DEVELOPMENT OF A STEADY RELIEF AT THE INTERFACE OF FLUIDS IN A VIBRATIONAL FIELD

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The vibrations of a vessel strongly influence the behavior of the interface of the fluids in it. Thus, vertical vibrations can lead both to the parametric excitation of waves (Faraday ripples) and to the suppression of the Rayleigh–Taylor instability [1-2]. At the present time, the influence of vertical vibrations on the behavior of a fluid surface have been studied in sufficient detail (see, for example, review [3]). The behavior of an interface of fluids in the case of horizontal vibrations has been studied less. An interesting phenomenon has been revealed in the experimental papers [4, 5]: in the case of fairly strong horizontal vibrations of a vessel containing a fluid with a free surface, the fluid collects near one of the vertical vessel walls, the free surface being practically plane and stationary with respect to the vessel, while its angle of inclination to the horizon depends on the vibration rate. But if there is a system of immiscible fluids with comparable but different densities in the vessel, horizontal vibrations lead to the formation of a steady wave relief at the interface. An explanation of the behavior of a fluid with a free boundary was given in [6] on the basis of averaged equations of fluid motion in a vibrational field. The present paper is devoted to an analysis of the behavior of the interface of fluids with comparable densities in a high-frequency vibrational field.

1. Let two immiscible incompressible fluids fill a vessel that executes vibrations in accordance with the law

$$\mathbf{r}=a\mathbf{k}\sin\omega t+\mathbf{r}_0 \tag{1.1}$$

where $\mathbf{r}$ is the coordinate of an arbitrary point of the vessel, $\mathbf{r}_0$ is its mean, ω is the frequency of the vibrations, a is their amplitude, $\mathbf{k}$ is the unit vector along the axis of the vibrations.

In the frame of reference associated with the vessel, the equations of fluid motion have the form

$$\frac{\partial\mathbf{v}_\beta}{\partial t}+(\mathbf{v}_\beta\nabla)\mathbf{v}_\beta=-\frac{1}{\rho_\beta}\nabla p_\beta+\nu_\beta\Delta\mathbf{v}_\beta-g\boldsymbol{\gamma}+a\omega^2\mathbf{k}\sin\omega t,\qquad \nabla\mathbf{v}_\beta=0,\qquad \beta=1,2 \tag{1.2}$$

where γ is the unit vector directed vertically upward, the subscript β labels the fluids, the remaining notation is standard.

The no-slip conditions are satisfied on the rigid walls of the vessel, while the following conditions of stress balance and rate continuity, and the kinematic condition are satisfied at the interface of the fluids $F(\mathbf{r},t)=0$:

$$-[\sigma_{ij}]n_j+[p]n_i=\alpha(\nabla\mathbf{n})n_i \tag{1.3}$$

$$[\mathbf{v}]=0,\qquad \frac{\partial F}{\partial t}+\mathbf{v}\nabla F=0,\qquad \mathbf{n}=\frac{\nabla F}{|\nabla F|} \tag{1.4}$$

Here σ_{ij} is the viscous stress tensor, α is the coefficient of surface tension, $\mathbf{n}$ is the vector of the normal to the surface, $[f]=f_1-f_2$.

If the vibration frequency is fairly great, so that $\omega \gg \nu/L^2$, where L is the characteristic dimension of the hydrodynamic structures, all the processes in the fluid can be divided into fast and slow. Effective decoupling of the problem into fast pulsation

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and average parts is possible if there is reason to discard the nonlinear terms in the equations of the pulsation component of the motion. The pulsation component of the fluid velocity is equal in order of magnitude to the velocity of the vessel itself in the laboratory frame of reference $v = a\omega$, therefore, the nonlinear terms can be discarded if $a^2\omega^2/L \ll a\omega^2$, this imposing constraints on the amplitude of the vibrations $a \ll L$. Subsequently, we shall assume that the conditions $\omega \gg \nu/L^2$ and $a \ll L$ are satisfied, so that we shall assume that the amplitude of the vibrations a is small, the amplitude of the vibration rate $b = a\omega$ is finite, while $b\omega$ is large.

