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Existence an regularity of constant mean curvature hypersurfaces in hyperbolic space

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1 Introduction and statement of results

In this paper, we will study the existence and regularity of complete constant mean curvature hypersurfaces in hyperbolic space with prescribed asymptotic behaviour at infinity. Let

$$\mathbf{H}^{n+1} = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}, y > 0\}$$

denote the standard hyperbolic space equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

In this model, $\mathbf{H}^{n+1}_\infty = \mathbf{R}^n \times \{0\} \cup \{*\}$ is the one-point compactification of $\mathbf{R}^n \times \{0\}$ which can be identified with the asymptotic infinity of $\mathbf{H}^{n+1}$. M. Anderson studied complete area-minimizing submanifolds in $\mathbf{H}^{n+1}$. One of his results is

Theorem 1.1 ([A1][A2]) *Let $N^{n-1} \subset \mathbf{H}^{n+1}_\infty$ be a closed embedded $n-1$ dimesional submanifold. Then there exits a complete absolutely area minimizing integral $n-$current Σ asymptotic to N^{n-1} at infinity.*

He notes that in case $n \leq 6$, currents obtained are smoothly embedded complete submanifolds. In case $n \geq 7$, as is the Euclidean case, there can be closed singular set of Hausdorff dimension at most $n-7$.

The question of the boundary behavior of such hypersurfaces was discussed by R. Hardt and F.-H. Lin in [HL],[L1] and [L2]. It is proved that

Theorem 1.2 ([L2]) *Suppose N is C^1 submanifold. Then there exists a positive ρ_N such that $(\Sigma \cup N) \cap \{y < \rho_N\}$ is a finite union of C^1 submanifolds with boundary. These have disjoint analytic interiors and meet $\mathbf{R}^n \times \{0\}$ orthogonally at N.*

The above boundary regularity implies that all the singularity remains at some finite distance, and the hypersurface has a finite genus in case $n \leq 6$. Concerning the higher order regularity, F.-H. Lin showed that

Theorem 1.3 ([L1]) *Suppose that N is $C^{k,\alpha}$ with either $1 \leq k \leq n-1$ and $0 \leq \alpha \leq 1$ or $n \leq k$ and $0 < \alpha < 1$. Then, there exists ρ_N such that $(\Sigma \cup N) \cap \{y < \rho_N\}$ is a finite union of $C^{k,\alpha}$ submanifolds with boundary.*

It is then natural to consider similar questions for constant mean curvature hypersurfaces, and investigate the differences and similarities between the area minimizing and non-zero constant mean curvature case. Let $M \subset \mathbf{H}^{n+1}$ be a smooth oriented hypersurface with continuous unit normal vector field ν. The mean curvature H of M at $x \in M$ is

$$H(x) = \frac{1}{n} \sum_{i=1}^{n} < \nabla^{\mathbf{H}}_{e_i} e_i, \nu >$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis for the tangent space of M at x and $\nabla^{\mathbf{H}}$ and $< \cdot, \cdot >$ are the covariant derivative and inner product respectively associated with hyperbolic space. Let $\Omega \subset \mathbf{R}^n \times \{0\} \subset \mathbf{H}^{n+1}_\infty$ be a bounded domain (in Euclidean metric) with boundary $\partial\Omega$, and suppose that there exists a smooth, oriented hypersurface M with $\bar{M} \setminus \mathbf{H}^{n+1} = \partial\Omega$ and constant mean curvature H, where the closure is taken in Euclidean topology. We first show that such M exists by assuming that $|H| < 1$ and $\partial\Omega$ is of class C^1 at least, with the possibility of M having a small singular set. The hypersurface M is realized as a reduced boundary of a set of locally finite perimeter (See [S], [F,4.5.11] for the definition) in $\mathbf{H}^{n+1}$ which locally minimizes a family of functionals

$$A_C(E) = nH \cdot \|E \cap C\|_{\mathbf{H}} + \|\partial^* E \cap C\|_{\mathbf{H}}$$

for all compact sets $C \subset \mathbf{H}^{n+1}$. Here, $\|\cdot\|_{\mathbf{H}}$ is the volume measure in hyperbolic space of appropriate dimension. Then, the existence and interior regularity follows from standard results of geometric measure theory and the use of appropriate barrior surfaces:

Theorem 1.4 *Suppose $\Omega \subset \mathbf{R}^n \times \{0\}$ is a C^1 bounded domain and $|H| < 1$. Then, there exists a set of locally finite perimeter $E \subset \mathbf{H}^{n+1}$ with $\bar{E} \setminus \mathbf{H}^{n+1} = \bar{\Omega}$ and whose reduced boundary $M = \partial^* E$ satisfies :*

(1)M is smooth outside a closed set Γ of Hausdorff dimension at most $n-7$.

(2) $M \setminus \Gamma$ has constant mean curvature H with respect to the inward normal vector field of $\partial^* E$.

(3)the closure $\bar{M}$ of M satisfies $\bar{M} \setminus \mathbf{H}^{n+1} = \partial\Omega$. (The closure in the above statement is taken in Euclidean topology of $\mathbf{R}^{n+1}$.)

The regularity near the boundary (in Euclidean metric) is discussed. One needs to examine the quasi-linear degenerate elliptic partial differential equation (or PDE)

$$(1.1)\qquad \begin{cases} y\left(\Delta u - \frac{u_i u_j}{1+|Du|^2} u_{ij}\right) - n\left(u_y - H\sqrt{1+|Du|^2}\right) = 0 & \text{on } B_1^+ \\ u(x,0) = \varphi(x) \end{cases}$$

where u is the non-parametric representation of the graph of M and the equation has degeneracy on $\{y = 0\}$. There have been numerous works on degenerate elliptic PDE, notably by Kohn and Nirenberg ([KN]), Baouendi and Goulaouic ([BG]). However, the above equation is not covered by these authors. F.-H. Lin discussed the case $H = 0$ in [L1], and the proof of this paper closely follows the idea there. We proved that

Theorem 1.5 *Suppose $\partial\Omega$ is in the class $C^{k,\alpha}$, $1 \le k \le n-1$ and $0 \le \alpha \le 1$ or $k = n$ and $0 \le \alpha < 1$. Then $M \cup \partial\Omega$ is a $C^{k,\alpha}$ submanifold with boundary near $\partial\Omega$.*

Hence in particular, $M \cup \partial\Omega$ has finite genus for $n \le 6$ and any interior singularity of M remains within some finite distance for $n \ge 7$. Higher order regularity exhibits certain differences depending on the dimension and the mean curvature H. For the area-minimizing case, i.e. $H = 0$, we show

Theorem 1.6 *Suppose that M is area-minimizing, i.e. $H = 0$, and $\partial\Omega$ is $C^{k,\alpha}$, $k \ge n+1$ and $0 < \alpha < 1$.*

(1) If n is even, then $M \cup \partial\Omega$ is a $C^{k,\alpha}$ submanifold with boundary near $\partial\Omega$.

(2) If n is odd, then $M \cup \partial\Omega$ may not be a C^{n+1} submanifold with boundary near $\partial\Omega$ in general.

For case (2), we will show that there is a necessary and sufficient condition that $\partial\Omega$ has to satisfy in the form of a non-trivial partial differential equation involving derivatives of up to $n+1$, to recover $C^{k,\alpha}$ regularity. The higher C^{n+1} regularity was claimed in [L1] for all dimensions, but the author found a gap here which led to the interesting dependence in theorem 1.6 on the parity of the dimensions. For $H \ne 0$ case, we can show regularity for $n = 2$.

Theorem 1.7 *Suppose that M is 2 dimensional surface of constant mean curvature H with $|H| < 1$, and $\partial\Omega$ is $C^{k,\alpha}$ with $k \ge n+1 = 3$, $0 < \alpha < 1$. Then $M \cup \partial\Omega$ is a $C^{k,\alpha}$ submanifold with boundary near $\partial\Omega$.*

We will exhibit that the case $H \ne 0$ is different somehow from the case $H = 0$ by showing

Theorem 1.8 *For $n = 4$, $H \ne 0$ and $|H| < 1$, there exists a smooth $\partial\Omega$ such that $M \cup \partial\Omega$ is not a $C^{n+1} = C^5$ submanifold with boundary.*

These two theorems show that $n = 2$ is special, and that we do not have the same kind of higher order regularity result for the $H \ne 0$ case as for the $H = 0$ case. Even though there are no such obstructions on the boundary regularities for minimal hypersurfaces in Euclidean space, we would like to note that the similar parity dependence of the boundary regularity has been observed in other problems such as the complex Monge-Ampère equations ([FE],[CY]), harmonic functions on hyperbolic spaces ([G]), and harmonic mappings between hyperbolic spaces ([LT1,2]). For example, in [LT1,2], Li and Tam studied harmonic mappings from $\mathbf{H}^m$ to $\mathbf{H}^n$ with prescribed boundary behaviors with various regularity assumptions and so called nowhere-vanishing energy-density hypothesis. Among the other interesting results, they showed that the map constructed by using the heat flow method is $C^{m-1,\beta}$ if the prescribed boundary is $C^{m-1,\alpha}$ for $\beta < \alpha$. The further higher order regularity seems to be difficult to prove, even though they conjecture for the positive answers ([LT2]). They also observed that there are obstructions for the smooth boundary regularity for odd dimension and none for even dimensions, assuming that the harmonic map is smooth up to the boundary. So the situation for the minimal surface here seems very similar, while the non-zero constant mean curvature breaks the parity.

