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RANDOM WALKS IN AN INHOMOGENEOUS ONE-DIMENSIONAL MEDIUM WITH REFLECTING AND ABSORBING BARRIERS

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A particle moving in inhomogeneous, one-dimensional media is considered. Its velocity changes direction at Poisson times. For such a random process, the backward and forward Kolmogorov equations are derived. The explicit formulas for the probability distributions of this process are obtained, as well as the formulas for similar processes in the presence of reflecting and absorbing barriers.

1. Introduction

Let $N(t)$, $t \geq 0$, be a homogeneous Poisson process with the rate $\mu > 0$ and ξ be a random variable independent of $N(t)$ with the values ± 1 taken with the probability 1/2. Let $c(x)$, $x \in (-\infty, \infty)$, be some positive continuous function. In the present paper, we consider the processes satisfying the equation

$$X(x,t) = x + \xi \int_0^t (-1)^{N(s)} c\big(X(x,s)\big)\, ds, \qquad x \in (-\infty,\infty), \qquad t > 0. \tag{1.1}$$

We also consider the analogous processes in the presence of reflecting and absorbing barriers. Process (1.1) describes the random walk of a particle in a one-dimensional, inhomogeneous media. The velocity of the particle performs instant changes of direction at Poisson times. Such models arise naturally in some physical problems, especially in diffusion problems where the finiteness of the propagation velocity is essential. There are many mathematical and physical articles written on processes of form (1.1) (see, e.g., [1–2], [3–5], and references therein). However, all known results in this field are concerned with random walks in homogeneous media, where $c(x) \equiv \text{const}$. Generalization of Orsingher's results [2] to the inhomogeneous case ($c(x) \not\equiv \text{const}$) is one of the aims of the present paper. Note that many known results look more natural for process (1.1) (in comparison with the homogeneous case).

The main results of this paper consist in the explicit calculation of the distribution of process (1.1), both in the case of "free" motion and in the presence of reflecting and absorbing barriers. Such a calculation is based on the fact that the distribution of process (1.1) is a solution of the telegrapher equation,

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2\mu \frac{\partial u(x,t)}{\partial t} = c(x)\frac{\partial}{\partial x}c(x)\frac{\partial}{\partial x}u(x,t), \qquad x \in (-\infty,\infty), \qquad t > 0. \tag{1.2}$$

This assertion was first formulated by Goldstein [6] and Kac [7]. It was proved later (see, e.g., [8, 9]) for the case $c(x) \equiv \text{const}$. The proofs in [8] and [9] were based on absolutely different ideas, but they both led to the same equation.

As follows from the results of Secs. 2 and 3 below, Orsingher's approach [8] for a random walk in inhomogeneous media (1.1) results in an equation that is identical to the forward Kolmogorov equation, while Kabanov's idea [9] leads to the backward Kolmogorov equation. The modifications that arise due to the presence of barriers are discussed in Sec. 4. Explicit formulas for the solutions of Eq. (1.2) are contained in Sec. 5.

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2. Deriving the backward Kolmogorov equation

In this section, we briefly repeat the arguments of [9] for the case of a variable motion velocity $c(x)$. Let $c \in C^1(-\infty,\infty)$, $c(x) > 0$, $\forall x \in (-\infty,\infty)$. Denote by $X^\pm$ the solutions of the following equations:

$$X^+(x,t) = x + \int_0^t (-1)^{N(s)} c\big(X^+(x,s)\big)\, ds \tag{2.1}$$

and

$$X^-(x,t) = x - \int_0^t (-1)^{N(s)} c\big(X^-(x,s)\big)\, ds, \tag{2.2}$$

where X^+ and X^- are the random walks of particles with initial velocities $c(x)$ and $-c(x)$, respectively.

