{-a}(\lambda_z^{03})^{-l} \left.\frac{\tau_{p_z-1}^{l,a}}{\tau_{p_z}^{l,a}}\right|_{p_z=0}. \tag{6.17}$$

Here $\zeta^a$ is a normalization factor to be specified below and $p_z$ is the time variable corresponding to the flow $\overrightarrow{\lambda_0 z}$ left adjacent to $l$. In the limit $z \to \lambda_1$, $p_z$ coincides with $u$. Substituting (6.17) with $\zeta = \lambda_z^{02}$ into (6.16), we reproduce the operator on the r.h.s. of Eq. (6.6).

## 7. On hierarchies of bilinear difference equations

Integrable partial differential equations can always be included in an *infinite hierarchy*. Infinite families of commuting flows generate infinite families of evolution equations. The hierarchies of discrete integrable equations have been less studied. First of all, it is not quite clear what "higher discrete flows" are on the space of pseudo-difference operators. An understanding of this matter is necessary if one is going to extend the Zakharov–Shabat formalism to the higher Hirota equations known in the literature [12, 30].

There are two "complimentary" points of view on this matter. First, one might consider the 3-term HBDE (4.20) as a counterpart of the entire infinite hierarchy. In this case, this equation should be understood as an infinite set of equations (continuously numbered by the labels $\lambda_\alpha$) for a function of infinitely many variables $l_{\alpha\beta}$ associated with $\overrightarrow{\lambda_\alpha\lambda_\beta}$. Second, one might expect that composite discrete flows are good candidates for true analogues of the higher continuous flows. This is justified by analyzing the continuum limit. Indeed, to obtain a higher continuous flow as a limiting case, one should start from a composite discrete flow with specially adjusted labels.

Our goal in this section is to show how these two approaches can be consistent with each other. A natural conjecture is that the $N$-term "higher" Hirota equations for a function of $N$ variables are consequences of the basic 3-term equations (4.20) treated *as a hierarchy*. This means that the 3-term equation is assumed to hold for each triad of $N$ variables with corresponding $\lambda_\alpha$. To support this conjecture, the case of the 4-term HBDE is considered in detail. In addition, an extension of the Zakharov–Shabat scheme to this case is suggested.

### 7.1. Higher equations of the hierarchy

Higher Hirota difference equations known in the literature [30] are written for a function $\tau(l_1,\ldots,l_N)$ of $N$ variables. These equations read

$$\begin{vmatrix} 1 & z_1 & z_1^2 & \ldots & z_1^{N-2} & \tau_1\hat{\tau}_1 \\ 1 & z_2 & z_2^2 & \ldots & z_2^{N-2} & \tau_2\hat{\tau}_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & z_N & z_N^2 & \ldots & z_N^{N-2} & \tau_N\hat{\tau}_N \end{vmatrix} = 0, \tag{7.1}$$

where $z_i$ are arbitrary constants and

$$\begin{aligned} \tau_i &\equiv \tau(l_1,l_2,\ldots,l_{i-1},l_i+1,l_{i+1},\ldots,l_N), \\ \hat{\tau}_i &= \tau(l_1+1,l_2+1,\ldots,l_{i-1}+1,l_i,l_{i+1}+1,\ldots,l_N+1). \end{aligned} \tag{7.2}$$

Expanding the determinant w.r.t. the last column, we can write these equations in a more compact form,

$$\sum_{j=1}^{N} \Lambda_j \tau_j \hat{\tau}_j = 0. \tag{7.3}$$

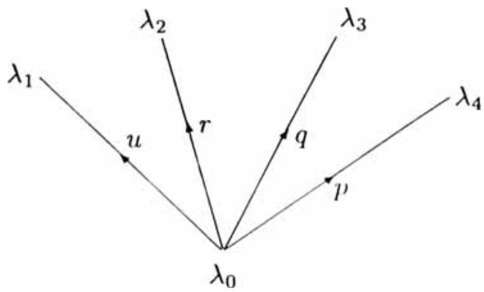



**Fig. 14**

The constants $\Lambda_j$ are subject to only one condition,

$$\sum_{j=1}^{N} \Lambda_j = 0.$$

For $N = 3$, we obtain the usual HBDE, where $z_i$ are certain rational functions of $\lambda_\alpha$. As in the 3-term case, the variables $l_i$ are identified with elementary discrete flows.

The transformation

$$\tau(l_1, \ldots, l_N) \to \exp\left[\frac{1}{2N-4} \sum_{k=1}^{N} \log \Lambda_k \Bigg( \sum_{j=1,\neq k}^{N} l_j \Bigg)^2 \right] \tau(l_1, \ldots, x_N) \tag{7.4}$$

changes Eq. (7.3) into the canonical form

$$\sum_{j=1}^{N} \tau_j \hat{\tau}_j = 0, \tag{7.5}$$

which does not contain any free parameters.

In Hirota's original notation, these equations are

$$\left( \sum_{j=1}^{N} \Lambda_j \exp(D_{x_j}) \right) \tau \cdot \tau = 0, \tag{7.6}$$

which are obtained by the linear change of variables

$$x_j = -l_j + \frac{1}{N-2} \sum_{i=1}^{N} l_i, \tag{7.7}$$

generalizing (2.15).

## 7.2. Zero curvature conditions for composite flows

We show that the zero curvature condition written for the composite discrete flows introduced in Sec. 3.1 lead to the "higher" bilinear equations of form (7.1). The "higher" $M$-operators are obtained as products of the elementary ones.

In this section, we deal with the graph of flows in Fig. 14. The reference flow is $u$; the other notation is clear from the picture. For simplicity, we only consider left adjacent flows, but all that follows can be easily reformulated in terms of right adjacent flows.

According to the definition of composite flows (Sec. 3.1), we introduce a "higher" $M$-operator $M_u^{qr}$ generating the evolution in the composite flow, labeled by the pair of vectors $\overrightarrow{\lambda_0\lambda_2}$, $\overrightarrow{\lambda_0\lambda_3}$, as the product of elementary $M$-operators of form (4.15),

$$M_u^{qr} = e^{\partial_r} M_u^q e^{-\partial_r} M_u^r. \tag{7.8}$$

It is useful to rewrite this equality by indicating the arguments explicitly,

$$M_u^{qr}(q,r) = M_u^q(q,r+1) M_u^r(q,r). \tag{7.9}$$

Due to the zero curvature condition for elementary flows, we have $M_u^{qr} = M_u^{rq}$. Then, the compatibility of this composite flow with an elementary flow $\overrightarrow{\lambda_0\lambda_4}$ reads as follows:

$$M_u^p(p,q+1,r+1) M_u^{qr}(p,q,r) = M_u^{qr}(p+1,q,r) M_u^p(p,q,r). \tag{7.10}$$

Clearly, this zero curvature condition follows directly from Eq. (4.9) and definition (7.8), provided Eq. (4.9) holds for any pair of flows from the triad $(p,q,r)$.

A few words about the notation. For the simplicity, we set

$$\lambda_2^{01} = \lambda_r, \qquad \lambda_3^{01} = \lambda_q, \qquad \lambda_4^{01} = \lambda_p.$$

Since we deal with a large number of variables in this section, it is also convenient to indicate the basic variable $u$ in the $\tau$-function by writing the argument $u$ in brackets: $\tau_u \to \tau(u)$. As above, the other variables are written as indices. We also use the notation

$$V_u^{(p)}(p,q,r) = \frac{\tau^{p,q,r}(u)\tau^{p+1,q,r}(u+1)}{\tau^{p+1,q,r}(u)\tau^{p,q,r}(u+1)}. \tag{7.11}$$

Now we are ready to elaborate Eq. (7.10) explicitly. This equation reads

$$\begin{aligned}\left(e^{\partial_u} - \lambda_p V_u^{(p)}(p,q+1,r+1)\right)\left(e^{\partial_u} - \lambda_q V_u^{(q)}(p,q,r+1)\right)\left(e^{\partial_u} - \lambda_r V_u^{(r)}(p,q,r)\right) = \\ = \left(e^{\partial_u} - \lambda_q V_u^{(q)}(p+1,q,r+1)\right)\left(e^{\partial_u} - \lambda_r V_u^{(r)}(p+1,q,r)\right)\left(e^{\partial_u} - \lambda_p V_u^{(p)}(p,q,r)\right).\end{aligned} \tag{7.12}$$

Comparing the coefficients in front of $e^{2\partial_u}$, we obtain

$$\begin{aligned}\lambda_p V_u^{(p)}(p,q+1,r+1) + \lambda_q V_{u+1}^{(q)}(p,q,r+1) + \lambda_r V_{u+2}^{(r)}(p,q,r) = \\ = \lambda_p V_{u+2}^{(p)}(p,q,r) + \lambda_q V_u^{(q)}(p+1,q,r+1) + \lambda_r V_{u+1}^{(r)}(p+1,q,r).\end{aligned} \tag{7.13}$$

This relation is a direct corollary of the 3-term HBDE. To see this, recall Eq. (4.18) from the proof of Proposition 4.1. In the present notation, it reads

$$\lambda_p V_u^{(p)}(p,q+1,r+1) + \lambda_q V_{u+1}^{(q)}(p,q,r+1) = \lambda_p V_{u+1}^{(p)}(p,q,r+1) + \lambda_q V_u^{(q)}(p+1,q,r+1), \tag{7.14}$$

where $r+1$ enters as a parameter. A similar equation can be written for the pair $(p,r)$, where the variable $q$ enters as a parameter,

$$\lambda_p V_{u+1}^{(p)}(p,q,r+1) + \lambda_r V_{u+2}^{(r)}(p,q,r) = \lambda_p V_{u+2}^{(p)}(p,q,r) + \lambda_r V_{u+1}^{(r)}(p+1,q,r). \tag{7.15}$$

Adding these equations, we obtain Eq. (7.13).