There are numerous works on the minimal surfaces as well as the constant mean curvature surfaces in hyperbolic space, but we only mension the works by Bryant [BR], Umehara and Yamada [UY] for the surfaces of constant mean curvature 1 in $\mathbf{H}^3$ and refer to the further references therein. In case $|H| > 1$, there have been investigations on the behavior near the infinity (see [KKMS]), and it seems to indicate that the analytic properties are totally different from the case of $|H| < 1$. We will discuss only $|H| < 1$ case in this note.

After this paper was accepted, we were informed that Alencar and Rosenberg ([AR]), Nelli and Spruck ([NS]) studied the constant mean curvature hypersurfaces in hyperbolic spaces and showed certain existence and uniqueness. Our work has a different approach and should be of independent interest.

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2 Proof of existence and regularity up to $C^{2,\alpha}$

In the following, we use Euclidean topology and metric associated with $\mathbf{R}^{n+1}$ unless steted specifically as "hyperbolic".

First, we consider the special case of $\Omega = B_r(x_0)$ and $|H| < 1$. Direct calculation shows that, for a set $E^H_{(x_0,r)} \subset \mathbf{H}^{n+1}$ with

$$E^H_{(x_0,r)} \equiv \left\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^+; |x-x_0|^2 + \left(y + \frac{rH}{\sqrt{1-H^2}}\right)^2 < \frac{r^2}{1-H^2}\right\}$$

$M^H = \partial E^H_{(x_0,r)} \cap \{y > 0\}$ has constant mean curvature H (in hyperbolic metric) with respect to the inward unit normal vector field, and $\partial M^H = \partial\Omega = \partial B_r(x_0)$. If $H = 0$, M^0 is a totally geodesic plane, which looks like a half sphere in the upper-half space model. If $1 > H \geq 0$, M^H is a graph over $B_r(x_0)$, which looks like a "cap of sphere" in the upper-half space model. We note that the boundary behaviors of these examples suggest that of the general case. In particular, if the boundary behavior of the surface is smooth, we expect that the angle θ_H of intersection of the constant mean curvature surface and the bottom plane $\mathbf{R}^n \times \{0\}$ at infinity is given, as is the case of the above example, by $\theta_H = \arctan\left(\frac{\sqrt{1-H^2}}{H}\right)$.

Suppose some bounded C^1 domain $\Omega \subset \mathbf{R}^n \times \{0\}$ is given and that there exists some open set $E \subset \mathbf{R}^n \times \mathbf{R}^+$ with $\bar{E} \setminus \{\mathbf{R}^n \times \mathbf{R}^+\} = \bar{\Omega}$ and that $M = \partial E \cap \{y > 0\}$ is a smooth submanifold with (hyperbolic) constant mean curvature $0 \leq H < 1$. Then, by using the family of aforementioned constant mean curvature surfaces, we can use a moving planes method adapted suitably to this setting (See [GNN]). It shows that surfaces with the same constant mean curvature cannot touch at a point away from (hyperbolic) infinity. One has to be careful with the direction of the normal vector field with which one measures the mean curvature of the surface in this argument. With these considerations, the followings are true:

(1) if $x \in \mathbf{R}^n$ and $\bar{B}_R(x) \subset \Omega$ for $R > 0$, then $E^H_{(x,R)} \subset E$.
(2) if $x \in \mathbf{R}^n$ and $B_R(x) \supset \bar{\Omega}$ for $R > 0$, then $E \subset E^H_{(x,R)}$.
(3) if $x \in \mathbf{R}^n$ and $\bar{B}_R(x) \cap \bar{\Omega} = \emptyset$ for $R > 0$, then $E \cap E^{-H}_{(x,R)} = \emptyset$.

Hence, the above $E^H_{(x,R)}$ serves as barriers for such E and are used to establish existence and regularity of the surfaces. With these remarks, we prove existence of a constant mean curvature surface for any given C^1 boundary at infinity and interior regularity.

Theorem 2.1 *Suppose $\Omega \subset \mathbf{R}^n \times \{0\}$ is a C^1 bounded domain and $|H| < 1$. Then, there exists a set of locally finite perimeter (in hyperbolic sense) $E \subset \mathbf{R}^n \times \mathbf{R}^+$ with $\bar{E} \setminus \{\mathbf{R}^n \times \mathbf{R}^+\} = \bar{\Omega}$ and whose reduced boundary (in hyperbolic sense) $M = \partial^* E$ satisfies:*

(1) M is smooth outside a closed set Γ of Hausdorff dimension at most $n-7$.

(2) $M \setminus \Gamma$ has constant mean curvature H with respect to the inward normal vector field of M.

(3)the closure $\bar{M}$ of M satisfies $\bar{M} \setminus \{\mathbf{R}^n \times \mathbf{R}^+\} = \partial\Omega$.

Proof. Without loss of generality, we can assume $H \geq 0$. If $0 > H > -1$, then we only need to invert $\mathbf{H}^{n+1}$ with a suitable isometry and change the direction of the normal vector field. Then define

$$\tilde{E}_\varepsilon = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^+; x \in \Omega, 0 < y \leq \varepsilon\}$$

for $\varepsilon > 0$ and choose $r_0 > 0$ and $x_0 \in \Omega$ so that $\bar{\Omega} \subset B_{r_0}(x_0)$. Also define

$$\tilde{D}_\varepsilon = \{(x,y); (x,y) \in E^0_{(x_0,r_0)}, y > \varepsilon\}$$

$$\mathcal{F}_\varepsilon = \{E \subset \mathbf{H}^{n+1}, E \text{ is a set of locally finite perimeter and } E \cap (\tilde{D}_\varepsilon)^c = \tilde{E}_\varepsilon\}$$

$$A^H_{\mathcal{O}} = nH \cdot |E|_{\mathcal{O}} + \|\partial^* E\|_{\mathcal{O}} \tag{2.1}$$

$$= nH \int_{E\cap\mathcal{O}} dV_{\mathbf{H}} + \sup_{\phi\in C^1_c(\mathcal{O}),|\phi|\leq 1} \int_{\mathcal{O}\cap E} div_{\mathbf{H}}\phi dV_{\mathbf{H}}$$

where volume, divergence and reduced boundary of the set of locally finite perimeter are associated with hyperbolic metric. With these, we solve a minimizing problem for all sufficiently small $\varepsilon > 0$, i.e. we find a $J_\varepsilon \in \mathscr{F}_\varepsilon$ such that

$$A^H_{\{y>\frac{\varepsilon}{2}\}}(J_\varepsilon) = \inf_{E\in\mathscr{F}_\varepsilon} A^H_{\{y>\frac{\varepsilon}{2}\}}(E)$$

This follows from BV function compactness (See [S]). Moreover, J_ε is in the (hyperbolic) convex hull of $\tilde{E}_\varepsilon$. Here, the convex hull of set A is defined to be

$$\cap_{A\subset E^0_{(x,r)}} E^0_{(x,r)}.$$

Since the normal projection map to a totally geodesic plane is Lipschitz continuous with Lipschitz constant 1, the minimizer should stay inside of the convex hull. Suppose $x_1 \in \Omega$ and $r_1 > 0$ satisfies $\bar{B}_{r_1}(x_1) \subset \Omega$. Then, by the barrier augument, $J_\varepsilon \supset E^H_{(x_1,r_1)}$ for any $\varepsilon > 0$. The use of moving planes method for J_ε can be justified by the fact that, if $E^H_{(x,r)}$ touches $\partial^* J_\varepsilon$ at $z \in \{R^n \times R^+\} \cap \tilde{D}_\varepsilon$ then $\partial^* J_\varepsilon$ is a smooth hypersurface in the neighborhood of z. This follows from the fact that the tangent cone exists at every point of $\partial^* J_\varepsilon \setminus \tilde{D}_\varepsilon$ and that it is actually tangent plane at this point due to the extremal nature of this point. With these and a suitable relative isoperimetric inequality, it follows that there exists a constant $C(n, \mathscr{O})$ for any open set $\mathscr{O} \subset\subset \mathbf{H}^{n+1}$ such that $|J_\varepsilon|_{\mathscr{O}} + \|\partial^* J_\varepsilon\|_{\mathscr{O}} \le C(n, \mathscr{O})$ for all small $\varepsilon > 0$. Thus, again by compactness, there exist a set of locally finite perimeter J and subsequence $\varepsilon_i \to 0$ such that $J_{\varepsilon_i} \to J$ in $L^1_{loc}$and a.e. and J minimizes functional $A^H_{\mathscr{O}}(\cdot)$ for any bounded open set $\mathscr{O}$. By a similar argument as in the smooth case, J has barriers of type (1 - 3) described above. This shows that $\bar{J} \setminus \{\mathbf{R}^n \times \mathbf{R}^+\} = \bar{\Omega}$ and $\partial^* J \setminus \{\mathbf{R}^n \times \mathbf{R}^+\} = \partial\Omega$. Standard regularity theory of geometric measure theory gives the stated interior regularity of $\partial^* J$. □

Remark 2.1 The open set Ω only needs to be a set of finite perimeter to carry out the above proof, and all the interior regularity holds as in the C^1 case.