Theorem 2.1. *Let $\varphi = \varphi(x)$, $x \in (-\infty,\infty)$ be a smooth function. Then, the functions*

$$u^\pm = u^\pm(x,t) = E\varphi(X^\pm(x,t)), \qquad x \in (-\infty,\infty), \qquad t > 0,$$

satisfy the telegrapher equation

$$\frac{\partial^2 u^\pm(x,t)}{\partial t^2} + 2\mu \frac{\partial u^\pm(x,t)}{\partial t} = c(x)\frac{\partial}{\partial x} c(x) \frac{\partial}{\partial x} u^\pm(x,t), \qquad x \in (-\infty,\infty), \qquad t > 0, \tag{2.3}$$

with the initial conditions

$$u^\pm\big|_{t=0} = \varphi(x), \qquad u_t^\pm\big|_{t=0} = \pm c(x)\varphi'(x). \tag{2.4}$$

Here and below, $E(\cdot)$ denotes the expectation value.

First, we prove the following lemma.

Lemma 2.1. *Let $s(t) = (-1)^{N(t)}$, where $N(t)$ is a Poisson process with the rate μ. Let $X(t)$ be some process predictable with respect to $N(t)$ and f be a measurable function (i.e., $f(X(t))$ is a random process). Then*

$$Ef\big(X(t)\big)\, ds(t) = -2\mu Ef\big(X(t)\big)s(t)\, dt. \tag{2.5}$$

Proof of Lemma 2.1 (cf. [9]). Consider $m(t) = s(t)e^{2\mu t}$. Using this notation, it is easy to see that the process $s(t)$ satisfies the equation $ds(t) = -2\mu m(t)e^{-2\mu t}dt + e^{-2\mu t}dm(t)$. Therefore,

$$Ef\big(X(t)\big)\, ds(t) = -2\mu Ef\big(X(t)\big)m(t)e^{-2\mu t}\, dt + e^{-2\mu t} Ef\big(X(t)\big)\, dm(t).$$

The last term vanishes because $m(t)$ is a martingale. Lemma 2.1 is proved.

Proof of Theorem 2.1. This proof consists of a direct calculation using Lemma 2.1. It follows from (2.1) and (2.2) that

$$\frac{\partial X^\pm}{\partial t}(x,t) = \pm(-1)^{N(t)} c\big(X^\pm(x,t)\big) \tag{2.6}$$

and

$$\int_x^{X^\pm} \frac{dy}{c(y)} = \pm \int_0^t (-1)^{N(s)}\, ds. \tag{2.7}$$

Differentiating (2.7) w.r.t. x, we obtain

$$\frac{\partial}{\partial x} X^\pm(x,t) = \frac{c\big(X^\pm(x,t)\big)}{c(x)}. \tag{2.8}$$

Accounting for (2.6), we calculate the derivatives of $u^{\pm}$ w.r.t. t,

$$\frac{\partial u^{\pm}}{\partial t} = \pm E\varphi'(X^{\pm})c(X^{\pm})s(t),$$

whence, formally (see the rigorous proof in [9]),

$$\frac{\partial^2 u^{\pm}}{\partial t^2} = E\varphi''(X^{\pm})c(X^{\pm})^2 + E\varphi'(X^{\pm})c'(X^{\pm})c(X^{\pm}) \pm E\varphi'(X^{\pm})c(X^{\pm})\dot{s}(t).$$

By Lemma 2.1, the last term is equal to

$$\mp 2\mu E\varphi'(X^{\pm})c(X^{\pm})s(t) = -2\mu\frac{\partial u^{\pm}}{\partial t}.$$

To find the derivative w.r.t. x, we use (2.8),

$$\begin{aligned} c(x)\frac{\partial u^{\pm}}{\partial x} &= E\varphi'(X^{\pm})c(X^{\pm}), \\ c(x)\frac{\partial}{\partial x}c(x)\frac{\partial u^{\pm}}{\partial x} &= E\big(\varphi''(X^{\pm})c(X^{\pm})^2 + \varphi'(X^{\pm})c'(X^{\pm})c(X^{\pm})\big). \end{aligned}$$

From here, it is evident that $u^{\pm}(x,t)$ satisfies (2.3). One can easily check initial conditions (2.4).

The following representation of solution to Eq. (2.3) follows from Theorem 2.1.

