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# Bounded orbits of flows on homogeneous spaces

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A well-known class of flows arises as follows: Let $G$ be a semisimple Lie group and $\Gamma$ be a lattice in $G$, that is, $\Gamma$ is a discrete subgroup such that $G/\Gamma$ admits a finite measure invariant under the action of $G$, on the left. Let $(g_t)$ be a one-parameter subgroup of $G$. The action of $(g_t)$ on $G/\Gamma$ defines a flow. Necessary and sufficient conditions are known, thanks to the work of C. C. Moore, for such a flow to be ergodic (with respect to the unique $G$-invariant probability measure); (cf. [13]). Thus, for instance, if $G$ is a noncompact simple Lie group with finite center then the action of $(g_t)$ on $G/\Gamma$ is ergodic if and only if $(g_t)$ is not contained in a compact subgroup of $G$.

When the flow induced by $(g_t)$, as above, is ergodic, the orbits of almost all points are dense in $G/\Gamma$. However, in general, all orbits of the flow are not dense. For instance, if $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ then the flow as above is the geodesic flow associated to the modular surface; in this case there exist periodic orbits, divergent orbits and also many other types of orbits which are not dense. A similar phenomenon occurs for most homogeneous spaces for flows induced by one-parameter subgroups $(g_t)$ such that $\mathrm{Ad}\, g_t$ is semisimple for all $t$ (cf. [2]). The situation is somewhat different when $\mathrm{Ad}\, g_t$, $t \in \mathbb{R}$ are unipotent; we shall however not concern ourselves with that here (cf. [8] and [3] for details).

In [2] we considered flows as above on noncompact homogeneous spaces $G/\Gamma$ and studied their trajectories (one-sided orbits $\{g_t g\Gamma \mid t \geq 0\}$) which are either divergent (that is, eventually leave every compact subset of $G/\Gamma$) or bounded (relatively compact). It was shown, in particular, that for flows on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, $n \geq 2$, induced by one-parameter subgroups of the form $\mathrm{diag}(e^{-t}, \ldots e^{-t}, e^{\lambda t}, \ldots, e^{\lambda t})$, where $\lambda$ is such that the determinant is 1, divergence or boundedness of a trajectory starting from $gSL(n, \mathbb{Z})$, $g \in SL(n, \mathbb{R})$, is equivalent to a certain system of linear forms associated to $g$, in a natural way, being singular or badly approximable (cf. [16] or [2] for definitions) respectively. In the particular case of $n = 2$, $g_t = \mathrm{diag}(e^{-t}, e^t)$, $g = p\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\gamma$ for some $\alpha \in \mathbb{R}$, an upper triangular matrix $p$ and $\gamma \in \Gamma$, the trajectory $\{g_t gSL(n, \mathbb{Z}) \mid t \geq 0\}$ is divergent if and only if $\alpha$ is rational and bounded if and only if $\alpha$ is badly approximable (cf. Remark 2.6 for the latter). Using the latter assertion and a theorem of W. M. Schmidt (cf. [1–5]) it was also deduced that for the flows on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ as above, the set of points on bounded trajectories is "large" in the sense that its Hausdorff dimension coincides with the dimension of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ as a manifold. We then raised the question whether an analogous assertion holds if $G = SL(2, \mathbb{R})$ and $\Gamma$ is any lattice in $G$ (not necessarily $SL(2, \mathbb{Z})$) for the flow induced by $\operatorname{diag}(e^{-t}, e^{t})$. In this paper we answer that question in the affirmative in the following more general form. (cf. Theorem 5.1 below).

THEOREM. *Let $G$ be a connected semisimple Lie group of $\mathbb{R}$-rank 1 and $\Gamma$ be a lattice in $G$. Let $(g_t)$ be a one-parameter subgroup of $G$ such that $\operatorname{Ad} g_1$ has an eigenvalue (possibly complex) of absolute value other than 1. Then for any nonempty open subset $\Omega$ of $G/\Gamma$*

$$\{g\Gamma \in \Omega \mid \text{the } (g_t)\text{-orbit of } g\Gamma \text{ is bounded}\}$$

*is of Hausdorff dimension equal to the dimension of $G$.*

Here $G/\Gamma$ is understood to be equipped with a metric obtained as a quotient of a right-invariant Riemannian metric on $G$. In particular, the theorem implies that if $M$ is a Riemannian manifold of constant negative curvature and finite Riemannian volume, then the set of line elements $(x, \xi)$, where $x \in M$ and $\xi$ is a tangent vector of unit norm at $x$, such that the geodesic through $x$ in the direction of $\xi$ is bounded, forms a subset of full Hausdorff dimension in the unit tangent bundle (cf. Corollary 5.2).

In the sequel, for convenience, we consider right actions of one-parameter subgroups $(g_t)$ on $\Gamma\backslash G$ rather than left actions on $G/\Gamma$. We first obtain a description of the set $E^+(\Gamma)$ of "endpoints" of the curves $\{gg_t \mid t \geq 0\}$ where $g \in G$ is such that $\{\Gamma gg_t \mid t \geq 0\}$ is bounded in $\Gamma\backslash G$ (cf. Proposition 2.5); here "endpoint" means the unique point on the Furstenberg boundary $B = G/P$, where $P$ is a minimal parabolic subgroup, to which a curve as above converges, as $t \to \infty$, in the Furstenberg compactification of $G$ (cf. §§1 and 2 for details). Corollaries 1.5 and 1.7 proved in the course of the above, using boundary theory, seem to be of independent interest.

In the particular case of $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and $g_t = \operatorname{diag}(e^{t}, e^{-t})$, $E^+(\Gamma)$ as above corresponds to the set of badly approximable numbers under a canonical identification of $\mathbb{R} \cup \{\infty\}$ with the Furstenberg boundary. (cf. Remark 2.6). Thus $E^+(\Gamma)$ may be viewed as an object generalising the set of badly approximable numbers.