We denote the set of locally finite perimeter obtained in the previous proof by E, and let $M = \partial^* E$, where the reduced boundary is considered in hyperbolic metric. Due to the barrier for the constant mean curvature hypersurfaces, we have

$$M \cap \{y \le \rho\} \subset W_\rho \equiv \mathbf{R}^n \times \{0 < y \le \rho\}$$

$$\setminus\{z \in E^{-H}_{(x,r)}; x \in \mathbf{R}^n \setminus \bar{\Omega}, 0 < r < d(x)\} \setminus \{z \in E^H_{(x,r)}; x \in \mathbf{R}^n \cap \Omega, 0 < r < d(x)\}$$

where $d(x) = dist(x, \partial\Omega)$. Let $\nu_{\partial\Omega}(x)$ be an inward unit normal vector field of $\partial\Omega$ at $x \in \partial\Omega$ in $\mathbf{R}^n$ and for $x \in \partial\Omega$ and $r > 0$, let $\delta(x,r) = \min\{d(x + r\nu_{\partial\Omega}(x)), d(x - r\nu_{\partial\Omega}(x))\}$. Simple calculations show the following.

Lemma 2.1

$$(1) \quad \sup_{x\in\partial\Omega} [1 - r^{-1}\delta(x,r)] \to 0 \quad \text{as } r \to 0 \qquad \text{if } \partial\Omega \text{ is } C^1$$

(2) $\sup_{x\in\partial\Omega} [1 - r^{-1}\delta(x,r)] \le c_0 r^{2\alpha/(1-\alpha)}$ for $r < \rho_1$ if $\partial\Omega$ is $C^{1,\alpha}$ with $0 < \alpha < 1$

Here, c_0 and ρ_1 depend on $\partial\Omega$, n and α.

(3) $r^{-1}\delta(x,r) = 1$ for all positive $r < \|\text{max. principal curv.}\|^{-1}_{L^\infty(\partial\Omega)}$ if $\partial\Omega$ is $C^{1,1}$.

Let $P_x \subset \mathbf{R}^n \times (\mathbf{R}^+ \cup \{0\})$ be a (Euclidean) half plane through $(x,0)$ whose intersection with the 'bottom plane' $\mathbf{R}^n \times \{0\}$ is tangent to $\partial\Omega$ at $(x,0)$, and makes an acute angle $\theta_H = \arctan(\frac{\sqrt{1-H^2}}{H})$ with the inward unit normal vector to $\partial\Omega$. Also let $d_{P_x}(x_1,y_1) = dist(P_x,(x_1,y_1))$ for any $(x_1,y_1) \in \mathbf{R}^n \times \mathbf{R}^+$. With this notation, direct computations using the previous lemma show

Lemma 2.2 *For* $(x_0,y_0) \in W_{\rho\partial\Omega}$,

(1)$d_{P_{\bar{x}_0}}(x_0,y_0)/y_0 \to 0$ *uniformly as* $y_0 \to 0$ *if* $\partial\Omega$ *is* C^1.

(2)$d_{P_{\bar{x}_0}}(x_0,y_0)/y_0 \le c_1 y_0^\alpha$ *for* $y_0 < \rho_2$ *if* $\partial\Omega$ *is* $C^{1,\alpha}$ *for* $0 < \alpha \le 1$. *Here,* c_1 *and* ρ_2 *depend on* $\partial\Omega$, n, α *and* H.

We next use the isometry of hyperbolic geometry under the scaling $(x,y) \mapsto (\frac{x}{\lambda},\frac{y}{\lambda})$. By the previous lemma, the current is squeezed in narrow strips parallel to the tilted plane P_x as one lets $\lambda \to 0$.

Theorem 2.2 *Suppose* $\partial\Omega$ *is* $C^{1,\alpha}$, $0 \le \alpha \le 1$. *Let* E *be a minimizer of the functional (2.1) and let* M *be* $\partial^* E$ *where reduced boundary is considered in hyperbolic metric and topology. Then there exists a positive* ρ *which depends on* $\partial\Omega$, H, n *and* α *such that* $M \cup \partial\Omega \cap \{y < \rho\}$ *is a* $C^{1,\alpha}$ *submanifold with boundary.*

Proof. We will show that there exists $\rho > 0$ such that $M \cap \{y < \rho\}$ has no singularity and ν_M is continuously extended to the boundary by defining $\nu_M(x,0)$ as a unit inward normal vector to the plane P_x. If this is false, then there exists a sequence (x_i,y_i) converging to $(x,0)$ such that either M has a singularity at (x_i,y_i) or $\lim_{i\to\infty} |\nu_M(x_i,y_i) - \nu_M(x,0)| > 0$. Let $\bar{x}_i \in \partial\Omega$ be such that $d(x_i,\partial\Omega) = |\bar{x}_i - x_i|$. For each i, we shift $\bar{x}_i$ to the origin and scale by $(x,y) \mapsto (\frac{x}{y_i},\frac{y}{y_i})$. With this scaling, (x_i,y_i) is shifted to $(0,1)$. Then, for any $\varepsilon > 0$ and for all large i, we have

$$d\left(M \cap \left\{(\hat{x},x_n,y) : |\hat{x}| \le 1, \frac{1}{2} \le y \le \frac{3}{2}, d_{P_0}(x,y) \le 1\right\}, P_0\right) \le \varepsilon$$

One can calculate that the generalized mean curvature (in Euclidean metric) of M (See [S] for the definition) is bounded by $C(n)(1+H)/y$ at (x,y), so that Allard's regularity theorem ([S],[AL],[BO]) combined with the small height conclude that M is a $C^{1,1}$ graph u_i over P_0 near $(0,1)$ and that $|u_i|_{C^{1,1}(B_{1/2})} \to 0$ as $i \to \infty$. Since the normal vector to $P_{\bar{x}_i}$ converges to P_x, $\lim_{i\to\infty} |\nu_M(x_i,y_i) - \nu_M(x,0)| = 0$. This concludes the proof of the C^1 case.

For $C^{1,\alpha}$ $0<\alpha\le 1$ regularity, we view M near $x\in\partial\Omega$ as a C^1 graph over the hodograph plane, i.e. let $x\in\partial\Omega$ be translated to the origin and $T_0(\partial\Omega)=\{x_n=0\}$. Then a C^1 function u defined on $B^{n-1}_\rho(0)\times(0,\rho)\subset\{x_n=0\}\times\mathbf{R}^+$ for suitably small ρ represents the surface M. Suppose that $u\le yL$ on $B^{n-1}_\rho(0)\times(0,\rho)\cap\{|x|\le\frac{1}{2}y\}$ for some constant L. Then u locally minimizes the corresponding functional

$$\int_K (1+u_x^2+u_y^2)^{\frac{1}{2}}/y^n\,dxdy+nH\int_K\frac{yL-u}{y^{n+1}}dx \tag{2.2}$$

for any compact set $K\subset\subset B^{n-1}_\rho(0)\times(0,\rho)\cap\{|x|\le\frac{1}{2}y\}$. The first term corresponds to the hyperbolic area of the graph and the second term to the enclosed volume times nH between the graph yL and u over K. The Euler-Lagrange equation for such functional is given by

$$\Delta u-\frac{u_iu_j}{1+|Du|^2}u_{ij}-\frac{n\left(u_y-H\sqrt{1+|Du|^2}\right)}{y}=0.$$

Also we have a uniform estimate for u on $B^{n-1}_\rho(0)\times(0,\rho)\cap\{|x|\le\frac{y}{2}\}$,

$$\left|u(x,y)-\frac{H}{\sqrt{1-H^2}}y\right|\le c(n,H,\Omega,\alpha)y^{1+\alpha}$$

if $\partial\Omega$ is $C^{1,\alpha}$. Consider a scaling

$$v_\lambda(x,y)=\lambda^{-1}u(\lambda x,\lambda y)\qquad\text{for }\lambda>0.$$

By the invariance of scaling, $v_\lambda(x,y)$ locally minimizes (2.2) on $B^{n-1}_{\rho/\lambda}(0)\times(0,\rho\lambda^{-1})\cap\{|x|\le\frac{y}{2}\}$ and

$$\left|v_\lambda(x,y)-\frac{H}{\sqrt{1-H^2}}y\right|\le c(n,H,\Omega,\alpha)\lambda^\alpha.$$

By the interior gradient estimate (See [GT]), we have

$$\left|D(v_\lambda-\frac{H}{\sqrt{1-H^2}}y)\right|\le c(n,H,\Omega,\alpha)\lambda^\alpha$$

$$|D^2v_\lambda|\le c(n,H,\Omega,\alpha)\lambda^\alpha$$

on $0<\lambda<\rho$, $\frac{1}{2}\le y\le\frac{2}{3}$ $|x|\le\frac{1}{4}$. After scaling v_λ back to u, we have

$$\left|D(u(x,y)-\frac{H}{\sqrt{1-H^2}}y)\right|\le c(n,H,\Omega,\alpha)(|x|+y)^\alpha$$

$$|D^2u(x,y)|\le c(n,H,\Omega,\alpha)(|x|+y)^{\alpha-1}$$

for all $(x,y)\in\{|x|\le\frac{y}{4}\}\cap\{y<\frac{\rho}{2}\}$. This shows, after the change of coordinates, that u is a $C^{1,\alpha}$ graph in the neighborhood we are considering. □

We now assume that $\partial\Omega$ is $C^{2,\alpha}$, $0 \leq \alpha < 1$. We showed above that the graph u over the vertical plane satisfies

$$\text{(2.3)} \qquad \begin{cases} y\left(\Delta u - \frac{u_i u_j}{1+|Du|^2} u_{ij}\right) - n\left(u_y - H\sqrt{1+|Du|^2}\right) = 0 & \text{on } B_1^+ \\ u(x,0) = \varphi(x) \end{cases}$$

where $B_1^+ = \{(x,y); x \in B_1^{n-1}(0), 0 < y < 1\}$ and $u \in C^{1,1}(B_1^{n+1}(0) \times [0,1))$. We can choose a coordinates so that $\varphi(0) = |\nabla\varphi(0)| = 0$, and $u_y(0,0) = \frac{H}{\sqrt{1-H^2}}$. Simple calculation also shows that $u_y(x,0) = \frac{H}{\sqrt{1-H^2}}\sqrt{1+|\nabla\varphi(x)|^2}$.