Theorem 2.2 (cf. [9]). *The solution of the Cauchy problem for Eq.* (2.3) *with the initial conditions*

$$u\big|_{t=0} = \varphi(x), \qquad u_t\big|_{t=0} = 0 \tag{2.9}$$

has the form

$$u(x,t) = \frac{1}{2}E\big[\varphi\big(X^{+}(x,t)\big) + \varphi\big(X^{-}(x,t)\big)\big]. \tag{2.10}$$

Remark 2.1. It is natural to call Eq. (2.3) the *backward Kolmogorov equation* for the process $X(x,t)$. The formulas resolving the problems (2.3), (2.4) and (2.3), (2.9) are contained in Sec. 5. Note that these formulas are well known in the case of constant coefficients (see, e.g., [10] or [2]).

Define the distribution densities $p(x,y,t)$ and $p^{\pm}(x,y,t)$ for the processes $X(x,t)$ and $X^{\pm}(x,t)$ by the relations

$$E\varphi\big(X(x,t)\big) = \langle p(x,\cdot,t),\varphi\rangle, \tag{2.11}$$

$$E\varphi\big(X^{\pm}(x,t)\big) = \langle p^{\pm}(x,\cdot,t),\varphi\rangle, \qquad x\in(-\infty,\infty), \quad t>0, \quad \varphi\in C_0^{\infty}(\mathbb{R}^1). \tag{2.12}$$

Here and below, $\langle\cdot,\cdot\rangle$ denotes the action of the distribution on a test function.

Corollary 2.1. *Functions $p(x,y,t)$ and $p^{\pm}(x,y,t)$ satisfy telegrapher equation* (2.3) *with the following initial conditions*:

$$p\big|_{t=0} = \delta(x-y), \qquad p_t\big|_{t=0} = 0, \tag{2.13}$$

$$p^{\pm}\big|_{t=0} = \delta(x-y), \qquad p_t^{\pm}\big|_{t=0} = \pm c(x)\delta'(x-y). \tag{2.14}$$

Here δ denotes the Dirac δ-function.

Remark 2.2. It is clear that the smoothness condition for the coefficients $c\in C^1(-\infty,\infty)$ may be replaced by a piecewise smoothness condition. Namely, it suffices to consider $c\in C(-\infty,\infty)$ and assume that the derivative $c'(x)$ exists everywhere except for a discrete set of points.

3. Deriving the forward Kolmogorov equation

Theorem 3.1. *The function $p(x, y, t)$ defined in* (2.11) *is the solution of the following Cauchy problem*:

$$\frac{\partial^2}{\partial t^2}p(x,y,t) + 2\mu\frac{\partial}{\partial t}p(x,y,t) = \frac{\partial}{\partial y}c(y)\frac{\partial}{\partial y}c(y)p(x,y,t), \tag{3.1}$$

$$p\big|_{t=0} = \delta(x-y), \qquad \frac{\partial p}{\partial t}\Big|_{t=0} = 0. \tag{3.2}$$

Proof. Following Orsingher (see, e.g., [1]), we introduce the following distribution functions:

$$F(x,y,t) = P\{X(x,t) < y, \quad V(x,t) > 0\}, \tag{3.3}$$

$$B(x,y,t) = P\{X(x,t) < y, \quad V(x,t) < 0\}. \tag{3.4}$$

(Here $V(x,t)$ denotes the current velocity, $V(x,t) = \xi(-1)^{N(t)}c(X(x,t))$.) Let $f(x,y,t)$ and $b(x,y,t)$ be the corresponding distribution densities.

Let $\lambda = \lambda(x,t)$ denote the solution of following equation:

$$\lambda(x,t) = x + \int_0^t c(\lambda(x,s))\,ds. \tag{3.5}$$

By definition of the process $X(x,t)$ (cf. [1]),

$$F(x,y,t+\Delta t) = (1-\mu\Delta t)F(x,\lambda(y,-\Delta t),t) + \mu\Delta t B(x,\lambda(y,\Delta t),t) + o(\Delta t), \quad \Delta t \to 0. \tag{3.6}$$

$$B(x,y,t+\Delta t) = (1-\mu\Delta t)B(x,\lambda(y,\Delta t),t) + \mu\Delta t F(x,\lambda(y,-\Delta t),t) + o(\Delta t), \quad \Delta t \to 0. \tag{3.7}$$