We now go over to a derivation of the averaged equations of motion and boundary conditions. In the case of high-frequency vibrations in all variables, we can separate the fast-changing pulsation component and the slow part whose characteristic times of variation are large in comparison with ω^{-1}. Thus, a time hierarchy arises, and this makes the application of the method of many scales natural in the averaging of the equations and the boundary conditions [7]. In accordance with the basic idea of this method, we introduce the time sequence

$$t_{-1}=\omega t,\quad t_0=t,\quad t_1=\omega^{-1}t,\ldots \tag{1.5}$$

and assume that all the variables in (1.2)-(1.4) depend on both the fast time t_{-1} and the slow times t_0, $t_1\ldots$. Then the derivative of any quantity $f(t)$ with respect to time, and the required velocity and pressure fields and the function F which determines the interface can be represented in the form of the series

$$\frac{\partial f}{\partial t}=\omega\frac{\partial f}{\partial t_{-1}}+\frac{\partial f}{\partial t_0}+\omega^{-1}\frac{\partial f}{\partial t_1}+\ldots,\quad \mathbf{v}_\beta=\mathbf{v}_\beta^{(0)}+\omega^{-1}\mathbf{v}_\beta^{(1)}+\ldots$$
$$p_\beta=\omega p_\beta^{(-1)}+p_\beta^{(0)}+\omega^{-1}p_\beta^{(1)}+\ldots,\quad F=F_0+\omega^{-1}F_1+\ldots \tag{1.6}$$

Equations (1.2) give the following leading orders in ω^{-1}:

$$\omega\frac{\partial \mathbf{v}_\beta^{(0)}}{\partial t_{-1}}=-\omega\frac{\nabla p_\beta}{\rho_\beta}+b\omega\mathbf{k}\sin t_{-1},\quad \nabla\mathbf{v}_\beta^{(0)}=0 \tag{1.7}$$

whence

$$\mathbf{v}_\beta^{(0)}=b\mathbf{V}_\beta\cos t_{-1}+\mathbf{u}_\beta,\quad p_\beta^{(-1)}=bP_\beta\sin t_{-1} \tag{1.8}$$

$$\rho_\beta(\mathbf{V}_\beta+\mathbf{k})=\nabla P_\beta,\quad \nabla\mathbf{V}_\beta=0,\quad \nabla\mathbf{u}_\beta=0 \tag{1.9}$$

where the fields $\mathbf{V}_\beta$, $\mathbf{u}_\beta$, P_β do not depend on the fast time t_{-1}.

Equations (1.7), which essentially determine the amplitude of the pulsation velocities $\mathbf{V}_\beta$, do not contain the viscosity and, therefore, the second derivatives of the velocity with respect to the coordinates. Therefore, their solution, generally speaking, does not satisfy the no-slip conditions on the rigid wall, so that for $\mathbf{V}_\beta$ on the vessel walls, we require only the no-flow condition $V_{\beta n}|_S = 0$, but the tangential component of the pulsation velocity on the wall can be nonvanishing. The substitution of the no-flow condition for the no-slip condition is admissible if the thickness of the viscous skin layer near the walls is small in comparison with the dimensions of the vessel. The equations are averaged on precisely the assumption that $\omega \gg \nu/L^2$.

It follows from the boundary condition (1.4) that the principal part of F does not depend on the fast time. Substituting the found fields (1.8) in Eqs. (1.2) and averaging them over the vibration period, we obtain equations for the mean velocities $\mathbf{u}_\beta$ with no-slip conditions on the rigid wall:

$$\frac{\partial\mathbf{u}_\beta}{\partial t}+(\mathbf{u}_\beta\nabla)\mathbf{u}_\beta=-\nabla\left(\frac{p_\beta}{\rho_\beta}+b^2V_\beta^2\right)+\nu_\beta\Delta\mathbf{u}_\beta-g\boldsymbol{\gamma},\quad \mathbf{u}_\beta|_s=0 \tag{1.10}$$

The boundary conditions (1.3)-(1.4) at the interface give, after averaging with allowance for (1.8),

$$-[\sigma_{ij}]n_j+[p]n_i+\frac{1}{2}b^2[\rho V_nW_n]n_i=\alpha(\nabla\mathbf{n})n_i \tag{1.11}$$

$$[\rho W_\tau]=0,\quad [W_n]=0,\quad \frac{\partial F}{\partial t}+\mathbf{u}\nabla F=0,\quad \mathbf{W}_\beta=\mathbf{V}_\beta+\mathbf{k} \tag{1.12}$$

In expressions (1.10)-(1.12), the subscript (superscript) zero is omitted from t_0, the mean pressure $p_\beta^{(0)}$, and the function F_0; the subscripts n and τ, respectively, denote the vector components normal and tangential to the interface. The tensor σ_{ij} in (1.11) is determined in the field of mean velocities $\mathbf{u}_\beta$.