Theorem 2.3 *Let u be a solution of (2.3) on B_1^+ and $\varphi \in C^{2,\alpha}(B_1)$. Then $u \in C^{2,\alpha}(B_{\frac{1}{2}}^+)$, for $0 \leq \alpha < 1$.*

Proof. The idea of proof is the same as [L1], so we only point out the difference in case of $H \neq 0$. First we consider the C^2 case. Let $(x_i, y_i) \in B_1^+$, $y_i > 0$ and $(x_i, y_i) \longrightarrow (x_0, 0)$ as $i \longrightarrow \infty$. We subtract a linear part of u and form a new sequence

$$u^i(x,y)$$
$$= y_i^{-2}\left[u(x_i + y_i x, y_i y) - \varphi(x_i) - \frac{H}{\sqrt{1-H^2}}\sqrt{1+|\nabla\varphi(x_i)|^2} y_i y - y_i x \cdot \nabla\varphi(x_i)\right].$$

By using $C^{1,1}$ estimate, one can conclude that there exists a subsequence $u^{m'}$ of u^m such that $u^{m'}$ converge locally uniformly in C^1 with uniform bound in $C^{1,1}$ on the half-space $\bar{\mathbf{R}}_+^n$ and converge locally uniformly in C^2 with uniform bound in $C^{2,\beta}$ ($\beta < 1$) on $\mathbf{R}_+^n$. The limit u^∞ of $u^{m'}$ satisfies the equation

$$y(\Delta u^\infty - H^2(u^\infty)_{yy}) - n(1-H^2)(u^\infty)_y = 0 \quad \text{on } \mathbf{R}_+^n$$

$$|u^\infty(x,y)| \leq c(|x|^2 + y^2)$$

Let $\tilde{y} = y(1-H^2)^{-\frac{1}{2}}$. Then the above equation is

$$\tilde{y}\Delta_{x,\tilde{y}} u^\infty - n(u^\infty)_{\tilde{y}} = 0,$$

so that it reduces to the same equation as that of the minimal surface. Hence, the subsequent arguments for C^2 and $C^{2,\alpha}$ are similarly done as the case $H = 0$ in [L1]. We omit the detail of the computations. □

3 Tangential derivative estimate and regularity up to $C^{n,\alpha}$

In this section, we obtain tangential derivative estimates as well as the $C^{n,\alpha}$ estimates. The idea of the proof is basically the same as for the $C^{2,\alpha}$ case, that is, we use the maximum principle to show a $C^{1,1}$ bound and a blow-up argument for the $C^{2,\alpha}$ bound. It is technically more involved, though, mainly due to the presence of non-vanishing mixed derivatives. The estimate will be used to obtain normal derivative estimates in the following sections.

Theorem 3.1 *Suppose that u is a solution of (2.3) with $|u| \leq 1$ on B_1^+. Suppose that $\varphi \in C^{k,\alpha}$ for $k \geq 2$, $0 \leq \alpha \leq 1$. Then $D_x^{k-2}u \in C^{2,\alpha}$ for $0 \leq \alpha < 1$ and $D_x^{k-1}u \in C^{1,1}$ for $\alpha = 1$.*

Proof. We prove by induction. Assume that $D_x^{k-1}u \in C^{2,\alpha}(B_1^+)$ have been proved for $\alpha < 1$ with the assumption that $\varphi \in C^{k+1,\alpha}(B_1)$, which is true for $k = 1$. Assume that $\varphi \in C^{k+1,1}(B_1)$ and we will show that $D_x^k u \in C^{1,1}$. We differentiate equation (2.3) k times in x. Then,

$$y\left(\Delta(D_x^k u) - \frac{u_i u_j}{1+|\nabla u|^2}(D_x^k u)_{ij}\right) - n(D_x^k u)_y + nHD_x^{k-1}\left(\sum_{i=1}^n \frac{u_i}{\sqrt{1+|\nabla u|^2}}D_x u_i\right)$$

$$= y\sum_{l=0}^{k-1}\binom{k}{l}\left(D_x^{k-l}\frac{u_i u_j}{1+|\nabla u|^2}\right)\left(D_x^l u_{ij}\right). \tag{3.1}$$

We will change the above equation into one for which we can find a suitable comparison function for $D_x^k u$ with quadratic decay at the origin. We will collect terms of order $O(y)$ and $O(|x|)$ with control of decay depending only on the derivatives of φ up to $C^{k+1,1}$. The last term on the left hand side of (3.1) is

$$nH\left(\sum_{i=1}^n \frac{u_i}{\sqrt{1+|\nabla u|^2}}\bigg|_{(x,0)} (D_x^k u)_i + B|_{(x,0)} + B|_{(x,0)}^{(x,y)} + \sum_{i=1}^n \frac{u_i}{\sqrt{1+|\nabla u|^2}}\bigg|_{(x,0)}^{(x,y)} (D_x^k u)_i\right)$$

where

$$B(x,y) = \sum_{l=0}^{k-2}\binom{k-1}{l}\left\{\sum_{i=1}^n\left(D_x^{k-1-l}\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)(D_x^{l+1}u_i)\right\}$$

and $B|_{(x,0)} = B(x,0)$, $B|_{(x,0)}^{(x,y)} = B(x,y) - B(x,0)$ and so on. By using $u_{x_i}(x,0) = \varphi_{x_i}(x)$, $u_y(x,0) = \frac{H}{\sqrt{1-H^2}}\sqrt{1+|\nabla\varphi|^2(x)}$, and $\frac{u_y}{\sqrt{1+|\nabla u|^2}}\Big|_{(x,0)} \equiv H$, $v = (D_x^k u)$ satisfies

$$y\left(\Delta v - \frac{u_i u_j}{1+|\nabla u|^2}v_{ij}\right) - n(1-H^2)v_y + \frac{nH\sqrt{1-H^2}\varphi_{x_i}(x)}{\sqrt{1+|\nabla\varphi|^2(x)}}v_{x_i}$$

$$= -nH\sqrt{1-H^2}\sum_{l=0}^{k-2}\binom{k-1}{l}\left(D_x^{k-1-l}\frac{\varphi_i}{\sqrt{1+|\nabla\varphi|^2}}\right)(D_x^{l+1}\varphi)_i(x) + yR$$

where

$$R(x,y) = -nHy^{-1}\left[\sum_{l=0}^{k-2}\binom{k-1}{l}\left\{\sum_{i=1}^{n}\left(D_x^{k-1-l}\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)(D_x^{l+1}u)_i\right\}\Bigg|_{(x,0)}^{(x,y)}\right.$$

$$\left.+\sum_{i=1}^{n}\frac{u_i}{\sqrt{1+|\nabla u|^2}}\Bigg|_{(x,0)}^{(x,y)}(D_x^k u)_i\right]+\sum_{l=0}^{k-1}\binom{k}{l}\left(D_x^{k-l}\frac{u_iu_j}{1+|\nabla u|^2}\right)(D_x^l u_{ij}).$$

The highest order of differentiations in $R(x,y)$ is k, and since we have control of derivatives $C^{k+1,\alpha}$ in the x direction and C^α for $D_x^{k-1}u_{yy}$ for $\alpha < 1$, we can check that $\|R\|_{C^\alpha(B_1^+)} \le c(n,H,\alpha,\|\varphi\|_{C^{k+1,\alpha}})$ for $\alpha < 1$. Let $\bar v$ be v minus a constant and linear term at the origin, i.e.

$$\bar v(x,y) = v(x,y)-(D_x^k\varphi)(0)-x_i(\nabla_{x_i}D_x^k\varphi)(0)-y\left(D_x^k\frac{H}{\sqrt{1-H^2}}\sqrt{1+|\nabla\varphi|^2}\right)(0).$$

