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Marcinkiewicz multipliers and multi-parameter structure on Heisenberg (-type) groups, II

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0 Introduction

This paper is the second of a series dealing with spectral multiplier operators on Heisenberg-like groups, in the setting where an underlying multi-parameter structure plays a key role. Here we refine the initial version of the Marcinkiewicz multiplier theorem considered in the first paper, and obtain a sharp theorem of this kind: one which requires the minimum differentiability hypotheses. We also broaden the scope of our results by extending the class of groups considered. This larger class, the H-type groups, includes all nilpotent Iwasawa sub-groups of real-rank one simple Lie groups, and has been of substantial interest recently, (see [K], [R], [Me], [CDKR], [DR]).

Starting with one of these groups, suppose $\mathscr{L}$ is the associated sub-Laplacian, and $U_1, U_2, \dots, U_n$ an orthonormal basis for the center of the Lie algebra. For an appropriate function m defined on $\mathbb{R}_+^* \times \mathbb{R}^n$, we consider the operator $m(\mathscr{L}, U_1/i, U_2/i, \dots, U_n/i) = m(\mathscr{L}, U/i)$, and its variants below.

Our main result is then a sharp version of Marcinkiewicz multiplier theorem with the following feature: that essentially $d/2+\varepsilon$ smoothness of the multiplier m suffices, where d is the (usual) dimension of the group.

More precisely, suppose we consider the following multi-parameter Sobolev norm on functions defined on $\mathbb{R} \times \mathbb{R}^n = \{(\tau, \mu)\}$:

$$(0.1) \qquad \|f\|_{L^2_{\alpha,\beta}} = \|(1+|\partial_\tau|)^\alpha \prod_{j=1}^{n} (1+|\partial_\tau| + |\partial_{\mu_j}|)^\beta f\|_{L^2} .$$

With this norm, if

$$\sup_r \|m^r \chi\|_{L^2_{\alpha,\beta}} < \infty, \quad \text{with } \alpha > (d-n)/2 \text{ and } \beta > 1/2 ,$$

then we can conclude that $m(\mathcal{L}, U/i)$ is bounded on L^p, $1 < p < \infty$. Here $m^r(\tau, \mu) = m(r_0\tau, r'\mu)$, $r \in (\mathbb{R}^+)^{n+1}$, and χ is a suitable cut-off function.

There are also corresponding results for functions of $(\Lambda^{-1}\mathcal{L}, U/i)$ instead of $(\mathcal{L}, U/i)$, with $\Lambda = [-(U_1^2 + \cdots + U_n^2)]^{1/2}$. Furthermore, we should note that the above theorem is an extension of the recent result for functions of $\mathcal{L}$ only, in [MS] and [He].

As in the first paper [MRS], the method of proof requires a reduction to appropriate Littlewood–Paley square functions. Here, however, a crucial difference is that we do not lift the problem to higher dimensions, since this would increase the needed differentiability; and thus we carry out our analysis on the group itself. There are two forms of the square functions studied in paragraphe 4, first g_1 and then g_2, each depending on a decomposition of the joint spectrum of $\mathcal{L}$ and U/i. The key estimate is the assertion

$$\|g_2(T(f))\|_{L^p} \leqq A\|g_2(f)\|_{L^p}, \quad 2 \leqq p < \infty\,, \tag{0.2}$$

where T is the operator $m(\Lambda^{-1}\mathcal{L}, U/i)$. This is demonstrated by showing that

$$\sup_{j,l} \int |N_{j,l}|^2 w_{j,l}\, dx < \infty\,, \tag{0.3}$$

where $N_{j,l}$ are the kernels of the components of T corresponding to the decomposition of the spectrum; also the $w_{j,l}$ are appropriate weights which have the property that

$$\sup_{j,l} \int |f(y)| 1/w_{j,l}(y^{-1}x)\, dy \leqq \mathcal{M}_s(|f|)\,,$$

with $\mathcal{M}_s$ a "strong" (i.e. multiparameter) maximal function.

The weights can be expressed in terms of products of powers of $|z|$ and $|u_1|, \ldots, |u_n|$, and the fact that (0.3) holds results from the relation between the kernels $N_{j,l}$ and their corresponding multipliers given by the Laguerre expansion. This is carried out in paragraphe 5.

The proofs of our results are preceded by a discussion of several applications and examples. A basic example is given by $\Lambda^{i\alpha}\mathcal{L}^{i\beta}$. In particular the cases $\alpha = -\beta$ and the case $\alpha = 0$ considered in [MS] allow one to see that our results cannot be improved.

As a final remark we note that in the $\mathbb{R}^n$ analogue (for spectral operators $m(\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \ldots, \frac{1}{i}\frac{\partial}{\partial x_n})$) our argument gives a corresponding variant of the classical Marcinkiewicz multiplier theorem; we have learned that this particular result has also been recently indicated in [CS].

1 Sub-Laplacians on H-type groups

Let us briefly recall the definition and some basic facts from harmonic analysis on H-type groups, also called groups of Heisenberg type.

Suppose $\mathfrak{g}$ is a 2-step nilpotent Lie algebra which decomposes into subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_2$ is central and $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2$. Suppose further that $\mathfrak{g}$ is endowed with an inner product $\langle , \rangle$ for which the above decomposition is orthogonal. This allows in particular to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$, and thus to endow $\mathfrak{g}^*$ with the induced inner product. For each $\mu \in \mathfrak{g}_2^*$, there exists a skew symmetric endomorphism J_μ of $\mathfrak{g}_1$ such that

$$\mu([z,z']) = \langle J_\mu z, z' \rangle, \quad z, z' \in \mathfrak{g}_1 .$$

$\mathfrak{g}$ is said to be of *H-type*, if J_μ is orthogonal for each $\mu \in \mathfrak{g}_2^*$ of unit length, i.e. if

$$J_\mu^2 = -|\mu|^2 \mathrm{Id}, \quad \mu \in \mathfrak{g}_2^* .$$

This implies that $\dim \mathfrak{g}_1 = 2m$ is even, that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and that $\mathfrak{g}_2$ is the center of $\mathfrak{g}$. Let $n := \dim \mathfrak{g}_2$.

In the sequel we assume that $\mathfrak{g}$ is of H-type, and we denote by $G = \exp \mathfrak{g}$ the corresponding group of H-type. The element $\exp(z+u)$, $z \in \mathfrak{g}_1$, $u \in \mathfrak{g}_2$, will be denoted by (z,u). The left- and right-invariant Haar measure on G is then given by the Lebesgue measure $dz\,du$.

A function f on G is said to be $\mathfrak{g}_1$*-radial*, if it depends only on $|z|$ and u. Let us denote by $\mathscr{A}$ the closed subspace of the group algebra $L^1(G)$ formed by the $\mathfrak{g}_1$-radial integrable functions on G. Analysis on H-type groups is often facilitated by the following facts [R] (see also [DR]):

$\mathscr{A}$ is a commutative, semisimple subalgebra of $L^1(G)$ whose Gelfand spectrum $\hat{\mathscr{A}}$ can be parametrized by $\mathbb{R}_+ \cup (\mathfrak{g}_2^* \setminus \{0\}) \times \mathbb{N}$, where $\mathbb{R}_+ := [0, +\infty[$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$.

For $\rho \in \mathbb{R}_+$, put $\chi_\rho(z,u) := e^{i\rho\langle z, z_0\rangle}$, where z_0 is a fixed unit vector in $\mathfrak{g}_1$, and for $(\mu, k) \in (\mathfrak{g}_2^* \setminus \{0\}) \times \mathbb{N}$ put

$$\chi_{\mu,k}(z,u) := \binom{k+m-1}{k}^{-1} e^{-i\mu(u)} e^{-\frac{|\mu||z|^2}{4}} L_k^{m-1}\left(\frac{1}{2}|\mu||z|^2\right) ,$$

where L_k^m denotes the Laguerre polynomial of type m and degree k.

Then the Gelfand transform of $f \in \mathscr{A}$ is given by $\mathscr{G}f(\omega) := \int_G f(x)\chi_\omega(x)\,dx$ for $\omega \in \mathbb{R}_+ \cup (\mathfrak{g}_2^* \setminus \{0\}) \times \mathbb{N}$.

Moreover, either from the representation theory of G [R] or directly from the orthogonality properties of Laguerre polynomials [E] one derives the following *Plancherel theorem* for G:

If $f \in \mathscr{A} \cap L^2(G)$, then

$$\|f\|_{L^2}^2 = \kappa \int\limits_{\mathfrak{g}_2^* \setminus \{0\}} \sum_{k=0}^{\infty} |\mathscr{G}f(\mu,k)|^2 \binom{k+m-1}{k} |\mu|^m \, d\mu , \tag{1.1}$$

where the constant κ depends only on $\dim \mathfrak{g}_1$ and $\dim \mathfrak{g}_2$.

Correspondingly there is an *inversion formula*

(1.2) $$f(z,u) = \kappa \int_{\mathfrak{g}_2^*\setminus\{0\}} \sum_{k=0}^{\infty} \mathscr{G}f(\mu,k)\overline{\chi}_{\mu,k}(z,u) \binom{k+m-1}{k} |\mu|^m \, d\mu \,,$$

valid for instance for f in the Schwartz space $S(G)$.

These formulas show that the part of $\hat{A}$ corresponding to $\mathbb{R}_+^*$ is of Plancherel measure zero.