We then determine the Hausdorff dimension of $E^+(\Gamma)$ employing the notion of winning sets of $(\alpha, \beta)$-games introduced by W. M. Schmidt, which was used by him to prove that the set of badly approximable numbers is of Hausdorff dimension 1. In §3 we prove a general result, Theorem 3.2, regarding winning sets of the $(\alpha, \beta)$-games in $\mathbb{R}^m$, $m \geq 1$. In §4 we show that all nontrivial orbits of a certain abelian Lie subgroup (exp $V$ as in §4 below) of positive dimension, on the Furstenberg boundary, intersect the set $E^+(\Gamma)$ in a set, which in $V$ corresponds to a winning set for the $(\alpha, \beta)$-game for any $\alpha, \beta$ such that $1-2\alpha+\alpha\beta>0$. This enables us to conclude that $E^+(\Gamma)$ has Hausdorff dimension equal to the dimension of the boundary. §5 contains the final deduction of the Theorem and the Corollary stated above. We conclude with some comments and questions.

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## §1. Preliminaries

Let $G$ be a connected semisimple Lie group of $\mathbb{R}$-rank 1. We fix a one-parameter subgroup $A = (\exp tY)_{t\in\mathbb{R}}$, where $Y$ is an element of the Lie algebra of $G$, such that the adjoint action of $A$ (on the Lie algebra of $G$) is diagonalisable over $\mathbb{R}$; $G$ being of $\mathbb{R}$-rank 1 such a subgroup is unique upto conjugacy. We denote by $N$ and $N^-$ the horospherical subgroups associated to $A$ (relative to the order determined by $Y$) defined by

$$N = \{n \in G \mid (\exp -tY)n(\exp tY) \to e \text{ as } t \to \infty\} \quad \text{and}$$
$$N^- = \{u \in G \mid (\exp tY)u(\exp -tY) \to e \text{ as } t \to \infty\}$$

$e$ being the identity element in $G$. Then $N$ and $N^-$ are connected Lie subgroups. We denote by $P$ and $P^-$ the normalisers of $N$ and $N^-$ respectively. Then $P$ and $P^-$ are parabolic subgroups of $G$ and $N$ and $N^-$ are their unipotent radicals. We fix a maximal compact subgroup $K$ of $G$ and denote by $M$ the centraliser of $A$ in $K$; viz. the subgroup consisting of those elements of $K$ which commute with all elements of $A$. We note that $M$ normalises $N$. We also fix an element $w$ of $K$ such that $waw^{-1} = a^{-1}$ for all $a \in A$; such an element exists and the coset $wM$ is unique. We recall the following standard facts (cf. [9] and [18]) which will be used frequently in the sequel.

1.1. PROPOSITION. i) *Iwasawa decomposition*: $G = NAK = KAN$; *further, the map of* $N \times A \times K$ *into* $G$ *which takes* $(n, a, k)$ *into* $nak$, *for all* $n \in N$, $a \in A$ *and* $k \in K$, *is a diffeomorphism.*

ii) *Langlands decomposition*: $P = NAM = MAN$

iii) *Bruhat decomposition*: $G = (PwP) \cup P = (NwP) \cup P$; further, *the map of $N$ into $G/P$ which takes $n$ into $nwP$ is a diffeomorphism of $N$ onto $NwP/P = (G - P)/P$.*

Let $\mathfrak{n}$ be the Lie subalgebra corresponding to $N$. Then $\mathfrak{n}$ is invariant under the adjoint action, denoted by Ad, of $A$ and all eigenvalues of $\mathrm{Ad}(\exp -Y)$ are in $(0, 1)$. From this observation it is easy to deduce the following (well-known) lemma needed in the sequel.

1.2. LEMMA. *If $t_i \to \infty$ then $(\exp -t_iY)n(\exp t_iY) \to e$ uniformly on compact subsets of $N$. If $F$ is a compact subset of $N$ then $\bigcup_{t \geq 0} (\exp -tY)F(\exp tY) \cup \{e\}$ is compact.*

A sequence $\{g_i\}_{i=1}^{\infty}$ in $G$ is said to be *divergent*, and we write $g_i \to \infty$, if for any compact subset $C$ there exists $i_0$ such that $g_i \in G - C$ for all $i \geq i_0$.

1.3. LEMMA. *Let $\{u_i\}$ be a divergent sequence in $N^-$ and let $u_i = n_ia_ik_i$ be the Iwasawa decompositions, where $n_i \in N$, $a_i \in A$ and $k_i \in K$ for all $i$. Let $t_i \in \mathbb{R}$ be such that $a_i^{-1} = \exp t_iY$. Then $t_i \to \infty$.*

*Proof.* Let $V = \bigwedge^l \mathfrak{g}$, the $l$th exterior power (as a vector space) of the Lie algebra $\mathfrak{g}$ of $G$, where $l$ is the dimension of $N$. Let $\rho$ be the $l$th exterior power of the adjoint (left) representation of $G$ and let $v_0$ be a non-zero vector contained in the one-dimensional subspace in $V$ corresponding to the Lie subalgebra of $N$. It is easy to see that $\rho(g)v_0 = v_0$ for $g \in G$ if and only if $g \in MN$ and that $\rho(\exp tY)v_0 = e^{t\mu}v_0$ for all $t \in \mathbb{R}$, where $\mu$ is a fixed positive number. Let $\|\cdot\|$ be a $\rho(K)$-invariant norm on $V$. Then for any $k \in K$, $a = \exp tY \in A$ and $n \in N$ we have $\|\rho(kan)v_0\| = e^{t\mu}\|v_0\|$. In particular, $\|\rho(u_i^{-1})v_0\| = e^{\mu t_i}\|v_0\|$ for all $i$. Hence it is enough to show that $\|\rho(u_i^{-1})v_0\| \to \infty$. Since $\rho(N^-)$ consists of unipotent elements $\rho(N^-)v_0$ is a closed subset of $V$. Further, since no non-trivial element of $N^-$ fixes $v_0$ under the action via $\rho$, the last assertion implies that the assignment $u \to \rho(u)v_0$ is a homeomorphism of $N^-$ onto $\rho(N^-)v_0$ (the latter equipped with the subspace topology). Since $\{u_i\}$ and in turn $\{u_i^{-1}\}$ are divergent sequences in $N^-$, $\rho(u_i^{-1})v_0$ is a divergent sequence $\rho(N^-)v_0$. Since the latter is a closed subset of $V$ the last condition implies that $\|\rho(u_i^{-1})v_0\| \to \infty$, as desired.