Then, $\bar v$ satisfies

$$y\left(\Delta\bar v-\frac{u_iu_j}{1+|\nabla u|^2}\bar v_{ij}\right)-n(1-H^2)\bar v_y+\frac{nH\sqrt{1-H^2}\varphi_i}{\sqrt{1+|\nabla\varphi|^2}}(x)\bar v_i \tag{3.2}$$

$$= yR - nH\sqrt{1-H^2}\left[\sum_{l=0}^{k-2}\binom{k-1}{l}\left(D_x^{k-1-l}\frac{\varphi_i}{\sqrt{1+|\nabla\varphi|^2}}\right)(D_x^{l+1}\varphi_i)\Bigg|_{(0)}^{(x)}\right.$$

$$\left.+\frac{\varphi_i}{\sqrt{1+|\nabla\varphi|^2}}(x)(D_{x_i}D_x^k\varphi)(0)\right]$$

$$\equiv yR - P(x).$$

Note that $|P(x)| \le c(n,H,\|\varphi\|_{C^{k+1}})|x|$. We show that $|\bar v(x,y)| \le c(|x|^2+y^2)$ on B_1^+ for some constant $c = c(n,H,\|\varphi\|_{C^{k+1,1}})$ by finding a suitable sub- and super-solution for $\bar v$ as follows. Let L be defined as

$$L\phi = y\left(\Delta\phi-\frac{u_iu_j}{1+|\nabla u|^2}\phi_{ij}\right)-n(1-H^2)\phi_y+\frac{nH\sqrt{1-H^2}\varphi_{x_i}}{\sqrt{1+|\nabla\varphi|^2}}(x)\phi_{x_i}.$$

Then,

$$L(|x|^2+c_1y^2)$$

$$\le 2y\left\{(n-1)+c_1\left(1-\frac{u_y^2}{1+|\nabla u|^2}\right)\right\}-2n(1-H^2)c_1y$$

$$+2nH\sqrt{1-H^2}\frac{\varphi_i}{\sqrt{1+|\nabla\varphi|^2}}(x)x_i$$

$$\le c_2|x|^2 \pm 2y\left\{(n-1)+c_1(1-n)(1-H^2)+c(n,H,\|\varphi\|_{C^{1,1}})y\right\}$$

$$\le -y(n-1)(1-H^2)c_1+c_2|x|^2.$$

The last line is true by assuming that c_1 is suitably large. Also the calculation shows that

$$L(|x|^2 y) \leq c_3 y - n(1 - H^2)|x|^2$$

$$L(P(x)y/n(1 - H^2)) \leq c_4 y - P(x)$$

where c_2,c_3,c_4 depend on n,H,$\|\varphi\|_{C^{k+1,1}}$, and note that the second derivative of $P(x)$ can be bounded by $\|\varphi\|_{C^{k+1,1}}$. Let

$$g(x,y) = c_5(|x|^2 + c_1 y^2) + c_6|x|^2 y + P(x)y/n(1 - H^2).$$

Then, by using the above calculation, one can choose c_1, c_5, c_6 depending only on n, H, $\|\varphi\|_{C^{k+1,1}}$ so that the following are true:

$$\bar{v}(x,y) \leq g(x,y) \qquad \text{on } \partial B_1^+$$

$$Lg \leq L\bar{v} \qquad \text{on } B_1^+.$$

Then by the maximum principle, we have

$$\bar{v} \leq g \qquad \text{on } B_1^+.$$

Since $|P(x)|y \leq c(|x|^2 + y^2)$, for a suitably large constant $c = c(n, H, \|\varphi\|_{C^{k+1,1}})$, we have

$$\bar{v} \leq c(|x|^2 + y^2)$$

on B_1^+. A similar argument shows that

$$|\bar{v}(x,y)| \leq c(|x|^2 + y^2) \qquad \text{on } B_1^+.$$

We return to equation (3.2) and note that

$$\|P(x)\|_{C^{1,1}} \leq c(n, H, \|\varphi\|_{C^{k+1,1}}).$$

By using the scaling

$$\bar{v}_\lambda(x,y) = \lambda^{-2}\bar{v}(\lambda x, \lambda y),$$

we have

$$\|\bar{v}_\lambda\|_{C^{2,\alpha}(|x| \leq \frac{1}{2}, \frac{1}{2} \leq y \leq \frac{2}{3})} \leq c(\|\bar{v}_\lambda\|_{C^0(\bar{B}_1^+)} + \|yR\|_{C^{0,\alpha}(B_1^+)} + \|P\|_{C^{1,1}})$$

for $\alpha < 1$, $0 < \lambda < 1$, and subsequently

$$\|\nabla^2 \bar{v}\|_{C^0(|x| \leq \frac{y}{2}, 0 < y < \frac{1}{2})} \leq c(n, H, \|\varphi\|_{C^{k+1,1}}).$$

Since a similar estimate is true for any $|x| \leq 1/2$ after a change of coordinates, we have $C^{1,1}$ estimate for $D_x^k u$ on $B_{1/2}^+$.

Here, one may not see immediately the necessity of taking $B|_{(x,0)}^{(x,y)}$ instead of $B|_{(0,0)}^{(x,y)}$. The reason is that, even though we obtain bound of the kind $c(|x| + y)$ for the right hand side of equation (3.2), it is not good enough to construct barrier functions since x changes sign in B_1^+.

For $\varphi \in C^{k+2}$ and $C^{k+2,\alpha}$, the method is the same as the C^2 and $C^{2,\alpha}$ estimate (just more computations), and we omit the proof. □

For $n \geq 3$ we can prove the following by employing the similar method.

Theorem 3.2 *Suppose that u satisfies the equation*

$$\begin{cases} y\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right) - n\left(u_y - H\sqrt{1+|\nabla u|^2}\right) = 0 & on \ B_1^+ \\ u(x,0) = \varphi(x) & on \ B_1, \end{cases}$$

$\varphi \in C^{k,\alpha}(B_1)$, $2 \leq k \leq n-1$,*and* $0 \leq \alpha \leq 1$. *Then* $u \in C^{k,\alpha}(\bar{B}^+_{1/2})$ *and*

$$\|u\|_{C^{k,\alpha}(\bar{B}^+_{1/2})} \leq c(H, k, \alpha, \|\varphi\|_{C^{k,\alpha}(B_1)}).$$

If $k = n$, *then the above is true for* $0 \leq \alpha < 1$.

4 Higher order regularity for dimension 2

In this section, we will show that constant mean curvature surfaces are as regular as the boundary near infinity in case the dimension is two. As we will see in section 5 and 6, unless we impose an extra condition on the boundary behavior, we may not have C^{n+1} regularity for certain cases depending on the dimension and value of H. Hence, dimension two can be considered special in the sense that it recovers all the regularity without any extra condition on the boundary behavior.

Theorem 4.1 *Suppose that u satisfies the equation*

$$(4.1) \qquad \begin{cases} y\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right) - 2\left(u_y - H\sqrt{1+|\nabla u|^2}\right) = 0 & on \ B_1^+ \\ u(x,0) = \varphi(x) & on \ B_1, \end{cases}$$

$\varphi \in C^{k,\alpha}(B_1)$, $k \geq 3$,*and* $0 < \alpha < 1$. *Then* $u \in C^{k,\alpha}(\bar{B}^+_{1/2})$ *and*

$$\|u\|_{C^{k,\alpha}(\bar{B}^+_{1/2})} \leq c(H, k, \alpha, \|\varphi\|_{C^{k,\alpha}(B_1)}).$$

Proof. Assume $\varphi \in C^{3,\alpha}(B_1)$, $0 < \alpha < 1$. We proved that $u \in C^{2,\beta}(\bar{B}^+_{1/2})$ for any $\beta < 1$ and $\nabla_x u \in C^{2,\alpha}(\bar{B}^+_{1/2})$ with the growth estimate for the higher order derivatives

$$(4.2) \qquad |D^j u|(x,y) \leq c + cy^{2+\alpha-j}, \qquad |D^j D_x u|(x,y) \leq c + cy^{2+\alpha-j}$$

for $j = 0, 1, \cdots$, $0 < y < 1/2$ and $|x| \leq 1/2$. For any $x \in B_{1/2}$, we can assume that $x = 0$ and $\varphi(0) = \nabla\varphi(0) = 0$ by a suitable transformation. Let

$$\bar{u}(y) = u(0,y) - u_y(0,0)y - u_{yy}(0,0)\frac{y^2}{2}$$

$$= u(0,y) - \frac{H}{\sqrt{1-H^2}}y - \frac{\varphi_{xx}(0)}{1-H^2}\frac{y^2}{2}.$$

We need to prove that $\bar{u} \in C^{3,\alpha}([0,1/2])$. From the equation (4.1), $\bar{u}$ satisfies

$$\text{(4.3)} \qquad y(\bar{u}_{yy} - 2\bar{u}_y)(1-H^2) = y\left[\left(\frac{u_y^2}{1+|\nabla u|^2} - H^2\right)u_{yy} + \varphi_{xx}(0)\right.$$

$$\left. -u_{xx} + \frac{2u_x u_y}{1+|\nabla u|^2}u_{xy} + \frac{u_x^2}{1+|\nabla u|^2}u_{xx}\right] + \left\{(1-H^2)\left(u_y - \frac{H}{\sqrt{1-H^2}}\right)\right.$$

$$\left. \times\left(\frac{1}{\sqrt{1-H^2}} - \sqrt{1+|\nabla u|^2}\right) - Hu_x^2\right\}\frac{2H}{u_y + H\sqrt{1+|\nabla u|^2}}$$

$$\equiv f_0(y).$$

By using estimate (4.2), we can check that

$$\frac{u_y^2}{1+|\nabla u|^2} - H^2 = 2H\sqrt{1-H^2}\varphi_{xx}(0)y + O(y^{1+\alpha})$$

$$\varphi_{xx}(0) - u_{xx} = -\frac{H}{\sqrt{1-H^2}}\varphi_{xx}^2(0)y + O(y^{1+\alpha})$$

$$\sqrt{1+|\nabla u|^2} - \frac{1}{\sqrt{1-H^2}} = \frac{H}{1-H^2}\varphi_{xx}(0)y + O(y^{1+\alpha})$$

$$\frac{u_x u_y}{1+|\nabla u|^2}u_{xy},\ \frac{u_x^2}{1+|\nabla u|^2}u_{xx} = O(y^2).$$