Next, let us fix an orthonormal basis $X_1,\ldots,X_{2m}$ of $\mathfrak{g}_1$, and consider the left-invariant sub-Laplacian

$$\mathscr{L} := -\sum_{j=1}^{2m} X_j^2$$

on G. Then one checks that $\mathscr{L}$ is $\mathfrak{g}_1$-radial in the sense that it maps $\mathscr{A} \cap C^\infty(G)$ into itself, and that $\mathscr{L}f = -(\Delta_z + \frac{|z|^2}{4}\Delta_u)f$ for $\mathfrak{g}_1$-radial f. Here Δ_z and Δ_u denote the Laplacians on $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Moreover, either by representation theory or by direct calculation based on the spectral properties of Laguerre polynomials [E, p. 188] one finds that

(1.3) $$\mathscr{G}(\mathscr{L}f)(\mu,k) = (2k+m)|\mu|\mathscr{G}f(\mu,k) \,.$$

Similarly, each $U \in \mathfrak{g}_2$ is $\mathfrak{g}_1$-radial, and

(1.4) $$\mathscr{G}(Uf)(\mu,k) = i\mu(U)\mathscr{G}f(\mu,k) \,.$$

2 Joint spectral multipliers

Besides the sub-Laplacian $\mathscr{L}$, let us also fix an orthonormal basis $U_1,\ldots,U_n$ of $\mathfrak{g}_2$. We denote by Λ the central pseudo-differential operator

$$\Lambda := [-(U_1^2 + \cdots + U_n^2)]^{1/2} \,.$$

By means of the bases $X_1,\ldots,X_{2m}$ of $\mathfrak{g}_1$ and $U_1,\ldots,U_n$ of $\mathfrak{g}_2$ we shall identify $\mathfrak{g}_1$ with $\mathbb{R}^{2m}$ and $\mathfrak{g}_2$ with $\mathbb{R}^n$.

Let us mention that the notation used throughout this paper differs partially from the one in [MRS] and is more adapted to the one in [MR]. If $G = H_m$ is the Heisenberg group of dimension $2m+1$, the variable u appearing here corresponds to the variable $t/4$ in [MRS], $U = U_1$ to $4T$, μ to 4λ, etc..

It is well-known that the operators $\mathscr{L}$ and $\frac{1}{i}U_1,\ldots,\frac{1}{i}U_n$ are essentially selfadjoint on $\mathscr{S}(G)$. Moreover, the sub-elliptic estimates for Rockland operators in [HN] show that $\mathscr{L}^{-1}\Lambda$ is a bounded symmetric operator on $L^2(G)$, so that, by spectral theory, $\Lambda^{-1}\mathscr{L}$ is essentially selfadjoint on $\mathscr{S}(G)$. Since all these operators commute, they admit a joint spectral resolution, and we can thus give meaning to expressions like $m(\mathscr{L},\frac{1}{i}U_1,\ldots,\frac{1}{i}U_n)$ or $m(\Lambda^{-1}\mathscr{L},\frac{1}{i}U_1,\ldots,\frac{1}{i}U_n)$ for each continuous function m defined on the corresponding joint spectrum.

Suppose $m \in C_0^\infty(\mathbb{R}_+^* \times \mathbb{R}^n)$ and $\operatorname{supp} m \subseteq \mathbb{R}_+^* \times (\mathbb{R}^n \backslash \{0\})$. For simplicity of notation we write

$$U/i := \left(\frac{1}{i} U_1, \ldots, \frac{1}{i} U_n\right) .$$

Since $m(\mathscr{L}, U/i)$ and $m(\Lambda^{-1}\mathscr{L}, U/i)$ are bounded, $\mathfrak{g}_1$-radial, left-invariant operators on $L^2(G)$, they are given by right convolution with $\mathfrak{g}_1$-radial distributions M_m and N_m on G. By (1.3) and (1.4) their Gelfand transforms are

$$\mathscr{G}M_m(\mu, k) = m((2k+m)|\mu|, \mu) , \tag{2.1}$$

$$\mathscr{G}N_m(\mu, k) = m((2k+m)\mu) ,$$

if we choose coordinates $\mu = (\mu_1, \ldots, \mu_n)$ for $\mathfrak{g}_2^*$ by putting $\mu_j := \mu(U_j)$. From (1.2) and (2.1) it can easily be seen that the support property of m implies that M_m and N_m are in fact Schwartz class functions (see also [Hu], [Ma]).

We shall also use the notation $M_m = m(\mathscr{L}, U/i)\delta$ and $N_m = m(\Lambda^{-1}\mathscr{L}, U/i)\delta$, where δ denotes the Dirac measure at the identity element $(0,0)$ of G.

Remark 2.1 (2.1) shows that the joint spectrum of $\mathscr{L}$ and U/i is contained in the "Heisenberg fan" $\Sigma_1 := \{(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}^n : \lambda = (2k+m)|\mu| \text{ for some } k \in \mathbb{N}\}^-$, and the joint spectrum of $\Lambda^{-1}\mathscr{L}$ and U/i in $\Sigma_2 := \{(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}^n : \lambda = (2k+m) \text{ for some } k \in \mathbb{N}\}$.

By means of representation theory and a reduction argument to the case of the Heisenberg group as in [R] one can prove that Σ_1 and Σ_2 are in fact exactly the joint spectra (compare also [St]).

Remark 2.2 The mappings $\delta_r(z,u) := (rz, r^2 u)$, $r > 0$, define automorphic "dilations" of G, and the operators $\mathscr{L}$, $\frac{1}{i}U_j$ and $\Lambda^{-1}\mathscr{L}$ are homogeneous of degree $2, 2$ and 0, respectively, with respect to these dilations. From this one infers the following *scaling properties*, which can also be read off directly from (1.2):

$$r^{-Q} M_m(\delta_{r^{-1}} x) = M_{m(r^2\,\cdot\,, r^2\,\cdot\,)}(x) , \tag{2.2}$$

$$r^{-Q} N_m(\delta_{r^{-1}} x) = N_{m(\,\cdot\,, r^2\,\cdot\,)}(x) , \tag{2.3}$$

where $Q := 2m + 2n$ is the *homogeneous dimension* of G. The *Euclidean dimension* $\dim G = 2m + n$ will be denoted by d.

For later use, let us also introduce the canonical *homogeneous norm* $|\cdot|$ (compare [FS]) on G given by

$$|(z,u)| := (|z|^4 + 16|u|^2)^{1/4} ,$$

which in particular satisfies $|\delta_r x| = r|x|$.

The identity (2.3) reveals in particular that if $m = \tilde{m}(\lambda)$ depends only on the first variable, then the corresponding kernel $N_m = \tilde{m}(\Lambda^{-1}\mathscr{L})\delta$ is *homogeneous of critical degree $-Q$*.

In order to formulate our main theorem, we need to introduce fractional difference respectively differentiation operators on the l.c. abelian group $\mathbb{Z} \times \mathbb{R}^n$.

If f is a suitable function on this group, we denote by Δ the *first order difference operator* in the $\mathbb{Z}$-variable

$$(\Delta f)(k,\mu) := f(k+1,\mu) - f(k,\mu)\,,$$

and by ∂_{μ_j} the partial derivative in the μ_j-variable. If

$$\hat{f}(t,u) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} f(k,\mu)e^{-i(kt+\mu\,\cdot\,u)}\,d\mu$$

denotes the Fourier transform of f, we have

$$\widehat{\Delta f}(t,u) = (e^{it}-1)\hat{f}(t,u)\,,$$

$$\widehat{\partial_{\mu_j} f}(t,u) = iu_j\hat{f}(t,u)\,.$$

Correspondingly, we define fractional powers $|\Delta|^\alpha$ and $|\partial_{\mu_j}|^\alpha$, $\alpha \in \mathbb{C}$, by

$$(|\Delta|^\alpha f)\,\hat{}\,(t,u) := |e^{it}-1|^\alpha\hat{f}(t,u)\,,$$

$$(|\partial_{\mu_j}|^\alpha f)\,\hat{}\,(t,u) := |u_j|^\alpha\hat{f}(t,u)\,.$$

The meaning of more general expressions like $(1+|r_0\Delta|)^\alpha$ or $(1+|r_0\Delta|+|r_j\partial_{\mu_j}|)^\alpha$ is then also evident.

Theorem 2.3 *Let $m = m(2k+m,\mu)$ be a continuous function defined on the joint spectrum $(2\mathbb{N}+m)\times\mathbb{R}^n$ of $\Lambda^{-1}\mathscr{L}$ and U/i. By putting*

$$\tilde{m}\,(k,\mu) := \begin{cases} m(2k+m,\mu) & \text{if } k \geqq 0\,,\\ 0 & \text{if } k < 0\,, \end{cases}$$

we identify m with a function $\tilde{m}$ on $\mathbb{Z}\times\mathbb{R}^n$.

Fix a non-trivial bump function $\eta_0 \in C_0^\infty(\mathbb{R})$ supported in $\mathbb{R}_+^*$, and define for $r = (r_0,r_1,\ldots,r_n) \in (\mathbb{R}_+^*)^{n+1}$ the function η_r by $\eta_r(k,\mu) := \eta_0(\frac{k+1}{r_0})\eta_1(\frac{\mu_1}{r_1})\ldots\eta_1(\frac{\mu_n}{r_n})$, where $\eta_1(x) := \eta_0(x)+\eta_0(-x)$. Suppose that

$$\|m\|^2_{l^2(L^2)_{\alpha,\beta,\mathrm{sloc}}} := \sup_{r_j>0}\frac{1}{r_0}\sum_{k\in\mathbb{Z}}\frac{1}{r_1\ldots r_n}$$

$$\times\int_{\mathbb{R}^n}\left|(1+|r_0\Delta|)^\alpha\prod_{j=1}^n(1+|r_0\Delta|+|r_j\partial_{\mu_j}|)^\beta(\tilde{m}\eta_r)(k,\mu)\right|^2 d\mu$$

is finite for some $\alpha > m$, $\beta > 1/2$. Then the operator $m(\Lambda^{-1}\mathscr{L},U/i)$ is bounded on $L^p(G)$ for $1 < p < \infty$, and

$$\|m(\Lambda^{-1}\mathscr{L},U/i)\|_{L^p,L^p} \leqq C_{p,\varepsilon}\|m\|_{l^2(L^2)_{\alpha,\beta,\mathrm{sloc}}}\,.$$

We postpone the proof of this result to later sections, but remark already here that, by standard partition of unity arguments, it can easily be shown that different bump functions η_0 lead to equivalent $l^2(L^2)_{\alpha,\beta,\mathrm{sloc}}$ norms (the symbol "sloc" stands for "scale invariant localized").

For Heisenberg groups, a weaker version, which did not specify the degree of smoothness of the multiplier, had already been proved in [MRS, Thm. 2.2].

