Math. Z. 221, 337–351 (1996)

Mathematische Zeitschrift

Semi-bornological spaces

T.A. Beatty, H.H. Schaefer

Florida Atlantic University, Boca Raton, FL 33431, USA

Received 14 July 1994; in final form 22 August 1994

Introduction

Continuity and sequential continuity were given their essential formulations in the nineteenth century. Their equivalence for spaces commonly studied during that period was attributable, of course, to the underlying metric nature of these spaces. The notion of metric space, in turn, was not fully articulated until the early twentieth century by Fréchet. The classical viewpoint provided no deeper insight into the relationship between these types of continuity or the possible extent of departure from full equivalence because it lacked the framework of modern topological ideas.

In contrast to the classical situation, modern analysis is rife with spaces which are non-metrizable and for which sequential continuity does not entail full continuity. Stonian spaces [4 Ch. V p. 255 or 5 Ch. II Sect. 7 p. 108] are typical in this respect. Convergent sequences on these spaces are stationary, hence all functions on the space are sequentially continuous but not necessarily continuous. Clearly spaces for which the two types of continuity are distinct must not be first countable. However, there exist spaces which are non-first countable for which the continuity notions coalesce for large classes of functions.

This paper explores the detailed relationship of full and sequential continuity in the restricted setting of linear functionals on locally convex topological vector spaces (LCS). As the starting point, a class of spaces is defined by precisely the condition that sequential continuity of any linear form implies full continuity. With notation following [4] define a Hausdorff LCS to be *weakly semi-bornological* (WSB) whenever full and sequential continuity of its linear forms are equivalent. If a WSB space has a Mackey topology [4 IV Sect. 3 p. 131], it is called *semi-bornological* (SB). The terminology stems from the fact that bornological spaces are always Mackey [4 IV Sect. 3.4 p. 132] and display equivalence of continuity types [4 II Sect. 8.3 p. 62] for linear functionals.

The non-degeneracy of the terminology follows from the fact that bornological spaces are not fully characterized by continuity type equivalence for linear forms. Clearly the bornological property implies SB, which in turn implies WSB [8 Ch. 12 p. 176 ff]. It develops that the equivalence of full and sequential continuity for linear forms on a given LCS is identical to such equivalence for linear maps into an arbitrary LCS.

Examples

We now exhibit two non-bornological SB spaces. The first involves the space of bounded real sequences, and the second involves the space of bounded Baire functions on a suitable compact set.

E.1 Space of bounded real sequences

The fact that weak* duals of separable Banach spaces are WSB spaces provides a ready source of non-bornological WSB spaces [1 Sect. 3.2.2 p. 65].

E.1.1 Proposition. $(l_\infty(\mathbf{N}), \tau(l_\infty(\mathbf{N}), l_1(\mathbf{N})))$ *is* SB *but not bornological.*

Proof. Let $E = l_1(\mathbf{N})$, then $l_1(\mathbf{N})' = l_\infty(\mathbf{N})$. It is well-known that E is norm separable. Since $l_1(\mathbf{N})$ is Banach, the remark above implies $l_\infty(\mathbf{N})$ is WSB. Claim: if (f_n) is a sequence in $l_\infty(\mathbf{N})$ that converges in the weak* topology $\sigma(l_\infty(\mathbf{N}), l_1(\mathbf{N}))$, then (f_n) converges in the Mackey topology $\tau(l_\infty(\mathbf{N}), l_1(\mathbf{N}))$. For the balance of this example, denote the weak* topology by σ and the Mackey topology by τ. Without loss of generality, assume that (f_n) σ-converges to 0. $\{(f_n): n \in \mathbf{N}\} \cup \{\mathbf{0}\}$ is σ-compact. σ is Hausdorff; hence $\{(f_n): n \in \mathbf{N}\} \cup \{0\}$ is σ-closed, and consequently closed for the finer topology τ. By Phillips' Lemma [5, Ch. 2, Lemma 10.3, Corollary] the bounded subsets of $l_\infty(\mathbf{N})$ are τ-relatively compact; hence $\{(f_n): n \in \mathbf{N}\} \cup \{0\}$ is τ-compact. It follows that (f_n) τ-converges to 0, for otherwise some τ-neighborhood of 0 would exclude infinitely many terms of (f_n), permitting an irreducibly infinite open cover, contrary to τ-compactness. Thus claim is true and the σ-convergent sequences are precisely the τ-convergent sequences in $l_\infty(\mathbf{N})$.

Now for linear forms on $(l_\infty(\mathbf{N}), \tau)$, τ-continuity certainly implies τ-sequential continuity. Conversely, τ-sequential continuity implies σ-sequential continuity by the equivalence for sequences of σ- and τ-convergence embodied in the claim. $l_\infty(\mathbf{N})$ is WSB by the above, hence σ-sequential continuity implies σ-continuity, and since τ refines σ, τ-continuity as well. It follows that $(l_\infty(\mathbf{N}), \tau)$ is SB. However, $(l_\infty(\mathbf{N}), \tau)$ is not bornological. If it were, then $B^o := \{f \in l_\infty(\mathbf{N}): |f(\xi)| \leqq 1, \xi \in l_1(\mathbf{N})\}$, which is bounded, balanced, and convex would have to be a τ-neighborhood of 0. This requires that $(l_\infty(\mathbf{N}), \tau)' = l_1(\mathbf{N})''$, the norm dual of $l_\infty(\mathbf{N})$, which is impossible, since $l_1(\mathbf{N})$ is irreflexive. Thus $(l_\infty(\mathbf{N}), \tau)$ is SB but not bornological. □

E.2 Space of bounded Baire functions

A classical construction due to Baire begins with an arbitrary infinite, compact, metrizable space K and forms the space $C(K)$ of continuous real functions on K. Set $B_0(K) = C(K)$, and for each countable ordinal α, define $B_{\alpha+1}(K)$ to be the space composed of all pointwise limits in $\mathbf{R}^K$ of sequences from $B_\alpha(K)$. This transfinite recursion results in an increasing nest of Baire classes whose union is sequentially closed. Those elements which are bounded make up the family $B(K)$ used below. Each member of $B(K)$ may be integrated against any measure in $M(K)$, the class bounded Baire measures, which by the Riesz Representation Theorem constitutes the dual of $C(K)$ (See [9] for additional details).