We recall that $G/P$ can be viewed as a boundary of $G/K$. Specifically, this is done as follows (cf. [6] and [12] for motivation and details). Let $\mathcal{P}$ be the space of probability measures on $G/P$ equipped with the weak* topology; a net $\{\mu_i\}$ in $\mathcal{P}$ converges to $\mu \in \mathcal{P}$ if $\int f\, d\mu_i \to \int f\, d\mu$ for all continuous functions. Since $G/P$ is compact and second countable, $\mathcal{P}$ is also a compact (Hausdorff) second countable space. The $G$-action on $G/P$ induces a $G$-action on $\mathcal{P}$ defined by $g\mu(E) = \mu(g^{-1}E)$. The action of $K$ on $G/P$ is transitive and consequently there exists a unique $K$-invariant probability measure on $G/P$; we denote this measure by $m$. The symmetric space $G/K$ is then viewed as a subset of $\mathcal{P}$ via the identification $gK \leftrightarrow gm$ for all $g \in G$. Also $G/P$ is viewed as a subset of $\mathcal{P}$ by identifying each $x \in G/P$ with the point mass $\delta_x$ based at $x$. It is easy to see that the identification maps are (well-defined) $G$-equivariant homeomorphisms. It is well-known that $G/P$ is contained the closure of $G/K$ in $\mathcal{P}$ and further that when $G$ is of $\mathbb{R}$-rank 1, as in our case, $(G/K) \cup (G/P)$ is compact (cf. Lemma 1.4 below for the latter and Theorem 7 of [12] for the general case).

In the sequel we use the following notation: $A^+ = \{\exp tY \mid t > 0\}$ and $A^- = \{\exp tY \mid t < 0\}$.

1.4. LEMMA. i) *If* $\{g_i\}$ *is a divergent sequence contained in* $NA^+$ *then* $g_i m \to \delta_p$.

ii) *If* $\{g_i\}$ *is a divergent sequence in* $G$ *then there exists a subsequence of* $\{g_i m\}$ *which converges to* $\delta_x$ *for some* $x \in G/P$; *hence* $(G/K) \cup (G/P)$ *is compact.*

*Proof.* i) Let $\{g_i\}$ be a divergent sequence in $NA^+$ and let $\nu$ be any limit point of $\{g_i m\}$ in $\mathcal{P}$. We shall show that $\nu = \delta_p$. Since $\mathcal{P}$ is compact, this would prove i). By passing to a subsequence we may assume that $g_i m \to \nu$. Let $g_i = n_i a_i$, where $n_i \in N$ and $a_i \in A^+$. Again by passing to a suitable subsequence we may assume that either $\{a_i\}$ is divergent or $a_i \to a$ for some $a \in A^+ \cup \{e\}$ and similarly that either $\{a_i^{-1} n_i a_i\}$ is divergent or $a_i^{-1} n_i a_i \to \bar{n}$ for some $\bar{n} \in N$.

We claim that for $x = nwP \in NwP/P$, where $n \in N$, $g_i x \to P$ unless $n = (\bar{n})^{-1}$, with $\bar{n}$ as above (no exception if $\{a_i^{-1} n_i a_i\}$ is divergent). Suppose this is not true, say for $x = nwP$, $n \in N$, $n \neq (\bar{n})^{-1}$. Since $G/P$ is compact, by passing to a subsequence we may assume that $g_i x \to n'wP \in NwP/P = (G-P)/P$, for some $n' \in N$. Since $g_i x = n_i a_i nwP = n_i(a_i n a_i^{-1})wP$, by Proposition 1.1, iii) it follows that $n_i(a_i n a_i^{-1}) \to n'$ as $i \to \infty$. Now suppose first that $\{a_i\}$ is divergent. Then $a_i^{-1} n'' a_i \to e$ as $i \to \infty$ uniformly for $n''$ in a neighbourhood of $n'$. Since $n_i(a_i n a_i^{-1}) \to n'$, this implies that $(a_i^{-1} n_i a_i)n \to e$ as $i \to \infty$. But this is a contradiction since $n \neq (\bar{n})^{-1}$. Next suppose that $a_i \to a$ as $i \to \infty$. Then $g_i n = n_i(a_i n a_i^{-1})a_i \to n'a$, which is a contradiction since $\{g_i\}$ is divergent. Hence the claim must hold. Since $m(P) = m(\bar{n}^{-1}wP) = 0$, in view of the bounded convergence theorem validity of the claim implies that $g_i m \to \delta_p$.

ii) Now let $\{g_i\}$ be any divergent sequence in $G$. As in [12], by Cartan decomposition we may write $g_i$ as $g_i = k_i a_i k_i'$ for some $a_i \in A^+ \cup \{e\}$ and $k_i$, $k_i' \in K$. By passing to a subsequence, we may assume that $k_i \to k \in K$. Then $\{a_i\}$ is divergent and hence by i) we have $a_i k_i' m = a_i m \to \delta_p$. But the $G$-action on $\mathcal{P}$ is continuous (cf. [12], Lemma 8). Hence $g_i m = k_i a_i k_i' m \to k\delta_p = \delta_{kP}$, so that ii) holds with $x = kP$.

It turns out, as pointed out by the referee, that an appropriate analogue of (ii) in Lemma 1.4 holds more generally for negatively curved manifolds and also that a similar assertion holds for any measure on $G/P$, not just $m$.

1.5. COROLLARY. *Let $\{u_i\}$ be a divergent sequence in $N^-$ and for all $i$ let $u_i = n_i a_i k_i$ be the Iwasawa decompositions, where $n_i \in N$, $a_i \in A$ and $k_i \in K$. Then $n_i \to e$, the identity, as $i \to \infty$.*

*Proof.* It is well known and easy to see that $N^- = wNw^{-1}$; thus for all $i$, $u_i$ can be written as $wn_i'w^{-1}$, where $\{n_i'\}$ is a divergent sequence in $N$. Hence $u_i m = wn_i'w^{-1}m = wn_1'm \to w\delta_p = \delta_{wP}$, by Lemma 1.4, i). Also, by Lemma 1.3 $a_i^{-1} = \exp t_i Y$, where $t_i \to \infty$. Hence $a_i k_i m = a_i wm = w(\exp t_i Y)m \to w\delta_P = \delta_{wP}$ by Lemma 1.4, i). We shall conclude from these two convergences that $n_i \to e$ as $i \to \infty$. Suppose this is not true. Then there exists a neighbourhood $\Omega_1$ of $e$ in $N$ such that $n_i \notin \Omega_1$ for infinitely many $i$. Let $\Omega$ be a neighbourhood of $e$ in $N$ such that $\Omega\Omega^{-1} \subset \Omega_1$. Since $u_i m \to \delta_{wP}$ and $a_i k_i m \to \delta_{wp}$ there exists $i_0$ such that $u_i m(\Omega wP/P) \geq \frac{2}{3}$ and $a_i k_i m(\Omega wP/P) \geq \frac{2}{3}$. Let $i \geq i_0$ be such that $n_i \notin \Omega_1$. Then $n_i^{-1}\Omega$ is contained in $N - \Omega$. Hence

$$\begin{aligned} u_i m((N-\Omega)wP/P) &= n_i a_i k_i m((N-\Omega)wP/P) \geq n_i a_i m(n_i^{-1}\Omega wP/P) \\ &= a_i m(\Omega wP/P) \geq \tfrac{2}{3}. \end{aligned}$$