If we define mixed L^2-Sobolev spaces $L^2_{\alpha,\beta} = L^2_{\alpha,\beta}(\mathbb{R} \times \mathbb{R}^n)$ by

$$\|f\|_{L^2_{\alpha,\beta}} := \|(1+|t|)^\alpha (\prod_{j=1}^n (1+|t|+|u_j|)^\beta \hat{f}(t,u)\|_{L^2}$$

$$= c\|(1+|\partial_\tau|)^\alpha \prod_{j=1}^n (1+|\partial_\tau|+|\partial_{\mu_j}|)^\beta f\|_{L^2} ,$$

we have

Corollary 2.4 *Fix a bump function η_0 as in Theorem 2.3, and let χ denote the corresponding bump function $\chi := \eta_0 \otimes \eta_1 \otimes \cdots \otimes \eta_1$ on $\mathbb{R} \times \mathbb{R}^n$.*

Let h be a bounded, continuous function on $\mathbb{R}_+^* \times \mathbb{R}$, and put $h^r(\tau,\mu) = h(r_0\tau, r_1\mu_1,\dots,r_n\mu_n)$ for $r=(r_0,\dots,r_n) \in (\mathbb{R}_+^*)^{n+1}$. If

$$\|h\|_{L^2_{\alpha,\beta,\mathrm{sloc}}} := \sup_r \|h^r\chi\|_{L^2_{\alpha,\beta}}$$

is finite for some $\alpha > m$, $\beta > 1/2$, then the operators $h(\Lambda^{-1}\mathscr{L}, U/i)$ and $h(\mathscr{L}, U/i)$ are bounded on $L^p(G)$ for $1 < p < \infty$, with norms controlled by $\|h\|_{L^2_{\alpha,\beta,\mathrm{sloc}}}$.

The proof of this corollary rests on the following

Lemma 2.5 *Let $\alpha > 1/2$, and suppose that $g \in L^2_\alpha(\mathbb{R})$ is continuous. Denote by γ the restriction of g to $\mathbb{Z}$. Then*

$$\|(1+|R\Delta|)^\alpha \gamma\|_{l^2(\mathbb{Z})} \leqq C_\varepsilon \|(1+|R\partial_t|)^\alpha g\|_{L^2(\mathbb{R})} \tag{2.4}$$

for every $R \geqq \varepsilon > 0$.

Proof. It suffices to establish (2.4) for $g \in \mathscr{S}(\mathbb{R})$. Then, by Poisson's summation formula,

$$\hat{\gamma}(\tau) = \sum_{k\in\mathbb{Z}} g(k)e^{-ik\tau} = \sum_{k\in\mathbb{Z}} \hat{g}(2\pi k+\tau) .$$

Therefore

$$|(1+R|e^{i\tau}-1|)^\alpha \hat{\gamma}(\tau)| \leqq (1+R|\tau|)^\alpha \sum_k |\hat{g}(2\pi k+\tau)|$$

$$\leqq (1+R|\tau|)^\alpha \left[\sum_k (1+R|2\pi k+\tau)|^{-2\alpha}\right]^{1/2}$$

$$\times \left[\sum_k |\hat{g}(2\pi k+\tau)(1+R|2\pi k+\tau|)^\alpha|^2\right]^{1/2} .$$

And, for $|\tau| \leqq \pi$ and $R \geqq \varepsilon$,

$$\sum_k (1+R|2\pi k+\tau|)^{-2\alpha} = (1+R|\tau|)^{-2\alpha} + O(R^{-2\alpha}) \leqq C_\varepsilon (1+R|\tau|)^{-2\alpha} .$$

So, by Plancherel's theorem,

$$\|(1+|R\Delta|)^\alpha \gamma\|^2_{l^2(\mathbb{Z})} \leqq C_\varepsilon \sum_k \int_{-\pi}^{\pi} |\hat{g}(2\pi k+\tau)(1+R|2\pi k+\tau|)^\alpha|^2 \, d\tau$$

$$= \int_{\mathbb{R}} |\hat{g}(s)(1+R|s|)^\alpha|^2 \, ds$$

$$= C_\varepsilon \|(1+|R\partial_t|)^\alpha g\|^2_{L^2(\mathbb{R})} \, . \qquad \square$$

For a partial converse to Lemma 2.5, see [MRS, proof of Thm. 2.2].

It is now easy to prove Corollary 2.4. Put $\tilde{m}(k,\mu) = h(2k+m,\mu)$, if $k \geqq 0, \tilde{m}(k,\mu) = 0$ for $k < 0$. We have

$$\tilde{m}\eta_r(k,\mu) = \left[h\chi_r \left(1 + \frac{(\,\cdot\,)-m}{2}, \,\cdot\, \right) \right] (2k+m,\mu) \, ,$$

for every $k \in \mathbb{Z}$, $\mu \in \mathbb{R}^n$, where $\chi_r(\tau,\mu) = \chi(\frac{r}{r_0}, \frac{\mu_1}{r_1}, \cdots, \frac{\mu_n}{r_n})$. Since $\tilde{m}\eta_r = 0$ for r_0 sufficiently small, and since $\alpha > m > 1/2$, it is clear by Lemma 2.5 that $\|m\|_{l^2(L^2)_{\alpha,\beta,\mathrm{sloc}}}$ is dominated by

$$\sup_r \frac{1}{r_0 \ldots r_n} \int_{\mathbb{R}^{n+1}} \left| (1+|r_0\partial_\tau|)^\alpha \prod_{j=1}^n (1+|r_0\partial_\tau| + |r_j\partial_{\mu_j}|)^\beta (h\chi_r) \right|^2 d\tau \, d\mu \, , \tag{2.5}$$

which equals $\|h\|_{L^2_{\alpha,\beta,\mathrm{sloc}}}$, as can be seen by scaling.

This proves the statement about $h(\Lambda^{-1}\mathscr{L}, U/i)$.

And, if we put $Sh(\tau,\mu) = \check{h}(\tau,\mu) := h(\tau|\mu|,\mu)$, then $h(\mathscr{L}, U/i) = \check{h}(\Lambda^{-1}\mathscr{L}, U/i)$, and

$$\|\check{h}\|_{L^2_{\alpha,\beta,\mathrm{sloc}}} \leqq C\|h\|_{L^2_{\alpha,\beta,\mathrm{sloc}}} \, . \tag{2.6}$$

In order to see this, it suffices to prove that the expression (2.5), with h replaced by $\check{h}$, is dominated by the corresponding expression for h, maybe with another bump function $\tilde{\chi}$ in place of χ. But, if we fix r and put $r_{\max} := \max_{j \geqq 1} r_j$, then $|\mu| \sim r_{\max}$ on $\operatorname{supp}\chi_r$, and thus

$$|\mu_j| \sim r_j, \quad |\tau| \sim r_{\max} r_0$$

on the support of $S^{-1}(\check{h}\chi_r)$. Thus, if we choose $\tilde{\eta}_0 \in C_0^\infty(\mathbb{R})$ supported in $\mathbb{R}_+^*$ and identically 1 on a sufficiently large neighborhood of supp η_0, and if we form $\tilde{\chi}_r$ from $\tilde{\eta}_0$ in analogy to the construction of χ_r from η_0, then

$$\check{h}\chi_r = (h\tilde{\chi}_{\tilde{r}})\check{\ }\chi_r \, , \tag{2.7}$$

where $\tilde{r} = (r_{\max} r_0, r_1, \ldots, r_n)$. Moreover, for any $\alpha \geqq 0$, $\beta \geqq 0$, one has

$$\text{(2.8)}\quad \frac{1}{r_0 \ldots r_n} \|(1+|r_0\partial_\tau|)^\alpha \prod_{j=1}^n (1+|r_0\partial_\tau|+|r_j\partial_{\mu_j}|)^\beta (\check{h}\chi_r)\|_{L^2}^2$$
$$\leqq C_{\alpha,\beta} \frac{1}{r_{\max} r_0 \ldots r_n} \|(1+|r_0\partial_\tau|)^\alpha \prod_{j=1}^n (1+|r_0\partial_\tau|+|r_j\partial_{\mu_j}|)^\beta h\|_{L^2}^2 .$$

For integer α and β, (2.8) can easily be checked by straight-forward calculation. For instance, we have

$$\int |r_0\partial_\tau(\check{h}\chi_r)(\tau,\mu)|^2\, d\tau\, d\mu$$
$$\leqq C\int |\tau|\mu|\partial_\tau h(\tau|\mu|,\mu)\chi_r(\tau,\mu)|^2\, d\tau\, d\mu + C\int |r_0 h(\tau|\mu|,\mu)\partial_\tau\chi_r(\tau,\mu)|^2\, d\tau\, d\mu$$
$$\leqq \frac{C}{r_{\max}} \left\{\int |r_0\partial_\tau h(\tau,\mu)|^2\, d\tau\, d\mu + \int |h(\tau,\mu)|^2\, d\tau\, d\mu\right\},$$

and

$$\int|r_j\partial_{\mu_j}(\check{h}\chi_r)(\tau,\mu)|^2\, d\tau\, d\mu \leqq C\left\{\int \left|r_j \frac{\mu_j}{|\mu|^2}\tau|\mu|\partial_\tau h(\tau|\mu|,\mu)\chi_r(\tau,\mu)\right|^2 d\tau\, d\mu\right.$$
$$\left. +\int |r_j\partial_{\mu_j}h(\tau|\mu|,\mu)\chi_r(\tau,\mu)|^2\, d\tau\, d\mu + \int |r_j h(\tau|\mu|,\mu)\partial_{\mu_j}\chi_r(\tau,\mu)|^2\, d\tau\, d\mu\right\}$$
$$\leqq \frac{C}{r_{\max}} \left\{\int |r_0\partial_\tau h(\tau,\mu)|^2\, d\tau\, d\mu + \int |r_j\partial_{\mu_j} h(\tau,\mu)|^2\, d\tau\, d\mu + \int |h(\tau,\mu)|^2\, d\tau\, d\mu\right\},$$

and the general term in (2.8) can be estimated similarly.

Interpolating the estimates for the operator $h \to \check{h}\chi_r$ between integer values of α and β, we obtain (2.8) for arbitrary $\alpha, \beta \geqq 0$.

Then (2.6) follows, since by (2.7) we may replace h in the right-hand side of (2.8) by $h\tilde{\chi}_{\tilde{r}}$, and from (2.6) we finally conclude that $\check{h}(\Lambda^{-1}\mathscr{L}, U/i) = h(\mathscr{L}, U/i)$ is L^p-bounded. □

If we apply Corollary 2.4 to multipliers depending only on the first variable, we retrieve the Hörmander–Mihlin multiplier theorem for functions of $\mathscr{L}$ of [MS], [He]. Since this result was shown to be essentially sharp with respect to the critical degree of differentiability of the multiplier, this indicates that also Theorem 2.3 is essentially optimal. For further evidence to this, see also Section 3.