Direct substitution shows that the coefficient of y^2 of the Taylor expansion of f_0 equals 0, and

$$y(\bar{u}_{yy} - 2\bar{u}_y)(1-H^2) = O(y^{2+\alpha}).$$

The vanishing of the coefficient is the decisive factor to obtain the higher order regularity, and it is uniquely so for dimension 2 with $H \neq 0$. Let

$$w = \frac{\bar{u}_y}{y^2}.$$

Then, $y^3 w_y = O(y^{2+\alpha})$, so that $w(0)$ is well defined by the integral and

$$|w(0)| \leq c(H, \|\varphi\|_{C^{3,\alpha}}, \alpha).$$

By integrating equation (4.3), we have

$$(1-H^2)\bar{u}_y(y) = y^2 w(0) + \int_0^1 f(ty)\frac{dt}{t^3}.$$

By using estimate (4.2), we can check that

$$|D_y^j f_0(y)| \leq c y^{2+\alpha-j}$$

for $j = 0, 1, \cdots$. This shows that

$$|D_y^j \bar{u}|(y) \leq c + cy^{3+\alpha-j}$$

which implies that $u \in C^{3,\alpha}(\bar{B}_{1/2}^+)$ with the desired estimate.

For $C^{k,\alpha}$, $k \geq 4$, $0 < \alpha < 1$, assume that the theorem is true for $k-1$, $0 < \alpha < 1$ with the estimate

$$|D^j u|(x,y) \leq c_1 + c_2 y^{k-1+\alpha-j} \tag{4.4}$$

for $|x| \leq 3/4$, $0 < y < 3/4$, c_i's depend only on H, k, α, j and $\|\varphi\|_{C^{k-1,\alpha}}$. By the tangential estimates, we have $D_x^{k-2}u \in C^{2,\alpha}(\bar{B}_{1/2}^+)$ and

$$|D^j D_x^{k-2} u|(x,y) \leq c + cy^{2+\alpha-j} \tag{4.5}$$

for $|x| \leq 1/2$, $0 < y < 1/2$. We prove $D_x^{k-3}u \in C^{3,\alpha}(\bar{B}_{1/2}^+)$ at first. To do so, differentiate the equation with respect to y. Then, we have

$$y\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij}\right)_{yy} + 2\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2}u_{ij}\right)_y$$
$$-2u_{yyy} + 2H\left(\frac{u_x u_{xy} + u_y u_{yy}}{\sqrt{1+|\nabla u|^2}}\right)_y = 0.$$

There is a cancelation of $2u_{yyy}$, and by solving for $yD_y^4 u$, we have

$$-(1-H^2)yD_y^4 u = y\left\{u_{xxyy} - \left(\frac{u_i u_j}{1+|\nabla u|^2}u_{ij} - H^2 D_y^2 u\right)_{yy}\right\} \tag{4.6}$$
$$+2u_{xxy} - 2\left(\frac{u_i u_j}{1+|\nabla u|^2}u_{ij} - H^2 D_y^2 u\right)_y$$
$$+2H\left(\frac{u_x u_{xy} + u_y u_{yy}}{\sqrt{1+|\nabla u|^2}} - HD_y^2 u\right)_y = 0$$

on $B_{1/2}^+ \times (0, 1/2)$. Since we have established $C^{3,\alpha}$ estimate, for each $x \in B_{1/2}$, we can let $y \to 0$. Then the equation above yields

$$2u_{xxy}(x,0) - 2\left(\frac{u_i u_j}{1+|\nabla u|^2}u_{ij} - H^2 D_y^2 u\right)_y (x,0) \tag{4.7}$$
$$+2H\left(\frac{u_x u_{xy} + u_y u_{yy}}{\sqrt{1+|\nabla u|^2}} - HD_y^2 u\right)_y (x,0) = 0$$

for $|x| \leq 1/2$. Note that expression (4.7) does not actually include $D_y^3 u(x,0)$ terms, since $\left.\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right|_{(x,0)} \equiv H$. Also note that each term is explicitly represented by the boundary value $D_x^j \varphi$ of order up to $j \leq 3$, so that it makes sense to take derivatives with respect to x $k-3$ times. Then, by using such identity, tangential derivative estimates and (4.7), we can check that

$$|yD_x^{k-3}D_y^4 u|(x,y) \le c(H,k,\alpha,\|\varphi\|_{C^{k,\alpha}})y^\alpha \tag{4.8}$$

for $|x| \le 1/2$, $0<y<1/2$. This shows that $D_x^{k-3}D_y^4 u$ has decay of order $y^{\alpha-1}$, which implies that $D_x^{k-3}u \in C^{3,\alpha}(\bar{u}^+_{1/2})$ with

$$\|D_x^{k-3}u\|_{C^{3,\alpha}(\bar{u}^+_{1/2})} \le c(H,k,\alpha,\|\varphi\|_{C^{k,\alpha}}),$$

$$|D^j D_x^{k-3}u|(x,y) \le c + cy^{3+\alpha-j} \tag{4.9}$$

on $|x| \le 1/2$, $0<y<1/2$ for any j.

Next, we show that $D_x^m u \in C^{k-m,\alpha}(\bar{u}^+_{1/2})$ for $1 \le m \le k-4$, with the estimate

$$|D_y^j D_x^m u|(x,y) \le c + cy^{k+\alpha-j-m} \tag{4.10}$$

Suppose that estimate (4.10) is proved for $m+1,\cdots,k-3$. The case $k-3$ is proved already in (4.9). Then we will prove it for m. Set v be $D_x^m u$ minus the Taylor expansion of up to the second order at x. By shifting x to the origin and assuming $\varphi(0)=\nabla\varphi(0)=0$, and restricting v on $x=0$, v satisfies

$$\begin{aligned}
&(yv_{yy}-2v_y)(1-H^2)\\
&= y\left\{-v_{xx}+\left(\frac{u_y^2}{1+|\nabla u|^2}-H^2\right)v_{yy}+2\frac{u_xu_y}{1+|\nabla u|^2}u_{xy}+\frac{u_x^2}{1+|\nabla u|^2}u_{xx}\right\}\\
&\quad +y\left\{\sum_{l=0}^{m-1}\binom{m}{l}\left(D_x^{m-l}\frac{u_iu_j}{1+|\nabla u|^2}\right)(D_x^l u_{ij})\right\}\Bigg|_{(0,0)}^{(0,y)}+G(0,y)\\
&-2H\sum_{i=1}^{2}\left[\frac{u_i}{\sqrt{1+|\nabla u|^2}}\Bigg|_{(0,0)}^{(0,y)}(D_x^m u)_i - y\left\{\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)_y(D_x^m u)_i\right\}(0,0)\right]\\
&\quad -2H\left\{A|_{(0,0)}^{(0,y)}-yA_y(0,0)\right\}\\
&\quad \equiv f^m(y)
\end{aligned}$$

for $x=0$, $0<y<1/2$, where

$$A(x,y)=\sum_{l=0}^{m-2}\binom{m-1}{l}\sum_{i=1}^{2}\left(D_x^{m-1-l}\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)(D_x^{l+1}u)_i$$

$$G(x,y)=y\left[\frac{u_x^2}{1+|\nabla u|^2}a_{11}+\left(\frac{u_y^2}{1+|\nabla u|^2}-H^2\right)c_1+2d_1\frac{u_xu_y}{1+|\nabla u|^2}\right]$$

with

$$c_1 = (D_x^m u)_{yy}(0,0)$$
$$d_1 = (D_x^m u)_{xy}(0,0)$$
$$a_{11} = (D_x^m u)_{xx}(0,0).$$

Since $m \le k-4$, $v \in C^{3,\alpha}([0,1/2])$ at least. Also,

$$|yv_{yy} - 2v_y|(0,y) \le c(H, m, \alpha, \|\varphi\|_{C^{m+3,\alpha}})y^{\alpha+2} \tag{4.11}$$

by the estimate (4.8), and

$$|D^j(yv_{yy} - 2v_y)|(0,y) \le c(H, m, \alpha, \|\varphi\|_{C^{m+3,\alpha}})y^{\alpha+2-j}.$$

By a similar procedure as in the $C^{3,\alpha}$ case, we have

$$(1-H^2)v_y(y) = y^2 w(0) + \int_0^1 f_m(ty)\frac{dt}{t^3} \tag{4.12}$$

where $w(y) = \frac{v_y}{y^2}$ and is well-defined at 0 due to the estimate

$$|D_y^{k-m} v|(y) \le c \qquad |D_y^{k-m+1} v|(y) \le cy^{\alpha-1} \tag{4.13}$$

We can estimate the order of growth of each term of $f_m(y)$ by using the estimates (4.2) and (4.10), which gives

$$|D^j f_m|(y) \le c + cy^{k+\alpha-m-1-j}.$$

By this and the expression (4.12) for v_y, the estimate (4.10) for $1 \le m \le k-3$ as well as the estimate (4.13) are proved.