E.2.1 Proposition. *Let K denote a compact metrizable space with* $\mathrm{card}(K) \geqq \aleph_0$, *$B(K)$ the bounded Baire functions on K, and $M(K)$ the bounded Baire measures on K. Then each $\sigma(M(K), B(K))$-compact subset of $M(K)$ is weakly (i.e. $\sigma(M(K), M(K)'))$-compact.*

Proof. Observe that the bilinear form ψ: $B(K) \times M(K) \to \mathbf{R}; (f, \mu) \mapsto \int_K f d\mu$ induces the duality $\langle B(K), M(K) \rangle$. Since K is metrizable, the space of continuous real functions on K, denoted by $C(K)$, is separable with respect to the $\| \cdot \|_\infty$ norm [5, Ch. 2, Proposition 7.5]. By the Riesz Representation Theorem, each continuous linear form on $C(K)$ is realized as a bounded Baire measure and vice versa, hence $C(K)' = M(K)$. The dual unit ball $B^o := \{\mu \in M(K)$: $\int_K f d\mu \leqq 1; \|f\|_\infty \leqq 1\}$ is weak*-compact (i.e. $\sigma(M(K), C(K))$-compact) by Banach–Alaoglu. Moreover, since $C(K)$ is separable, B^o is metrizable. Now let A be a $\sigma(M(K), B(K))$-compact subset of $M(K)$. A is compact for the coarser (Hausdorff) topology $\sigma(M(K), C(K))$, and hence absorbed by B^o. But then $\lambda B^o \supset A$ for some $\lambda \in \mathbf{R}$, and λB^o, hence A, is clearly metrizable for the topology $\sigma(M(K), C(K))$, so $\sigma(M(K), C(K))$ and $\sigma(M(K), B(K))$ must coincide on A. Thus $\sigma(M(K), B(K))$ restricted to A is metrizable as well. Suppose (μ_n) is a sequence of measures in A. Since A is $\sigma(M(K), B(K))$-compact and metrizable, A is $\sigma(M(K), B(K))$-sequentially compact, whence (μ_n) contains a $\sigma(M(K), B(K))$-convergent subsequence, say $(\mu_{n(k)})$. Since $B(K)$ is a Dedekind σ-complete unital AM-space, it follows that $(\mu_{n(k)})$ $\sigma(M(K), M(K)')$-converges [5, Ch. 2, Theorem 10.4] in $M(K)$. Since A is $\sigma(M(K), C(K))$-closed, it is $\sigma(M(K), M(K)')$-closed as well. Finally, by Eberlein's Theorem [4, Ch. 4, Theorem 11.1, Corollary 2] A is $\sigma(M(K), M(K)')$-compact. □

E.2.2 Proposition. *The topology induced on $B(K)$ by $\tau(M(K)', M(K))$ is the Mackey topology $\tau(B(K), M(K))$. Also, $(M(K)', \tau(M(K)', M(K)))$ can be regarded as the completion of $(B(K), \tau(B(K), M(K)))$.*

Proof. For a given duality $\langle F, G \rangle$, a zero neighborhood base for $\tau(F, G)$ is given by the family of polars of the τ-equicontinuous sets in G, namely the class of subsets of all $\sigma(G, F)$-compact, circled, convex sets. In the case at hand, a zero neighborhood base for $\tau(M(K)', M(K))$ is given by the polars of all $\sigma(M(K),M(K)')$-compact, circled, convex sets. These sets are $\sigma(M(K), B(K))$-compact as well, since $B(K) \subseteq M(K)'$. By E.2.1 every $\sigma(M(K),B(K))$-compact set is $\sigma(M(K),M(K)')$-compact. Denote by $\mathscr{F}$ the family of convex, circled $\sigma(M(K),M(K)')$-compact subsets of $M(K)$. $\{A^o: A \in \mathscr{F}\}$ is a zero neighborhood base for $\tau(M(K)',M(K))$, and $\{A^o: A \in \mathscr{F}\} \cap B(K)$ is a zero neighborhood base for its restriction to $B(K)$, equivalently for $\tau(B(K),M(K))$. Denote by B^- the completion of $(B(K),\tau(B(K),M(K)))$, and let $\mathfrak{C}$ be the family of all convex, circled, $\sigma(M(K),B(K))$-compact sets $C \subset M(K)$. B^- is the set of all linear forms u on $M(K)$ with the property that for each $C \in \mathfrak{C}, u|_C$ is $\sigma(M(K),B(K))$-continuous [4, Ch. 4, Theorem 3.2, Corollary 1]. Claim: $B^- = M(K)'$. Certainly if $u \in M(K)'$, then $u|_C$ is $\sigma(M(K),M(K)')$-continuous for each $C \in \mathfrak{C}$. E.2.1 establishes that the $\sigma(M(K),M(K)')$-compact sets are the same as the $\sigma(M(K),B(K))$-compact sets. It follows that the $\mathfrak{C}$-topology on $B(K)$ is the relativization of the $\mathfrak{C}$-topology on $M(K)'$ to $B(K)$. Hence u is $\sigma(M(K),B(K))$-continuous, and $u \in B^-$.

Conversely, if $u \in B^-$, then u is $\sigma(M(K),M(K)')$-continuous on each $C \in \mathfrak{C}$, by Grothendieck's theorem [4, Ch. 4, Theorem 6.2, Corollary 1]. Accordingly, u is bounded on the unit ball of $M(K)'$. If not, there exists a sequence in $M(K)$, say (μ_n), such that $\lim_{n\to\infty} \int_K d|\mu_n| \to 0$ and $|u(\mu_n)| > n$. But $\{\mu_n\}_{n\in\mathbb{N}} \cup \{0\}$ is norm-compact, hence its closed, convex, circled hull is norm-compact, hence $\sigma(M(K),M(K)')$-compact. By $\sigma(M(K),M(K)')$-continuity of u on such a set, $\{|u(\mu_n)|\}_{n\in\mathbb{N}}$ must be bounded, which is absurd. Since u is bounded on the unit ball, it is continuous on $M(K)$, hence $u \in M(K)'$, establishing the claim. By Grothendieck's theorem once more, $M(K)'$ endowed with the $\mathfrak{C}$-topology is a complete locally convex space in which $B(K)$ is dense. Since the $\mathfrak{C}$-topology was selected to induce $\tau(B(K),M(K))$, the result follows. □

E.2.3 Proposition. *$B(K)$ and $M(K)'$ are topological Riesz spaces under their respective Mackey topologies $\tau(B(K),M(K))$ and $\tau(M(K)',M(K))$. Moreover, the respective sets of sequentially continuous linear forms constitute ideals in their respective Riesz duals* [5 Ch II Sect. 5 p. 81 ff].

Proof. It must be shown that $(B(K),\tau(B(K),M(K)))$ and $(M(K)',\tau(M(K)', M(K)))$ have bases at zero consisting of solid sets. Both Mackey topologies are formed by taking polars (in the respective space) of all compact disks in $M(K)$, where compactness is understood to be relative to the appropriate weak topology. $M(K)$ is an abstract Lebesgue (AL) space under the integral norm coming from the bilinear form defining the duality under consideration. Accordingly, if A is a $\sigma(M(K),B(K))$-compact disk, the closed, solid hull of A is likewise $\sigma(M(K),B(K))$-compact [5, Ch. 2, Proposition 8.8, Corollary]. It follows that the solid hulls of weakly compact disks in $M(K)$ generate the same $\mathfrak{C}$-topology as the disks themselves, namely $\tau(B(K),M(K))$. But the polar of a solid set is solid [5, Ch. 2, Proposition 4], hence $(B(K),\tau(B(K),M(K)))$ has a neighborhood base at zero consisting of solid sets and thus is a topological Riesz space.