From (3.6), (3.7), we have the following differential equations:

$$\frac{\partial F}{\partial t} = -c(y)\frac{\partial F(x,y,t)}{\partial y} + \mu\big(B(x,y,t) - F(x,y,t)\big),$$

$$\frac{\partial B}{\partial t} = c(y)\frac{\partial B(x,y,t)}{\partial y} + \mu\big(F(x,y,t) - B(x,y,t)\big).$$

Differentiating these equations w.r.t. y, we obtain

$$\frac{\partial f}{\partial t} = -\frac{\partial c(y)f(x,y,t)}{\partial y} + \mu\big(b(x,y,t) - f(x,y,t)\big),$$

$$\frac{\partial b}{\partial t} = \frac{\partial c(y)b(x,y,t)}{\partial y} + \mu\big(f(x,y,t) - b(x,y,t)\big).$$

Note that $p(x,y,t) = f(x,y,t) + b(x,y,t)$. Denoting $w(x,y,t) = f(x,y,t) - b(x,y,t)$, we obtain

$$\frac{\partial p}{\partial t} = -\frac{\partial c(y)w(x,y,t)}{\partial y}, \qquad \frac{\partial w}{\partial t} = -\frac{\partial c(y)p(x,y,t)}{\partial y} - 2\mu w(x,y,t),$$

which leads to (3.1). Initial conditions (3.2) coincide with (2.13).

Remark 3.1. We call Eq. (3.1) the forward Kolmogorov equation (or the Fokker–Planck equation). Note that the operator on the r.h.s. of (3.1) is formally conjugate to the operator in (2.3).

Remark 3.2. The following obvious relations between f, b, and $p^{\pm}$ are valid (see (2.12)):

$$p^+(x,y,t) = 2f(y,x,t), \qquad p^-(x,y,t) = 2b(y,x,t), \tag{3.8}$$

$$p^+(x,y,t) = p^-(y,x,t), \qquad p(x,y,t) = p(y,x,t). \tag{3.9}$$

4. Telegraph processes with barriers

4.1. Reflecting barriers. Assume that the particle walking in accordance with law (1.1) changes the direction of its velocity at the point $y = a > x$. In other words, consider the process

$$X^{\text{ref}}(x,t) = \begin{cases} X(x,t), & t \le \tau = \tau^1, \\ a - (-1)^{N(\tau^k)} \int_{\tau^k}^t (-1)^{N(s)} c\big(X^{\text{ref}}(x,s)\big)\, ds, & \tau^k < t \le \tau^{k+1}, \end{cases} \qquad k = 1, 2, \dots .$$

Here $\tau^k = \tau^k(x,a)$, $k = 1, 2, \dots$, denotes the time of the kth reflection and

$$\tau = \tau^1 = \inf\{t > 0 : X^{\text{ref}}(x,t) = a\} \tag{4.1}$$

is the first moment when the particle reaches the point $y = a$,

$$\tau^k = \inf\{t > \tau^{k-1} : X^{\text{ref}}(x,t) = a\}, \qquad k \ge 2. \tag{4.2}$$

Note that after each collision, the particle velocity $c(a)$ becomes negative regardless of the sign of the initial velocity. Furthermore, the process starts afresh after each impact with the wall $y = a$. Indeed, at $t \in (\tau^k, \tau^{k+1}]$, we have

$$\begin{aligned} X^{\text{ref}}(x,t) &= a - (-1)^{N(\tau^k)} \int_{\tau^k}^t (-1)^{N(s)} c\big(X^{\text{ref}}(x,s)\big)\, ds = \\ &= a - \int_0^{t-\tau^k} (-1)^{N(s+\tau^k) - N(\tau^k)} c\big(X^{\text{ref}}(x, s+\tau^k)\big)\, ds = \\ &= a - \int_0^{t-\tau^k} (-1)^{N'(s)} c\big(X^{\text{ref}}(x, s+\tau^k)\big)\, ds. \end{aligned} \tag{4.3}$$

Here $N'(s) = N(s+\tau^k) - N(\tau^k)$ is the number of Poisson events that have occurred since the moment τ^k and has the same distribution as $N(s)$. In this section, we assume that $c = c(x)$, $x < a$, is a function from the class C^1. Let $\varphi = \varphi(x)$, $x < a$, be some smooth function with $\varphi'|_{x=a-} = 0$. Consider the function $u^{\text{ref}}(x,t) = E\varphi\big(X^{\text{ref}}(x,t)\big)$.