Comparing the coefficients in front of $e^{\partial_u}$, we obtain

$$\lambda_p\lambda_q V_u^{(q)}(p,q,r+1)V_u^{(p)}(p,q+1,r+1) + \lambda_q\lambda_r V_{u+1}^{(r)}(p,q,r)V_{u+1}^{(q)}(p,q,r+1)+$$
$$+ \lambda_p\lambda_r V_u^{(p)}(p,q+1,r+1)V_{u+1}^{(r)}(p,q,r) =$$
$$= \lambda_p\lambda_q V_u^{(q)}(p+1,q,r+1)V_{u+1}^{(p)}(p,q,r) + \lambda_q\lambda_r V_u^{(r)}(p+1,q,r)V_u^{(q)}(p+1,q,r+1)+$$
$$+ \lambda_p\lambda_r V_{u+1}^{(p)}(p,q,r+1)V_{u+1}^{(r)}(p,q,r). \tag{7.16}$$

In the same way, it is easy to show that this equality also follows from the HBDE. Of course, this result is trivial since the zero curvature conditions (4.9) hold for both elementary $M$-operators in the product (7.9).

The fact which is *not* obvious from the very beginning is that Eqs. (7.14)–(7.16) imply one of the *higher Hirota equations*, namely, (7.1) at $N = 4$. Proceeding as in the proof of Proposition 4.1, we rewrite (7.16) in the form

$$\frac{\tau(u+1)}{\tau(u+2)}\left(\lambda_q\lambda_r\frac{\tau^{q+1,r+1}(u+2)}{\tau^{q+1,r+1}(u+1)} - \lambda_p\lambda_r\frac{\tau^{p+1,r+1}(u+2)}{\tau^{p+1,r+1}(u+1)} - \right.$$
$$\left. - \lambda_p\lambda_q\frac{\tau^{p+1,r+1}(u)\tau^{p+1,q+1,r+1}(u+1)\tau^{p+1}(u+2)}{\tau^{p+1,q+1,r+1}(u)\tau^{p+1,r+1}(u+1)\tau^{p+1}(u+1)}\right) =$$
$$= \frac{\tau^{p+1,q+1,r+1}(u+1)}{\tau^{p+1,q+1,r+1}(u)}\left(\lambda_q\lambda_r\frac{\tau^{p+1}(u)}{\tau^{p+1}(u+1)} - \lambda_p\lambda_q\frac{\tau^{r+1}(u)}{\tau^{r+1}(u+1)} - \right.$$
$$\left. - \lambda_p\lambda_r\frac{\tau^{q+1,r+1}(u)\tau(u+1)\tau^{r+1}(u+2)}{\tau^{r+1}(u+1)\tau^{q+1,r+1}(u+1)\tau(u+2)}\right) \tag{7.17}$$

(cf. (4.18)). Multiplying both sides by $(\lambda_q-\lambda_r)\tau^{q+1,r+1}(u+1)\tau^{p+1}(u+1)$ and using the HBDE in the form

$$(\lambda_q-\lambda_r)\tau^{q+1,r+1}(u)\tau(u+1) = \lambda_q\tau^{r+1}(u)\tau^{q+1}(u+1) - \lambda_r\tau^{q+1}(u)\tau^{r+1}(u+1), \tag{7.18}$$

we obtain

$$\frac{\tau(u+1)\tau^{p+1,q+1,r+1}(u)}{\tau(u+2)\tau^{p+1,q+1,r+1}(u+1)}A^{p,q,r}(u) = B^{p,q,r}(u), \tag{7.19}$$

where

$$A^{p,q,r}(u) = (\lambda_q-\lambda_r)\lambda_q\lambda_r\tau^{q+1,r+1}(u+2)\tau^{p+1}(u+1)-$$
$$- \lambda_p\lambda_q\frac{\tau^{p+1,q+1}(u+2)}{\tau^{p+1,q+1,r+1}(u)}\big(\lambda_q\tau^{p+1,r+1}(u)\tau^{q+1,r+1}(u+1)\big)+$$
$$+ \lambda_p\lambda_r\frac{\tau^{p+1,r+1}(u+2)}{\tau^{p+1,q+1,r+1}(u)}\big(\lambda_r\tau^{p+1,q+1}(u)\tau^{q+1,r+1}(u+1)\big),$$
$$B^{p,q,r}(u) = (\lambda_q-\lambda_r)\lambda_q\lambda_r\tau^{p+1}(u)\tau^{q+1,r+1}(u+1)-$$
$$- \lambda_p\lambda_q\frac{\tau^{r+1}(u)}{\tau(u+2)}\big(\lambda_q\tau^{p+1}(u+1)\tau^{q+1}(u+2)\big)+$$
$$+ \lambda_p\lambda_r\frac{\tau^{q+1}(u)}{\tau(u+2)}\big(\lambda_r\tau^{p+1}(u+1)\tau^{r+1}(u+2)\big).$$

The last two terms can be further transformed using Eq. (7.18). The result is

$$A^{p,q,r}(u) = h^{p,q,r}(u+1) - \lambda_p^2(\lambda_q-\lambda_r)\frac{\tau^{q+1,r+1}(u)\tau^{p+1,q+1,r+1}(u+1)\tau^{p+1}(u+2)}{\tau^{p+1,q+1,r+1}(u)},$$
$$B^{p,q,r}(u) = h^{p,q,r}(u) - \lambda_p^2(\lambda_q-\lambda_r)\frac{\tau^{q+1,r+1}(u)\tau(u+1)\tau^{p+1}(u+2)}{\tau(u+2)},$$

where

$$h^{p,q,r}(u) = \lambda_q \lambda_r(\lambda_q - \lambda_r)\tau^{p+1}(u)\tau^{q+1,r+1}(u+1)+ \\ + \lambda_r \lambda_p(\lambda_r - \lambda_p)\tau^{q+1}(u)\tau^{p+1,r+1}(u+1)+ \\ + \lambda_p \lambda_q(\lambda_p - \lambda_q)\tau^{r+1}(u)\tau^{p+1,q+1}(u+1). \tag{7.20}$$

Finally, Eq. (7.19) becomes

$$\frac{\tau(u+1)\tau^{p+1,q+1,r+1}(u)}{\tau(u+2)\tau^{p+1,q+1,r+1}(u+1)} = \frac{h^{p,q,r}(u)}{h^{p,q,r}(u+1)}, \tag{7.21}$$

which leads to the equation

$$h^{p,q,r}(u) + (\lambda_p - \lambda_q)(\lambda_q - \lambda_r)(\lambda_r - \lambda_p)\tau(u+1)\tau^{p+1,q+1,r+1}(u) = 0 \tag{7.22}$$

having form (7.1) for $N = 4$. This completes the calculation.

The $M$-operators for arbitrary composite flows can be defined as a straightforward generalization of (7.8):

$$M_u^{p_N \cdots p_2 p_1} = \exp\left(\sum_{j=1}^{N} \partial_{p_j}\right) \prod_{i=1}^{N} (e^{-\partial_{p_i}} M_u^{p_i}). \tag{7.23}$$

Note that the order of the operators in the product is not essential since the operators $e^{-\partial_{p_i}} M_u^{p_i}$ commute due to the zero curvature condition (4.9). For simplicity, it is assumed that all flows $p_i$ are left adjacent to the reference flow $u$. Operators (7.23) generate discrete analogues of higher flows of the KP hierarchy (see the corresponding graph of the flows in Fig. 3). Now, there is a straightforward way to write similar operators for right adjacent flows that would generate higher flows of the discrete 2DTL hierarchy.

**Postulate.** *All higher HBDE* (7.1) *follow from the compatibility of the composite flows generated by the $M$-operators* (7.23) *and elementary discrete flows.*

The calculation given above shows that the postulate is true for the 4-term bilinear equation. Unfortunately, we are not aware of any proof other than this sophisticated calculation, which is hard to perform in the general case. The postulate claims that all of the higher bilinear equations are corollaries of Eq. (4.20) considered as a hierarchy, i.e., applied to all triads of adjacent flows.

## 8. Reductions of the Hirota equations

The hierarchy of the discrete Hirota equations admits several important reductions. A *reduction* means imposing a constraint compatible with the hierarchy such that the number of independent variables becomes reduced. In this way, one is able to construct discrete analogues of the KdV, sine-Gordon, and other interesting equations.

The simplest way to impose a constraint is to require that the $\tau$-function be stationary with respect to a particular flow $s$ (possibly up to a "gauge" transformation (2.2)). Nontrivial examples emerge when the stationary flow is a composite. As for the commutation representation, there are two possibilities.

First, the stationary flow can be the reference flow. Then $M$-operators become free of differentiation because the symbol $\partial_s$ commutes with all of the coefficients. In other words, $\partial_s$ can be considered as a $c$-number and can be identified with a *spectral parameter*. This is the natural origin of $M$-operators depending on a (rational) spectral parameter.

Alternatively, one may take any flow $y$ other than $s$ as the reference flow. Then any $M$-operator $M^{(f)}$ generating a flow $f$ contains the operators $\partial_y$. Since coefficients of the operator $M^{(f)}$ do not depend on $s$, the compatibility condition for the flows $s$ and $f$ acquires the Lax-type form

$$M^{(s)}(f+1)M^{(f)} = M^{(f)}M^{(s)}(f),$$

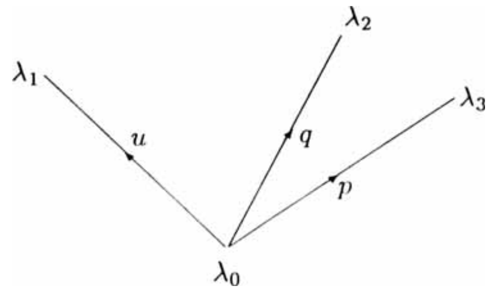



**Fig. 15**

where $M^{(f)}$ plays the role of the *Lax operator*. Unlike the Zakharov–Shabat scheme, where each zero curvature condition involves two different time flows (apart from the reference flow), Lax equations are written for each flow separately. The Lax equation represents the time flow as a similarity (i.e., isospectral) transformation of the Lax operator. It is natural to call $M^{(s)}$ *the L-operator of the reduced hierarchy*. This is the natural origin of $L$-operators, which are difference (differential) operators rather than pseudo-difference (pseudo-differential) operators.[8] To illustrate this general scheme, we give some examples below.