But since $\Omega wP/P$ and $(N-\Omega)wP/P$ are disjoint subsets and $u_i m$ is a probability measure both cannot be assigned measure $\geq \frac{2}{3}$; the contradiction shows that the corollary must hold.

Let $\pi: G \to N$ be the map defined by $\pi(nak) = n$ for all $n \in N$, $a \in A$ and $k \in K$, every element of $G$ being expressed uniquely as such by Iwasawa decomposition.

1.6. LEMMA. *For any $a \in A$, $\pi(N^-a) = \pi(aN^-) = a\pi(N^-)a^{-1}$.*

*Proof.* Let $a \in A$. Note that $a$ normalises $N^-$ and hence $aN^- = N^-a$. Now let $n \in \pi(aN^-)$. Then there exist $u \in N^-$, $b \in A$ and $k \in K$ such that $au = nbk$. Hence $u = a^{-1}nbk = (a^{-1}na)(a^{-1}b)k$ so that $a^{-1}na = \pi(u) \in \pi(N^-)$. Therefore $n \in a\pi(N^-)a^{-1}$ for all $n \in \pi(aN^-)$, so that $\pi(aN^-) \subset a\pi(N^-)a^{-1}$. Similar argument also yields the other way inclusion.

1.7. COROLLARY. *$\pi(N^-A^-)$ is a bounded subset of $N$; viz. it has compact closure.*

*Proof.* If $\Omega$ is a compact neighbourhood of the identity then by Corollary 1.5 there exists a compact subset $F$ of $N^-$ such that $\pi(N^- - F) \subset \Omega$; hence $\pi(N^-) \subset \pi(F) \cup \Omega$, which implies that $\pi(N^-)$ is bounded. Therefore by Lemmas 1.2 and 1.6, $\pi(N^-A^-) = \bigcup_{a \in A^-} a\pi(N^-)a^{-1}$ is a bounded subset of $N$.

1.8. LEMMA. *For any $t_0 > 0$, $\pi(N^-A^-)$ contains a neighbourhood $\Omega$ of e in N such that $nN^-A^-K \cap (\exp tY)N^-A^-K$ is nonempty for all $n \in \Omega$ and all t such that $|t| \le t_0$.*

*Proof.* Let $t_1 > t_0$ and let $D = \{\exp tY \mid t \in [-t_1 - t_0, -t_1 + t_0]\}$. Then $D$ is a compact subset contained in the open set $N^-A^-K$. Since $G = NAK$ (Iwasawa decomposition) is topologically a Cartesian product of the component subspaces, we can conclude from the above that there exists a neighbourhood $\Omega$ of the identity in $N$ such that $\Omega DK$ is contained in $N^-A^-K$. Passing to a smaller neighbourhood we may also assume $\Omega$ to be symmetric; that is, $\Omega = \Omega^{-1}$. Then for any $n \in \Omega$ and $t \in [-t_0, t_0]$, $N^-A^-K \cap n^{-1}(\exp tY)N^-A^-K$ contains $n^{-1}\exp(t - t_1)Y$, and hence, in particular, it is nonempty; therefore $nN^-A^-K \cap (\exp tY)N^-A^-K$ is also nonempty. From the choice of $\Omega$ it is evident that it is contained in $\pi(N^-A^-)$.

## §2. A characterisation of bounded trajectories

Let the notation be as in §1. Recall that $G$ is a connected semisimple Lie group of $\mathbb{R}$-rank 1. Let $\Gamma$ be a lattice in $G$; that is, $\Gamma\backslash G$ admits a finite (Borel) measure invariant under the action of $G$ (on the right). For obvious reasons we assume $\Gamma\backslash G$ to be noncompact. In this section we obtain a characterisation of the set of $x$ in $G$ such that $\{\Gamma x(\exp tY) \mid t \ge 0\}$ is a bounded trajectory (that is, it has compact closure) in $\Gamma\backslash G$.

2.1. LEMMA. *Let $x \in G$ and $p \in P$ be arbitrary. Then $\{\Gamma x(\exp tY) \mid t \ge 0\}$ is bounded if and only if $\{\Gamma xp(\exp tY) \mid t \ge 0\}$ is bounded.*

*Proof.* Let $p = nam$, where $n \in N$, $a \in A$ and $m \in M$, be the Langlands decomposition of $p$. Then $\Gamma xp(\exp tY) = \Gamma xnam(\exp tY) = \Gamma x(\exp tY)\ \{(\exp - tY)n(\exp tY)\}am$. Since $(\exp -tY)n(\exp tY) \to e$, the identity, as $t \to \infty$, the relation evidently implies the Lemma.

In view of the lemma it is enough for us to characterise the subset $E^+(\Gamma)$ of the boundary $G/P$ defined by

$$E^+(\Gamma) = \{xP \in G/P \mid \{\Gamma x(\exp tY) \mid t \geqslant 0\} \text{ is bounded}\}.$$

For this purpose we recall a well-known fundamental domain for the $\Gamma$-action. For $s \in \mathbb{R}$ let $A_s = (\exp sY)A^+ = \{\exp ty \mid t > s\}$. A subset of the form $\Sigma A_s K$, where $\Sigma$ is a compact subset of $N$ and $s \in \mathbb{R}$ is called a Siegel set. We need the following result on fundamental domains, due to Garland and Raghunathan (cf. [7], Theorem 0.6; note that we consider the $G$-action on the right and hence must employ the inverses of the relevant subsets as in [7]).