3 Some applications

Before we come to the proof of Theorem 2.3, let us discuss some applications. The first one is obvious.

Corollary 3.1 *Let $m = m(2k+m)$ be a multiplier defined on the spectrum of $\Lambda^{-1}\mathscr{L}$, and put*

$$\|m\|^2_{l^2_{\alpha,\mathrm{sloc}}} := \sup_{r>0} \frac{1}{r} \sum_{k\in\mathbb{Z}} \left| (1+|r\Delta|^\alpha)\left(\tilde{m}\eta_0\left(\frac{1+\,\cdot}{r}\right)\right)(k)\right|^2 .$$

If $\|m\|_{l^2_{\alpha,\mathrm{sloc}}} < \infty$ for some $\alpha > d/2$, then $m(\Lambda^{-1}\mathscr{L})$ is bounded on $L^p(G)$, $1 < p < \infty$, with norm controlled by $\|m\|_{l^2_{\alpha,\mathrm{sloc}}}$.

We remark that $\|\cdot\|_{l^2_{\alpha,\mathrm{sloc}}}$ is related to the *wbv*, α-norm studied in [GT].

Example 3.2 For $k \in \mathbb{N}$ let $\mathscr{P}_k = \chi_{\{2k+m\}}(\Lambda^{-1}\mathscr{L})$ denote the orthogonal projection onto the $(2k+m)$-eigenspace of $\Lambda^{-1}\mathscr{L}$. Then

$$\|\mathscr{P}_k\|_{L^p,L^p} \leqq C_{p,\varepsilon}(1+k)^{(d-1)|\frac{1}{p}-\frac{1}{2}|+\varepsilon}, \quad 1 < p < \infty, \tag{3.1}$$

for every $\varepsilon > 0$.

This follows easily from Corollary 3.1. Since $\tilde{m}(l) = \chi_{\{k\}}(l)$, we have $\tilde{m}\eta_0(\frac{1+\,\cdot}{r}) \equiv 0$ unless $r \approx k+1$. And if $r \approx k+1$, then $\tilde{m}\eta_0(\frac{1+\,\cdot}{r}) = a_r\tilde{m}$, with $|a_r| = |\eta_0(\frac{k+1}{r})| \leqq \|\eta_0\|_{L^\infty}$. Moreover, Plancherel's theorem implies

$$\sum_l |(1+|r\Delta|^\alpha)\tilde{m}(l)|^2 = C\int_{-\pi}^{\pi} |(1+r^\alpha|e^{it}-1|^\alpha)e^{-ikt}|^2\,dt$$

$$\leqq C(1+r)^{2\alpha} \approx C(1+k)^{2\alpha},$$

so that

$$\|\chi_{\{2k+m\}}\|_{l^2_{\alpha,\mathrm{sloc}}} = O((1+k)^{\alpha-1/2}),$$

hence by Corollary 3.1

$$\|\mathscr{P}_k\|_{L^p,L^p} = O((1+k)^{\frac{d-1}{2}+\varepsilon})$$

for every $\varepsilon > 0$. Interpolating with $\|\mathscr{P}_k\|_{L^2,L^2} = 1$ then leads to (3.1).

For the Heisenberg group H_m, for which $d-1 = 2m$, estimate (3.1) is due to Strichartz [St]. He derived it, however, by a completely different method, making use of the explicit formulas for the kernels $P_{k,\varepsilon}$ as defined below (respectively their analogues on H_m). Theorem 2.3 could indeed easily be applied to obtain the same estimates as for $\mathscr{P}_k$ also for the corresponding projections $\mathscr{P}_{k,\varepsilon}$.

Example 3.3 Let $\alpha, \beta \in \mathbb{R}$. The operator $\Lambda^{i\alpha}\mathscr{L}^{i\beta}$ is bounded on $L^p(G)$, $1 < p < \infty$, and for every $\varepsilon > 0$

$$\|\Lambda^{i\alpha}\mathscr{L}^{i\beta}\|_{L^p,L^p} \leqq C_\varepsilon[(1+|\beta|)^{2m}(1+|\alpha|+|\beta|)^n]^{(1+\varepsilon)|\frac{1}{p}-\frac{1}{2}|}. \tag{3.2}$$

We apply Corollary 2.4. If we put $h_{\alpha,\beta}(\tau,\mu) := \tau^{i\beta}|\mu|^{i\alpha}$, then, by some standard interpolation argument,

$$\|\Lambda^{i\alpha}\mathscr{L}^{i\beta}\|_{L^p,L^p} \leqq C\|h_{\alpha,\beta}\|_{L^2_{m(1+\varepsilon),\frac{1+\varepsilon}{2},\mathrm{sloc}}} \leqq C(1+|\beta|)^{m(1+\varepsilon)}(1+|\beta|+|\alpha|)^{\frac{n}{2}(1+\varepsilon)}.$$

Interpolation with $\|\Lambda^{i\alpha}\mathscr{L}^{i\beta}\|_{L^2,L^2} = 1$ leads to (3.2).

Notice that for $|\alpha| \leqq C|\beta|$ we have

$$\|\Lambda^{i\alpha}\mathscr{L}^{i\beta}\|_{L^p,L^p} \leqq C_\varepsilon(1+|\beta|)^{(1+\varepsilon)d|\frac{1}{p}-\frac{1}{2}|}.$$

In particular $\|\mathscr{L}^{i\beta}\|_{L^p,L^p} \leqq C_\varepsilon(1+|\beta|)^{(1+\varepsilon)d|\frac{1}{p}-\frac{1}{2}|}$ and $\|(\Lambda^{-1}\mathscr{L})^{i\beta}\|_{L^p,L^p} \leqq C_\varepsilon(1+|\beta|)^{(1+\varepsilon)d|\frac{1}{p}-\frac{1}{2}|}$. It was shown in [MS] that the first estimate is essentially sharp for p close to 1 and $|\beta|$ large (and G a Heisenberg group), and the same is in fact also true of the second estimate.

In order to see this, let us look more closely at the case of the 3-dimensional *Heisenberg group* H_1, for which $\mathfrak{g}_1 = \mathbb{R}^2$, $\mathfrak{g}_2 = \mathbb{R}$ and $J_\mu = \left(\begin{smallmatrix} 0 & \mu \\ -\mu & 0\end{smallmatrix}\right)$, $\mu \in \mathbb{R}$.

The discussion to follow will also illuminate the meaning of the spectral parameters k.

For H_1, the results in [St] imply the following spectral decomposition of $m(\Lambda^{-1}\mathscr{L})$:

$$m(\Lambda^{-1}\mathscr{L})\delta = N_m = \sum_{k=0}^{\infty} m(2k+1)(P_{k,-1}+P_{k,1}),$$

where, away from the origin, $P_{k,\varepsilon}(\varepsilon = \pm 1)$ is given by the Calderon–Zygmund kernel

$$P_{k,\varepsilon}(z,u) = \frac{(-1)^k}{\pi^2}\left(k+1+k\frac{|z|^2+4i\varepsilon u}{|z|^2-4i\varepsilon u}\right)\frac{(|z|^2-4i\varepsilon u)^k}{(|z|^2+4i\varepsilon u)^{k+2}}$$

($P_{k,\varepsilon}$ is in fact the sum of this kernel and a multiple of the Dirac measure δ).

From these formulas it is again evident that $m(\Lambda^{-1}\mathscr{L})\delta$ is a kernel of critical degree -4, so that it is determined by the *angular parameter* θ defined by

$$4u + i|z|^2 =: |(z,u)|^2 e^{i\theta/2}, \quad 0 \leqq \theta \leqq 2\pi,$$

where $|(z,u)|$ denotes the homogeneous norm defined in Section 2. Then, in *polar coordinates* $r := |(z,u)|$ and θ, we may write $P_{k,\varepsilon}$ as

$$P_{k,\varepsilon}(r,\theta) = \frac{-1}{\pi^2 r^4}((k+1)e^{i\varepsilon(k+1)\theta} - ke^{i\varepsilon k\theta}).$$

Consequently, whenever convergence is guaranteed (say in the sense of distributions),

$$N_m(r,\theta) = \frac{2}{\pi^2 r^4}\sum_{k=0}^{\infty}(k+1)(\Delta\tilde{m})(k)\cos((k+1)\theta), \quad r>0, \tag{3.3}$$

where we have again put $\tilde{m}(k) := m(2k+1)$.

The numbers $(k+1)(\Delta\tilde{m})(k)$ are thus nothing but the *Fourier coefficients for the Fourier series expansion of* N_m *with respect to the angular variable* θ.

Corollary 3.1 thus presents a condition on the Fourier coefficients of the angular Fourier development of a homogeneous multiplier of degree 0 which ensures L^p-boundedness. For corresponding results on $\mathbb{R}^n$, see e.g. [CS].