Theorem 4.1. *The function u^{ref} is the solution to the mixed boundary-value problem*

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2\mu \frac{\partial u(x,t)}{\partial t} = c(x) \frac{\partial}{\partial x} c(x) \frac{\partial}{\partial x} u(x,t), \qquad x < a, \qquad t > 0, \tag{4.4}$$

$$u\big|_{t=0} = \varphi(x), \qquad u_t\big|_{t=0} = 0, \qquad x < a, \tag{4.5}$$

$$u_x\big|_{x=a-0} = 0, \qquad t > 0. \tag{4.6}$$

Proof. Let $\bar{c} = \bar{c}(x)$, $x \in (-\infty, \infty)$, be the symmetric (under reflection with respect to a) extension of the function $c(x)$ initially defined for $x < a$. Denote $\overline{X}(x,t)$, $x \in (-\infty, \infty)$, $t > 0$, as a process of form (1.1) with respect to the velocity field $\bar{c}(x)$.

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.1. *The distributions $\overline{X}(x,t)$ and $X^+(a, t - \tau^{2k-1})$ coincide for $\tau^{2k-1} \le t < \tau^{2k}$, while the distributions $\overline{X}(x,t)$ and $X^-(a, t - \tau^{2k})$, $k \in \mathbf{N}$, coincide for $\tau^{2k} \le t < \tau^{2k+1}$.*

Here the processes X^+ and X^- are defined by equalities (2.1), (2.2), and the moments τ^k are determined in (4.1), (4.2).

Proof of Lemma 4.1. We present the proof for the case $\tau = \tau^1 < t \le \tau^2$ (in the general case, the arguments are analogous). Observe that by the definition of τ^1 (4.1),

$$\overline{X}(x,t) = a + \xi \int_\tau^t (-1)^{N(s)} \bar{c}(\overline{X}(x,s))\, ds = a + \xi(-1)^{N(\tau)} \int_0^{t-\tau} (-1)^{N'(s)} \bar{c}(\overline{X}(x,s+\tau))\, ds.$$

To complete the proof, it suffices to observe that $\xi(-1)^{N(\tau)} = 1$ since, in the case $\xi = +1$, the number of Poisson events occurring up to the moment τ is even, whereas, for $\xi = -1$, this number is odd.

Let us turn to the proof of Theorem 4.1. Observe that

$$u^{\mathrm{ref}}(x,t) = E\varphi(X^{\mathrm{ref}}(x,t)) = E\Big[\varphi(X^{\mathrm{ref}}(x,t)) I_{\{t<\tau^1\}} + \varphi(X^{\mathrm{ref}}(x,t)) I_{\{t\ge\tau^1\}}\Big].$$

Denote by $\overline{\varphi} = \overline{\varphi}(x)$ the symmetric extension of the function φ to the right of the point $y = a$. For $t < \tau^1$, $X^{\mathrm{ref}}(x,t) = X(x,t)$, and by virtue of (4.3), $X^{\mathrm{ref}}(x,t) = X^-(a, t-\tau^k)$ for each of the intervals $\tau^k \le t < \tau^{k+1}$, $k \in \mathbf{N}$. Hence, by Lemma 4.1 and due to the symmetry of the extension of the function φ, we have

$$u^{\mathrm{ref}}(x,t) = E\big[\bar{\varphi}(\overline{X}(x,t)) I_{\{t<\tau^1\}} + \bar{\varphi}(\overline{X}(x,t)) I_{\{t\ge\tau^1\}}\big] = E\bar{\varphi}(\overline{X}(x,t)).$$

By Theorem 2.2, the r.h.s. of this equality, $E\overline{\varphi}(\overline{X}(x,t)) \equiv u(x,t)$, is the solution of the Cauchy problem (2.3), (2.9). Since $\bar{c}$ and $\overline{\varphi}$ are symmetric functions, problems (2.3), (2.9) and (4.4)–(4.6) are equivalent.