## 8.1. KdV-type reductions

1. *Discrete d'Alembert equation* (*a trivial example*). Let $u$ be an elementary discrete flow from Sec. 4. The stationarity condition with respect to this flow, $\tau(u+1) = \tau(u)$, immediately leads (see (4.20)) to the relation

$$\tau^{l+1,m}\tau^{l,m+1} = \tau^{l,m}\tau^{l+1,m+1}, \tag{8.1}$$

where $l, m$ are any other elementary flows. This is the discrete two-dimensional d'Alembert equation written in the "light cone" coordinates. The general solution is $\tau^{l,m} = \chi_+(l)\chi_-(m)$, with arbitrary functions $\chi_\pm$. However, this is just the allowed "gauge" freedom (2.8) of the $\tau$-function, such that when related to the HBDE, this solution is equivalent to the constant solution $\tau^{l,m} = \text{const}$. Thus, we can see that this reduction is too strong because it only contains trivial solutions. To obtain nontrivial examples, one should either impose stationarity conditions with respect to the composite ("higher") flow or periodic conditions in $u$ with periods $N > 1$ (e.g., $\tau(u+2) = \tau(u)$).

2. *Discrete KdV equation*. Consider the graph of flows depicted in Fig. 15 and set $\lambda_2^{01} = \lambda_q$, $\lambda_3^{01} = \lambda_p$ for brevity. In this notation, Eq. (4.20) takes the form

$$\lambda_q \tau_u^{p+1,q}\tau_{u+1}^{p,q+1} - \lambda_p \tau_u^{p,q+1}\tau_{u+1}^{p+1,q} + (\lambda_p - \lambda_q)\tau_u^{p+1,q+1}\tau_{u+1}^{p,q} = 0. \tag{8.2}$$

To obtain the discrete KdV reduction, we impose the constraint

$$\tau_u^{p+1,q+1} = \tau_u^{p,q}, \tag{8.3}$$

i.e., the $\tau$-function is stationary with respect to the composite flow labeled by the pair of vectors $\overrightarrow{\lambda_0\lambda_2}$, $\overrightarrow{\lambda_0\lambda_3}$. This condition converts the three-dimensional equation (8.2) into the following two-dimensional one:

$$\lambda_q \tau_u^{p+1}\tau_{u+1}^{p-1} - \lambda_p \tau_u^{p-1}\tau_{u+1}^{p+1} + (\lambda_p - \lambda_q)\tau_u^p\tau_{u+1}^p = 0. \tag{8.4}$$

This is the discrete KdV equation in bilinear form [2, 30]. The discrete KdV equation is also known in the form [30]

$$V(u,p) - V(u-1,p-1) = \kappa\big(V^{-1}(u,p-1) - V^{-1}(u-1,p)\big). \tag{8.5}$$

[8] In the general case, the Lax operator for a hierarchy without any reduction is an infinite series in negative powers of the first-order difference operators. The theory based on the Lax representation with Lax operators of this kind is not considered here.

The equivalence of (8.4) and (8.5) follows from the identification

$$V(u,p) = \frac{\tau_u^p \tau_{u+1}^{p+1}}{\tau_u^{p+1}\tau_{u+1}^p}, \qquad \kappa = \frac{\lambda_q}{\lambda_p}.$$

Let us turn to the $M$- and $L$- operators. Following the history of the KdV equation, we begin with the difference operators with scalar coefficients. Let $u$ be the reference variable. Then the (composite) stationary flow is generated by an $M$-operator of type (7.9),

$$\begin{aligned} M_u^{pq} &= M_u^p(p, q+1) M_u^q(p,q) = L^{(\mathrm{KdV})} = \\ &= e^{2\partial_u} - \left( \lambda_p \frac{\tau_u^{p-1}\tau_{u+1}^p}{\tau_u^p \tau_{u+1}^{p-1}} + \lambda_q \frac{\tau_{u+1}^p \tau_{u+2}^{p-1}}{\tau_{u+1}^{p-1}\tau_{u+2}^p} \right) e^{\partial_u} + \lambda_p \lambda_q. \end{aligned} \tag{8.6}$$

We call this second-order difference operator the *Lax operator of the discrete KdV equation*. The spectral problem $L^{(\mathrm{KdV})}\psi = E\psi$ is a discrete analogue of the stationary Schrödinger equation that is an auxiliary linear problem for KdV. The $p$-evolution generated by the $M$-operator of type (4.15) is isospectral: $L^{(\mathrm{KdV})}(p+1) = M_u^p L^{(\mathrm{KdV})}(p)(M_u^p)^{-1}$.

If the reference flow coincides with the stationary one, we obtain $2\times 2$ matrix $M$-operators depending on a spectral parameter $z$. The spectral parameter is an eigenvalue of the shift operator along the stationary flow acting on the $\psi$-function: $\exp(\partial_p + \partial_q)\psi = z^2\psi$. Consider the vector

$$\begin{pmatrix} \psi_u^p \\ \chi_u^p \end{pmatrix} = \begin{pmatrix} \psi^p(u) \\ \frac{\tau_u^{p+1}}{\tau_u^p}\psi_u^{p+1} \end{pmatrix}.$$

Repeating the argument of Sec. 5.5, we obtain the following linear problems for shifts in $u$ and $p$, respectively:

$$\begin{pmatrix} \psi_{u+1}^p \\ \chi_{u+1}^p \end{pmatrix} = \begin{pmatrix} \lambda_p \frac{\tau_u^p \tau_{u+1}^{p+1}}{\tau_u^{p+1}\tau_{u+1}^p} & \frac{\tau_u^p}{\tau_u^{p+1}} \\ z^2 \frac{\tau_{u+1}^{p+1}}{\tau_{u+1}^p} & \lambda_q \end{pmatrix} \begin{pmatrix} \psi_u^p \\ \chi_u^p \end{pmatrix}, \tag{8.7}$$

$$\begin{pmatrix} \psi_u^{p+1} \\ \chi_u^{p+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\tau_u^p}{\tau_u^{p+1}} \\ z^2 \frac{\tau_u^{p+2}}{\tau_u^{p+1}} & \lambda_q - \lambda_p \end{pmatrix} \begin{pmatrix} \psi_u^p \\ \chi_u^p \end{pmatrix}. \tag{8.8}$$

The compatibility of these linear problems yields the discrete matrix Zakharov–Shabat equations with the spectral parameter $z$.

Equations for the first component $\psi_u^p$ read

$$\psi_{u+1}^p + \frac{1}{\lambda_p - \lambda_q}\left( \lambda_q^2 \frac{\tau_{u-1}^{p+1}\tau_{u+1}^{p-1}}{\tau_{u-1}^p \tau_{u+1}^p} - \lambda_p^2 \frac{\tau_{u-1}^{p-1}\tau_{u+1}^{p+1}}{\tau_{u-1}^p \tau_{u+1}^p} \right)\psi_u^p = (z^2 - \lambda_p\lambda_q)\psi_{u-1}^p, \tag{8.9}$$

$$\psi_u^{p+1} + (\lambda_p - \lambda_q)\frac{(\tau_u^p)^2}{\tau_u^{p-1}\tau_u^{p+1}}\psi_u^p = z^2 \psi_u^{p-1}. \tag{8.10}$$

Equation (8.9) coincides with the spectral problem for the Lax operator (8.6), provided the $\tau$-function obeys the bilinear relation (8.4).

3. *Discrete Boussinesq equation* [13]. In this example, the $\tau$-function $\tau_u^{p,q}$ satisfies the stationary equation $\tau_{u+1}^{p+1,q+1} = \tau_u^{p,q}$. Then, Eq. (8.2) is reduced to

$$\lambda_q \tau^{p+1,q}\tau^{p-1,q} - \lambda_p \tau^{p,q+1}\tau^{p,q-1} + (\lambda_p - \lambda_q)\tau^{p+1,q+1}\tau^{p-1,q-1} = 0, \tag{8.11}$$

where $u$ implicitly enters as a parameter. Different kinds of Lax and Zakharov Shabat representations for this equation can be written straightforwardly, but we do not discuss that here.

In a similar way, it is possible to define more general $A_n$-type reductions. In this case, the $\tau$-function obeys the condition

$$\exp\left(\sum_{\alpha=1}^{n+1} \partial_{p_\alpha}\right)\tau = \tau.$$

Note that this becomes an actual reduction only for higher Hirota equations where the number of variables is greater than $n$.

## 8.2. Discrete time 1D Toda chain and its relatives

This group of examples includes the discrete time 1D Toda chain (1DTC), the discrete AKNS system (in particular, the discrete nonlinear Schrödinger equation), the discrete time relativistic Toda chain, and the discrete Heisenberg ferromagnet (HF). These models differ in the choice of dependent and independent variables, while the type of reduction is essentially the same.

Let the graph of flows and the notation be the same as in Sec. 4.3. Now the $\tau$-function is required to be stationary with respect to the composite flow labeled by the pair of vectors $\overrightarrow{\lambda_0\lambda_3}$, $\overrightarrow{\lambda_1\lambda_2}$, i.e.,

$$\tau_n^{l+1,\bar{l}+1} = \tau_n^{l,\bar{l}}. \tag{8.12}$$

The stationary flow is generated by the "composite" $M$-operator $e^{\partial_l}\overline{M}_n^{\bar{l}}e^{-\partial_l}M_n^l$ with the reference flow $n$. This operator should be identified with the Lax operator of the discrete time 1DTC,

$$L^{(\mathrm{TC})} = e^{\partial_n} - \left(\nu\frac{\tau_n^l\tau_{n+1}^{l+1}}{\tau_n^{l+1}\tau_{n+1}^l} + \mu\frac{\tau_{n-1}^{l+1}\tau_{n+1}^l}{\tau_n^l\tau_n^{l+1}}\right) + \nu\mu\frac{\tau_{n-1}^l\tau_{n+1}^l}{(\tau_n^l)^2}e^{-\partial_n}. \tag{8.13}$$

This is a second-order difference operator in $n$ with scalar coefficients. The spectral problem $L^{(\mathrm{TC})}\psi = E\psi$ is a 1D discrete *stationary* Schrödinger equation (cf. (5.43)). The $l$-dynamics preserve the spectrum of $L^{(\mathrm{TC})}$.