Let us specialize (3.3) for the case of the multiplier m_γ defined as follows: $\tilde{m}_\gamma(k) = \hat{f}(k)$, where for $\gamma \in \mathbb{R}$ the function f on $\mathbb{R}$ is defined as

$$f(x) = \begin{cases} |x|^{-1+i\gamma} e^{-|x|}, & x > 0\,, \\ |x|^{-1+i\gamma} e^{-(1-i)|x|}, & x < 0\,. \end{cases}$$

Since

$$\int_0^\infty x^{-1+i\gamma} e^{-x} e^{-ixu}\, dx = \Gamma(-i\gamma)(1+iu)^{-i\gamma}\,,$$

one has

$$\hat{f}(u) = \Gamma(-i\gamma)\{(1+iu)^{-i\gamma} + (1-i(u+1))^{-i\gamma}\}\,.$$

In particular we see that $\hat{f}(-k) = \hat{f}(k-1)$, $k \in \mathbb{Z}$. This relation implies that

$$N_{m_\gamma}(r,\theta) = \frac{1}{\pi^2 r^4} \sum_{k \in \mathbb{Z}} (k+1) \Delta \hat{f}(k) e^{i(k+1)\theta}\,.$$

And, by Poisson's summation formula,

$$\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ik\theta} = 2\pi \sum_{k \in \mathbb{Z}} f(\theta + 2\pi k) = F(\theta)\,,$$

where

$$F(\theta) = 2\pi |\theta|^{-1+i\gamma} e^{-|\theta| - i\varepsilon(\theta)\theta} + E(\theta)\,,$$

with $E(\theta) = 2\pi \sum_{k \neq 0} f(\theta + 2\pi k)$. Here we have put $\varepsilon(\theta) = 0$, if $\theta > 0$, and $\varepsilon(\theta) = 1$, if $\theta < 0$. It is then easily seen that, say for $|\theta| < \pi$,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (k+1) \Delta \hat{f}(k) e^{i(k+1)\theta} &= \frac{d}{d\theta}[(1-e^{i\theta})F(\theta)] \\ &= 2\pi(\gamma + i)|\theta|^{-1+i\gamma} e^{-|\theta|} + O(|\gamma|) \quad \text{as } |\gamma| \to \infty\,, \end{aligned}$$

i.e.

$$N_{m_\gamma}(r,\theta) = \frac{2}{\pi r^4}\{(\gamma + i)|\theta|^{-1+i\gamma} e^{-|\theta|} + O(|\gamma|)\} \quad \text{as } |\gamma| \to \infty\,.$$

This shows that the weak type (1,1) norm of $m_\gamma(\Lambda^{-1}\mathscr{L})$ is grows at least like a multiple of $|\gamma|$ (and the norm on L^p at least like a multiple of $|\gamma|^{1-\varepsilon}(\varepsilon > 0)$ for p sufficiently close to 1) as $|\gamma| \to \infty$.

On the other hand, since $|\Gamma(-i\gamma)| \approx (2\pi)^{1/2} e^{-\pi|\gamma|/2} |\gamma|^{-1/2}$ as $|\gamma| \to \infty$ and $(1+ik)^{-i\gamma} = O(e^{\pi|\gamma|/2})$, one verifies easily that $\|m_\gamma\|_{L^2_{3/2+\delta,\,\mathrm{sloc}}} \leqq C_\delta |\gamma|^{1+\delta}$ as $|\gamma| \to \infty$.

This shows that Corollary 3.1 is essentially sharp with respect to the critical index $d/2$, at least for the Heisenberg group (which however seems to be representative for the general case here).

The preceding example allows also to give a lower bound for the operator $(\Lambda^{-1}\mathscr{L})^{-i\gamma}$.

In fact, we may write

$$|\gamma|^{1/2}\hat{f}(k) = \psi_\gamma(2k+1)(2k+1)^{-i\gamma}, \quad k \in \mathbb{N},$$

where

$$\tilde{\psi}_\gamma(k) = |\gamma|^{1/2}\Gamma(-i\gamma)\left\{\left(\frac{1+ik}{2k+1}\right)^{-i\gamma} + \left(\frac{1-i(k+1)}{2k+1}\right)^{-i\gamma}\right\}.$$

One checks that for any $\alpha > 0$ $\|\psi_\gamma\|_{l^2_{\alpha,\mathrm{sloc}}} \leqq C_\alpha$ for $|\gamma|$ sufficiently large, and thus Corollary 3.1 shows that, for p close to 1,

$$c|\gamma|^{\frac{3}{2}-\varepsilon} \leqq \||\gamma|^{1/2} m_\gamma(\Lambda^{-1}\mathscr{L})\|_{L^p,L^p} \leqq C\|(\Lambda^{-1}\mathscr{L})^{-i\gamma}\|_{L^p,L^p}.$$

Our discussions show that the operators $\mathscr{L}^{i\beta}$ and $(\Lambda^{-1}\mathscr{L})^{i\beta}$ essentially do have comparable norms on L^p for p close to 1. This somewhat surprising result reflects the fact that the operators $\mathscr{L}$ and Λ are not independent operators, like for instance ∂_{x_1} and ∂_{x_2} on $\mathbb{R}^2$, which act on separate coordinates, but both act on the u-variables. Also this indicates why one should expect a "non-standard" Marcinkiewicz condition in Theorem 2.3.

Corollary 2.4 could of course also be used to estimate operators like $|U_1|^{i\alpha_1}\dots|U_n|^{i\alpha_n}\mathscr{L}^{i\beta}$. Another immediate consequence is

Example 3.4 Let $a, b, \gamma, r \in \mathbb{R}$, $r \geqq 0$. The operator $(\mathscr{L}^a + r\Lambda^b)^{i\gamma}$ is bounded on $L^p(G)$, $1 < p < \infty$, with norm of order $O((1+|\gamma|)^{(1+\varepsilon)d|\frac{1}{2}-\frac{1}{p}|})$. This includes for example powers of the full Laplacian $\mathscr{L} - (U_1^2 + \cdots U_n^2)$ on G.

As a last example, let us consider the Heisenberg group H_m, for which $\mathfrak{g}_1 \cong \mathbb{R}^{2m}$, $\mathfrak{g}_2 \cong \mathbb{R}$ and $J_\mu = \mu\begin{pmatrix} 0 & I_m \\ -I_m & 0\end{pmatrix}$, $\mu \in \mathbb{R}$. There is only one central vector field U_1 in this case, which we shall denote by U. For $r \geqq 0$, $a \in \mathbb{R}$, let us consider (formally)

$$T := |\mathscr{L} - rU^2 - iaU|^{i\beta}, \quad \beta \in \mathbb{R}.$$

The corresponding multiplier on $\mathbb{R}_+^* \times \mathbb{R}$ is $h(\tau,\mu) = |\tau|\mu| - a\mu + r\mu^2|^{i\beta}$, i.e. $T = h(\Lambda^{-1}\mathscr{L}, \frac{1}{i}U)$. If $a \neq 0$, h will not be continuous, so that Corollary 2.4 does not apply. However, if we consider the restriction to the Heisenberg fan, i.e. if we look at $\tilde{m}$ defined on $\mathbb{Z} \times \mathbb{R}^m$ by

$$\tilde{m}(k,\mu) = |(2k+m)|\mu| - a\mu + r\mu^2|^{i\beta}$$

for $k \in \mathbb{N}$, and $\tilde{m} = 0$ for k negative, we see that $\tilde{m}$ is continuous if $r = 0$, and if in addition a avoids the singular set $\{\pm(2k+m) : k \in \mathbb{N}\}$, i.e. if $\mathscr{L} - iaU$ is hypoelliptic. Notice that for a in the singular set T is not defined. And, if $r = 0$ and $k \in \mathbb{N}$, then

$$\tilde{m} = \begin{cases} |(2k+m) - a|^{i\beta}\mu^{i\beta}, & \mu > 0, \\ |(2k+m) + a|^{i\beta}|\mu|^{i\beta}, & \mu < 0, \end{cases}$$

which by Theorem 2.3 easily implies

Example 3.5 If $a \notin \{\pm(2k+m) : k \in \mathbb{N}\}$, then $|\mathscr{L} - iaU|^{i\beta}$ is L^p-bounded on H_m for $1 < p < \infty$, and $\||\mathscr{L} - iaU|^{i\beta}\|_{L^p,L^p} \leqq C_\varepsilon(1+|\beta|)^{(1+\varepsilon)(2m+1)|\frac{1}{p}-\frac{1}{2}|}$.

In contrast, if for instance $r = 1$, i.e. if we consider $|L - iaU|^{i\beta}$, where $L = \mathscr{L} - U^2$ is the full Laplacian on H_m, and if $|a| > m$, then $(2k+m)|\mu| - a\mu + \mu^2$ will always vanish away from $\mu = 0$. This will, however, happen on a set of Plancherel measure zero, so that T is still defined on L^2, and one expects that T is also bounded on L^p for $1 < p < \infty$. However such a result does not seem to be attainable with our methods.

4 Littlewood–Paley decompositions

Fix $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \geqq 0$, $\operatorname{supp}\chi \subseteq [1/2, 2]$ and $\sum_{j=-\infty}^{+\infty} \chi_j^2(x) = 1$ for $x > 0$, where we have put $\chi_j(x) := \chi(2^{-j}x)$. Let $\tilde{\chi}_j(x) := \chi_j(x) + \chi_j(-x)$, and for $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ let us write $\tilde{\chi}_l(\mu) := \tilde{\chi}_{l_1}(\mu_1) \ldots \tilde{\chi}_{l_n}(\mu_n)$. Then one has

$$\sum_{j,l} \chi_j^2(\lambda)\tilde{\chi}_l^2(\mu) = 1 \quad \text{for } \lambda > 0,\ \mu \in \mathbb{R}^n \backslash \{0\}\,. \tag{4.1}$$

Set

$$\varphi_j := \chi_j(\mathscr{L})\delta\,,$$

$$\psi_l := \tilde{\chi}_l(U/i)\delta\,.$$

Notice that if $G_2 := \exp \mathfrak{g}_2$ denotes the center of G, then we may form $\tilde{\chi}_l(U/i|_{G_2})\delta =: \psi_l^0$ on G_2. Then $\psi_l^0 \in \mathscr{S}(G_2)$, and

$$\begin{aligned} \psi_l^0(u) &= \psi_{l_1}^0(u_1) \ldots \psi_{l_n}^0(u_n)\,, \\ \psi_l(z,u) &= \delta(z) \otimes \psi_l^0(u)\,, \\ \varphi_j * \psi_l &= (\chi_j \otimes \tilde{\chi}_l)(\mathscr{L}, U/i)\delta\,, \end{aligned} \tag{4.2}$$

where $\psi_k^0(t) := \tilde{\chi}_k(\frac{1}{i}\frac{\partial}{\partial t})\delta$ on $\mathbb{R}$.

If we replace $\mathscr{L}$ by $\Lambda^{-1}\mathscr{L}$, we may formally also define

$$\Phi_j = \chi_j(\Lambda^{-1}\mathscr{L})\delta\,.$$

By Remark 2.2, Φ_j is a kernel of critical degree which, in the sense of distributions, can be defined correctly for instance by $\Phi_j := \sum_l \Phi_j * \psi_l$, where

$$\Phi_j * \psi_l := (\chi_j \otimes \tilde{\chi}_l)(\Lambda^{-1}\mathscr{L}, U/i)\delta\,. \tag{4.3}$$

We define two *g-functions* as follows:

$$g_1(f)(x) := \left(\sum_{j,l} |f * (\varphi_j * \psi_l)(x)|^2\right)^{1/2},$$

$$g_2(f)(x) := \left(\sum_{j,l} |f * (\Phi_j * \psi_l)(x)|^2\right)^{1/2},$$

for $f \in \mathscr{S}'(G)$. Observe that the Littlewood–Paley decomposition appearing in $g_1(f)$ is just the "projection" to the Heisenberg (type) group of the one defined in [MRS].