2.2. PROPOSITION. *There exists a Siegel set* $\Sigma A_\rho K$ *and a finite subset* $\Lambda$ *of* $G$ *such that the following conditions are satisfied.*

i) $G = \Gamma\Lambda\Sigma A_\rho K$

ii) *for any* $\lambda \in \Lambda$, $(\lambda^{-1}\Gamma\lambda) \cap NM$ *is a (cocompact) lattice in* $NM$.

iii) *for any compact subsets* $D$ *and* $D'$ *of* $N$ *there exists* $\sigma \in \mathbb{R}$ *such that the following holds: if* $\lambda$, $\lambda' \in \Lambda$ *and* $\gamma \in \Gamma$ *are such that* $\gamma\lambda D A_\rho K \cap \lambda' D' A_\sigma K$ *is nonempty then* $\lambda' = \lambda$ *and* $\lambda^{-1}\gamma\lambda \in NM$.

It may be noted that the proposition would continue to hold, for suitably modified $\Sigma$ and $\rho$, if any $\lambda \in \Lambda$ is replaced by an element of the form $\gamma\lambda p$, where $\gamma \in \Gamma$ and $p \in P$. However, any set $\Lambda$ for which the proposition holds, for a suitable $\Sigma$ and $s$, is a set of representatives for a fixed class of double cosets of the form $\Gamma g P$, $g \in G$. It may be worthwhile recalling that these double cosets consist precisely of elements $g \in G$ such that $\{\Gamma g(\exp tY) \mid t \geq 0\}$ is a divergent trajectory in $\Gamma\backslash G$; namely, for any compact subset $C$ of $\Gamma\backslash G$ there exists $T \geq 0$ such that $\Gamma g(\exp tY) \notin C$ for $t \geq T$. (cf. [2], Corollary 6.2).

2.3. *Remark.* Let the notation be as in Proposition 2.2. Then there exists $\sigma \in \mathbb{R}$ such that the following holds: if $\lambda$, $\lambda' \in \Lambda$ and $\gamma \in \Gamma$ are such that either $\gamma\lambda\Sigma A_\rho K \cap \lambda' N A_\sigma K$ or $\gamma\lambda N A_\sigma K \cap \lambda' N A_\sigma K$ is nonempty then $\lambda = \lambda'$ and $\lambda^{-1}\gamma\lambda \in NM$.

*Proof.* In view of ii) in Proposition 2.2 and finiteness of $\Lambda$ there exists a compact subset $D$ of $NM$ such that $NM = (NM \cap \lambda^{-1}\Gamma\lambda)D$ for all $\lambda \in \Lambda$. Hence for any $s$, $\lambda N A_s K = \lambda N M A_s K = \lambda(NM \cap \lambda^{-1}\Gamma\lambda) D A_s K = (\lambda NM\lambda^{-1} \cap \Gamma)\lambda D A_s K$ for all $\lambda \in \Lambda$. Let $\sigma \geq \rho$ be such that iii) of Proposition 2.2 holds for $\Sigma \cup D$ ($\Sigma$ as in Proposition 2.2, i)) and $D$ in the place of $D$ and $D'$ respectively. Now let $\lambda$, $\lambda' \in \Lambda$ be such that $\gamma\lambda\Sigma A_\rho K \cap \lambda' N A_\sigma K$ is nonempty. Then by the preceding observation there exists $\gamma' \in \lambda' NM(\lambda')^{-1} \cap \Gamma$ such that $\gamma\lambda\Sigma A_\rho K \cap \gamma'\lambda' D A_\sigma K$ is nonempty. By our choice of $\sigma$ this implies that $\lambda' = \lambda$ and $\lambda^{-1}\gamma^{-1}\gamma'\lambda \in NM$. But since $\lambda^{-1}\gamma'\lambda \in NM$ (as $\lambda = \lambda'$), this implies that $\lambda^{-1}\gamma\lambda \in NM$. A similar argument shows that if $\gamma\lambda N A_\sigma K \cap \lambda' N A_\sigma K$ is nonempty for some $\lambda$, $\lambda' \in \Lambda$ and $\gamma \in \Gamma$ then $\lambda = \lambda'$ and $\lambda^{-1}\gamma\lambda \in NM$.

Through the rest of the section, in characterising bounded trajectories, we use the notation as in Proposition 2.2 and fix $\sigma \in \mathbb{R}$, $\sigma \geq \rho$ for which Remark 2.3 holds.

2.4. PROPOSITION. *Let* $x \in G$. *Then* $\{\Gamma x(\exp tY) \mid t \geq 0\}$ *is a bounded trajectory in* $\Gamma\backslash G$ *if and only if there exists* $s \in \mathbb{R}$ *such that* $x(\exp tY) \in G - \Gamma\Lambda N A_s K$ *for all* $t \geq 0$.

*Proof.* Observe that $G - \Gamma\Lambda NA_sK$ is contained in $\Gamma\Lambda\Sigma(A_\rho - A_s)K$ whose image in $\Gamma\backslash G$ has compact closure. This implies the 'if' part of the Proposition. Next suppose that $\{\Gamma x(\exp tY) \mid t \geq 0\}$ is bounded. Then evidently there exists $s \in \mathbb{R}$ such that $x(\exp tY) \in \Gamma\Lambda\Sigma(A_\rho - A_s)K$ for all $t \geq 0$. Further, without loss of generality we may assume $s \geq \sigma$. Suppose the proposition is not true; then there must exist $t \geq 0$ such that $x(\exp tY) \in \Gamma\Lambda\Sigma(A_\rho - A_s)K \cap \Gamma\Lambda NA_sK$. Hence, in particular, there exist $\lambda$, $\lambda' \in \Lambda$ and $\gamma \in \Gamma$ such that $\gamma\lambda\Sigma(A_\rho - A_s)K \cap \lambda' NA_sK$ is nonempty. Since $s \geq \sigma$, by Remark 2.3 this implies that $\lambda = \lambda'$ and $\lambda^{-1}\gamma\lambda \in NM$. But then we find that $\Sigma(A_\rho - A_s)K \cap (\lambda^{-1}\gamma^{-1}\lambda)NA_sK = \Sigma(A_\rho - A_s)K \cap NA_sK$ is nonempty, which is absurd by uniqueness of expression in Iwasawa decomposition. Hence the proposition.