For the larger space $M(K)'$, observe that every $\sigma(M(K),M(K)')$-compact set is trivially $\sigma(M(K),B(K))$-compact, and conversely by E.2.1. It follows that $\tau(M(K)',M(K))$ is the $\mathfrak{C}$-topology induced by the same class of solid, $\sigma(M(K),B(K))$-conpact sets as in the preceding; hence $(M(K)',\tau(M(K)', M(K)))$ is also a topological Riesz space.

Denote $(B(K),\tau(B(K),M(K)))$ by (B,τ) and the family of τ-sequentially continuous linear forms on B, or *sequential dual*, by $(B,\tau)'_C$. Certainly $(B,\tau)'_C$ is a subspace of the Riesz dual, denoted by $(B,\tau)^b$; hence to show it is an ideal it is enough to verify that $(B,\tau)'_C$ is both Riesz and solid.

Claim 1. If $f \in (B,\tau)'_C$, then $|f| \in (B,\tau)'_C$, where $|f|$ is $\sup\{-f,f\}$ taken in $(B,\tau)^b$. For the sake of contradiction, suppose not. Then for pre-assigned $\varepsilon > 0$ there exists a τ-null sequence $\{x_n\}$ in the positive cone B^+ such that $|f|(x_n) > \varepsilon$ for all $n \in \mathbf{N}$. Now $|f|(x) := \sup\{|f(z)|: |z| \leqq x;\ x \geqq 0\}$, so for each $n \in \mathbf{N}$ there exists a $z_n \in B$ such that $|z_n| \leqq x_n$ and $|f(z_n)| > \varepsilon/2$ by the supremum property. By the local solidity of τ established in the preceding, it follows that $\{z_n\}$ is a τ-null sequence as well. But $\{f(z_n)\}$ cannot converge to 0, contrary to sequential continuity. Thus $|f|$ is sequentially continuous and $(B,\tau)'_C$ is Riesz.

Claim 2. If $g \in (B,\tau)^b$ and $|g| \leqq |f|$ for $f \in (B,\tau)'_C$, then $g \in (B,\tau)'_C$. Suppose $\{x_n\}$ is a τ-null sequence. It is true that $|g(x_n)| \leqq |g|(|x_n|) \leqq |f|(|x_n|)$. The sequential continuity of $|f|$ forces the sequence $\{|g(x_n)|\}$ to converge to 0; hence $|g|$ is sequentially continuous as well, establishing solidity. It follows that $(B,\tau)'_C$ is an ideal in $(B,\tau)^b$.

For the larger space $(M(K)',\tau(M(K)',M(K)))$, the foregoing argument applies mutatis mutandis. □

E.2.4 Proposition. *$B(K)$ endowed with the Mackey topology induced by the duality $\langle B(K),M(K)\rangle$ is* SB *but not bornological. Moreover, $B(K)$ is both Dedekind σ-complete and $\tau(B(K),M(K))$-sequentially complete.*

Proof. Let $\tau = \tau(B(K),M(K))$ and $\mu \in (B(K),\tau)'_C$. To show $(B(K),\tau)$ is SB, it is clearly enough to establish $\mu \in M(K)$. First observe that μ is bounded on bounded sets. Otherwise, by the usual argument, there exists a sequence $\{f_n\}$ of Baire functions in the $\|\cdot\|_\infty$-norm unit ball such that $|\mu(f_n)| > n$. But $\{n^{-1}f_n\}$ is a norm-null sequence in $B(K)$, hence a fortiori a τ-null sequence as well, hence μ cannot be sequentially continuous, contrary to supposition. Denote by Σ the σ-algebra of Baire sets in K (the minimal σ-algebra in $\wp(K)$ which contains every compact G_δ-set). In the terminology of Dunford and Schwartz [2], $\mu \in \mathrm{ba}(\Sigma)$, the bounded finitely additive Baire measures on K, and it must be established that $\mu \in \mathrm{ca}(\Sigma)(\equiv M(K))$, the countably additive Baire measures on K. To this end, let $A_n \downarrow \phi$ in Σ, that is to say $m > n$ implies $A_m \subset A_n$, and $\bigcap_{n\in\mathbf{N}} A_n = \phi$. This forces the characteristic functions χ_{A_n} to converge monotonically to the null function on K. Certainly for every measure $\nu \in M(K)$, $\lim_{n\to\infty}\nu(A_n) = 0$; otherwise ν cannot be countably additive. Since

$\chi_{An} \to 0$ pointwise on $M(K)$, it follows that $\chi_{An} \to 0$ for $\sigma(B(K), M(K))$ by Lebesgue's Dominated Convergence Theorem. E.2.3 has established that $(B(K), \tau)$ is locally solid, hence it follows [4, Ch. 5, Theorem 7.1] that $B(K)^+$ is a normal cone. But under this condition, any subset S of $B(K)$ that is directed for the canonical order has the property that if the section filter $\mathscr{F}(S)$ converges weakly, then it converges with respect to the original topology, τ [4, Ch. 5, Theorem 4.3]. Applied to the set $\{\chi_{An}\}$, it is clear that $\chi_{An} \to 0$ for τ. Returning to μ, since μ is τ-sequentially continuous on $B(K)$, it follows that $\lim_{n\to\infty}\mu(\chi_{An}) = 0$. But then μ is countably additive, or $\mu \in M(K)$, and $(B(K), \tau)$ is SB.

$(B(K), \tau)$ is not bornological, however. If it were, each bornivorous disk would necessarily be a neighborhood of zero. Consider the unit ball $U := \{f \in B(K) : \|f\|_\infty \leqq 1\}$. U certainly absorbs any τ-bounded set. Suppose, for the sake of contradiction, that U is a τ-neighborhood of zero. By Kolmogorov's normability criterion, since U is a bounded, convex zero neighborhood for τ, τ must actually be the norm topology. Now $C(K)$ is a subspace of $B(K)$ which separates $M(K)$; hence $C(K)$ is $\sigma(B(K), M(K))$-dense in $B(K)$ [4, Ch. 4, Theorem 1.3]. $C(K)$ is convex, so its closure for any consistent topology, in particular τ, is the same as its weak closure. But this means the norm closure of $C(K)$ is $B(K)$. However, $C(K)$ is norm-complete, hence norm-closed. It follows that $C(K) = B(K)$, which is absurd if the ground space K is infinite, which is assumed. Therefore, U is a bornivorous set which is not a zero neighborhood; whence $(B(K), \tau)$ is not bornological.

For Dedekind σ-completeness, it must be shown that if A is a countable subset of $B(K)$, and A is majorized in $B(K)$, then sup A exists in $B(K)$. Let $\{a_i\}$ be an enumeration of the elements of A, and define $b_n := \sup_{i \leqq n}\{a_i\}$. The finite suprema b_n are certainly in $B(K)$, and letting $b := \lim_{n\to\infty} b_n$, it can be seen that $b = \sup A$. But b is a pointwise sequential limit, and $B(K)$ is sequentially closed, hence $b \in B(K)$, as required.