By applying the Gelfand transform and (4.1), we observe that

$$\lim_{N\to\infty} \sum_{|j|\leqq N,\, |l|\leqq N} \langle (\phi_j * \psi_l) * ((\phi_j * \psi_l)^*, f\rangle = f(0) \tag{4.4}$$

for each $f \in \mathscr{S}(G)$.

In fact, by polarization of (1.1) and (1.2), this is clear for $\mathfrak{g}_1$-radial f, and the general case follows by "radializing" f.

(4.4) implies that $\|g_1(f)\|_{L^2} = \|f\|_{L^2}$ for $f \in \mathscr{S}(G)$, i.e. that g_1 is an isometry of $L^2(G)$.

Similarly one proves that also g_2 is an isometry of $L^2(G)$.

Proposition 4.1 *For each* $p \in]1, +\infty[$ *there exists a constant* $c_p > 0$ *such that*

$$c_p\|f\|_{L^p} \leqq \|g_1(f)\|_{L^p} \leqq c_p^{-1}\|f\|_{L^p},$$

$f \in \mathscr{L}^p(G)$. *One may choose* $c_2 = 1$.

Proof. By a standard duality argument [S], it suffices to prove the second inequality.

Moreover, in order to simplify the notation, we shall assume that $n = 1$, the extension of the argument to the general case being straight-forward.

If $\varepsilon = ((\varepsilon_j^1, \varepsilon_l^2))_{j,l}$ is a double sequence of numbers $\varepsilon_j^i = \pm 1$, we denote by K_ε the singular kernel

$$K_\varepsilon := \sum_{j,l\in\mathbb{Z}} \varepsilon_j^1 \varepsilon_l^2 \varphi_j * \psi_l\,.$$

By some standard randomization argument based on Khintchin's inequality [S, p. 276] it then suffices to prove that

$$\|f * K_\varepsilon\|_{L^p} \leqq c_p^{-1}\|f\|_{L^p}, \tag{4.5}$$

independently of ε.

But, if we put

$$K_{\varepsilon^1}^1 := \sum_j \varepsilon_j^1 \varphi_j = m_{\varepsilon^1}(\mathscr{L})\delta\,,$$

$$K_{\varepsilon^2}^2 := \sum_l \varepsilon_l^2 \psi_l = m_{\varepsilon^2}\left(\frac{1}{i}U\right)\delta\,,$$

with

$$m_{\varepsilon^1}(\lambda) := \sum_j \varepsilon_j^1 \chi_j(\lambda)\,,$$

$$m_{\varepsilon^2}(\mu) := \sum_l \varepsilon_l^2 \tilde{\chi}_l(\mu)\,,$$

then m_{ε^1} and m_{ε^2} satisfy a Hörmander–Mihlin condition of any fixed degree, uniformly for all ε^1 and ε^2. Therefore, by any of the known Hörmander–Mihlin type multiplier theorems for stratified groups, right convolution with $K^1_{\varepsilon^1}$ and $K^2_{\varepsilon^2}$ is bounded on $L^p(G)$, uniformly in ε^1 and ε^2. This implies (4.5), since $K_\varepsilon = K^1_{\varepsilon_1} * K^2_{\varepsilon^2}$. $\square$

As a corollary, we shall obtain a weak Marcinkiewicz type multiplier theorem for G. For Heisenberg groups, this had already been proved in [MRS], however by a somewhat different method. The proof of this corollary will also make use of the following

Lemma 4.2 *Let $a > 0, M \in \mathbb{N}$. There exists a constant $C = C_{n,a,M}$ and $N \in \mathbb{N}$, such that each function $g \in C_0^N(]-a,a[^n)$ admits a development into a tensor series*

$$g = \sum_{\nu=1}^{\infty} \gamma_1^\nu \otimes \cdots \otimes \gamma_n^\nu\,,$$

where the γ_j^ν are in $C_0^M(]-a,a[)$ and

$$\sum_{\nu=1}^{\infty} \|\gamma_1^\nu\|_{C^M} \cdots \|\gamma_n^\nu\|_{C^M} \leqq C\|g\|_{C^N}\,.$$

Proof. This follows for instance easily from a Fourier series expansion of g on $]-a,a[^n$ (compare [MRS]). $\square$

Corollary 4.3 *For $N \in \mathbb{N}$ and $m \in C^N(\mathbb{R}_+^* \times \mathbb{R}^n)$ put*

$$\|m\|_{(N)} := \sup_{|\alpha|\leqq N}\ \sup_{\lambda,\mu} \left| \left(\lambda \frac{\partial}{\partial \lambda}\right)^{\alpha_0} \left(\mu_1 \frac{\partial}{\partial \mu_1}\right)^{\alpha_1} \cdots \left(\mu_n \frac{\partial}{\partial \mu_n}\right)^{\alpha_n} m(\lambda,\mu) \right|\,.$$

There exists $N \in \mathbb{N}$ such that if $\|m\|_{(N)} < \infty$, then $m(\mathcal{L}, U/i)$ is bounded on $L^p(G)$ for $1 < p < \infty$, with norms controlled by $\|m\|_{(N)}$.

Proof. Choose $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta \geqq 0$, $\operatorname{supp}\eta \subseteq [1/4,4]$ and $\eta = 1$ on $\operatorname{supp}\chi$. Define η_j and $\tilde{\eta}_l$ in analogy with χ_j and $\tilde{\chi}_l$, and put

$$M_{j,l} := [m(\eta_j \otimes \tilde{\eta}_l)](\mathcal{L}, U/i)\delta\,.$$

In order to control the kernel $M_m = m(\mathcal{L}, U/i)\delta$, we observe that

$$M_m * (\varphi_j * \psi_l) = (\varphi_j * \psi_l) * M_{j,l}\,,$$

and thus by Proposition 4.1 and standard Littlewood–Paley theory it suffices to prove that the maximal operator

$$\mathcal{M}(g) := \sup_{j,l} |g * M_{j,l}|$$

is bounded on $L^q(G)$ for $1 < q \leqq \infty$, with norms controlled by $\|m\|_{(N)}$. Put

$$\tilde{m}_{j,l}(\lambda, \mu) := m(2^j\lambda, 2^{l_1}\mu_1, \dots, 2^{l_n}\mu_n)\eta_0(\lambda)\tilde{\eta}_0(\mu)\,.$$

Then, for every j and l,

$$\|\tilde{m}_{j,l}\|_{C^N} \leqq \|m\|_{(N)}\,. \tag{4.6}$$

Fix $M \in \mathbb{N}$. By Lemma 4.2 and (4.6) there is an $N \in \mathbb{N}$ such that each $\tilde{m}_{j,l}$ admits a representation

$$\tilde{m}_{j,l}(\lambda, \mu) = \sum_{\nu=1}^{\infty} \gamma_{j,l}^{\nu,0}(\lambda)\gamma_{j,l}^{\nu,1}(\mu_1)\dots\gamma_{j,l}^{\nu,n}(\mu_n)\,,$$

where each term is supported in $[1/8, 8]^{n+1}$ and

$$\sum_{\nu} \|\gamma_{j,l}^{\nu,0}\|_{C^M} \dots \|\gamma_{j,l}^{\nu,n}\|_{C^M} \leqq C\|m\|_{(N)}\,. \tag{4.7}$$

Put

$$R_{j,l}^{\nu} := \gamma_{j,l}^{\nu,0}(2^{-j}\mathcal{L})\delta\,,$$

$$S_{j,l}^{\nu,k} := \gamma_{j,l}^{\nu,k}\left(2^{-l_k}\frac{1}{i}U_k\right)\delta\,.$$

Then

$$M_{j,l} = \sum_{\nu} R_{j,l}^{\nu} * S_{j,l}^{\nu,1} * \dots * S_{j,l}^{\nu,n}\,.$$

Moreover, for M sufficiently large, well-known functional calculus [H], [Ma] for $\mathcal{L}$ yields uniform size estimates

$$|(\gamma_{j,l}^{\nu,0}(\mathcal{L})\delta)(x)| \leqq C\|\gamma_{j,l}^{\nu,0}\|_{C^M}(1 + |x|)^{-Q-1}\,,$$

which in combination with (2.2) imply

$$|g * |R_{j,l}^{\nu}|| \leqq C\|\gamma_{j,l}^{\nu,0}\|_{C^M}\mathcal{M}_0(g)\,, \tag{4.8}$$

where $\mathcal{M}_0$ denotes the anisotropic Hardy–Littlewood maximal operator on G (see [FS]).

Similarly, if $\mathcal{M}_1$ denotes the strong maximal operator with respect to the central variables $u_1, \dots, u_n$ on G, then

$$|g * |S_{j,l}^{\nu,1}| * \dots * |S_{j,l}^{\nu,n}|| \leqq C\|\gamma_{j,l}^{\nu,1}\|_{C^M} \dots \|\gamma_{j,l}^{\nu,n}\|_{C^M}\mathcal{M}_1(g)\,. \tag{4.9}$$

Combining (4.7) to (4.9) yields

$$|g * M_{j,l}| \leqq C\|m\|_{(N)}\mathcal{M}_1(\mathcal{M}_0(g))\,,$$

hence

$$\mathcal{M}(g) \leqq C\|m\|_{(N)}\mathcal{M}_1(\mathcal{M}_0(g))\,.$$

But $\mathcal{M}_1$ and $\mathcal{M}_0$ are bounded on $L^q(G)$, $1 < q \leqq \infty$ [FS], [S], and thus the Corollary is proved. □

Let us finally turn to $g_2(f)$.

Proposition 4.4 *For each* $p \in]1, +\infty[$ *there exists a constant* $c_p > 0$ *such that*

$$c_p \|f\|_{L^p} \leqq \|g_2(f)\|_{L^p} \leqq c_p^{-1} \|f\|_{L^p} \, ,$$

$f \in \mathcal{L}^p(G)$. *One may choose* $c_2 = 1$.