4.2. Absorbing barriers. Now we assume that the particle is absorbed at the point $y = a > x$. This means that the corresponding process denoted as $X^{\mathrm{abs}}(x,t)$, $x \in (-\infty, a]$, $t > 0$, coincides with $X(x,t)$ for $t < \tau^1$. If $t \ge \tau^1$, then $X^{\mathrm{abs}}(x,t) = a$. Denote $u^{\mathrm{abs}}(x,t) \equiv E\varphi(X^{\mathrm{abs}}(x,t))$. Also, consider the distribution functions $F^{\mathrm{abs}}(x,y,t)$ and $B^{\mathrm{abs}}(x,y,t)$, which are defined for the process X^{abs} similarly to (3.3), (3.4). Let f^{abs} and b^{abs} be the corresponding probability densities. It is obvious that $p^{\mathrm{abs}} = f^{\mathrm{abs}} + b^{\mathrm{abs}}$, where p^{abs} is the density of X^{abs}.

Proposition.
(1) *f^{abs} is the solution of the problem*

$$\frac{\partial^2}{\partial t^2} u + 2\mu \frac{\partial}{\partial t} u = \frac{\partial}{\partial y} c(y) \frac{\partial}{\partial y} c(y) u, \qquad y < a, \tag{4.7}$$

$$u\big|_{t=0} = \frac{1}{2}\delta(x-y), \qquad u_t\big|_{t=0} = -\frac{1}{2}\frac{\partial}{\partial y} c(y)\delta(x-y), \qquad x, y < a, \tag{4.8}$$

$$\left(\frac{\partial u}{\partial t} + c(y)\frac{\partial u}{\partial y} + \mu u\right)\bigg|_{y=a-0} = 0; \tag{4.9}$$

(2) *b^{abs} is the solution of the problem*

$$\frac{\partial^2}{\partial t^2} u + 2\mu \frac{\partial}{\partial t} u = \frac{\partial}{\partial y} c(y) \frac{\partial}{\partial y} c(y) u, \qquad y < a, \tag{4.7$'$}$$

$$u\big|_{t=0} = \frac{1}{2}\delta(x-y), \qquad u_t\big|_{t=0} = \frac{1}{2}\frac{\partial}{\partial y} c(y)\delta(x-y), \qquad x, y < a, \tag{4.8$'$}$$

$$u\big|_{y=a-0} = 0. \tag{4.9$'$}$$

Hence, the following theorem is valid.

Theorem 4.2. *Let $\varphi \in C(-\infty, a]$. Then*

$$u^{\mathrm{abs}} = \langle f^{\mathrm{abs}}(x,\cdot,t), \varphi\rangle + \langle b^{\mathrm{abs}}(x,\cdot,t), \varphi\rangle,$$

where f^{abs} and b^{abs} are the solutions of problems (4.7)–(4.9) and (4.7$'$)–(4.9$'$), respectively.

5. Solutions of the telegrapher equations

In this section, we write the formulas for the solutions to problems (2.3), (2.9), (4.4)–(4.6), (4.7)–(4.9), and (4.7′)–(4.9′). First, we construct the solution to the Cauchy problem (2.3), (2.9). To this end, we consider the following equation, together with telegrapher equation (2.3):

$$\frac{\partial^2 v}{\partial t^2} = c(x)\frac{\partial}{\partial x}c(x)\frac{\partial v}{\partial x} + \mu^2 v. \tag{5.1}$$

For $\mu^2 < 0$, this equation is known as the Klein–Gordon equation.

Consider the bi-characteristics $\lambda^{\pm}$ common to Eqs. (2.3), (3.1), and (5.1). Let $\lambda^+ = \lambda(x,t)$ and $\lambda^- = \lambda(x,-t)$, $t \in (-\infty,\infty)$, where the function $\lambda(x,t)$ is defined in (3.5). Observe that $\lambda^{\pm}$ are solutions to the following (ordinary) equations:

$$\lambda^+(x,t) = x + \int_0^t c\big(\lambda^+(x,s)\big)\,ds, \tag{5.2}$$

$$\lambda^-(x,t) = x - \int_0^t c\big(\lambda^-(x,s)\big)\,ds. \tag{5.3}$$

Theorem 5.1. *The solution to problem* (2.3), (2.9) *has the form*

$$u(x,t) = \frac{1}{2}e^{-\mu t}\Big[\varphi\big(\lambda^+(x,t)\big) + \varphi\big(\lambda^-(x,t)\big) + \int_{-t}^{t} \varphi\big(\lambda(x,s)\big)\Big(\mu I_0\big(\mu\sqrt{t^2-s^2}\,\big) + \frac{\partial}{\partial t}I_0\big(\mu\sqrt{t^2-s^2}\,\big)\Big)\,ds\Big]. \tag{5.4}$$