Changing $e^{\partial_\nu} = e^{\partial_l+\partial_{\bar{l}}}$ by $z^2$ in (4.23), we obtain an $(L-M)$-pair for the discrete time 1DTC realized by $2\times 2$ matrices depending on the spectral parameter $z$,

$$L_n^{(\mathrm{TC})}(z) = \begin{pmatrix} z^2 + \nu\dfrac{\tau_n^l\tau_{n+1}^{l+1}}{\tau_n^{l+1}\tau_{n+1}^l} + \mu\dfrac{\tau_{n+1}^l\tau_{n-1}^{l+1}}{\tau_n^l\tau_n^{l+1}} & -\mu\nu\dfrac{\tau_{n+1}^l}{\tau_n^l} \\ \dfrac{\tau_n^l}{\tau_{n+1}^l} & 0 \end{pmatrix}, \tag{8.14}$$

$$M_n^{(\mathrm{TC})}(z) = \begin{pmatrix} \nu & -\mu\nu\dfrac{\tau_{n+1}^{l-1}}{\tau_n^{l-1}} \\ \dfrac{\tau_{n-1}^l}{\tau_n^l} & -z^2 - \mu\dfrac{\tau_{n+1}^{l-1}\tau_{n-1}^l}{\tau_n^{l-1}\tau_n^l} \end{pmatrix}. \tag{8.15}$$

Correspondingly, the 2D discrete Schrödinger equation (5.44) becomes a 1D spectral problem in the variable $l$,

$$\psi_n^{l+1} + (\nu - \mu)\frac{\tau_n^l\tau_{n+1}^{l-1}}{\tau_n^{l+1}\tau_{n+1}^{l-1}}\psi_n^l = z^2\left(\psi_n^l + \nu\frac{\tau_n^{l-1}\tau_{n+1}^l}{\tau_n^l\tau_{n+1}^{l-1}}\psi_n^{l-1}\right). \tag{8.16}$$

Let us pay attention to the nonstandard dependence on the spectral parameter.

Another useful form of this equation is

$$\overline{\psi}_n^{l+1} - \nu \frac{\tau_n^{l-1}\tau_{n+1}^l}{\tau_n^l \tau_{n+1}^{l-1}} \overline{\psi}_n^{l-1} = \left( z + (\mu - \nu) z^{-1} \frac{\tau_n^l \tau_{n+1}^l}{\tau_n^{l+1} \tau_{n+1}^{l-1}} \right) \overline{\psi}_n^l, \tag{8.17}$$

which can be obtained from the previous one by substituting $\overline{\psi}_n^l = z^{-l}\psi_n^l$.

The equation of motion in bilinear form is the following two-dimensional reduction of Eq. (2.11):

$$\nu \tau_n^{l+1}\tau_n^{l-1} - \mu \tau_{n+1}^{l-1}\tau_{n-1}^{l+1} = (\nu - \mu)(\tau_n^l)^2. \tag{8.18}$$

After the linear change $l \to m = l + n$, $\tau_n^l \to \tau_n(m)$, this equation coincides with the discrete Toda chain in Hirota's original bilinear form [3],

$$(1 + g^{-2})\tau_n(m+1)\tau_n(m-1) - \tau_{n+1}(m)\tau_{n-1}(m) = g^{-2}\big(\tau_n(m)\big)^2, \tag{8.19}$$

where

$$g^{-2} = \frac{\nu}{\mu} - 1.$$

In terms of the new dependent variable

$$\phi_n(m) = \log \frac{\tau_{n+1}(m)}{\tau_n(m)}, \tag{8.20}$$

Eq. (8.19) acquires the form

$$\exp\big(\phi_n(m+1) + \phi_n(m-1) - 2\phi_n(m)\big) = \frac{1 + g^2 \exp\big(\phi_{n+1}(m) - \phi_n(m)\big)}{1 + g^2 \exp\big(\phi_n(m) - \phi_{n-1}(m)\big)}, \tag{8.21}$$

which is nothing more than the discrete time 1DTC equation studied by Suris [31].

The continuum limit in $m$ is straightforward. Set $m \to m/\epsilon$, $g^2 = -\epsilon^2$, then

$$\phi_n(m \pm 1) \to \phi_n \pm \epsilon\phi_n' + \frac{1}{2}\epsilon^2\phi_n'' \qquad \text{for} \qquad \epsilon \to 0.$$

Developing Eq. (8.21) into a series in $\epsilon$, we obtain the well-known 1DTC equation,

$$\phi_n'' = e^{\phi_n - \phi_{n-1}} - e^{\phi_{n+1} - \phi_n}. \tag{8.22}$$

It is interesting to note that Eq. (8.21) possesses another continuum limit that yields the sine-Gordon (SG) equation. To see this, let us redefine the field $\phi$ before taking the limit,

$$\phi_n(m) = i(-1)^{n-m}\varphi_n(m), \tag{8.23}$$

such that the equation reads

$$\exp\big(i\varphi_{n+1}(m) + i\varphi_{n-1}(m) - i\varphi_n(m+1) - i\varphi_n(m-1)\big) = \\ = \frac{1 + g^{-2}\exp\big(i\varphi_{n+1}(m) + i\varphi_n(m)\big)}{1 + g^{-2}\exp\big(-i\varphi_{n-1}(m) - i\varphi_n(m)\big)}. \tag{8.24}$$

Now it is the field $\varphi_n(m)$ that is assumed to have a smooth continuum limit in $n, m$. Setting $m \to m/\epsilon$, $n \to n/\epsilon$, and $g^{-2} = -i\epsilon^2$, $\epsilon \to 0$, and expanding in $\epsilon$ as before, we obtain the SG equation

$$(\partial_n^2 - \partial_m^2)\varphi = 2\sin(2\varphi). \tag{8.25}$$

(The same limit for the field $\phi$ in Eq. (8.21) would give the d'Alembert equation $(\partial_n^2 - \partial_m^2)\varphi = 0$.) A discrete analogue of the SG equation of a different kind is given below in Sec. 8.3.

**Remark 8.1.** The discrete KdV reduction discussed in Sec. 8.1 is formally a particular case of the present reduction where $\lambda_0$ and $\lambda_1$ merge. However, it is more convenient to consider them separately. Note that there also exists a continuum limit of Eq. (8.18) leading to the KdV equation. In this sense, the discrete time 1DTC is sometimes considered as a discretization of the KdV equation.

The discrete version of the decoupled nonlinear Schrödinger equation (also called the AKNS system) possesses essentially the same reduction, i.e., the stationary flow is as above. The difference is in the choice of the other independent flows to be involved in the equations. Specifically, instead of the $n$-flow, one may consider any other elementary discrete flow $p$ left adjacent to $l$.

We show how to derive the discrete AKNS system directly from the bilinear Hirota equations. The basic bilinear equations in question are

$$\lambda_p \tau_n^{p,\bar{l}+1}\tau_n^{p+1,\bar{l}} + (\mu - \lambda_p)\tau_n^{p,\bar{l}}\tau_n^{p+1,\bar{l}+1} = \mu\tau_{n+1}^{p,\bar{l}+1}\tau_{n-1}^{p+1,\bar{l}}, \tag{8.26}$$

$$\nu\tau_n^{p+1,\bar{l}+1}\tau_{n+1}^{p,\bar{l}} - \lambda_p\tau_n^{p,\bar{l}}\tau_{n+1}^{p+1,\bar{l}+1} = (\nu - \lambda_p)\tau_n^{p+1,\bar{l}}\tau_{n+1}^{p,\bar{l}+1}. \tag{8.27}$$

The first of these equations is Eq. (4.22) written for the triad $(p, n, \bar{l})$. The second is obtained from Eq. (4.20) written for the triad $(l, p, n)$ by taking into account the stationary condition $\tau^{l+1,\bar{l}+1} = \tau^{l,\bar{l}}$. It is easy to see that in terms of the quantities

$$Q^{p,\bar{l}} = \frac{\tau_{n+1}^{p,\bar{l}}}{\tau_n^{p,\bar{l}}}, \qquad R^{p,\bar{l}} = \frac{\tau_{n-1}^{p,\bar{l}}}{\tau_n^{p,\bar{l}}} \tag{8.28}$$

($n$ is fixed), these equations can be rewritten as follows:

$$(\nu - \lambda_p)(\lambda_p - \mu)Q^{p,\bar{l}+1} = (\lambda_p - \mu Q^{p,\bar{l}+1}R^{p+1,\bar{l}})(\nu Q^{p,\bar{l}} - \lambda_p Q^{p+1,\bar{l}+1}), \tag{8.29}$$

$$(\nu - \lambda_p)(\lambda_p - \mu)R^{p+1,\bar{l}} = (\lambda_p - \mu Q^{p,\bar{l}+1}R^{p+1,\bar{l}})(\nu R^{p+1,\bar{l}+1} - \lambda_p R^{p,\bar{l}}). \tag{8.30}$$

In this form, the system is equivalent to the discrete AKNS system from [13].