Proof. If $\varepsilon = (\varepsilon_j^0 \varepsilon_{l_1}^1, \ldots, \varepsilon_{l_n}^n)_{j,l}$ is an $(n+1)$-fold sequence of numbers ± 1, let us put

$$m_\varepsilon(\lambda, \mu) := \sum_{j,l} \varepsilon_j^0, \varepsilon_{l_1}^1 \ldots \varepsilon_{l_n}^n \chi_j(\lambda) \tilde{\chi}_l(\mu) \, .$$

A comparison with the proof of Proposition 4.1 reveals that it here suffices to prove a uniform (in ε) estimate

$$\|m_\varepsilon(\Lambda^{-1}\mathcal{L}, U/i) f\|_{L^p} \leqq c_p^{-1} \|f\|_{L^p} \, . \tag{4.10}$$

To this end, let n_ε be the multiplier

$$n_\varepsilon(\lambda, \mu) := m_\varepsilon \left(\frac{\lambda}{|\mu|}, \mu \right) \, .$$

Then clearly $m_\varepsilon(\Lambda^{-1}\mathcal{L}, U/i) = n_\varepsilon(\mathcal{L}, U/i)$. Moreover, one readily checks that

$$\|m_\varepsilon\|_{(N)} \leqq C_N$$

for every ε, which easily implies also

$$\|n_\varepsilon\|_{(N)} \leqq C_N' \, .$$

Thus (4.10) is a consequence of Corollary 4.3. □

5. Proof of Theorem 2.3

By analogy with the proof of Corollary 4.3, we need here to consider the kernels

$$N_{j,l} := [m(\eta_j \otimes \tilde{\eta}_l)](\Lambda^{-1}\mathcal{L}, U/i)\delta \, ,$$

with η as before.

Observe that $N_{j,l} = 0$ if $j \leqq 1$.

The main problem will be to derive precise size estimates for these kernels. However, even with these at hand, one still has to argue more carefully than in the proof of Corollary 4.3 in order to obtain Theorem 2.3.

Let us begin with this part of the proof, which will be based on an adaptation of a method from [S] to the present setting of "discrete" Littlewood–Paley theory. If $l \in \mathbb{Z}^n$, with some slight risk of confusion $|l|$ will denote the integer

$|l| := l_1 + \cdots + l_n \in \mathbb{Z}$; notice that $|l|$ may be negative. Also, we put

$$l_{\max} := \max_{j=1,\dots,n} l_j .$$

Lemma 5.1 *Let $w^{\varepsilon}_{j,l}$ be the weight*

$$w^{\varepsilon}_{j,l}(z,u) := 2^{-m(j+l_{\max})}(1 + 2^{\frac{j+l_{\max}}{2}}|z|)^{2m(1+\varepsilon)} \prod_{i=1}^{n} 2^{-l_i}(1 + 2^{l_i}|u_i|)^{1+\varepsilon} .$$

Suppose there is some $\varepsilon > 0$ and $A > 0$ such that

$$\int_G |N_{j,l}(x)|^2 w^{\varepsilon}_{j,l}(x)\, dx \leqq A^2 \tag{5.1}$$

for every $j \geqq 2$, l. Then for $1 < p < \infty$

$$\|m(\Lambda^{-1}\mathcal{L}, U/i)\|_{L^p,L^p} \leqq C_p A .$$

Proof. By Proposition 4.4, we have

$$\|m(\Lambda^{-1}\mathcal{L}, U/i)f\|_{L^p} \leqq c_p^{-1} \left\| \left(\sum_{j,l} |f_{j,l} * N_{j,l}|^2 \right)^{1/2} \right\|_{L^p} ,$$

where

$$f_{j,l} := f * (\Phi_j * \psi_l) .$$

By Hölder's inequality and (5.1),

$$\begin{aligned} |f_{j,l} * N_{j,l}(x)|^2 &= \left|\int f_{j,l}(xy^{-1}) N_{j,l}(y)\, dy\right|^2 \\ &\leqq A^2 \int |f_{j,l}(xy^{-1})|^2 \frac{1}{w^{\varepsilon}_{j,l}(y)}\, dy . \end{aligned}$$

Assume now that $p \geqq 2$ (the case $p < 2$ follows by duality). Then there exists $g \in L^{(p/2)'} = L^{p/(p-2)}$ with $g \geqq 0$ and $\|g\|_{L^{p/(p-2)}} = 1$ such that

$$\|m(\Lambda^{-1}\mathcal{L}, U/i)f\|^2_{L^p} \leqq \frac{2}{c_p^2} \int \sum_{j,l} |f_{j,l} * N_{j,l}|^2 g\, dx ,$$

hence, by Fubini's theorem,

$$\begin{aligned} \|m(\Lambda^{-1}\mathcal{L}, U/i)f\|^2_{L^p} &\leqq C_p A^2 \int\int \sum_{j,l} |f_{j,l}(xy^{-1})|^2 \frac{1}{w^{\varepsilon}_{j,l}(y)}\, dy\, g(x)\, dx \\ &= C_p A^2 \int \left(\sum_{j,l} |f_{j,l}(y)|^2 \right) \left(\int g(x) \frac{1}{w^{\varepsilon}_{j,l}(y^{-1}x)}\, dx \right) dy . \end{aligned}$$

Denote by $\mathcal{M}_s$ the following *strong maximal operator* on G:

$$\mathcal{M}_s(g)(y) := \sup_{r_j > 0} \frac{1}{r_0^{2m} r_1 \dots r_n} \int_{|z| < r_0} \int_{|u_i| < r_i} |g(y(z,u))|\, dz du .$$

Then clearly

$$\int g(x)\frac{1}{w_{j,l}^{\varepsilon}(y^{-1}x)}\,dx \leqq \mathcal{M}_s g(y)\,.$$

Since by [C] (here also the simple argument sketched in the introduction to [RS] would apply) $\mathcal{M}_s$ is bounded on $L^q(G)$, $1 < q \leqq \infty$, we obtain, again by Hölder's inequality and Proposition 4.4,

$$\begin{aligned}\|m(A^{-1}\mathcal{L}, U/i)f\|_{L^p}^2 &\leqq C_p A^2 \int \left(\sum_{j,l} |f_{j,l}(y)|^2\right) \mathcal{M}_s g(u)\,dy \\ &\leqq C_p A^2 \|g_2(f)\|_{L^p}^2 \|\mathcal{M}_s g\|_{L^{p/(p-2)}} \\ &\leqq C_p A^2 \|f\|_{L^p}^2 \|g\|_{L^{p/(p-2)}} \\ &= C_p A^2 \|f\|_{L^p}^2\,. \qquad \square\end{aligned}$$

It remains to prove that the conditions in Theorem 2.3 imply the estimate (5.1) for some $\varepsilon > 0$. This will be accomplished by means of the Plancherel formula (1.1).

If $p(z,u)$ is a $\mathfrak{g}_1$-radial polynomial on G, it multiplies the subalgebra of radial Schwartz class functions of $\mathcal{A}$, and we can therefore define an operator ∂_p by

$$\mathcal{G}(pf)(\mu,k) =: \partial_p\,(\mathcal{G}f)(\mu,k), \quad (\mu,k) \in (\mathfrak{g}_2^*\backslash\{0\}) \times \mathbb{N},\ f \in \mathcal{S} \cap \mathcal{A}\,.$$

If R is a function on $(\mathfrak{g}_2^*\backslash\{0\}) \times \mathbb{N}$, we define translation operators τ_l, $l \in \mathbb{Z}$, by

$$(\tau_l R)(\mu,k) := \begin{cases} R(\mu,k+l), & \text{if } k+l \geqq 0\,, \\ 0, & \text{else}\,. \end{cases}$$

Notice that the τ_l can be considered as ordinary translation operators, if we extend R to $\mathfrak{g}_2^*\backslash\{0\} \times \mathbb{Z}$ by setting it 0 for $k < 0$.

Straight-forward computations, based on well-known properties of Laguerre polynomials [E] yield

(5.2a) $$\partial_{|z|^2} = \frac{2}{|\mu|}((2k+m)\tau_0 - k\tau_{-1} - (k+m)\tau_1)\,,$$

(5.2b) $$\partial_{-iu_j} = \frac{\partial}{\partial\mu_j} + \frac{\mu_j}{2|\mu|^2}(m\tau_0 + k\tau_{-1} - (k+m)\tau_1)\,.$$

Let us denote by Δ the first order difference operator $\Delta := \tau_1 - \tau_0$. Then the following commutation relations between Δ, τ_l and multiplication by k,

(5.3) $$[\tau_l, \Delta] = 0, \qquad [\tau_l, k] = l\tau_l, \qquad [\Delta, k] = \tau_1\,,$$

allow us to re-write (5.2) as

$$\partial_{|z|^2} = \frac{-1}{|\mu|}(\tau_{-1}k\Delta^2 + ((2m+1)\tau_0 - \tau_{-1})\Delta), \tag{5.4a}$$

$$\partial_{-iu_j} = \frac{\partial}{\partial\mu_j} - \frac{\mu_j}{2|\mu|^2}((\tau_0 + \tau_{-1})k\Delta + (m\tau_1 - (m-1)\tau_0 - \tau_{-1})). \tag{5.4b}$$

If one applies these operators iteratedly, induction and (5.3) easily lead to the following lemma (we omit the details):

Lemma 5.2

(1) *For* $p \in \mathbb{N}\backslash\{0\}$ *one has*

$$\partial^p_{|z|^2} = |\mu|^{-p}\sum_{\nu=0}^{p} a_{p,\nu} \cdot \tilde{\tau} k^\nu \Delta^{p+\nu},$$

where each $a_{p,\nu} \cdot \tilde{\tau}$ *denotes a finite sum of terms* $a_l\tau_l$, $a_l \in \mathbb{R}$.