To finish, we need to prove $u \in C^{k,\alpha}(\bar{B}^+_{1/2})$, or $m = 0$ case. This is accomplished by using estimates for $m = 1, \cdots, k-2$, and one can show that

$$|D^j f_0|(y) \le c + cy^{k-1+\alpha-j}.$$

This completes the proof that $u \in C^{k,\alpha}(\bar{B}^+_{1/2})$. □

5 Higher order regularity for area-minimizing case

For $n \ge 3$ and $H = 0$, we show the following.

Theorem 5.1 *Suppose that u satisfies the equation*

$$\begin{cases} y\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right) - n u_y = 0 & \text{on } B_1^+ \\ u(x,0) = \varphi(x) & \text{on } B_1 \end{cases} \tag{5.1}$$

with $\varphi \in C^{k,\alpha}(B_1)$, $k \ge n+1$, and $0 < \alpha < 1$. Then

(1) (n is even) $u \in C^{k,\alpha}(\bar{B}^+_{1/2})$ and

$$\|u\|_{C^{k,\alpha}(\bar{B}^+_{1/2})} \le c(n,k,\alpha,\|\varphi\|_{C^{k,\alpha}(B_1)}).$$

(2) *(n is odd) u does not belong to $C^{n+1}(\bar{B}^+_{1/2})$ if φ does not satisfy a partial differential equation $P_n \equiv 0$ on $B_{1/2}$. P_n involves derivatives of φ of order $n+1$, and the form of P_n depends only on n. If the equation $P_n \equiv 0$ is satisfied on B_1, then $u \in C^{k,\alpha}(\bar{B}^+_{1/2})$ with*

$$\|u\|_{C^{k,\alpha}(\bar{B}^+_{1/2})} \le c(n,k,\alpha,\|\varphi\|_{C^{k,\alpha}(B_1)}).$$

To prove the above theorem, we need the following

Lemma 5.1. *Suppose u is a solution for equation (5.1) and $\varphi \in C^{n,\alpha}(B_1)$ for some $0 < \alpha < 1$. Then*

$$\begin{array}{ll} D_y^j u|_{y=0} \equiv 0 & \text{for odd number } j \text{ with } 1 \le j \le n \\ D_y^j \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)\Big|_{y=0} \equiv 0 & \text{for odd number } j \text{ with } 1 \le j \le n-2 \end{array}.$$

If $\varphi \in C^{n+1,\alpha}$ and n is even, then

$$D_y^{n-1}\left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)(x,y) \to 0$$

as $y \to 0$ uniformly in x.

Proof of Lemma.

$D_y u|_{y=0} \equiv 0$ follows from the equation. Then all the mixed derivatives of the form $D_x^\beta D_y u$ is identically 0 on $y = 0$. From this follows

$$D_y\left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)\Big|_{y=0} \equiv 0.$$

Suppose that

$$D_y^l u|_{y=0} \equiv 0$$

$$D_y^l \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)\Big|_{y=0} \equiv 0$$

for all the odd numbers l less than or equal to some odd number s, and $s \le n-2$. Then, take $(s+1)$ derivative of the equation (5.1) with respect to y. By rearranging, we have

$$(n-1-s)u^{(s+2)} = y\left(\Delta u^{(s+1)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(s+1)}\right) \tag{5.2}$$

$$+s\left(\Delta_x u^{(s)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(s)}\right)$$

where $u^{(i)} \equiv D_y^i u$. By the $C^{n,\alpha}$ estimate,

$$|D^k u|(x,y) \le c y^{n+\alpha-k} + c. \tag{5.3}$$

Hence, the first term of the right-hand-side of (5.2) goes to 0 as $y \to 0$ uniformly in x. Since $u^{(s)}|_{y=0} \equiv 0$, we have $\Delta_x u^{(s)}|_{y=0} \equiv 0$, and also $\left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(s)}\Big|_{y=0} \equiv 0$ by the inductive assumption. This shows that $u^{(s+2)}|_{y=0} \equiv 0$. If $s+2 \ge n-1$, this proves the first part of the Lemma. If $s+2 \le n-2$, then we need to show

$$\left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(s+2)}\Bigg|_{y=0} \equiv 0.$$

We can check that

$$\left(\frac{1}{1+|\nabla u|^2}\right)^{(l)}\Bigg|_{y=0} \equiv 0$$

for odd l with $1 \le l \le s+2$. Since

$$\left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(s+2)} = \sum_{l=0}^{s+2} \binom{s+2}{l} \left(\frac{1}{1+|\nabla u|^2}\right)^{(l)} (D_y^{s+2-l} u_i u_j u_{ij})$$

we only need to prove

$$(D_y^{s+2-l} u_i u_j u_{ij})\big|_{y=0} \equiv 0$$

for even l. But this follows from the inductive assumption and s being an odd number. To show the last statement, we use the tangential estimate for $C^{n+1,\alpha}$ and we can conclude the proof. □

We first derive a necessary condition for φ to have u in C^{n+1} up to the boundary. So assume $u \in C^{n+1}(\bar{B}^+_{1/2})$. Take $(n-1)$ derivative of equation (5.1) with respect to y. Then,

$$\begin{aligned} y u^{(n+1)} - u^{(n)} = &-y\left(\Delta_x u^{(n-1)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(n-1)}\right) \\ &-(n-1)\left(\Delta_x u^{(n-2)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(n-2)}\right). \end{aligned} \tag{5.4}$$

By Taylor expansion at any $x \in B_{1/2}$ and by substracting a suitable constant term depending on x from both side, one can see that the left-hand-side of

$$\left\{\Delta_x u^{(n-1)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(n-1)}\right\}\Bigg|_{y=0} \equiv 0 \tag{5.5}$$

has to be satisfied. From the equation (5.1), we have

$$u^{(m)}(x,0) = \frac{m-1}{n-m+1}\left\{\Delta_x u^{(m-2)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right)^{(m-2)}\right\}\Bigg|_{(x,0)} \tag{5.6}$$

for $2 \le m \le n$. If n is odd, then, one can show that (5.5) combined with (5.6) impose a condition on φ in the form of non-trivial partial differential equation of order $n+1$. This shows the necessary condition for odd dimension n. By the previous lemma, (5.5) is satisfied by the even dimension n with an extra assumption that $\varphi \in C^{n+1,\alpha}$.

Proof of theorem.

We first prove $\varphi \in C^{n+1,\alpha}$ implies $u \in C^{n+1,\alpha}$. Also, assume that condition (5.5) is satisfied for the odd dimensional case, which is necessary as was noted. By the tangential derivative estimate, we have

$$D_x^{n-1} u \in C^{2,\alpha}(\bar{B}^+_{1/2})$$

$$\vdots$$

$$D_y^{n-2} D_x u \in C^{2,\alpha}(\bar{B}^+_{1/2}).$$

Hence, we only need to prove that $D_y^{n-2} u \in C^{3,\alpha}(\bar{B}^+_{1/2})$ with the appropriate estimate. Take $(n-2)$ derivative of (5.1). Then,

$$(5.7) \qquad y u_{yy}^{(n-2)} - 2u_y^{(n-2)} = -y \left(\Delta_x u^{(n-2)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij} \right)^{(n-2)} \right)$$
$$-(n-2) \left(\Delta_x u^{(n-3)} - \left(\frac{u_i u_j}{1+|\nabla u|^2} u_{ij} \right)^{(n-3)} \right).$$

By subtracting a suitable constant and linear term, we can assume that both sides of (5.7) vanish at least of order $O(y^2)$ at a fixed x. The crucial point here is that the quadratic term of the right-hand-side of (5.7) also vanishes due the the condition (5.5). With the help of tangential derivatives, we can conclude that

$$(5.8) \qquad |(y u_{yy}^{(n-2)} - 2u_y^{(n-2)})(x,y) - (-2u_y^{(n-2)} - u_{yy}^{(n-2)})(x,0)|$$
$$\le c(n, \|\varphi\|_{C^{n+1,\alpha}}, \alpha) y^{2+\alpha}.$$

This, combined with an argument similar to the one used for $n=2$ shows that

$$D_y^{n-2} u \in C^{3,\alpha}(\bar{B}^+_{1/2}).$$

This concludes the $C^{n+1,\alpha}$ estimate.

For $C^{k,\alpha}$ with $k \ge n+2$, the proof is similar to the argument described for $n=2$ with $H=0$, so we will only sketch the proof. Suppose that the theorem is proved for $C^{k-1,\alpha}$, with the estimate

$$|D^j u|(x,y) \le c + c y^{k-1+\alpha-j}.$$

Assume that $\varphi \in C^{k,\alpha}$. By the tangential derivative estimate, we have

$$(5.9) \qquad |D^j D_x^i u|(x,y) \le c + c y^{k-i+\alpha-j}$$

for any j and $i = k-n, \cdots, k-2$. Then we show that

$$|D^j D_x^{k-n-1} u|(x,y) \le c + c y^{n+1+\alpha-j}$$

by examining the equation satisfied by $D_y^{n-2} D_x^{k-n-1} u$, as was done for $n = 2$. Then, assume that (5.9) is true for any j and for $i = m+1, \cdots, k-2$ with $m+1 \le k-n-1$. Then a similar computation can be carried out to show that (5.9) is satisfied for $i = m$, and this concludes the proof of higher order regularity.