There is another choice of dependent variables which converts Eqs. (8.18), (8.26), and (8.27) into a discrete analogue of the relativistic Toda chain (RTC) [32]. Passing, again, to the variable $m = l + n$, as in Eq. (8.20), we set

$$x_m(p) = -\log\frac{\tau_n^p(m+1)}{\tau_n^p(m)}. \tag{8.31}$$

The equation of motion for $x_m(p)$ has the form

$$\frac{\big(1 - \alpha\exp(x_{m+1}(p) - x_m(p))\big)}{\big(1 - \alpha\exp(x_m(p) - x_{m-1}(p))\big)} \times \\ \times\frac{\big(1 - \beta\exp(x_m(p) - x_{m-1}(p-1))\big)\big(1 - \gamma\exp(x_m(p) - x_m(p+1))\big)}{\big(1 - \beta\exp(x_{m+1}(p+1) - x_m(p))\big)\big(1 - \gamma\exp(x_m(p-1) - x_m(p))\big)} = 1, \tag{8.32}$$

where

$$\alpha = \frac{\nu}{\nu - \mu}, \qquad \beta = \frac{\nu(\mu - \lambda_p)}{\lambda_p(\mu - \nu)}, \qquad \gamma = \frac{\lambda_p}{\lambda_p - \mu}. \tag{8.33}$$

This equation differs only slightly from the discrete time RTC equation suggested in [33].

Let us outline the method of deriving Eq. (8.32).[9] The basic bilinear relations (8.18), (8.26), and (8.27) read

$$(\nu - \mu)\big(\tau_n^p(m)\big)^2 - \nu\tau_n^p(m+1)\tau_n^p(m-1) = -\mu\tau_{n+1}^p(m)\tau_{n-1}^p(m), \tag{8.34}$$

$$(\mu - \lambda_p)\tau_n^p(m+1)\tau_n^{p+1}(m) + \lambda_p\tau_n^{p+1}(m+1)\tau_n^p(m) = \mu\tau_{n+1}^p(m+1)\tau_{n-1}^{p+1}(m), \tag{8.35}$$

$$\nu\tau_{n+1}^p(m+1)\tau_n^{p+1}(m-1) - \lambda_p\tau_n^p(m)\tau_{n+1}^{p+1}(m) = (\nu - \lambda_p)\tau_n^{p+1}(m)\tau_{n+1}^p(m), \tag{8.36}$$

respectively. It is straightforward to show that the following two bilinear relations are direct corollaries of the basic relations:

$$(\nu - \lambda_p)\tau_{n+1}^p(m)\tau_n^{p+1}(m+1) - (\mu - \lambda_p)\tau_{n+1}^{p+1}(m)\tau_n^p(m+1) = \\ = (\nu - \mu)\tau_{n+1}^p(m+1)\tau_n^{p+1}(m), \tag{8.37}$$

$$\lambda_p(\nu - \mu)\tau_n^p(m)\tau_n^{p+1}(m) + \nu(\mu - \lambda_p)\tau_n^p(m+1)\tau_n^{p+1}(m-1) = \\ = \mu(\nu - \lambda_p)\tau_{n+1}^p(m)\tau_{n-1}^{p+1}(m). \tag{8.38}$$

Equation (8.37) is obtained by eliminating $\tau_n^p(m)$ from (8.35), (8.36) (i.e., by dividing them by $\tau_n^{p+1}(m+1)$, $\tau_{n+1}^{p+1}(m)$, respectively, and adding the results) and making use of Eq. (8.34). Equation (8.38) is obtained analogously by eliminating $\tau_n^{p+1}(m)$ from (8.35), (8.36) and making use of Eq. (8.37). Now Eq. (8.32) easily follows from (8.34), (8.35), and (8.38).

**Remark 8.2.** Bilinearization of the usual (continuous time) RTC was suggested in [34]. The equivalence of the RTC and the "semi-discretized" AKNS system (with discrete "space" and continuous time variables) was recently proved in [35].

We conclude this subsection with a note about the discrete Heisenberg ferromagnet (HF) [12]. This equation fits the scheme in the following way and the reduction is the same. However, the choice of the independent variables is different. In addition to the flow $p$ from the previous example, one should introduce yet another elementary discrete flow $q$ left adjacent to $l$. The $\tau$-function now depends on four independent variables: $\tau = \tau_n^{\bar{l}}(p, q)$. Fix $n$, $\bar{l}$ and consider the following four functions of $p, q$: $\tau_n^{\bar{l}+1}(p, q)$, $\tau_n^{\bar{l}-1}(p, q)$, $\tau_{n+1}^{\bar{l}+1}(p, q)$, and $\tau_{n-1}^{\bar{l}-1}(p, q)$. It follows from the bilinear equations that certain combinations of these functions satisfy a system of nonlinear difference equations in the variables $p, q$, which play the role of discrete space-time coordinates. This system is equivalent to the discrete HF model discussed in detail in [12], where it was treated in a slightly different manner as a part of the reduced 2-component 2DTL hierarchy. As in the case of the discrete AKNS system, the aforementioned embedding into the one-component discrete 2DTL hierarchy leads to equivalent equations of motion. We omit the details.

## 8.3. Periodic reductions

Periodic reductions of the continuous 2DTL hierarchy give rise to a number of very important equations. For example, the 2-periodic reduction $\tau_{n+2} = \tau_n$ contains the sine-Gordon (SG) equation. The same periodic constraint can be imposed in the discretized setup, thus providing us with a discrete analogue of the SG equation.

We call attention to the fact that periodic reductions can be treated on equal footing with stationary reductions. Indeed, the flow $p \to p + 2$ is, formally, a degenerate case of a composite flow where the

[9] The idea for this derivation belongs to S. Kharchev (unpublished).

corresponding labels pairwise merge on the complex plane. The periodicity $\tau^{p+2} = \tau^p$ means stationarity with respect to this degenerate "composite" flow. However, this point of view does not appear to be useful in practice. Usually, it is more convenient to treat periodic reductions separately.

Let us consider the 2DTL-like form of HBDE (2.11) with the additional constraint

$$\tau_{n+2}^{l,\bar{l}} = \tau_n^{l,\bar{l}}. \tag{8.39}$$

Then, the three-dimensional HBDE becomes the following system of two-dimensional equations:

$$\begin{aligned} \nu\tau_0^{l,\bar{l}+1}\tau_0^{l+1,\bar{l}} - (\nu-\mu)\tau_0^{l,\bar{l}}\tau_0^{l+1,\bar{l}+1} &= \mu\tau_1^{l,\bar{l}+1}\tau_1^{l+1,\bar{l}}, \\ \nu\tau_1^{l,\bar{l}+1}\tau_1^{l+1,\bar{l}} - (\nu-\mu)\tau_1^{l,\bar{l}}\tau_1^{l+1,\bar{l}+1} &= \mu\tau_0^{l,\bar{l}+1}\tau_0^{l+1,\bar{l}}. \end{aligned} \tag{8.40}$$

The SG field $\Phi^{l,\bar{l}}$ on the square lattice $(l,\bar{l})$ is given by the formula

$$\Phi^{l,\bar{l}} = \frac{1}{2}\log\frac{\tau_0^{l,\bar{l}}}{\tau_1^{l,\bar{l}}}. \tag{8.41}$$

Rearranging Eqs. (8.40), one obtains a closed equation for $\Phi^{l,\bar{l}}$,

$$\nu\sinh(\Phi^{l,\bar{l}} + \Phi^{l+1,\bar{l}+1} - \Phi^{l,\bar{l}+1} - \Phi^{l+1,\bar{l}}) = \mu\sinh(\Phi^{l,\bar{l}} + \Phi^{l+1,\bar{l}+1} + \Phi^{l,\bar{l}+1} + \Phi^{l+1,\bar{l}}), \tag{8.42}$$

which is known as the discrete SG equation [4] written in light-cone coordinates.

Let us mention another useful form of the discrete SG equation [36, 37]. Set

$$S^{l,\bar{l}} = \exp\left(-2\Phi^{l+1,\bar{l}} - 2\Phi^{l,\bar{l}+1}\right) = \frac{\tau_1^{l+1,\bar{l}}\tau_1^{l,\bar{l}+1}}{\tau_0^{l+1,\bar{l}}\tau_0^{l,\bar{l}+1}}, \tag{8.43}$$

$$\widetilde{S}^{l,\bar{l}} = \exp\left(-2\Phi^{l,\bar{l}} - 2\Phi^{l+1,\bar{l}+1}\right) = \frac{\tau_1^{l,\bar{l}}\tau_1^{l+1,\bar{l}+1}}{\tau_0^{l,\bar{l}}\tau_0^{l+1,\bar{l}+1}}, \tag{8.44}$$

then

$$\widetilde{S}^{l,\bar{l}+1}\widetilde{S}^{l+1,\bar{l}} = S^{l,\bar{l}}S^{l+1,\bar{l}+1}.$$

On the other hand, the discrete SG equation implies that

$$\widetilde{S}^{l,\bar{l}} = \frac{\mu - \nu S^{l,\bar{l}}}{\mu S^{l,\bar{l}} - \nu}, \tag{8.45}$$

such that (8.42) (cf. (2.5)) becomes

$$S^{l,\bar{l}}S^{l+1,\bar{l}+1} = \frac{(\mu - \nu S^{l,\bar{l}+1})(\mu - \nu S^{l+1,\bar{l}})}{(\mu S^{l,\bar{l}+1} - \nu)(\mu S^{l+1,\bar{l}} - \nu)}. \tag{8.46}$$

We now turn to the zero curvature representation. Let $l$ be the reference flow. The shift $n \to n+2$ is generated by the scalar $L$-operator

$$L^{(\mathrm{SG})} = e^{2\partial_l} + \nu\left(\frac{\tau_1^{l,\bar{l}}\tau_0^{l+1,\bar{l}}}{\tau_0^{l,\bar{l}}\tau_1^{l+1,\bar{l}}} + \frac{\tau_0^{l+1,\bar{l}}\tau_1^{l+2,\bar{l}}}{\tau_1^{l+1,\bar{l}}\tau_0^{l+2,\bar{l}}}\right)e^{\partial_l} + \nu^2. \tag{8.47}$$

However, this representation is not convenient for describing the evolution in $\bar{l}$. The matrix $(L-M)$-pair with a spectral parameter is more appropriate here. To derive it, one should take the "stationary" flow $2n$ as the reference flow and repeat the arguments given in Sec. 5.5 with necessary modifications. The operator $e^{2\partial_n}$ should be substituted by the spectral parameter $z^2$. Omitting the details, we present the result.