Here

$$I_0(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{2n}(n!)^2}$$

is the Bessel function of imaginary argument.

Remark 5.1. Passing to the variable $y = \lambda(x,s)$ in the integral on the r.h.s. of (5.4), we obtain

$$u(x,t) = \frac{1}{2}e^{-\mu t}\Big[\varphi\big(\lambda^+(x,t)\big) + \varphi\big(\lambda^-(x,t)\big) + \int_{\lambda^-(x,t)}^{\lambda^+(x,t)} \frac{\varphi(y)}{c(y)}\Big(\mu I_0\big(\mu\sqrt{t^2-\sigma(y,x)^2}\,\big) + \frac{\partial}{\partial t}I_0\big(\mu\sqrt{t^2-\sigma(y,x)^2}\,\big)\Big)\,dy\Big]. \tag{5.4′}$$

Here $\sigma = \sigma(y,x)$ is the solution to the equation $\lambda(x,\sigma) = y$.

Proof of Theorem 5.1. The proof is based on the following lemmas.

Lemma 5.1. *The function $u = u(x,t)$ is the solution to problem* (2.3), (2.9) *if $u = e^{-\mu t}v$, where $v = v(x,t)$ is the solution to Eq.* (5.1) *with the initial conditions*

$$v\big|_{t=0} = \varphi, \qquad v_t\big|_{t=0} = \mu\varphi. \tag{5.5}$$

Denote

$$Z(x,t;\psi) = \frac{1}{2}\int_0^t \big(\psi(\lambda^+(x,s)) + \psi(\lambda^-(x,s))\big)I_0\big(\mu\sqrt{t^2-s^2}\,\big)\,ds \equiv \frac{1}{2}\int_{-t}^{t}\psi\big(\lambda(x,s)\big)I_0\big(\mu\sqrt{t^2-s^2}\,\big)\,ds.$$

Lemma 5.2. *The solution $v(x,t)$ to Eq.* (5.1) *with the initial conditions*

$$u\big|_{t=0} = \varphi, \qquad u_t\big|_{t=0} = \psi$$

has the form

$$v(x,t) = Z(x,t;\psi) + Z_t(x,t;\varphi). \tag{5.6}$$

To prove Theorem 5.1, it suffices to substitute $\mu\varphi$ instead of ψ into (5.6) and to differentiate w.r.t. t. Lemmas 5.1 and 5.2 are proved by simple direct calculations, which are omitted here.

Theorem 5.1 leads to explicit formulas both for the distribution $p = p(x,y,t)$ of the process $X(x,t)$ ("free" motion) and for $p^{\text{ref}}(x,y,t)$ or $p^{\text{abs}}(x,y,t)$ (for cases of motion with reflecting or absorbing barriers). Corollary 2.1 and Theorem 5.1 result in the following formula for the distribution of the process $X(x,t)$.

Corollary 5.1. *The probability density $p(x,y,t)$ of the process $X(x,t)$ has the form*

$$p(x,y,t) = \frac{1}{2}e^{-\mu t}\Big[\delta\big(y-\lambda^+(x,t)\big) + \delta\big(y-\lambda^-(x,t)\big) + \\ + \frac{1}{c(y)}\Big(\mu I_0\big(\mu\sqrt{t^2-\sigma(y,x)^2}\,\big) + \frac{\partial}{\partial t}I_0\big(\mu\sqrt{t^2-\sigma(y,x)^2}\,\big)\Big)\,\chi_{[\lambda^-(x,t),\lambda^+(x,t)]}(y)\Big]. \tag{5.7}$$

Here $\chi_\Delta(y) = 1$ if $y \in \Delta$ and $\chi_\Delta(y) = 0$ if $y \notin \Delta$ (cf. formula (2.5) from [2]).