Equations (1.9)-(1.10) with the no-flow and no-slip boundary conditions and (1.11)-(1.12) fully determine the fields of the amplitudes of the pulsation velocities $\mathbf{V}_\beta$, $\mathbf{u}_\beta$, and the pressures p_β, and the position of the averaged interface of the fluids.

We consider the "equilibrium" conditions of fluids in a vibrating vessel, understanding by equilibrium the state in which there is no averaged motion ($\mathbf{u}_\beta=0$), while the interface is steady (i.e., does not depend on the slow time t_0). In this case, the equations and boundary conditions for the pulsation velocities can be written in the form

$$\operatorname{rot} \mathbf{W}_\beta=0, \quad \nabla \mathbf{W}_\beta=0, \quad W_{\beta n}|_s=k_n \tag{1.13}$$

On the interface of the media $F(\mathbf{r})=0$, relations (1.12) are satisfied, while condition (1.11) takes the form

$$[p]+\frac{1}{2}b^2[\rho V_n W_n]=\alpha(\nabla \mathbf{n}) \tag{1.14}$$

The equilibrium pressures are determined from Eqs. (1.10), in which the mean velocities $\mathbf{u}_\beta$ must be set equal to zero.

We note that for the pulsation velocities at the interface, only the conditions of balance of normal stresses and equality of normal velocity components are satisfied. The tangential components of the pulsation velocities are different and there is no condition of balance of tangential stresses. Thus, the conditions written down are correct only if the thickness of the viscous skin layer near the interface is small in comparison with the characteristic dimensions of the surface structures.

2. We will apply the obtained equations and boundary conditions to the development of an undulating relief on the interface of media in a horizontally vibrating vessel. Such a relief was observed experimentally [4, 5].

Let fluids with densities ρ_1 and ρ_2 ($\rho_1 > \rho_2$) fill a horizontal layer of thickness 2h. We direct the z axis of the Cartesian coordinate system vertically upward. For simplicity, we assume that the fluids occupy equal volumes. We select the origin so that in the absence of vibrations the heavy fluid occupies the region $-h < z < 0$ and the light fluid $0 < z < h$. The horizontal dimensions of the vessel are assumed large (in comparison with h), so that the effects associated with the presence of vertical walls can be disregarded and the vessel assumed unbounded in the horizontal directions.

We consider the behavior of fluids in the presence of horizontal vibrations with frequency ω and amplitude a. We direct the x axis along the axis of the vibrations, so that $\mathbf{k}$ in (1.1) is the unit vector along the x axis.

We formulate the problem for the determination of the "equilibrium" interface of the fluids $z = \zeta(x, y)$:

$$\operatorname{rot} \mathbf{W}_\beta=0, \quad \nabla \mathbf{W}_\beta=0 \tag{2.1}$$

$$W_{1z}=0 \quad (z=-h), \quad W_{2z}=0 \quad (z=h), \quad W_{1n}=W_{2n}, \quad \rho_1 W_{1\tau}=\rho_2 W_{2\tau} \tag{2.2}$$

$$\frac{b^2}{4}[W_{1n}^2(\rho_1-\rho_2)-(\rho_1 W_{1\tau}^2-\rho_2 W_{2\tau}^2)]-(\rho_1-\rho_2)g\zeta+\alpha(\nabla \mathbf{n})=\text{const} \tag{2.3}$$

Condition (2.3) is obtained from (1.14) after the substitution of the pressures determined from (1.10) with $u_\beta = 0$ in (1.14). Since the fluids are incompressible, it is still necessary to require the satisfaction of the condition of normalization of the volume.

It is easy to verify that problem (2.1)-(2.3) admits an equilibrium solution with a plane boundary that satisfies the following condition of closure of the pulsation flow:

$$\zeta=0, \quad W_{1x}=\frac{2\rho_2}{\rho_1+\rho_2}, \quad W_{2x}=\frac{2\rho_1}{\rho_1+\rho_2}, \quad W_{1y}=W_{2y}=W_{2z}=W_{1z}=0, \quad \int_{-h}^{\zeta} V_{1x}\,dz+\int_{\zeta}^{h} V_{2x}\,dz=0 \tag{2.4}$$

We consider the bifurcations of solution (2.4). Since the vibrations are directed along the x axis, they do not influence the perturbations polarized along the y axis and, therefore, it is sufficient to consider the plane perturbations that depend only on x