For τ-sequential completeness, suppose $\{f_n\}$ is a τ-Cauchy sequence in $B(K)$. τ is the $\mathfrak{S}$-topology induced by the class of solid, convex, $\sigma(M(K), B(K))$-compact sets in $M(K)$; hence it may be viewed as being generated by the Minkowski functionals defined by the sets in this class, namely semi-norms of the form $p_A(f) = \sup_{\mu \in A} \int_K |f| d\mu$ for $A \in \mathfrak{S}$. The τ-Cauchy condition is then expressible as follows: for pre-assigned $\varepsilon > 0$ and for all $A \in \mathfrak{S}$, there exists an $N \in \mathbf{N}$, which depends on both ε and A, such that for $n, m > \mathbf{N}$, $p_A(f_n - f_m) \leqq \varepsilon$. The Dirac measure δ_t is a Baire measure on K, since singletons are Baire sets in compact, metrizable spaces. δ_t must be in some $A \in \mathfrak{S}$, hence $\int_K |f_n(t) - f_m(t)| d\delta_t = |f_n(t) - f_m(t)| \leqq \varepsilon$, provided $m, n > \mathbf{N}$. It follows that $\lim_{n\to\infty} f_n(t)$ exists pointwise. Additionally, the sequence $\{f_n\}$ is τ-bounded, hence $\|\cdot\|_\infty$-bounded [1 Sect. 5.3.8 p. 108]. Thus for all $n \in \mathbf{N}$, $\|f_n\|_\infty \leqq c$, for some constant c, and $f(t) := \lim_{n\to\infty} f_n(t)$ is in $B(K)$. Claim: f is the τ-limit for $\lim_{n\to\infty} f_n(t)$ as well. Clearly, $|f_n(t) - f_m(t)| < 2c$; hence by Lebesgue's Dominated Convergence Theorem, $\lim_{m\to\infty} \int_K |f_n(t) - f_m(t)| d\mu = \int_K |f_n(t) - f(t)| d\mu$. It follows $\sup_{\mu \in A} \int_K |f_n(t) - (t)| \leqq \varepsilon$, or equivalently, $p_A(f_n(t) - f(t)) \leqq \varepsilon$. But A and ε were arbitrary; hence $\{f_n\}$

converges sequentially with respect to the entire family of semi-norms $\{p_A\colon A \in \mathfrak{S}\}$, and accordingly with respect to τ. □

It is not mandatory in the preceding that $M(K)'$ strictly include $B(K)$. The inclusion is proper whenever the compact space K contains a (non-void) perfect subset. In the event K is scattered (i.e. contains no non-void perfect subset) and $B(K)$ is Dedekind complete, $M(K)'$ collapses to precisely $B(K)$ [6].

The significance of the example developed in E.2.1–E.2.4 is twofold. First, as advertised, a non-bornological SB space is presented. Second, the space $(B(K), \tau)$ is a natural object, namely the space of bounded Baire functions outfitted with the finest locally convex topology that preserves the validity of the Riesz Representation Theorem. Any finer locally convex topology would necessarily introduce continuous linear forms which would not be realizable as bounded Baire measures.

Stability properties

We turn now to the question of stability of SB and WSB spaces under common topological constructions. It would be reasonable to expect that inductive topological constructions involving SB spaces would be stable since such constructions and the SB notion are both defined in terms of a finest locally convex topology which preserves a given property. In the former case, the continuity of a family of linear maps is preserved, and in the latter case, the family of linear maps which are a priori continuous is invariant. Intuitively, it is plausible that a topology cannot be refined indefinitely without ruining the continuity of linear maps into the space or without conferring continuity upon originally discontinuous linear maps defined on the space. That the amount of refinement tolerable for either case is precisely the same is the key to the next result.

S.1 Proposition. *Let $\{L_\alpha, T_\alpha\}_{\alpha \in A}$ be a family of* SB *spaces and $\{f_\alpha\}_{\alpha \in A}$ a corresponding family of linear maps into a vector space L (i.e. $f_\alpha\colon L_\alpha \to L$). Endow L with the finest locally convex topology $\mathfrak{F}$ such that each f_α is continuous. Then $(L, \mathfrak{F})$ is* SB.

Proof. It must be shown that continuity and sequential continuity of linear forms on L are equivalent, and that $\mathfrak{F}$ is a Mackey topology, i.e. the finest locally convex topology consistent with the continuous dual. For the former it clearly suffices to show that sequential continuity entails full continuity. Suppose $u\colon L \to \mathbf{R}$ is a sequentially continuous linear form. Then u induces a sequentially continuous linear form on any L_α given by $u \circ f_\alpha$. By hypothesis $u \circ f_\alpha$ is continuous, showing u to be a continuous linear form. It remains to verify that $\mathfrak{F} = \tau(L, L')$. Each map f_α is T_α-$\tau(L, L')$-continuous [4, Ch. 4, Theorem 7.4(b$_1$)], since T_α is already Mackey. It follows that $\mathfrak{F}$, which is induced by the family $\{f_\alpha\}_{\alpha\in A}$, must refine $\tau(L,L')$. But $\tau(L,L')$ refines $\mathfrak{F}$ by definition; hence $\mathfrak{F} = \tau(L,L')$, and the result is established. □

S.2 Proposition. *Let (L,T) be a semi-bornological space, and M a closed subspace of L. Then the quotient L/M is semi-bornological.*

Proof. Apply S.1 to the space (L,T) and canonical quotient map $q\colon L \to L/M$. Note that the closedness of M ensures the quotient is Hausdorff. □

S.3 Proposition. *Let $\{L_\alpha, T_\alpha\}_{\alpha\in A}$ be a family of semi-bornological subspaces of the vector space $L = \bigcup_{\alpha\in A} L_\alpha$. Suppose the index set A is partially ordered by " $<$ ", and whenever $\alpha,\beta \in A$ and $\alpha<\beta$, there exists a continuous embedding of L_α into L_β. Equip L with the finest topology $\mathfrak{F}$ that makes each embedding $h_\alpha\colon L_\alpha \to L$ continuous. Then $(L,\mathfrak{F})$ is semi-bornological.*

Proof. Apply S.1 to the family of spaces $\{L_\alpha, T_\alpha\}_{\alpha\in A}$ with maps $\{h_\alpha\colon L_\alpha \to L\}_{\alpha\in A}$ where $L = \bigcup_{\alpha\in A} L_\alpha$. □

For projective constructions, the situation is not so tidy. Permanence under formation of very general products is available, but subspaces of SB spaces may fail to be SB (or even WSB). This parallels the situation for fully bornological spaces.