Remark 5.2. It is clear from the assertions proved above that the solutions to problems (2.3), (2.9) and (3.1), (3.2) exist for all $t \in [0,+\infty)$ if the bi-characteristics λ^+ and λ^- tend to infinity during an infinite time. To provide this condition, it suffices to assume that

$$c(x) \ge c_0 > 0, \qquad x \in (-\infty,\infty), \tag{5.8}$$

$$\int_0^{+\infty}\frac{dy}{c(y)} = \infty, \qquad \int_{-\infty}^0 \frac{dy}{c(y)} = \infty. \tag{5.9}$$

Formulas for the probability densities $p^+(x,y,t)$, $p^-(x,y,t)$, $p^{\text{ref}}(x,y,t)$, and $p^{\text{abs}}(x,y,t)$ are based on formula (5.4), Theorem 2.1, and Lemmas 5.1 and 5.2.

Corollary 5.2. *The probability densities $p^\pm(x,y,t)$ of the processes $X^\pm$ have the form (cf. (2.17) from [2])*

$$p^\pm(x,y,t) = p(x,y,t) \mp \frac{e^{-\mu t}}{2}\chi_{[\lambda^-(x,t),\lambda^+(x,t)]}(y)\frac{\partial}{\partial y}I_0\big(\mu\sqrt{t^2-\sigma(y,x)^2}\,\big). \tag{5.10}$$

Using (5.7), one can obtain the explicit formulas for $p^\pm(x,y,t)$.

To write the formulas for $p^{\text{ref}}(x,y,t)$ and $p^{\text{abs}}(x,y,t)$, we again consider the process $\overline{X}(x,t)$ generated by the symmetric (with respect to the point $x = a$) velocity field $\bar{c}(x)$. Below, we denote by $\bar{p} = \bar{p}(x,y,t)$ the corresponding probability density.

Corollary 5.3. *The probability density $p^{\text{ref}}(x,y,t)$ of the process with reflections $X^{\text{ref}}(x,t)$ has the form*

$$p^{\text{ref}}(x,y,t) = \bar{p}(x,y,t) + \bar{p}(2a-x,y,t), \qquad x,y < a. \tag{5.11}$$

Proof. As follows from Theorem 4.1, the function $p^{\text{ref}}(x,y,t)$ satisfies problem (2.3), (2.9) with $c(x) = \bar{c}(x)$ and $\varphi(x) = \delta(x-y) + \delta(2a-x-y)$. Hence, by Corollary 2.1, equality (5.11) follows.

From (5.7), it is easy to obtain the explicit formulas for p^{ref}.

Corollary 5.4. *The probability density $p^{\text{abs}}(x,y,t)$ of the process with the absorption X^{abs} has the form (cf. formulas* (3.3), (3.4) *from* [2])

$$p^{\text{abs}}(x,y,t) = \bar{p}(x,y,t) - \bar{p}^{-}(x,2a-y,t), \qquad x,y<a. \tag{5.12}$$

Explicit formulas for p^{abs} follow from (5.7) and (5.10).

Proof. First, we observe that

$$b^{\text{abs}}(x,y,t) = \frac{1}{2}\big(\bar{p}^{-}(x,y,t) - \bar{p}^{-}(x,2a-y,t)\big), \qquad x,y<a, \tag{5.13}$$

is the solution to problem (4.7′)–(4.9′) and

$$f^{\text{abs}}(x,y,t) = \frac{1}{2}\big(\bar{p}^{+}(x,y,t) - \bar{p}^{-}(x,2a-y,t)\big), \qquad x,y<a, \tag{5.14}$$

is the solution to problem (4.7)–(4.9).

These assertions are verified directly by substituting (5.13) into (4.7′)–(4.9′) and, respectively, (5.14) into (4.7)–(4.9). In this case, we use Corollary 2.1, formula (3.8), and the identity

$$-\frac{\partial}{\partial y}c(y)\delta(x-y) = c(x)\delta'(x-y).$$

Adding (5.13) and (5.14), we obtain (5.12).

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