The auxiliary linear problems read

$$-\nu^{-1}\begin{pmatrix}\psi^{l+1,\bar{l}}\\ \chi^{l+1,\bar{l}}\end{pmatrix}=\begin{pmatrix}\dfrac{\tau_0^{l,\bar{l}}\tau_1^{l+1,\bar{l}}}{\tau_0^{l+1,\bar{l}}\tau_1^{l,\bar{l}}} & -\dfrac{z}{\nu}\dfrac{\tau_0^{l,\bar{l}}}{\tau_1^{l,\bar{l}}}\\ -\dfrac{z}{\nu}\dfrac{\tau_1^{l+1,\bar{l}}}{\tau_0^{l+1,\bar{l}}} & 1\end{pmatrix}\begin{pmatrix}\psi^{l,\bar{l}}\\ \chi^{l,\bar{l}}\end{pmatrix}, \tag{8.48}$$

$$\mu\begin{pmatrix}\psi^{l,\bar{l}+1}\\ \chi^{l,\bar{l}+1}\end{pmatrix}=\begin{pmatrix}1 & -\dfrac{\mu}{z}\dfrac{\tau_1^{l,\bar{l}+1}}{\tau_0^{l,\bar{l}+1}}\\ -\dfrac{\mu}{z}\dfrac{\tau_0^{l,\bar{l}}}{\tau_1^{l,\bar{l}}} & \dfrac{\tau_0^{l,\bar{l}}\tau_1^{l,\bar{l}+1}}{\tau_0^{l,\bar{l}+1}\tau_1^{l,\bar{l}}}\end{pmatrix}\begin{pmatrix}\psi^{l,\bar{l}}\\ \chi^{l,\bar{l}}\end{pmatrix}, \tag{8.49}$$

which is similar to (8.7). Denoting the matrices on the r.h.s. of Eqs. (8.48), (8.49) by $M^{(+)}$, $M^{(-)}$, respectively, we can write the compatibility condition

$$M^{(+)}(l,\bar{l}+1)M^{(-)}(l,\bar{l})=M^{(-)}(l+1,\bar{l})M^{(+)}(l,\bar{l}), \tag{8.50}$$

whence the discrete SG equation follows.

The $N$-periodic reductions ($\tau_{n+N}=\tau_n$) can be treated in a similar way. They correspond to $N$-periodic Toda lattices in discrete time. It is also possible to impose periodic conditions with respect to any of the composite flows. In the remaining part of this section, we briefly comment on an important class of such reductions, which are discrete analogues of the intermediate long wave (ILW) equations.

The universal form of reductions from the 2DTL to the family of continuous ILW equations is most transparently written in terms of the $\tau$-function of the 2DTL hierarchy. The reduction to the ILW$_k$ equation reads (38)

$$\tau_{n+k}(t_1+h,t_2,\ldots;\bar{t}_1,\bar{t}_2,\ldots)=\tau_n(t_1,t_2,\ldots;\bar{t}_1,\bar{t}_2,\ldots), \tag{8.51}$$

where $h$ is a fixed parameter. This parameter interpolates between the $k$-periodic reduction ($h=0$) and the Benjamin–Ono equation ($h\to\infty$). This means that the $\tau$-function should not depend on a particular combination of $n$ and $t_1$, which suggests a discretization of the ILW$_k$ equation. According to our general rules of discretization, one should substitute $t_1$ by an elementary discrete flow $p$. Then it is natural to substitute Eq. (8.51) by

$$\tau_{n+k}^{p+l}=\tau_n^p, \tag{8.52}$$

where $l$ and $k$ are integer parameters. The particular cases are the discrete KdV equation ($l=k=1$) and the $k$-periodic reduction ($l=0$). In the continuum limit, we obtain the continuous ILW$_k$ equation.

## 8.4. Discrete Liouville equation

The discrete Liouville equation (DLE) and its $A_n$-generalizations [39] (discrete time 2DTL with open boundaries) form a very important special class of discrete integrable systems, which, in general, does not fit the reduction scheme discussed in this section. We include it here because the DLE is, formally, a degenerate case of the discrete SG equation. The relationship between these two integrable systems deserves further study.

The DLE can be obtained from the discrete SG equation as a result of a certain scaling limit. Let us rescale $S^{l,\bar{l}} \to \mu S^{l,\bar{l}}$ in Eq. (8.46). Clearly, this rescaling means a constant shift of the field,

$$\Phi^{l,\bar{l}} \to \Phi^{l,\bar{l}} - \frac{1}{4}\log\mu.$$

Then, taking the limit $\mu \to 0$ in Eq. (8.46) (keeping the shifts in $\bar{l}$ untouched!), one arrives at the DLE

$$S_L^{l,\bar{l}} S_L^{l+1,\bar{l}+1} = \left(\nu^{-1} - S_L^{l,\bar{l}+1}\right)\left(\nu^{-1} - S_L^{l+1,\bar{l}}\right). \tag{8.53}$$

Here,

$$S_L^{l,\bar{l}} = \lim_{\mu\to 0}\left(\mu^{-1} S^{l,\bar{l}}\right). \tag{8.54}$$

Setting

$$S_L^{l,\bar{l}} = \exp\left(-2\Phi_L^{l+1,\bar{l}} - 2\Phi_L^{l,\bar{l}+1}\right), \tag{8.55}$$

we obtain the DLE written in terms of the discrete Liouville field [6] (cf. (8.42)),

$$2\nu\sinh\left(\Phi_L^{l,\bar{l}} + \Phi_L^{l+1,\bar{l}+1} - \Phi_L^{l,\bar{l}+1} - \Phi_L^{l+1,\bar{l}}\right) = \exp\left(\Phi_L^{l,\bar{l}} + \Phi_L^{l+1,\bar{l}+1} + \Phi_L^{l,\bar{l}+1} + \Phi_L^{l+1,\bar{l}}\right), \tag{8.56}$$

or, in a simpler form,

$$\exp\left(-2\Phi_L^{l+1,\bar{l}} - 2\Phi_L^{l,\bar{l}+1}\right) - \exp\left(-2\Phi_L^{l,\bar{l}} - 2\Phi_L^{l+1,\bar{l}+1}\right) = \nu^{-1}. \tag{8.57}$$

In the continuum limit, one should set $l \to \nu^{1/2}x_+$, $\bar{l} \to \nu^{1/2}x_-$, and $S_L^{l,\bar{l}} \to \exp\left(-4\Phi(x_+,x_-)\right)$. Then, expanding in $\nu^{-1} \to 0$, we obtain, in the leading order, the continuous Liouville equation

$$2\partial_{x_+}\partial_{x_-}\Phi(x_+,x_-) = e^{4\Phi(x_+,x_-)}. \tag{8.58}$$

The bilinear form of Eq. (8.53) is obtained by the substitution

$$S_L^{l,\bar{l}} = \frac{T^1(l+1,\bar{l})T^1(l,\bar{l}+1)}{T^0(l+1,\bar{l})T^2(l,\bar{l}+1)}, \tag{8.59}$$

after which the DLE becomes equivalent (up to a "gauge freedom," see below) to the bilinear relation

$$T^a(l+1,\bar{l})T^a(l,\bar{l}+1) - T^a(l,\bar{l})T^a(l+1,\bar{l}+1) = \nu^{-1}T^{a-1}(l+1,\bar{l})T^{a+1}(l,\bar{l}+1) \tag{8.60}$$

with the condition

$$T^a(l,\bar{l}) = 0 \tag{8.61}$$

for all $a$, except $a = 0,1,2$. This condition implies the discrete d'Alembert equation (8.1) for $T^0$ and $T^2$. Thus, $T^0$ and $T^2$ need to have a factorized form $T^0(l,\bar{l}) = \chi^0(l)\overline{\chi}^0(\bar{l})$, $T^2(l,\bar{l}) = \chi^2(l)\overline{\chi}^2(\bar{l})$ with arbitrary (and independent) functions $\chi^{0,2}$, $\overline{\chi}^{0,2}$. This is the aforementioned gauge freedom.

The striking similarity between Eqs. (8.43) and (8.56) is clear after the replacing $T^a(l,\bar{l}) \to \tau_a^{l,\bar{l}}$. Furthermore, taking into account the periodicity $\tau_2^{l,\bar{l}} = \tau_0^{l,\bar{l}}$, these equations become formally identical. (Equivalently, using the gauge freedom, one can set $T^2(l,\bar{l}) = T^0(l,\bar{l})$ in Eq. (8.56).) It would be interesting to link them directly on the level of solutions, i.e., to trace what happens to solutions when taking the continuum limit.

The author is grateful to A. Gorsky, D. Lebedev, O. Lipan, A. Marshakov, A. Mironov, A. Orlov, P. Wiegmann and, especially, to S. Kharchev and I. Krichever for the many interesting and helpful discussions. It is a pleasure to thank CMAT de l'Ecole Polytechnique, where this work was initiated, for the hospitality.

This work was supported in part by RFBR Grant No. 97-02-19085.

# Appendix
# Bilinear difference equations from continuous hierarchies

In this Appendix, we give an alternative point of view to the difference Hirota equations. It relies on the famous Miwa transformation (3.3), which, so far, was obscure in our presentation. Given a continuous integrable hierarchy (such as KP or 2DTL), this relation can be used as a *definition* of the elementary discrete flows. This definition leads to the same discrete flows as in Sec. 4.2. This approach has as many advantages as disadvantages. The main advantage is a much more direct and instructive connection with the Grassmannian approach to continuous hierarchies and their $\tau$-functions. The main disadvantage is the misleading and less invariant formulation, which is inconvenient in some cases.