2.5. PROPOSITION. *Let $\lambda \in \Lambda$ and let $\{g_i\}$ be an enumeration of the countable set $\lambda^{-1}\Gamma\Lambda - P$. Let $g_i = n_iwa_iz_i$, where $n_i \in N$, $a_i \in A$ and $z_i \in NM$, be their Bruhat decompositions. Then*

$$E^+(\Gamma) = \bigcup_{a\in A} \bigcap_{i=1}^{\infty} \{\lambda nwP \mid n \notin \pi(n_ia_i^{-1}a^{-1}N^-A^-)\}$$

*Proof.* Since for any $s$, $\lambda(\exp tY) \in \Gamma\Lambda NA_sK$ for all $t > s$, in view of Proposition 2.4, $\lambda P \notin E^+(\Gamma)$. Hence by Bruhat decomposition every $x \in E^+(\Gamma)$ is of the form $\lambda nwP$ for some $n \in N$. Using Proposition 2.4 and the fact that $\lambda^{-1}\Gamma\lambda \cap P$ is contained in $NM$ we deduce that $\lambda nwP \in E^+(\Gamma)$ if and only if there exists $s \in \mathbb{R}$ such that for all $t \geq 0$, $nw(\exp tY) \notin \cup g_iNA_sK$. Now, for any $i = 1, 2, \ldots,$ we have $g_iNA_sK = n_iwa_iz_iNA_sK = n_iwa_iNA_sK = n_iwa_iaNA^+K$, where $a = \exp sY$. Further, since $w\xi w^{-1} = \xi^{-1}$ for all $\xi \in A$ and $wNw^{-1} = N^-$, we have $n_iwa_iaNA^+K = n_ia_i^{-1}a^{-1}N^-A^-K$. Thus $\lambda nwP \in E^+(\Gamma)$ if and only if there exists $a \in A$ such that for all $t \geq 0$, and $i = 1, 2, \ldots,$ $nw(\exp tY) \notin n_ia_i^{-1}a^{-1}N^-A^-K$, or equivalently, $n(\exp -tY) \notin n_ia_i^{-1}a^{-1}N^-A^-K$. If $n \notin \pi(n_ia_i^{-1}a^{-1}N^-A^-) = n_ia_i^{-1}a^{-1}\pi(N^-A^-)aa_i$, or equivalently if $aa_i(n_i^{-1}n)a_i^{-1}a^{-1} \notin \pi(N^-A^-)$, for any $i$, then the condition evidently holds. This shows that the set on the right hand side in the equation as in the proposition is contained in $E^+(\Gamma)$.

For proving the other way inclusion we need the following observations. In view of Lemma 1.3 there exists a $s_0 \in \mathbb{R}$, such that if $u \in N^-$ and $u = n(\exp tY)k$, where $n \in N$, $k \in K$ and $t \in \mathbb{R}$ is the Iwasawa decomposition, then $t \leq s_0$. Further, if $u \in N^-$ and $b = \exp(-sY) \in A^-$, where $s > 0$, and $ub = n(\exp t'Y)k$ is the Iwasawa decomposition of $ub$, then $b^{-1}ub = (b^{-1}nb)$ $(b^{-1}\exp t'Y)k = (b^{-1}nb)$ $(\exp(s + t')Y)k$ is the Iwasawa decomposition of $b^{-1}ub$ and hence $s + t' \leq s_0$; in particular, $t' \leq s_0$. Secondly, by Remark 2.3, for any $i$, $g_iNA_\sigma K$ is disjoint from $NA_\sigma K$. Since $g_i = n_iwa_iz_i$ this implies that $wa_iNA_\sigma K$ is disjoint from $NA_\sigma K$. In particular, $w(a_iA_\sigma)w^{-1}$ must be disjoint from $A_\sigma$. Hence, for any $i$, if $t_i \in \mathbb{R}$ is such that $a_i^{-1} = \exp t_iY$ then $t_i \le 2\sigma$.

Now let $n \in N$ be such that for all $a \in A$ there exists $i$ such that $n \in \pi(n_ia_i^{-1}a^{-1}N^-A^-)$. We shall show then that for every $b \in A$ there exist an index $i$ and $t \ge 0$ such that $n(\exp -tY) \in n_ia_i^{-1}b^{-1}N^-A^-K$; as noted earlier, this would imply that $\lambda nwP \notin E^+(\Gamma)$, thereby completing the proof of the proposition. Let $b \in A$ be given. Choose $a = \exp sY$ such that $s \ge s_0 + 2\sigma$ and $ab^{-1} \in A^+$. Let $i$ be an index such that $n \in \pi(n_ia_i^{-1}a^{-1}N^-A^-) = n_ia_i^{-1}a^{-1}\pi(N^-A^-)aa_i$. Then $aa_in_i^{-1}na_i^{-1}a^{-1} \in \pi(N^-A^-)$ and hence there exist $y \in N^-A^-$, $a' \in A$ and $k \in K$ such that $y = (aa_in_i^{-1}na_i^{-1}a^{-1})a'k$. Then $n(a_i^{-1}a^{-1}a') = n_ia_i^{-1}a^{-1}yk^{-1} \in n_ia_i^{-1}a^{-1}N^-A^-K \subset n_ia_i^{-1}b^{-1}N^-A^-K$, where the last inclusion follows from the fact that $(ba^{-1})N^-A^-K = N^-(ba^{-1})A^-K \subset N^-A^-K$ as $(ba^{-1}) = (ab^{-1})^{-1} \in A^-$. Let $t_i$ and $t'$ be such that $a_i^{-1} = \exp t_iY$ and $a' = \exp t'Y$. By the observations made earlier, $t_i \le 2\sigma$ and $t' \le s_0$. Then $a_i^{-1}a^{-1}a' \in A^- \exp -(s - s_0 - 2\sigma)Y$. Since $n(a_i^{-1}a^{-1}a') \in n_ia_i^{-1}b^{-1}N^-A^-K$ and $s - s_0 - 2\sigma \ge 0$, this completes the proof.