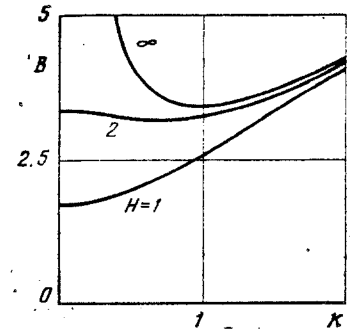

Fig. 1

and z. Since the vectors $\mathbf{W}_\beta$ are solenoidal in the two-dimensional problem, we can introduce the stream functions Ψ, Φ for the perturbations which, in accordance with (2.1), are harmonic functions:

$$W_{1x}=\frac{2\rho_2}{\rho_1+\rho_2}+\frac{\partial\Psi}{\partial z},\quad W_{1z}=-\frac{\partial\Psi}{\partial x},\quad W_{2x}=\frac{2\rho_1}{\rho_1+\rho_2}+\frac{\partial\Phi}{\partial z},\quad W_{2z}=-\frac{\partial\Phi}{\partial x}$$
$$\Delta\Psi=0,\quad \Delta\Phi=0 \tag{2.5}$$

It is convenient to solve the problem in dimensionless variables. We select the combination $[\alpha/(\rho_1 - \rho_2)g]^{1/2}$ as the unit of length. We will measure Ψ and Φ in the same units. With such a selection of units, Eqs. (2.5) are written as formerly, while the boundary conditions take the following form in terms of Ψ and Φ:

$$\Psi=0\quad (z=-H),\quad \Phi=0\quad (z=H),\quad z=\zeta(x):$$
$$\Psi-\Phi=2\frac{\rho-1}{\rho+1}\zeta,\quad \rho(\Psi_z-\Psi_x\zeta_x)=\Phi_z-\Phi_x\zeta_x \tag{2.6}$$

$$B\left[\frac{2\rho}{\rho+1}\Psi_z+\frac{2}{\rho+1}\Phi_z+\Psi_z\Phi_z+\Psi_x\Phi_x\right]-\zeta+\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}=\text{const}$$
$$H=h\left[\frac{(\rho_1-\rho_2)g}{\alpha}\right]^{1/2},\quad \rho=\frac{\rho_1}{\rho_2},\quad B=\frac{b^2}{4}\left(\frac{\rho_1-\rho_2}{\alpha g}\right)^{1/2} \tag{2.7}$$

Here H is the dimensionless mean thickness of the fluid layers, B is the dimensionless parameter characterizing the vibrations, the subscripts to Ψ, Φ, ζ denote differentiation with respect to the corresponding variable.

For an analysis of the bifurcations of the plane interface of the fluids, we linearize problem (2.5)-(2.7) near the equilibrium solution (2.4) and, considering the x-periodic perturbations (Ψ, Φ, ζ ~ cos kx), we obtain the following bifurcation curve in the coordinates B, k or the dimensional coordinates b, k:

$$B=\frac{(\rho+1)^3}{8\rho(\rho-1)}(k+k^{-1})\,\mathrm{th}\,kH \tag{2.8}$$

$$b^2=\frac{(\rho_1+\rho_2)^3}{2\rho_1\rho_2(\rho_1-\rho_2)^2}[\alpha k+(\rho_1-\rho_2)gk^{-1}]\,\mathrm{th}\,kh \tag{2.9}$$

If the amplitude of the vibration rate exceeds the critical value (2.9), the plane fluid interface becomes unstable and a wave relief develops.

As can be seen from (2.9), for $\rho_2 > \rho_1$ (heavy fluid above), perturbations (with fairly long wavelengths) leading to a loss of stability will always be found. Thus, in this case, the plane interface is absolutely unstable, i.e., the horizontal vibrations do not hinder the development of the Rayleigh–Taylor instability, in contrast to the vertical vibrations which suppress its development under certain conditions [2].

It also follows from (2.9) that a wave relief (for $\rho_1 > \rho_2$) is possible only on the surface of fluids with comparable densities, but not for a free surface, since the critical value b^2 becomes infinite as $\rho_2 \to 0$. This fact is also noted in the experimental papers [4, 5].

Figure 1 gives the neutral curves for different values of H. The relief with a finite wavelength is not possible for every fluid thickness. In fact, for thin layers ($kH \ll 1$), the neutral curve (2.8) takes the form

$$B=\frac{(\rho+1)^3 H}{8\rho(\rho-1)}(k^2+1),$$

hence, it follows that, in this case, the perturbations with $k = 0$ are the most dangerous. Analysis of (2.8) indicates that a relief with a finite wavelength develops only in fairly thick layers for $H > \sqrt{3}$, i.e., $h > [3\alpha/(\rho_1 - \rho_2)g]^{1/2}$.