**The Miwa transformation.** Let $\tau(t_1, t_2, t_3, \dots) \equiv \tau(t)$ be the $\tau$-function of the continuous KP hierarchy. It is a function of an infinite number of "times" $t_i$ and it satisfies infinitely many bilinear equations. The $\tau$-function solves all of the equations of the hierarchy simultaneously.

In general, the $\tau$-function can be represented as an infinite-dimensional determinant [15–17]. It turns out that there exists a choice of independent variables such that the determinant reduces to a finite-dimensional one. This choice is provided by the Miwa transformation [11],

$$t_k = t_k^{(0)} - \frac{1}{k} \sum_{\alpha \in I} p_\alpha \mu_\alpha^{-k}, \qquad k = 1, 2, \dots . \tag{A.1}$$

Here, the summation runs over a finite set $I$, while $t_k^{(0)}$ are "background values" of the times, $\mu_\alpha$ are arbitrary complex numbers (called Miwa's variables), and $p_\alpha$ are integers (sometimes called multiplicities of $\mu_\alpha$).

**Remark.** The Miwa transformation plays an important role in revealing the integrable structures of matrix models of 2D gravity. In particular, the easiest proof of the fact that the partition functions of the Kontsevich model [40] and its generalizations [41] are $\tau$-functions of the KP hierarchy relies on Miwa's transformation.

In what follows, we use the concise notation of (3.5).

**Important fact.** The $\tau$-function of the KP hierarchy obeys the identity

$$\tau\left(t^{(0)} + \sum_{\alpha=1}^{N} \left([\nu_\alpha^{-1}] - [\mu_\alpha^{-1}]\right)\right) = \frac{\tau(t^{(0)}) \prod_{\alpha,\beta}^{N} (\nu_\alpha - \mu_\beta)}{\prod_{\alpha>\beta}^{N} (\nu_\alpha - \nu_\beta) \prod_{\alpha<\beta}^{N} (\mu_\alpha - \mu_\beta)} \det_{1 \le \alpha,\beta \le N} K(\nu_\alpha, \mu_\beta), \tag{A.2}$$

where

$$K(\nu, \mu) = \frac{\tau\left(t^{(0)} + [\nu^{-1}] - [\mu^{-1}]\right)}{(\nu - \mu)\tau(t^{(0)})}. \tag{A.3}$$

Here $N \ge 1$ and $\mu_\alpha, \nu_\alpha$ are arbitrary complex numbers. A useful particular case of this formula is

$$\tau\left(t^{(0)} - \sum_{\alpha=1}^{N} [\mu_\alpha^{-1}]\right) = \frac{\det_{1 \le \alpha,\beta \le N} \left(\varphi_\alpha(\mu_\beta)\right)}{\prod_{\alpha<\beta}^{N} (\mu_\alpha - \mu_\beta)}, \tag{A.4}$$

where

$$\varphi_m(\mu) = \frac{1}{(m-1)!} \lim_{\nu \to \infty} \nu^{2m-1} \frac{\partial^{m-1}}{\partial \nu^{m-1}} K(\nu, \mu). \tag{A.5}$$

When one translates the KP theory into the language of free fermions [15], formula (A.2) becomes nothing more than the Wick theorem, while $K(\nu, \mu)$ becomes the fermionic propagator on a Riemannian surface.

Instead of treating Eq. (A.2) as an identity, one can go another way. Given a function $K(\nu, \mu)$ with a simple pole at $\nu = \mu$, this equation can be used as a *definition* of the left-hand side. This simply means that we disregard the dependence on background times $t_k^{(0)}$, assuming they are fixed. The $\tau$-function in the Miwa variables satisfies certain bilinear relations for which formula (A.2) gives a solution in the form of a finite-dimensional determinant.

In the case of the 2DTL hierarchy, the Miwa transformation works in a similar way. The $\tau$-function $\tau_n(t_1, t_2, \ldots; \bar{t}_1, \bar{t}_2, \ldots) \equiv \tau_n(t; \bar{t})$ depends on the discrete time $n$ and two infinite sets of continuous times $t_i$ and $\bar{t}_i$. We set

$$\begin{aligned} t_k &= t_k^{(0)} - \frac{1}{k}\sum_{\alpha\in I} p_\alpha \mu_\alpha^{-k}, \\ \bar{t}_k &= \bar{t}_k^{(0)} - \frac{1}{k}\sum_{\alpha\in \bar{I}} \bar{p}_\alpha \bar{\mu}_\alpha^{k}, \qquad k = 1, 2, \ldots, \end{aligned} \tag{A.6}$$

where $\bar{\mu}_\alpha$ is an independent set of Miwa variables with multiplicities $\bar{p}_\alpha$.

The following analogue of Eq. (A.2) holds:

$$\begin{aligned} &\tau_{n-N}\left(t^{(0)} - \sum_{\alpha=1}^{N}[\mu_\alpha^{-1}]; \bar{t}^{(0)} + \sum_{\alpha=1}^{N}[\bar{\mu}_\alpha]\right) = \\ &\qquad = \frac{\tau_n(t^{(0)}; \bar{t}^{(0)}) \prod_{\alpha=1}^{N} \mu_\alpha^{N-1}}{\prod_{\alpha<\beta}^{N}(\mu_\alpha - \mu_\beta)(\bar{\mu}_\beta - \bar{\mu}_\alpha)} \det_{1\le\alpha,\beta\le N} J_n(\mu_\alpha, \bar{\mu}_\beta), \end{aligned} \tag{A.7}$$

where

$$J_n(\mu, \bar{\mu}) = \frac{\tau_{n-1}\big(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)} + [\bar{\mu}]\big)}{\tau_n(t^{(0)}; \bar{t}^{(0)})}. \tag{A.8}$$

Note that in this case, the function $J_n(\mu, \bar{\mu})$ does not necessarily have a first-order pole at $\mu = \bar{\mu}$.

**Discrete flows.** Discrete equations for the $\tau$-function listed in Sec. 2 are obtained if one fixes the Miwa variables $\mu_\alpha$ and considers the dependence on their multiplicities $p_\alpha$. We give a few examples.

**Example 1.** Set

$$\tau^{p_1, p_2, p_3} = \tau\left(t^{(0)} - \sum_{\alpha=1}^{3} p_\alpha[\mu_\alpha^{-1}]\right) \tag{A.9}$$

and consider

$$\tilde{t}^{(0)} = t^{(0)} - \sum_{\alpha=1}^{3} p_\alpha[\mu_\alpha^{-1}]$$

as a new "background" field. According to Eq. (A.4), we have

$$\begin{aligned} \tau^{p_1+1} &= \tilde{\varphi}_1(\mu_1), \\ \tau^{p_1+1, p_2+1} &= \frac{\begin{vmatrix} \tilde{\varphi}_1(\mu_1) & \tilde{\varphi}_1(\mu_2) \\ \tilde{\varphi}_2(\mu_1) & \tilde{\varphi}_2(\mu_2) \end{vmatrix}}{\mu_1 - \mu_2} \end{aligned} \tag{A.10}$$

with some functions $\tilde{\varphi}_1, \tilde{\varphi}_2$. Combining the zero determinant with two identical lines,

$$0 = \begin{vmatrix} \tilde{\varphi}_1(\mu_1) & \tilde{\varphi}_1(\mu_2) & \tilde{\varphi}_1(\mu_3) \\ \tilde{\varphi}_1(\mu_1) & \tilde{\varphi}_1(\mu_2) & \tilde{\varphi}_1(\mu_3) \\ \tilde{\varphi}_2(\mu_1) & \tilde{\varphi}_2(\mu_2) & \tilde{\varphi}_2(\mu_3) \end{vmatrix}, \tag{A.11}$$

and expanding it in the first line using (A.10), we obtain an equation of form (2.7). Since its coefficients do not depend on the chosen background, the equation holds for all values of $p_1$, $p_2$, and $p_3$.

**Example 2.** Repeating the previous argument for

$$\tau_{p_0}^{p_1,p_2,p_3} = \tau\left(t^{(0)} + p_0\big([\mu_0^{-1}] - [\nu^{-1}]\big) + \sum_{\alpha=1}^{3} p_\alpha\big([\nu^{-1}] - [\mu_\alpha^{-1}]\big)\right)$$

and making use of Eq. (A.2), we obtain Eq. (2.10).

**Example 3.** In (A.7), let us set $N = 2$, $\mu_1 = \mu$, $\bar{\mu}_1 = \bar{\mu}$, $\mu_2 \to \infty$, and $\bar{\mu}_2 \to 0$,

$$\tau_{n-2}\big(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)} + [\bar{\mu}]\big)\tau_n\big(t^{(0)}; \bar{t}^{(0)}\big) =$$
$$= \frac{\mu}{\bar{\mu}} \begin{vmatrix} \tau_{n-1}\big(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)} + [\bar{\mu}]\big) & \tau_{n-1}\big(t^{(0)} - [\mu^{-1}]; \bar{t}^{(0)}\big) \\ \tau_{n-1}\big(t^{(0)}; \bar{t}^{(0)} + [\bar{\mu}]\big) & \tau_{n-1}\big(t^{(0)}; \bar{t}^{(0)}\big) \end{vmatrix}. \tag{A.12}$$

Denoting

$$\tau_n^{l,\bar{l}} = \tau_n\big(t^{(0)} - l[\mu^{-1}]; \bar{t}^{(0)} - \bar{l}[\bar{\mu}]\big), \tag{A.13}$$

we obtain the equation

$$\tau_n^{l,\bar{l}+1}\tau_n^{l+1,\bar{l}} - \tau_n^{l,\bar{l}}\tau_n^{l+1,\bar{l}+1} = (\bar{\mu}/\mu)\tau_{n+1}^{l,\bar{l}+1}\tau_{n-1}^{l+1,\bar{l}}. \tag{A.14}$$

**Example 4.** Example 1 can be generalized in the following way. Consider an $N \times N$ matrix with the lines $\varphi_1(\mu_i), \varphi_1(\mu_i), \varphi_2(\mu_i), \varphi_3(\mu_i), \ldots, \varphi_{N-1}(\mu_i)$, $i = 1, 2, \ldots, N$, such that the first two lines coincide and the determinant of this matrix is zero. Then, expanding in the first row, as in Example 1, we obtain the "higher" bilinear difference equation of form (7.1).