2.6. *Remark.* Consider the particular case $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and $Y = \operatorname{diag}(1, -1)$, so that $\exp tY = \operatorname{diag}(e^t, e^{-t})$. Then $P = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \,\middle|\, a,b \in \mathbb{R},\ a \ne 0\right\}$ and $G/P$ may be identified with $\mathbb{R} \cup \{\infty\}$, via the correspondence $\begin{pmatrix} a & b \\ c & d\end{pmatrix}P \leftrightarrow a/c$, where $a/c$ is taken to be $\infty$ if $c = 0$. Similarly we also identify $G/K$ with the upper half-plane $\mathbb{H}^+$ via the usual action of $SL(2, \mathbb{R})$ on $\mathbb{H}^+$, $K$ being chosen to be the isotropy subgroup of $i = \sqrt{-1}$. The topology on the compactification $G/K \cup G/P$ corresponds canonically to the usual topology on $\mathbb{H}^+ \cup (\mathbb{R} \cup \{\infty\})$. In this case the identity is a cusp element (cf. [17] for a fundamental domain) and it is straightforward to verify that the set as on the right hand side of the equation in Proposition 2.5 is precisely the set of badly approximable numbers in $\mathbb{R}$. (It may be recalled that a real number $x$ is said to be *badly approximable* if there exists $\delta > 0$ such that $|x - k/l| > \delta/l^2$ for all integers $k,l$ with $l \ne 0$.) Thus by Proposition 2.5, $E^+(SL(2, \mathbb{Z}))$ is the set of badly approximable real numbers.

## §3. Winning sets of $(\alpha, \beta)$-games

In this section we prove a general result, Theorem 3.2 about winning sets of the $(\alpha, \beta)$-games introduced by W. M. Schmidt [14]. Applying it to the set $E^+(\Gamma)$ as in §2 together with a result from [14] enables us to conclude that $E^+(\Gamma)$ is of Hausdorff dimension equal to the dimension of $G/P$.

The game in question goes as follows: Let $\mathscr{A}$ and $\mathscr{B}$ be two players, $X$ be a complete metric space and let $\alpha, \beta \in (0, 1) = \{t \in \mathbb{R} \mid 0 < t < 1\}$ be given. $\mathscr{B}$ starts the game by picking a closed ball $B_0$ in $X$ with arbitrary positive radius. Then $\mathscr{A}$ picks a closed ball $A_1$ contained in $B_0$ and having radius $\alpha$ times that of $B_0$. Next $\mathscr{B}$ chooses a closed ball contained in $A_1$ of radius $\beta$ times that of $A_1$ and so on; the game proceeds inductively by $\mathscr{A}$ choosing a closed ball $A_k$ contained in $B_{k-1}$ with radius $\alpha$ times that of $B_{k-1}$ and then $\mathscr{B}$ choosing a closed ball contained in $A_k$ and having radius $\beta$ times that of $A_k$. Since $X$ is a complete metric space there is a unique point of $X$ which belongs to $A_k$ for all $k$, and hence also to $B_k$ for all $k$. A subset $S$ of $X$ is called an $(\alpha, \beta)$*-winning set* (for $\mathscr{A}$) if, irrespective of what choices $\mathscr{B}$ makes during his turns, $\mathscr{A}$ can make his choices in such a way as to ensure that the point of intersection belongs to $S$; $S$ is said to be an $\alpha$*-winning set* if it is an $(\alpha, \beta)$–winning set for all $\beta \in (0, 1)$. Evidently $X$ itself is always an $(\alpha, \beta)$-winning set for all $\alpha, \beta \in (0, 1)$. It turns out that if $1 - 2\alpha + \alpha\beta \leq 0$ then $X$ is the only $(\alpha, \beta)$-winning set (cf. [14], Lemma 5). On the other hand if $1 - 2\alpha + \alpha\beta > 0$ then there exist proper subsets which are $(\alpha, \beta)$-winning sets (cf. [14], Theorem 3, for example). However, Schmidt shows (in particular) that in a $m$-dimensional euclidean space, that is, $X = \mathbb{R}^m$ for some $m \geq 1$, any $\alpha$-winning set is "large" in the sense that its "Hausdorff dimension" (see below for definition) is $m$ (cf. [14], Corollary 2 to Theorem 6).

We recall that the Hausdorff dimension of a metric space $X$ is defined as follows (cf. [5] and [10] for motivation and general reference): For any ball $U$ in $X$ let $r(U)$ denote the radius of $U$. For $\varepsilon > 0$ let $\mathscr{C}(\varepsilon)$ be the class of open balls of radius less than $\varepsilon$. For $s \geq 0$ let

$$h(s, \varepsilon) = \inf\left\{\sum_1^\infty r(U_i)^s \mid U_i \in \mathscr{C}(\varepsilon),\ i = 1, 2, \ldots \text{ such that } X = \bigcup_1^\infty U_i\right\}$$

Evidently, as $\varepsilon$ decreases $h(s, \varepsilon)$ increases monotonically. The quantity $h(s) = \lim_{\varepsilon \to 0} h(s, \varepsilon)$ (possibly $\infty$) is called the $s$-dimensional *Hausdorff measure* of $X$. It is easy to see that there exists a (unique) $d \geq 0$ (possibly $\infty$) such that $h(s) = \infty$ for all $s < d$ and $h(s) = 0$ for all $s > d$; $d$ is called the *Hausdorff dimension* of $X$. The Hausdorff dimension of a subspace of $X$ is understood to be with respect to the induced metric; evidently, it is at most as much as that of $X$. We note that $\mathbb{R}^m$ with the usual metric is of Hausdorff dimension $m$ and that more generally any $m$-dimensional Riemannian manifold metrized by the Riemannian metric has Hausdorff dimension $m$.

A slight modification of the proof of Schmidt's result alluded above yields the following.

3.1. PROPOSITION. *If $S$ is an $\alpha$-winning set in $\mathbb{R}^m$, where $\alpha \in (0, 1)$ and $m \geq 1$, then for any nonempty open subset $\Omega$ of $\mathbb{R}^m$, $S \cap \Omega$ has Hausdorff dimension $m$.*

It will hardly serve any purpose to give details of the proof except to note that given $\Omega$ as above, for all sufficiently small $\beta$ the sets $C_1(i_j)$ and hence $S^*$ as in the proof of Theorem 6 in [14] can be assumed to be contained in $\Omega$; the rest of the proof of that theorem and Corollary 2 in [14] goes through word for word and implies Proposition 3.1 as above.

In [14], Schmidt also proved that the set of badly approximable numbers is an $(\alpha, \beta)$-winning set in $\mathbb{R}$ for any $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$ (cf. [14], Theorem 3). In this section we shall prove a general result, Theorem 3.2, on $(\alpha, \beta)$-winning sets in euclidean spaces. The idea of the proof is motivated by that of Schmidt's theorem.