We shall subsequently consider layers with $kH \gg 1$. In this case, the critical amplitude of (2.8) is determined from the expression

$$B=\frac{(\rho+1)^3}{8\rho(\rho-1)}(k+k^{-1})$$

with a minimum at $k = 1$, while $B_{\min}={}^1/_4(\rho+1)^3/\rho(\rho-1)$. Thus, for fairly thick fluid layers in the field of horizontal vibrations with a velocity amplitude exceeding the critical value

$$b_{\min}^2=\frac{(\rho_1+\rho_2)^3}{2\rho_1\rho_2(\rho_1-\rho_2)}[\alpha(\rho_1-\rho_2)g]^{1/2},$$

a wave relief with the following wavelength develops at the interface of the fluids:

$$\lambda=2\pi\left[\frac{\alpha}{(\rho_1-\rho_2)g}\right]^{1/2}$$

Before proceeding to a nonlinear analysis of problem (2.5)-(2.7), we note one interesting fact. Problem (2.1)-(2.3) and, therefore, problem (2.5)-(2.7) obtained from it is equivalent (excepting for the notation) to the problem for the development of a steady relief on the surface of a fluid insulator (a magnet) in a vertical constant electric (magnetic) field. The analogy between the problems is incomplete. In the problem considered here for the appearance of the wave relief in the vibration field, there is a defined direction (the axis of the vibrations). But in the electric problem, all the horizontal directions are on an equal footing. However, if we solve the problem for the instability of the surface of the insulator in the electric field in a two-dimensional formulation, the substitution of $(\rho_1 - \rho_2)g/\rho_1$ for g, $\mathbf{E}\times\mathbf{k}/\sqrt{4\pi}$ for ${}^1/_2 b^2 \mathbf{W}_1$, and ε for ρ (E is the electric field strength, ε is the permittivity of the fluid) converts the problem under discussion here to the problem for the stability of the plane surface of a fluid insulator in a vertical electric field, solved previously in [7].

Nonlinear analysis of (2.5)-(2.7), carried out by the standard method of expansion with respect to the amplitude of the perturbations, gives the following expression for the amplitude of the wave A:

$$A^2=-\frac{64\rho(\rho^2-1)}{11\rho^2-42\rho+11}(B-B_{\min}) \tag{2.10}$$

whence it follows that for $\rho < 3.535$ the wave develops through soft excitation, and for $\rho > 3.535$ through hard excitation. (We recall that the wave relief is only possible for $\rho > 1$.) The same value ($\varepsilon = 3.535$) for the boundary of the change of relief excitation regimes is obtained in [8], Eq. (2.10) of the above substitution being converted into the expression for the amplitude of the wave obtained in the same paper.

The vibrational—electrical analogy is not complete. All horizontal directions are on an equal footing in an analysis of the stability of the surface of an insulator in an electric field, therefore, a three-dimensional relief is possible besides the two-dimensional relief. As shown in [9, 10], the two-dimensional relief (at least for $(\varepsilon - 1)/(\varepsilon + 1) \ll 1$) on the surface of the insulator is unstable and the relief is realized with either a hexagonal or a square structure. But in the problem considered here, there is a distinguished direction — the axis of the vibrations — and, therefore, only the two-dimensional solution has the necessary symmetry. Precisely this two-dimensional structure was observed in the experiments of [4, 5].

Thus, the theory developed in the present paper qualitatively describes the experiments on the development of a steady wave relief at the interface of the fluids in the case of horizontal vibrations. It is not possible to carry out a quantitative comparison with the experiment, since papers [4, 5] do not contain the necessary data for such a comparison.

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DISSIPATIVE INSTABILITY OF FLUID FLOWS WITH PIECEWISE LINEAR VELOCITY PROFILES

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The viscous dissipative instability of two flows with continuous spectrum of neutrally-stable perturbations in the absence of dissipation is investigated. Ranges of wave numbers in which viscosity leads to flow destabilization are determined for a shear discontinuity in a smoothly stratified fluid. A shear flow with a velocity in the transition layer that depends linearly on the coordinate has a continuum of neutral modes even in the case of an unstratified fluid. When viscosity is present in one of the layers with constant velocity, one of the branches of the spectrum becomes unstable. When the viscosity is the same above and below the shear layer, dissipation only leads to the damping of the perturbations.

1. Formulation of the Problem

Study of flows with piecewise linear velocity profiles is of interest in gas dynamic stability theory. The greatest number of results have been obtained for flows of the shear discontinuity type and shear flows with linear transition layers (see the reviews in [1-3]):

$$u(z)=u\,\mathrm{sgn}\,(z),\quad u=\mathrm{const} \tag{1.1}$$

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