**Example 5.** At last, we show how to derive the HBDE in a KP-like form from Eq. (A.4) in a direct way.[10] When two or more variables $\mu_\alpha$ coincide, both the numerator and denominator on the r.h.s. of Eq. (A.4) equal zero. Resolving the indeterminacy, we have

$$\tau\left(t^{(0)} - \sum_{\alpha=1}^{N} p_\alpha[\mu_\alpha^{-1}]\right) = \frac{\det(M_{ij}^{(\mathcal{N})})}{\prod_{\alpha<\beta}^{N}(\mu_\alpha - \mu_\beta)^{p_\alpha p_\beta}}, \qquad 1 \le i, j \le \mathcal{N}, \tag{A.15}$$

where all $\mu_\alpha$ are now distinct. Here

$$\mathcal{N} \equiv \sum_{\alpha=1}^{N} p_\alpha$$

and $M_{ij}^{(\mathcal{N})}$ is the $\mathcal{N} \times \mathcal{N}$ matrix having the rows

$$\begin{array}{lllll} \varphi_i(\mu_1), & \varphi_i'(\mu_1), & \varphi_i''(\mu_1), & \ldots, & \varphi_i^{(p_1-1)}(\mu_1), \\ \varphi_i(\mu_2), & \varphi_i'(\mu_2), & \varphi_i''(\mu_2), & \ldots, & \varphi_i^{(p_2-1)}(\mu_2), \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ \varphi_i(\mu_N), & \varphi_i'(\mu_N), & \varphi_i''(\mu_N), & \ldots, & \varphi_i^{(p_N-1)}(\mu_N), \qquad 1 \le i \le \mathcal{N}. \end{array} \tag{A.16}$$

We need the well-known Jacobi identity for determinants,

$$D[i_1 \mid j_1]D[i_2 \mid j_2] - D[i_1 \mid j_2]D[i_2 \mid j_1] = D[i_1, i_2 \mid j_1, j_2]D, \qquad i_1 < i_2, j_1 < j_2. \tag{A.17}$$

---

[10] The formulas below were taken from [41].

Here $D$ is the determinant of a square matrix and $D[i_1, i_2 | j_1, j_2]$ denotes the minor of this matrix with the $i_{1,2}$th rows and $j_{1,2}$th columns removed. Applying this identity to the matrix $M_{ij}^{(\mathcal{N})}$ in (A.15) for

$$i_1 = \mathcal{N} - 1, \qquad i_2 = \mathcal{N}, \qquad j_1 = \sum_{\alpha=1}^{a} p_\alpha, \qquad j_2 = j_1 + \sum_{\alpha=a+1}^{b} p_\alpha, \qquad 1 < a < b \le N,$$

we obtain, in short hand notation,

$$(\mu_a - \mu_b)\tau\tau^{p_a-1,p_b-1} = \tau^{p_b-1}\hat{\tau}^{p_a-1} - \tau^{p_a-1}\hat{\tau}^{p_b-1}, \tag{A.18}$$

where $\hat{\tau}$ is defined by the same formula (A.15) with the matrix $\widehat{M}_{ij}^{(\mathcal{N}-1)} = M_{ij}^{(\mathcal{N}-1)}$ for $1 \le i \le \mathcal{N} - 2$, $\widehat{M}_{\mathcal{N}-1,j}^{(\mathcal{N}-1)} = M_{\mathcal{N},j}^{(\mathcal{N})}$.

Let $\mu_c$ be a third Miwa variable (different from $\mu_a$, $\mu_b$) with the multiplicity $p_c$ not shown explicitly in Eq. (A.18). Multiplying this equality by $\tau^{p_c-1}/\tau$ and then writing a couple of similar equations obtained by cyclic permutations of the indices $a, b, c$, we can see that the sum of these three equations coincides with Eq. (2.7).

**Remark.** The discrete flows discussed here coincide with those introduced in the main body of the paper if one fixes the following choice of the labels $\lambda_0$ and $\lambda_1$: $\lambda_0 = \infty$, $\lambda_1 = 0$. (To remove a label to infinity, one should use a different normalization.)

**Continuum limit.** As is clear from Eq. (A.1), the inverse Miwa variables $\mu_\alpha^{-1}$ play the role of lattice spacings for the discrete flows. Therefore, to perform the limit to continuous equations, it is necessary for $\mu_\alpha$ to tend to infinity with a simultaneous rescaling of $p_\alpha$.

Here is a typical example (the KP hierarchy). (In this example, we follow [30].) Introduce three (a priori independent) lattice spacings $\varepsilon_i = \mu_i^{-1}$, $i = 1, 2, 3$, and rescale $p_i \to p_i/\varepsilon_i$. Thus, it is convenient to rewrite the KP-like form (2.7) of the HBDE in terms of Hirota's $D$-operator (1.2):

$$\begin{aligned}\Big(&\varepsilon_1(\varepsilon_2 - \varepsilon_3)e^{-(\varepsilon_1/2)D_{p_1} + (\varepsilon_2/2)D_{p_2} + (\varepsilon_3/2)D_{p_3}} + \\ &+ \varepsilon_2(\varepsilon_3 - \varepsilon_1)e^{(\varepsilon_1/2)D_{p_1} - (\varepsilon_2/2)D_{p_2} + (\varepsilon_3/2)D_{p_3}} + \\ &+ \varepsilon_3(\varepsilon_1 - \varepsilon_2)e^{(\varepsilon_1/2)D_{p_1} + (\varepsilon_2/2)D_{p_2} - (\varepsilon_3/2)D_{p_3}}\Big)\tau \cdot \tau = 0.\end{aligned} \tag{A.19}$$

This equation serves as a "generating function" for part of the continuous KP hierarchy. To see this, we express the operators $D_{p_i}$ through the Hirota derivatives with respect to the continuous flows $t_k$,

$$D_{p_i} = -\sum_{k=1}^{\infty} \frac{1}{k}\varepsilon_i^{k-1} D_{t_k}, \qquad i = 1, 2, 3$$

(see (A.1)). Substituting this into Eq. (A.19) and expanding in a power series in $\varepsilon_i$, we have

$$\begin{aligned}\Bigg(&\varepsilon_1(\varepsilon_2 - \varepsilon_3) \sum_{j,k,l=0}^{\infty} \varepsilon_1^j \varepsilon_2^k \varepsilon_3^l \mathcal{P}_j\left(\frac{1}{2}\tilde{D}\right)\mathcal{P}_k\left(-\frac{1}{2}\tilde{D}\right)\mathcal{P}_l\left(-\frac{1}{2}\tilde{D}\right) + \\ &+ \varepsilon_2(\varepsilon_3 - \varepsilon_1) \sum_{j,k,l=0}^{\infty} \varepsilon_1^j \varepsilon_2^k \varepsilon_3^l \mathcal{P}_j\left(-\frac{1}{2}\tilde{D}\right)\mathcal{P}_k\left(\frac{1}{2}\tilde{D}\right)\mathcal{P}_l\left(-\frac{1}{2}\tilde{D}\right) + \\ &+ \varepsilon_3(\varepsilon_1 - \varepsilon_2) \sum_{j,k,l=0}^{\infty} \varepsilon_1^j \varepsilon_2^k \varepsilon_3^l \mathcal{P}_j\left(-\frac{1}{2}\tilde{D}\right)\mathcal{P}_k\left(-\frac{1}{2}\tilde{D}\right)\mathcal{P}_l\left(\frac{1}{2}\tilde{D}\right)\Bigg)\tau \cdot \tau = 0,\end{aligned} \tag{A.20}$$

where $\widetilde{D} \equiv (D_{t_1}, D_{t_2}/2, \ldots, D_{t_k}/k, \ldots)$ and $\mathcal{P}_j(t)$ are the Schur polynomials defined by the formula

$$\exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \sum_{m=0}^{\infty} \mathcal{P}_m(t) z^m. \tag{A.21}$$

Comparing the coefficients in front of $\varepsilon_1^j \varepsilon_2^k \varepsilon_3^l$, we obtain an infinite set of bilinear equations,

$$\begin{vmatrix} \mathcal{P}_{j-1}\left(\frac{1}{2}\widetilde{D}\right) & \mathcal{P}_{j-1}\left(-\frac{1}{2}\widetilde{D}\right) & \mathcal{P}_j\left(-\frac{1}{2}\widetilde{D}\right) \\ \mathcal{P}_{k-1}\left(\frac{1}{2}\widetilde{D}\right) & \mathcal{P}_{k-1}\left(-\frac{1}{2}\widetilde{D}\right) & \mathcal{P}_k\left(-\frac{1}{2}\widetilde{D}\right) \\ \mathcal{P}_{l-1}\left(\frac{1}{2}\widetilde{D}\right) & \mathcal{P}_{l-1}\left(-\frac{1}{2}\widetilde{D}\right) & \mathcal{P}_l\left(-\frac{1}{2}\widetilde{D}\right) \end{vmatrix} \tau \cdot \tau = 0, \tag{A.22}$$

which, for $1 \le j < k < l$, form a subset of the entire KP hierarchy in bilinear form.

The leading term as $\varepsilon_i \to 0$ in (A.20) corresponds to $(j, k, l) = (1, 2, 3)$ in (A.22). In this case, Eq. (A.22) produces the bilinear form of the KP equation itself,

$$(D_{t_1}^4 - 4D_{t_1}D_{t_3} + 3D_{t_2}^2)\tau \cdot \tau = 0. \tag{A.23}$$

This example shows, once again, that the discrete hierarchy has a more transparent structure than the continuous one. The continuum limit brings artificial complications.

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