S.4 Proposition. *There exist non-*SB *subspaces of* SB *spaces.*

Proof. Suppose V is any real vector space, and $B = \{x_\alpha\colon \alpha\in A\}$ is a Hamel basis for V. V is algebraically isomorphic to the direct sum $\bigoplus_{\alpha\in A}\mathbf{R}$, and its algebraic dual V^* is $\prod_{\alpha\in A}\mathbf{R}$. The weak* topology in this setting is the product topology on $\prod_{\alpha\in A}\mathbf{R}$. The Mackey–Ulam Theorem asserts that for index set cardinality less than the smallest strongly inaccessible cardinal d_0, $\prod_{\alpha\in A}\mathbf{R}$ is bornological, hence a fortiori SB. Thus $(V^*,(\sigma(V^*,V)))$ is SB assuming $\mathrm{card}(A) < d_0$. Take V to be l_∞, and let $E = (l_\infty^*, \sigma(l_\infty^*, l_\infty))$. By definition, $E' = l_\infty$. It is well-known that l_1 is irreflexive [2, IV.15]; hence l_∞ is as well. In fact $l'_\infty \cong \mathrm{ba}(\mathbf{N})$, the bounded additive measures on $\mathbf{N}$. $l_1 \cong \mathrm{ca}(\mathbf{N})$, the countably additive measures on $\mathbf{N}$, and $\mathrm{ba}(\mathbf{N}) \supsetneq \mathrm{ca}(\mathbf{N})$. Consider the continuous bidual E_0 of l_1 endowed with the weak topology induced by l_∞. Claim: $E_0 := (\mathrm{ba}(\mathbf{N}), \sigma(\mathrm{ba}(\mathbf{N}), l_\infty))$ is not WSB. Suppose it were. Every linear form that is $\sigma(\mathrm{ba}(\mathbf{N}), l_\infty)$-sequentially continuous is $\sigma(\mathrm{ba}(\mathbf{N}), \mathrm{ba}(\mathbf{N})')$-sequentially continuous because the predual of $\mathrm{ba}(\mathbf{N})$, namely l_∞, is a Grothendieck space (i.e. weak* sequential convergence implies weak sequential convergence with respect to the bidual). Then by assumption every functional in $\mathrm{ba}(\mathbf{N})'$ is $\sigma(\mathrm{ba}(\mathbf{N}), l_\infty)$-continuous. It follows that $\sigma(\mathrm{ba}(\mathbf{N}), \mathrm{ba}(\mathbf{N})') = \sigma(\mathrm{ba}(\mathbf{N}), l_\infty)$, which is absurd since $\mathrm{ba}(\mathbf{N})' \supsetneq l_\infty$, and the claim is established. Now $E(= l_\infty^*)$ is the completion of E_0 with respect to $\sigma(\mathrm{ba}(\mathbf{N}), l_\infty)$, and E is SB by the above argument. However, the subspace $E_0 \subset E$ is not SB nor even WSB. □

The foregoing example extends the well-known result that subspaces of bornological spaces need not be bornological [3 Sect. 28.4].

As a further illustration of the special role played by sequentially continuous linear forms, consider the curious fact that such forms on a product of SB spaces turn out to be almost identically zero in the sense that a given functional can be non-zero only on a finite number of co-ordinates. This result is put to use in the following analogue to the well-known theorem that bornology is preserved for products over arbitrary index sets, as long as they are not "too large", i.e. their cardinality must be constructible within the framework of the Zermelo–Fraenkel + Choice axiom system.

S.5 Proposition. *Let $\{L_\alpha, T_\alpha\}_{\alpha\in A}$ be a family of* SB *spaces indexed by A, where* card$(A) < d_0$ *the smallest strongly inaccessible cardinal. If $X := \prod\{L_\alpha : \alpha \in A\}$, then X is* SB.

Proof. Since an arbitrary product of Mackey spaces is Mackey [4, Ch. 4, Theorem 4.3, Corollary 2] and product continuity implies product sequential continuity, it remains only to show that product sequential continuity of an arbitrary linear form is equivalent to full continuity. Suppose u is a sequentially continuous linear form on X. Let u_α be the restriction of u to L_α. Claim: u_α is identically zero for all but finitely many co-ordinates. For the sake of contradiction, suppose $u_\alpha \neq 0$ for indices in the set B, where $\aleph_0 \leqq \text{card}(B) \leqq \text{card}(A) < d_0$. For $\alpha \in B$, $u_\alpha(x_\alpha) \neq 0$ for some $x_\alpha \in L_\alpha$. Form $Y := \prod\{\mathbf{R}x_\alpha : \alpha \in B\}$. Y is certainly a linear subspace of X; moreover it is topological under the relative topology coming from X. The map $\zeta : Y \to \mathbf{R}^B$; $(\lambda_\alpha x_\alpha) \mapsto (\lambda_\alpha)$ is bijective. Viewing Y as $\prod_{\alpha\in B}\{\mathbf{R}x\{x_\alpha\}\} \cong \mathbf{R}^B x \prod_{\alpha\in B} x_\alpha = \mathbf{R}^B x(x_\alpha)$, where (x_α) is a fixed vector, it is clear that ζ is bicontinuous as well; hence Y is linearly homeomorphic to $\mathbf{R}^B$. But for product index cardinality less than d_0, $\mathbf{R}^B$ is bornological [3, Sect. 28.8]; hence so is Y. Now u is sequentially continuous on X by hypothesis; hence its restriction to Y, denoted by u_Y, is likewise. The bornological property of Y implies that u_Y is continuous on Y, in particular at zero. Accordingly, u_Y must be bounded for some 0-neighborhood V in Y. A technical argument [1 p. 82] may be adduced to show that the infinite cardinality of B contradicts this boundedness condition on u_Y. It must be concluded that B cannot be an infinite set, and the claim is establised.

For a given u, the finite set B is determined for which $\alpha \in A\backslash B$ implies $u_\alpha = 0$. Write $x \in X$ as the finite sum of vectors x_α with support only at the α-th co-ordinate for $\alpha \in B$ plus a complementary vector $\hat{x}$ supported on co-ordinates in $A\backslash B$. Let $Z := \prod\{L_\alpha : \alpha \in A\backslash B\}$. Denote the family of finite subsets of $A\backslash B$ by $\mathscr{F}$. $\mathscr{F}$ is directed upwards by inclusion and induces a net $\{x_H\}_{H\in\mathscr{F}}$ where x_H is zero on co-ordinates in $A\backslash(B \cup H)$ and co-incides with $\hat{x}$ on those in H. $u_z(x_H) = 0$ by hypothesis. By the earlier argument, Z is homeomorphic to $\mathbf{R}^{A\backslash B}$, hence bornological. It follows that the image net $\{u_z(x_H)\}_{H\in\mathscr{F}}$ converges to $u_z(\lim_{\mathscr{F}}\{x_H\}_{H\in\mathscr{F}}) = u_z(\hat{x})$. But the image net is null; hence $u_z(\hat{x}) = 0$. Thus $u(x) = u(\sum_{\alpha\in B} x_\alpha + \hat{x}) = u(\sum_{\alpha\in B} x_\alpha) + u(\hat{x}) = u(\sum_{\alpha\in B} x_\alpha) = \sum_{\alpha\in B} u(x_\alpha) = \sum_{\alpha\in B} u_\alpha(x_\alpha)$. Clearly $u = \bigoplus_{\alpha\in B} u_\alpha$ is continuous as the finite sum of elements of L'_α. Thus X is SB. □

It should be noted that the above yields $(\prod\{L_\alpha : \alpha \in A\})' = \bigoplus\{L'_\alpha : \alpha \in A\}$.