(2) *Introducing auxiliary variables* $T_1, \ldots, T_n \in \mathbb{R}$, *we have for* $q \in \mathbb{N}$

$$\partial^q_{[(T_1^2+u_1^2)\cdots(T_n^2+u_n^2)]} = \sum_{\substack{\alpha,\beta,\gamma,\delta\in\mathbb{N}^n \\ \alpha_j+\beta_j\leqq 2q \\ \alpha_j+2\gamma_j+\delta_j=2q}} T_1^{2\gamma_1}\cdots T_n^{2\gamma_n} b_{q,\alpha,\beta,\gamma,\delta}(\mu)\cdot\tilde{\tau}\,|\mu|^{-|\delta|}\frac{\partial^\alpha}{\partial\mu^\alpha}k^{|\beta|}\Delta^{|\beta|},$$

where each $b_{q,\alpha,\beta,\gamma,\delta}(\mu) \cdot \tilde{\tau}$ *is a finite sum of terms* $b_l(\mu)\tau_l$, *with* $b_l \in C^\infty(\mathfrak{g}_2^*\backslash \{0\})$ *being homogeneous of degree* 0, *hence in particular bounded.*

Let us put (compare (2.1))

$$m_{j,l}(k,\mu) := \begin{cases} \mathcal{G}N_{j,l}(\mu,k) = (m(\eta_j \otimes \tilde{\eta}_l))(2k+m,\mu), & k \geqq 0, \\ 0, & k < 0, \end{cases}$$

and let us denote by Δ also the ordinary difference operator on $\mathbb{Z}$. With the aid of Lemma 4.2 and the Plancherel formula (1.1) we shall now prove the crucial

Proposition 5.3 *For* $p \in \mathbb{R}_+$, $l \in \mathbb{Z}^n$ *and* $j \geqq 2$ *we have*

$$\begin{aligned} &\int\int |[(1+2^{j+l_{\max}}|z|^2)^{2m}(1+2^{2l_1}u_1^2)\ldots(1+2^{2l_n}u_n^2)]^p N_{j,l}(z,u)|^2\,dz\,du \\ &\leqq C2^{m(j+l_{\max})+|l|}\frac{1}{2^j}\sum_{k\in\mathbb{Z}}\frac{1}{2^{|l|}} \\ &\times\int\left|(1+|2^j\Delta|)^{2mp}\prod_{i=1}^{n}(1+|2^j\Delta|+|2^{l_i}\partial_{\mu_i}|)^{2p}m_{j,l}(k,\mu)\right|^2 d\mu\,. \end{aligned} \tag{5.5}$$

Before we enter the proof, let us remark that the estimate (5.5) can be reduced to the case $l = 0$ by means of the scaling identity (2.3), if

$\dim \mathfrak{g}_2 = 1$. This reduction, which would simplify the argument considerably, will however fail in the general case, since the group of "dilating" automorphisms of G will in general be too small for this purpose.

Let us put

$$w(z,u) := 2^{2|l|p}[1+(2^{j+l_{\max}}|z|^2)^{2mp}][(2^{-2l_1}+u_1^2)\dots(2^{-2l_n}+u_n^2)]^p\,,$$

where again $l_{\max} = \max l_j$. To prove Proposition 5.3, we have to show that

$$\int_G |w(x)N_{j,l}(x)|^2\,dx$$

is dominated by the right hand side of (5.5). Moreover, it will suffice to prove this for $p \in \mathbb{N}$, since the general case will then follow by interpolation.

So, assume $p \in \mathbb{N}$. By Lemma 5.2 and (5.3) we see that

(5.6)

$$\begin{aligned}\partial_{w(z,u)} &= \sum_{\mathscr{I}} \sum_{v=0}^{2np-|\alpha|} b_{\alpha,\gamma,\delta,v}(\mu)\cdot\tilde{\tau}\left(2^{2|l|p-2l\cdot\gamma}|\mu|^{-|\delta|}\frac{\partial^\alpha}{\partial\mu^\alpha}\right)k^v\Delta^v\\ &+\sum_{\mathscr{I}}\sum_{v=0}^{2(m+n)p-|\alpha|} c_{\alpha,\gamma,\delta,v}(\mu)\cdot\tilde{\tau}\left(2^{2|l|p+2(mpl_{\max}-l\cdot\gamma)}|\mu|^{-(|\delta|+2mp)}\frac{\partial^\alpha}{\partial\mu^\alpha}\right)\\ &\times(2^j)^{2mp}k^v\Delta^{2mp+v}\,,\end{aligned}$$

where $\mathscr{I}$ is the index set

$$\mathscr{I} := \{\alpha,\gamma,\delta \in \mathbb{N}^n : \alpha_j \leqq 2p,\ \alpha_j+2\gamma_j+\delta_j = 2p\}\,,$$

and where the "coefficients" of the $b_{\alpha,\gamma,\delta,v}(\mu)\cdot\tilde{\tau}$ and $c_{\alpha,\gamma,\delta,v}(\mu)\cdot\tilde{\tau}$ are bounded functions. Observing that $|\mu_i| \sim 2^{l_i}$ on $\operatorname{supp} m_{j,l}$, one sees that

$$|\mu| \sim 2^{l_{\max}} \quad \text{on } \operatorname{supp} m_{j,l}\,, \tag{5.7}$$

and hence in the first double sum of (5.6)

$$2^{2|l|p-2l\cdot\gamma}|\mu|^{-|\delta|} \sim 2^{2|l|p-2l\cdot\gamma-|\delta|l_{\max}} \lessapprox 2^{l\cdot\alpha}\,, \tag{5.8}$$

since $2|l|p-2l\cdot\gamma = l\cdot\alpha+l\cdot\delta$ and $l\cdot\delta \leqq (l_{\max},\dots,l_{\max})\cdot\delta = l_{\max}|\delta|$.

Similarly, in the second double sum

$$2^{2|l|p+2(mpl_{\max}-l\cdot\gamma)}|\mu|^{-(|\delta|+2mp)} \lessapprox 2^{l\cdot\alpha}\,. \tag{5.9}$$

On $\operatorname{supp} m_{j,l}$ one also has

$$k+1 \sim 2^j, \qquad \binom{k+m-1}{k} \sim 2^{(m-1)j}\,,$$

in particular

$$k^v \sim 2^{vj}, \qquad (2^j)^{2mp}k^v \sim 2^{(2mp+v)j}\,. \tag{5.10}$$

The statements (5.6) to (5.10) in combination with (1.1) imply the following estimate:

$$(5.11)\quad \int_G |w(x)N_{j,l}(x)|^2\,dx \leqq C2^{m(j+l_{\max})+|l|} \sum_{\alpha\in\mathbb{N}^n,\,\alpha_j\leqq 2p}\ \sum_{v=0}^{2dp-|\alpha|} \frac{1}{2^j}\sum_k \frac{1}{2^{|l|}}$$

$$\times \int \left|(2^j\Delta)^v\left(2^{l\cdot\alpha}\frac{\partial^\alpha}{\partial\mu^\alpha}\right) m_{j,l}(k,\mu)\right|^2 d\mu\,.$$

An application of the following lemma to the terms in (5.11) will finally conclude the proof of Proposition 5.3.

Lemma 5.4 *Suppose that $f = f(k,\mu)$ is a square integrable function on the l.c. abelian group $\mathbb{Z}\times\mathbb{R}^n$. Let $R,\rho_1,\ldots,\rho_n \geqq 0$, $v,\alpha_1,\ldots,\alpha_n \in \mathbb{N}$ be such that $\alpha_j \leqq p$, $j = 1,\ldots,n$, and $v + |\alpha| \leqq q$, where $q \geqq np$. Then, with $\rho^\alpha := \rho_1^{\alpha_1}\ldots\rho_n^{\alpha_n}$,*

$$\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n} \left|(R\Delta)^v\left(\rho^\alpha\frac{\partial^\alpha}{\partial\mu^\alpha}\right) f(k,\mu)\right|^2 d\mu$$

$$\leqq C\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^n}\left|(1+|R\Delta|)^{q-np}\prod_{j=1}^n(1+|R\Delta|+|\rho_j\partial_{\mu_j}|)^p f(k,\mu)\right|^2 d\mu\,.$$

Proof. By Plancherel's theorem for the group $\mathbb{Z}\times\mathbb{R}^n$ this will follow from the estimate

(5.12)

$$r_1^{\alpha_1}\cdots r_n^{\alpha_n}r^v \leqq \sum_{\mathcal{J}\subseteq\{1,\ldots,n\}}(1+r)^{q-|\mathcal{J}|p}\prod_{i\in\mathcal{J}} r_i^p = (1+r)^{q-np}\prod_{j=1}^n(1+r+r_j)^p\,,$$

with $r_i = \rho_i|u_i|$ and $r = R|e^{it}-1|$, if u_i and t denote the variables dual to μ_i and k, respectively.

But, if $r \geqq 1$ and $\mathcal{J} := \{i : r_i > r\}$, then

$$\begin{aligned} r_1^{\alpha_1}\cdots r_n^{\alpha_n}r^v &= r^{v+|\alpha|}\left(\frac{r_1}{r}\right)^{\alpha_1}\cdots\left(\frac{r_n}{r}\right)^{\alpha_n} \\ &\leqq r^{v+|\alpha|}\prod_{i\in\mathcal{J}}\left(\frac{r_i}{r}\right)^p \\ &= r^{v+|\alpha|-|\mathcal{J}|p}\prod_{i\in\mathcal{J}} r_i^p \\ &\leqq r^{q-|\mathcal{J}|p}\prod_{i\in\mathcal{J}} r_i^p\,. \end{aligned}$$

And, if $r < 1$, then

$$r_1^{\alpha_1} \dots r_n^{\alpha_n} r^\nu \leqq (1 + r_1)^p \dots (1 + r_n)^p ,$$

and again this is bounded by the right-hand side of (5.12). □

If we now choose $p = \frac{1+\varepsilon}{4}$ in Proposition 5.3, we obtain the following estimate of type (5.1) as required in Lemma 5.1:

$$\text{(5.13)} \quad \int_G |N_{j,l}(x)|^2 w_{j,l}^\varepsilon(x)\, dx \leqq C \sup_{j,l} \frac{1}{2^j} \sum_{k \in \mathbb{Z}} \frac{1}{2^{|l|}}$$

$$\times \int \left| (1 + |2^j \Delta|)^{(1+\varepsilon)m} \prod_{i=1}^n (1 + |2^j \Delta| + |2^{l_i} \partial_{\mu_i}|)^{\frac{1+\varepsilon}{2}} m_{j,l}(k, \mu) \right|^2 d\mu ,$$

which by Lemma 5.1 proves Theorem 2.3.

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