6 Remark on the regularity of non-zero constant mean curvature case

The discussion in the previous section may motivate a conjecture that one can establish similar regularity results for $H \neq 0$. In particular, we have already established the higher order regularity for dimension $n = 2$ without any extra condition on the boundary value φ, so that one may speculate that the regularity for even dimension may be obtained without any extra condition on φ for $H \neq 0$ as well. In this section, we show that there exists a smooth φ in $n = 4$ such that constant mean curvature surface u with $H \neq 0$ is not $C^5 = C^{n+1}$ up to the boundary. This shows that φ has to satisfy some condition to assure that u is as smooth as the boundary value φ. For $n \ge 5$ and even, we do not prove that it is necessary for φ to satisfy extra condition. But the example strongly suggests that constant mean curvature surfaces with $H \neq 0$ may not be as regular as the asymptotic boundary value even if the dimension of the surface is even and $n \ge 6$ unless φ satisfies some extra condition. More precisely,

Theorem 6.1 *For dimension $n = 4$ and $H \neq 0$, there exists a smooth boundary value φ defined on B_1 and a solution to the equation*

$$(6.1)\qquad \begin{cases} y\left(\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij}\right) - n\left(u_y - H\sqrt{1+|\nabla u|^2}\right) = 0 & on\ B_1^+ \\ u(x,0) = \varphi(x) & on\ x \in B_1 \end{cases}$$

such that u does not belong to $C^5(\bar{B}^+_{1/2})$.

Note that the equation (6.1) can be written in divergence form

$$y\left\{\sum_{i=1}^{n-1} D_{x_i}\left(\frac{D_{x_i} u}{\sqrt{1+|\nabla u|^2}}\right) + D_y\left(\frac{D_y u}{\sqrt{1+|\nabla u|^2}}\right)\right\} - n\left(\frac{D_y u}{\sqrt{1+|\nabla u|^2}} - H\right) = 0.$$

By taking the $n-2$ derivative with respect to y and by comparing the coefficient of the Taylor expansion at x, we can obtain a necessary condition for $u \in C^{n+1}(\bar{B}^+_{1/2})$, that is,

$$(6.2)\qquad D_y^{n-1} \sum_{i=1}^{n-1} D_{x_i}\left(\frac{D_{x_i} u}{\sqrt{1+|\nabla u|^2}}\right)\Bigg|_{(x,0)} \equiv 0 \quad \text{for } |x| \le 1/2.$$

Notice that in view of the $C^{n,\alpha}$ estimate, all the normal derivatives of u up to order n are determined in terms of φ, so that the condition involves derivatives of order up to $n+1$.

Proof of Theorem 6.1. We only need to find φ such that the condition coordinate so that $\varphi(0) = \nabla\varphi(0) = 0$. We choose φ such that the second derivatives of φ vanish at 0, i.e., $\nabla^2\varphi(0) = 0$, but at the same time, some of the third derivatives are non zero at the origin. With this choice of boundary value, we will prove that

$$D_y^3 \sum_{i=1}^{3} D_{x_i}\left(\frac{D_{x_i}u}{\sqrt{1+|\nabla u|^2}}\right)(0,0) \neq 0.$$

We recall that

$$u_y(x,0) = \frac{H}{\sqrt{1-H^2}}\sqrt{1+|\nabla\varphi|^2(x,0)} \tag{6.3}$$

$$u_{yy}(x,0) = \frac{1}{3(1-H^2)}\Delta\varphi(x,0) + \frac{3H^2-1}{3(1-H^2)}\frac{\varphi_i\varphi_j}{1+|\nabla\varphi|^2}\varphi_{ij}(x,0).$$

The second identity can be obtained by taking first derivative of the equation we eliminate terms which vanish at the origin. Then,

$$D_y^3 \sum_{i=1}^{3} D_{x_i}\left(\frac{D_{x_i}u}{\sqrt{1+|\nabla u|^2}}\right)(0,0) \tag{6.4}$$

$$= D_y^3\left(\frac{\Delta_x u}{\sqrt{1+|\nabla u|^2}}\right)(0,0) - 3\frac{u_y u_{yyx_j}}{\sqrt{1+|\nabla u|^2}^3}u_{yyx_j}(0,0)$$

the third line follows by using $u_{yx_i}(0,0) = 0$. From (6.3),

$$3\frac{u_y u_{yyx_j}}{\sqrt{1+|\nabla u|^2}^3}u_{yyx_j}(0,0) = \frac{H}{3(1-H^2)}|\nabla(\Delta\varphi)|^2(0,0). \tag{6.5}$$

Next, we have

$$D_y\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right)(0,0) = 0$$

by $u_{yy}(0,0) = 0$ and

$$D_y\Delta_x u(0,0) = \Delta_x u(0,0) = 0.$$

Thus,

$$D_y^3\left(\frac{\Delta_x u}{\sqrt{1+|\nabla u|^2}}\right)(0,0) = \frac{D_y^3\Delta_x u}{\sqrt{1+|\nabla u|^2}}(0,0). \tag{6.6}$$

Hence, we need to find $\Delta_x u_{yyy}(0,0)$. To do so, take second derivative of the equation and rearrange to obtain

$$(n-2)(1-H^2)u_{yyy} = 2\Delta_x u_y - 2\left(\sum_{i,j=1}^{n-1} \frac{u_{x_i}u_{x_j}}{1+|\nabla u|^2}u_{x_ix_j}\right)_y$$

$$-4\left(\sum_{i=1}^{n-1} \frac{u_{x_i}u_y}{1+|\nabla u|^2}u_{x_iy}\right)_y - 2\left(\frac{u_y^2}{1+|\nabla u|^2}\right)_y u_{yy}$$

$$+nH\left(\sum_{i=1}^{n-1} \frac{u_{x_i}u_{x_iy}}{\sqrt{1+|\nabla u|^2}}\right)_y + nH\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)_y u_{yy}$$

where the above is evaluated at $(x,0)$. We have some nice cancelations in case $n=4$, that is,

$$nH\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)_y u_{yy} - 2\left(\frac{u_y^2}{1+|\nabla u|^2}\right)_y u_{yy}$$

$$= 4H\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)_y u_{yy} - 4\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)_y u_{yy} \equiv 0$$

since $\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)(x,0) \equiv H$. Also,

$$nH\left(\sum_{i=1}^{n-1} \frac{u_{x_i}u_{x_iy}}{\sqrt{1+|\nabla u|^2}}\right)_y - 4\left(\sum_{i=1}^{n-1} \frac{u_{x_i}u_y}{1+|\nabla u|^2}u_{x_iy}\right)_y$$

$$= -4\left(\frac{u_y}{\sqrt{1+|\nabla u|^2}}\right)_y \left(\sum_{i=1}^{3} \frac{u_{x_i}u_{x_iy}}{\sqrt{1+|\nabla u|^2}}\right).$$

The second derivative in the tangential direction for the last identity vanishes at the origin, hence we can eliminate this term as well. With these remarks, we have

$$\Delta_x u_{yyy}(0,0)$$

$$= \frac{1}{1-H^2}\Delta_x\Delta_x u_y(0,0) - \frac{1}{1-H^2}\Delta_x\left(\sum_{i,j=1}^{3} \frac{u_{x_i}u_{x_j}}{1+|\nabla u|^2}u_{x_ix_j}\right)_y (0,0).$$

One can check that the second term of the above also vanishes due to the assumption $\nabla^2\varphi(0,0) = 0$. Thus, finally,

$$\Delta_x u_{yyy}(0,0) = \frac{1}{1-H^2}\Delta_x\Delta_x u_y(0,0) \tag{6.7}$$

$$= \frac{H}{\sqrt{1-H^2}^3}\left(\sum_{i,j,k=1}^{3} (\varphi_{ijk})^2 + |\nabla(\Delta\varphi)|^2\right)(0).$$

Combining (6.4), (6.5), (6.6) and (6.7), we obtain

$$D_y^3\left(\sum_{i=1}^{3} D_{x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right)\right)(0,0)$$

$$= \frac{H}{3(1-H^2)}\left(3\sum_{i,j,k=1}^{3}(\varphi_{ijk})^2 + 2|\nabla(\Delta\varphi)|^2\right)(0).$$

If the third derivatives of φ at the origin are non-zero at the origin and $H \neq 0$, then u does not belong to $C^5(\bar{B}^+_{1/2})$ with this choice of φ. □

Remark 6.1 The proof also indicates that u is not in $C^5(\bar{B}^+_{1/2})$ if $|\nabla^2\varphi|(0)$ is sufficiently small relative to the size of $|\nabla^3\varphi|(0)$. This follows from the continuity of φ.

Remark 6.2 It is possible to prove higher order regularity of a non-zero constant mean curvature surface beyond $C^{n,\alpha}$ by assuming condition (6.2). But we will not pursue further in this note.

Remark 6.3 The calculation for dimension n larger than 4 seems quite complicated, even though it is possible in principle. It would be nice if one could prove a similar theorem for any even dimension. It is not proved that there is some boundary value φ which does not permit the surface u to be in C^{n+1} for odd dimension n, but it is most likely that it is so in view of the area-minimizing case.

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