We conclude the survey of stability properties with a result on the sequential completion of an SB space. Recall that a topological vector space L admits a uniform completion $\tilde{L}$. Define the *sequential completion* $\hat{L}$ to be the sequential closure of L taken within $\tilde{L}$ (Cf [1 p. 84]).

S.6 Proposition. *Let (L,T) be an* SB *space. Then its sequential completion $(\hat{L},\hat{T})$ is* SB.

Proof. Denote the uniform completion of (L,T) by $(\tilde{L},\tilde{T})$. Observe that $\hat{T}$ is $\tilde{T}$ relativized of $\hat{L}$. It is necessary to show that $\hat{T}$ is the Mackey topology, and sequential continuity of linear forms on $\hat{L}$ is equivalent to full continuity. For the former, note that the topological duals of $\hat{L}$ and $\tilde{L}$ co-incide (with appropriate restriction of domain) with L' by denseness of L in $\tilde{L}$, hence $\hat{L}$. $\tilde{T}$ is a Mackey topology [4, Ch. 4, Theorem 3.5], so a fundamental system of equicontinuous sets consists of the convex, circled, $\sigma(L',\hat{L})$-compact subsets, denoted by $\mathfrak{C}$. These sets are certainly compact for the coarser topology $\sigma(L',L)$. T is Mackey, so $\mathfrak{C}$ is an equicontinuous family for T.

Conversely, denseness requires any T-equicontinuous set to be $\hat{T}$-equicontinuous as well. It follows that the equicontinuous sets for each topology coincide and that $\mathfrak{C}$ is the (saturated) family of all equicontinuous sets for $\hat{T}$. But the Mackey topology on $\hat{L}$ is precisely the $\mathfrak{C}$-topology. Hence $\hat{T}$ is the Mackey topology on $\hat{L}$.

For the latter, it suffices to show sequential continuity entails full continuity. If u is a $\hat{T}$-sequentially continuous linear form on $\hat{L}$, then its restriction to L is certainly T-sequentially continuous. Since (L,T) is semi-bornological, $u|_L$ is T-continuous. By denseness, $u|_L$ has a unique $\tilde{T}$-continuous extension to $\tilde{L}$, say $\tilde{u}$. Clearly $\tilde{u}|_{\hat{L}}$ and u are sequentially continuous on $\hat{L}$ and their restrictions to L agree. It follows that they co-incide on $\hat{L}$. But $\tilde{u}|_{\hat{L}}$ is continuous on $\hat{L}$; hence so is u. □

For the sake of comparison, it should be noted that the strictly bornological version of the preceding theorem, wherein the full uniform completion is taken instead of the sequential completion, is not true.

The sequential dual

We continue with the problem of characterizing the sequential dual of a given LCS (see E.2.3 for definition). It will emerge that the nature of the sequential dual is a litmus test for the WSB property. The application hinges on construction of the topology of uniform convergence on a distinguished class of subsets of the LCS derived from vector null sequences. The following lemma sets out the details.

SD.1 Lemma. Let (L,T) be a Hausdorff, locally convex space which is complete relative to its Mackey topology. Denote by $\mathfrak{C}$ the collection of all subsets of L which are the closed, convex, circled hull of the range of some null sequence $\{x_n\}$ in L. Then every $C \in \mathfrak{C}$ can be expressed in the form

$C = \{\sum_{n \in N} \zeta_n x_n : \{\zeta_n\} \in B\}$, *where B is the closed unit ball of the Banach space* l_1.

Proof. Each set C is weakly compact by Krein's Theorem [4, Ch. 4, Theorem 11.4]; hence $\mathfrak{C}$ is a subfamily of the family of sets whose polars induce the Mackey topology on L', namely the family of all weakly compact, convex, circled sets in L. Accordingly, the $\mathfrak{C}$-topology on L' is coarser than $\tau(L',L)$, hence consistent with the duality $\langle L, L' \rangle$.

Consider the map $\phi : (L', T_{\mathfrak{C}}) \to c_0$ given by $\phi(f) := \{\langle x_n, f \rangle\}$, where c_0 is the space of all real null sequences (equipped with the supremum norm). Now c_0 has norm-continuous dual l_1. The canonical bilinear form associated with these spaces is $\langle \cdot , \cdot \rangle : c_0 x l_1 \to \mathbf{R}$; $\langle \xi, \zeta \rangle = \sum_{n \in \mathbf{N}} \xi_n \zeta_n$, where $\xi = \{\xi_n\}$ and $\zeta = \{\zeta_n\}$. The adjoint map ϕ' is implicitly defined by the relation $\langle \phi'(\xi), \zeta \rangle = \langle \xi, \phi(\zeta) \rangle$. Then $\langle \phi(f), \{\zeta_n\} \rangle = \sum_{n \in \mathbf{N}} \langle x_n, f \rangle \zeta_n$. This sum is well-defined by Dirichlet's Convergence Theorem since $\{\langle x_n, f \rangle\}$ is a null sequence by continuity of f, and the series $\sum_{n \in \mathbf{N}} \zeta_n$ has bounded partial sums (even absolutely) since the l_1 norm of ζ is finite by assumption. By linearity of $\langle \cdot , f \rangle$, $\sum_{n \in \mathbf{N}} \langle x_n, f \rangle \zeta_n = \sum_{n \in \mathbf{N}} \langle \zeta_n x_n, f \rangle$. Claim: $\sum_{n \in \mathbf{N}} \langle \zeta_n x_n, f \rangle = \langle \sum_{n \in \mathbf{N}} \zeta_n x_n, f \rangle$. Again by linearity, $\langle \cdot , f \rangle$ commutes with finite summation. It suffices to show that there exists an $M \in \mathbf{N}$ such that for pre-assigned $\varepsilon > 0$, $|\sum_{n > M} \langle \zeta_n x_n, f \rangle| < \varepsilon$. But this is immediate from the established convergence, and the claim is valid. It follows that $\phi'(\{\zeta_n\}) = \sum_{n \in \mathbf{N}} \zeta_n x_n$. Note that the partial sums of the preceding form a Cauchy sequence in L, and by assumption L is complete, so $\sum_{n \in \mathbf{N}} \zeta_n x_n \in L$. Clearly the image under ϕ of any set C^0, $C \in \mathfrak{C}$, is contained in the closed unit ball of c_0, since $\sup_n |\langle x_n, f \rangle| \leqq 1$ by definition of the polar. Thus ϕ is $T_{\mathfrak{C}} - \beta(c_0, l_1)$-continuous. It follows [4, Ch. 4, Theorem 7.4 $(b_1) \Rightarrow (b_4)$] that ϕ' is $\sigma(l_1, c_0) - \sigma(L, L')$ continuous. By Banach–Alaoglu, the closed unit ball B in l_1 is $\sigma(l_1, c_0)$-compact; hence its adjoint image is $\sigma(L, L')$-compact by weak continuity. This image contains the closed, convex, circled hull of $\{x_0\}$, that is, C. On the other hand $C \supset \phi'(B)$. Thus $C = \phi'(B)$, and C is expressible as $\{\sum_{n \in \mathbf{N}} \zeta_n x_n : \{\zeta_n\} \in B\}$. □

We note without proof the following corollary.

SD.2 Corollary. SD.1 *remains valid when the assumption of Mackey completeness is replaced by the assumption of T-sequential completeness.*

SD.3 Proposition. *Let (L, T) be a Hausdorff, locally convex space which is T-sequentially complete. Denote by $\mathfrak{C}$ the collection of all subsets of L which are closed, convex, circled hulls of some null sequence in L. Then $f \in L^*$ is sequentially continuous if and only if the restriction of f to each $C \in \mathfrak{C}$ is weakly continuous.*

Proof. Suppose $f|_C$ is $\sigma(L, L')$-continuous for each $C \in \mathfrak{C}$. It must be shown that if $\{x_n\}$ T-converges to $0 \in L$, then $\{f(x_n)\}$ converges to $f(0)$. Consider the closed, convex, circled hull of $\{x_n\}$, denoted by C_0. Clearly $C_0 \in \mathfrak{C}$ and $0 \in C_0$. By assumption, f restricted to C_0 is weakly continuous, hence continuous for the finer topology T. In particular, f is T-continuous at 0 on C_0. It follows that $\{f(x_n)\}$ converges to $f(0)$, as required.

Conversely, suppose f is T-sequentially continous on L. Consider the space $(l_1, \sigma(l_1, c_0))$. As noted in SD.1, the closed unit ball B in this space is weakly compact by Banach–Alaoglu. Since c_0 is separable, B is metrizable. It has been verified in SD.1 that every null sequence in L defines both a $C \in \mathfrak{C}$ and a map ϕ_C. Moreover every such set C is expressible as the image of B under the adjoint map ϕ'_C. This image is likewise compact by continuity, and furthermore metrizable by a standard technical argument [1 p. 107]. Thus the sequential continuity of f on each $C \in \mathfrak{C}$ is equivalent to weak continuity. □

The following theorem presents a fortuitous equivalence of the WSB property for a given sequentially complete space with the completeness of its dual for the topology of uniform convergence on sets defined by null sequences. It is necessary to introduce the notion of associated sequential topology to facilitate the argument. Given a topological space (X, T), the *associated sequential topology* T_S is formed by taking all subsets of X whose complements are sequentially closed relative to T. This construction does not commute, in general, with formation of the subspace topology. A distinction must be drawn between induced and intrinsic associated sequential topologies. The *induced* topology is the relativization of T_S to the subspace, and the *intrinsic* topology results from forming T_S after relativizing T to the subspace. For first countable (a fortiori metrizable) spaces, T agrees with T_S. In the context of locally convex spaces, T_S may be coarsened to ensure local convexity. The finest such locally convex topology refined by T_S is denoted by T_C (See [1] for detailed discussion).

SD.4 Theorem. *Let (L, T) be a Hausdorff, locally convex space which is T-sequentially complete. Denote by $\mathfrak{C}$ the family of sets defined in* SD.1. *Then (L, T) is weakly semi-bornological if and only if $(L', T_{\mathfrak{C}})$ is complete.*

Proof. Claim: $T_{\mathfrak{C}}$ is consistent with the duality $\langle L, L'_C \rangle$. It must be shown that $T_{\mathfrak{C}}$ is coarser than the Mackey topology $\tau(L'_C, L)$. Now $\tau(L'_C, L)$ is induced by the polars of the fundamental family of all $\sigma(L, L'_C)$-compact, convex, circled sets in L. The claim would follow if $\mathfrak{C}$ were shown to be a subfamily of this fundamental family. Each $C \in \mathfrak{C}$ is convex and balanced by definition, so it remains only to show that each C is $\sigma(L, L'_C)$-compact as well. The converse case argument of SD.3 establishes that each $C \in \mathfrak{C}$ is a T-compact, metrizable set. T and the induced T_S agree on any metrizable set a fortiori [1 Sect. 1.2.7]. T is Hausdorff, so each C is T-closed. The intrinsic T_S on each C is identical to the T_S topology on L relativized to that C [1 Sect. 1.3.4]. But then each $C \in \mathfrak{C}$ is T_S-compact. Since the associated convex topology T_C on L is coarser than T_S by definition, each C is T_C-compact as well. Observe that T_C refines

$\sigma(L, L'_C)$; hence each C is $\sigma(L, L'_C)$-compact, and the claim is established. The validity of the claim guarantees that L is the topological dual of L'_C endowed with $T_{\mathfrak{C}}$; hence the fundamental family $\mathfrak{C}$ of subsets of L is appropriate for applying the $\mathfrak{S}$-topology construction.

For necessity, suppose L is WSB, so $L' = L'_C$. Grothendieck has shown [4, Ch. 4, Theorem 6.2 et seq.] that for L' to be complete under the $\mathfrak{S}$-topology induced by a saturated family $\mathfrak{S}$ of bounded sets covering L, it is (necessary and) sufficient that every linear form f on L which is T-continuous on each member of $\mathfrak{S}$ be T-continuous on all of L. The family $\mathfrak{C}$ is saturated and covers L (every $x \in L$ is in the closed, convex, circled hull of the null sequence $\{n^{-1}x\}$). Moreover, if f is a T-continuous linear functional on any $C \in \mathfrak{C}$, then it is T-sequentially continuous on L, hence fully continuous on L by the WSB condition. It is now apparent that L', hence L'_C, is complete for $T_{\mathfrak{C}}$, or more exactly, for the canonical uniformity associated with $T_{\mathfrak{C}}$.

For sufficiency, suppose $(L', T_{\mathfrak{C}})$ is complete. Then L' is certainly closed in $(L'_C, T_{\mathfrak{C}})$. By a corollary to Grothendieck's completeness theorem, the vector space of all $f \in L^*$ which have weakly continuous restrictions to the sets in $\mathfrak{C}$, given the $\mathfrak{C}$-topology, is a complete locally convex space in which L' is dense. But the linear forms which are weakly continuous when restricted to the members of $\mathfrak{C}$ are precisely the sequentially continuous linear forms. This is immediate from SD.3. Hence L'_C topologized by $T_{\mathfrak{C}}$ is a complete locally convex space in which L' is dense. But since L' is closed, $L' = L'_C$, and L is WSB. □

It has been known [7 Cor. 3.6 p. 352] that the WSB property in the setting of SD.4 is sufficient to conclude Mackey completeness of the dual. SD.4 strengthens this result by weakening the topology for which completeness obtains and, more interestingly, provides the converse.

With the addition of the Mackey property the following is clear.

SD.5 Corollary. *Let (L, T) be a Mackey space. Then L is* SB *if and only if $(L', T_{\mathfrak{C}})$ is complete.*

SD.6 Proposition. *Let (L, T) be a Hausdorff, locally convex space which is T-sequentially complete. If each bounded subset of L is metrizable, then L is* WSB *if and only if the strong dual $(L', \beta(L', L))$ is complete.*

Proof. SD.4 establishes necessity verbatim. For sufficiency, suppose the strong dual is complete. Consider the class $\mathfrak{B}$ of all T-bounded sets in L whose polars induce the strong topology on L'. Again by Grothendieck's completeness theorem [4, Ch.4, Theorem 6.2], it is necessary that every functional on L which is T-continuous on $B \in \mathfrak{B}$ be T-continuous on all of L. By metrizability, the T-continuous forms on the sets in $\mathfrak{B}$ are exactly the sequentially continuous linear forms and it follows that $L' = L'_C$. □

Extension of sequentially continuous functionals

We conclude by examining the general question of what sort of extension theorem might be available for sequentially continuous linear forms? More specifically, if (E_0, T_0) is a topological vector subspace of (E, T), and $\phi_0: E_0 \to \mathbf{R}$ is T-sequentially continuous, is there a sequentially continuous linear form $\phi: E \to \mathbf{R}$ such that $\phi|_{E_0} = \phi_0$. Regrettably, but not unexpectedly, the answer is "generally not." The following example illustrates what can go awry.

E.3 Example. In E.1 the non-SB subspace $E_0 := (\mathrm{ba}(\mathbf{N}), \sigma(\mathrm{ba}(\mathbf{N}), l_\infty))$ of the SB space $E := (l_\infty^*, (\sigma(l_\infty^*, l_\infty)))$ was introduced. Let ϕ_0 be any sequentially continuous but not fully continuous linear form on the subspace, specifically a functional in the set $\mathrm{ba}(\mathbf{N})' \backslash l_\infty$. By irreflexivity, this set is not void. ϕ_0 has no extension to the SB superspace E that respects sequential continuity. If it did, then by the semi-bornological proper the extension ϕ to E would be fully continuous, and ϕ restricted back to E_0 would be continuous as well, contrary to the original specification of ϕ_0. □

It is always possible, however, to extend a sequentially continuous linear functional on a topological vector space M to a superspace E, provided M is sequentially closed and finite co-dimensional in the containing space. This holds without the assumption of local convexity. If E is weakly semi-bornological, these conditions force the original space to be WSB as well, and the type of difficulty embodied in E.3 is avoided. The following lemma is essential.

SE.1 Lemma. *Let E be any topological vector space; M a sequentially closed subspace of finite co-dimension. Then any projection $P: E \to M$ is sequentially continuous.*

Proof. It is sufficient to show that the complementary operator $(\mathrm{id}_E - P)$ is sequentially continuous. Note that $(\mathrm{id}_E - P)$ has finite rank. For arbitrary $x \in E$, $(\mathrm{id}_E - P)x$ may be written $\sum_{1 \leqq i \leqq n} \alpha_i x_i$, where $\{x_i\}_{(1 \leqq i \leqq n)}$ is a basis for the range. Claim: The coefficient functional $c_i(x) = \alpha_i$ for each $i = 1, \dots, n$ is sequentially continuous. For the sake of contradiction, suppose not. Then there exists a null sequence $\{x_k\}_{k \in \mathbf{N}}$ with $x_k = y_k + \sum_{1 \leqq i \leqq n} \alpha_{k,i} x_i$, $y_k \in M$, such that for all k, $\sum_{1 \leqq i \leqq n} |\alpha_{k,i}| = 1$. By considering subsequences, if necessary, it can be assumed that $\lim_{k \to \infty} \alpha_{k,i} = \alpha_i$. By sequential closure of M, $\lim_{k \to \infty} y_k = y \in M$. Hence $0 = y + \sum_{1 \leqq i \leqq n} \alpha_i x_i$ where $\sum_{1 \leqq i \leqq n} |\alpha_{k,i}| = 1$, which is absurd. The claim is established and the result follows. □

SE.2 Proposition. *Let (E, T) be a Hausdorff topological vector space. Suppose M is a T-sequentially closed subspace of finite co-dimension in E. Then $M_C' \approx E_C'/M^0$.*

Proof. Consider the map $\rho\colon E'_C \to M'_C$ which orders to each sequentially continuous linear form on E its restriction to M. The induced map $\rho_0\colon E'_C/M^0 \to M'_C$ is clearly injective. Moreover, ρ_0 is surjective by SE.1, establishing the result. □

SE.3 Proposition. *Let (L, T) be a locally convex space and M a sequentially closed subspace of finite codimension in L. If L is (resp. weakly) semi-bornological, then M is closed and (resp. weakly) semi-bornological.*

Proof. Let P be any projection of L onto M. $\mathrm{id}_L - P$ is sequentially continuous and has form $\sum_{(1 \leqq i \leqq n)} \alpha_i x_i$ by SE.1. Moreover, the coefficient functionals α_i are all sequentially continuous, hence by the (W)SB property of L, continuous as well. Thus $\mathrm{id}_L - P$ is continuous and $M = (\mathrm{id}_L - P)^{-1}(0)$ is closed.

For the assertion regarding semi-bornological spaces, it remains to be shown that if L is Mackey, so is M. The preceding shows that M is a direct summand of L; hence it can be viewed as the topological quotient of L by the complementary summand. But the Mackey topology on M is precisely the quotient topology on L [4, Ch. 4, Theorem 4.1, Corollary 4]; hence M is Mackey. □

References

1. Beatty, Thomas A.: Sequential Topologies and Their Application to a New Class of Locally Convex Spaces. Doctoral Dissertation, Florida Atlantic University, Boca Raton, Florida 1993
2. Dunford, N., Schwartz, J.T.: Linear Operators. Part 1: General Theory. 4th printing, New York: Interscience Publishers, 1967
3. Köthe, Gottfried: Topologische lineare Räume. I Berlin-Heidelberg-New York: Springer 1960
4. Schaefer, H.H.: Topological Vector Spaces. 5th printing, Berlin-Heidelberg-New York: Springer 1986
5. Schaefer, H.H.: Banach Lattices and Positive Operators (Grundlehren Math. Wiss. Vol. 215). Berlin-Heidelberg-New York: Springer 1974
6. Schaefer, H.H.: Radon, Baire, and Borel Measures on Compact Spaces. II Hokkaido Mathematical Journal XIX, 151–163 (1990)
7. Webb, J.H.: Sequential Convergence in Locally Convex Spaces. Proc. Cambridge Phil. Soc., 64–73 (1968)
8. Wong, Yau-Chuen.: Introductory Theory of Topological Vector Spaces. New York: Marcel Dekker (1992)
9. Yosida, Kosaku.: Functional Analysis. Berlin-Heidelberg-New York: Springer 1980