We consider $\mathbb{R}^m$ equipped with the usual Hilbert norm which we denote by $\|\cdot\|$. If $x_1, x_2 \in \mathbb{R}^m$ and $S_1$ and $S_2$ are subset of $\mathbb{R}^m$ then $d(x_1, x_2)$, $d(x_1, S_1)$, $d(S_1, S_2)$ etc. denote the distances between the respective pairs, with respect to the norm; e.g. $d(S_1, S_2) = \inf\{\|x - y\| \mid x \in S_1, y \in S_2\}$. For any subset $S$ the *thickness* of $S$ is defined to be

$$\tau(S) = \inf_V \sup_{x,y \in S} d(x - y, V)$$

where the infimum is taken over all hyperplanes $V$ in $\mathbb{R}^m$.

For $x \in \mathbb{R}^m$ and $r > 0$, $B(x, r)$ denotes the open ball of radius $r$ with center at $x$. For any ball $B$, whether open or closed, we denote by $z(B)$ and $r(B)$ the center and the radius of $B$ respectively.

3.2. THEOREM. *Let $\{S(p, t)\}$ be a family of subsets of $\mathbb{R}^m$ (doubly) indexed over $p \in \mathbb{N}$ and $t \in (0, 1)$. Suppose that for any compact subset $C$ of $\mathbb{R}^m$ and $\mu \in (0, 1)$ there exist $M \geq 1$, $\varepsilon \in (0, 1)$ and a sequence $\{\tau_p)$ of positive numbers such that the following conditions are satisfied:*

a) *if $p \in \mathbb{N}$ and $t \in (0, \varepsilon)$ are such that $S(p, t) \cap C$ is nonempty, then $\tau_p \leq M$ and $\tau(S(p, t)) \leq t\tau_p$*

b) *if $p, q \in \mathbb{N}$ and $t \in (0, \varepsilon)$ are such that $S(p, t) \cap C$ and $S(q, t) \cap C$ are nonempty and $\mu\tau_p \leq \tau_q \leq \mu^{-1}\tau_p$, then either $p = q$ or $d(S(p, t), S(q, t)) \geq \varepsilon(\tau_p + \tau_q)$*

*Let*

$$F = \bigcup_{\delta > 0} \left(\mathbb{R}^m - \bigcup_1^\infty S(p, \delta)\right)$$

*Then $F$ is an $(\alpha, \beta)$-winning set for all $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$.*

3.3. *Remark.* The proof below shows that given $\alpha, \beta \in (0, 1)$ as above, $\mathscr{A}$ can ensure the point of intersection to be in the set $F$ as above if conditions a) and b) are satisfied (for suitable choices of $M$, $\varepsilon$ and $\{\tau_p\}$) in the particular case of $C = B_0$, the closed ball chosen by $\mathscr{B}$ to start the game, and $\mu = (\alpha\beta)^h$, where $h$ is the smallest integer such that $(\alpha\beta)^h < \frac{1}{2}(1 - 2\alpha + \alpha\beta)$.

Though we have not put this condition, typically for each $p$, the sets $\{S(p, t)\}$, $t \in (0, 1)\}$ may be thought of as a shrinking family. The following particular case which is less technical may be worth pointing out.

3.4. COROLLARY. *Let $\{x_p\}$ be a sequence of (distinct) points in $\mathbb{R}^m$ and let $\{r_p\}$ be a bounded sequence of positive numbers. Suppose that for any $p$ and $q$, $p \neq q$, we have*

$$\|x_p - x_q\| \geq \sqrt{(r_p r_q)}.$$

*Let*

$$F = \bigcup_{\delta>0} \left(\mathbb{R}^m - \bigcup_1^\infty B(x_p, \delta r_p)\right)$$

*Then $F$ is an $(\alpha, \beta)$-winning set for all $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$.*

*Proof.* For all $p \in \mathbb{N}$ and $0 < t \leq 1$, put $S(p, t) = B(x_p, tr_p)$. Then $\tau(S(p, t)) = tr_p$. Put $\tau_p = r_p$. Since $\{r_p\}$ is bounded, condition a) of the theorem is satisfied irrespective of the compact set $C$. Now let $\mu \in (0, 1)$ be given. Choose $\varepsilon = \frac{1}{4}\mu\sqrt{\mu}$. Let $p$ and $q$ be such that $\mu r_p \leq r_q \leq \mu^{-1} r_p$ and let $t \in (0, \varepsilon)$. Then

$$\begin{aligned} d(B(x_p, tr_p), B(x_q, tr_q)) &= \|x_p - x_q\| - t(r_p - r_q) \geq \sqrt{(r_p r_q)} - t(r_p + r_q) \\ &\geq \sqrt{\mu} r_p - t(r_p + r_q) \geq (\tfrac{1}{2}\mu\sqrt{\mu} - t)(r_p + r_q) > \varepsilon(r_p + r_q), \end{aligned}$$

which shows that condition b) is also satisfied. Hence the theorem implies the corollary.

It is evident that the corollary would be true for various other expressions in the place of $\sqrt{(r_p r_q)}$. The particular expression is, however, significant in view of the following lemma.

3.5. LEMMA. *Let $\mathbb{R}^m$ be viewed as a hyperplane in $\mathbb{R}^{m+1}$ in a natural way. Let $\{x_p\}$ be a sequence in $\mathbb{R}^m$. Let $D_p$ be a sequence of balls in $\mathbb{R}^{m+1}$ such that the boundary of $D_p$, $p \in \mathbb{N}$, is tangential to $\mathbb{R}^m$ at $x_p$. Suppose also that the interiors of the balls $D_p$, $p \in \mathbb{N}$, are pairwise disjoint and are all contained in the same connected component of $\mathbb{R}^{m+1} - \mathbb{R}^m$. Let $r_p$ be the radius of $D_p$. Then*

$$\|x_p - x_q\| \geq 2\sqrt{(r_p r_q)}$$

*for all $p \neq q$.*

*Proof* is immediate from Pythagoras theorem!

We note that the set $F$ as in Theorem 3.2 or Corollary 3.4 could be of zero measure. Applying Corollary 3.4 to the particular case when $m = 1$, $\{x_p\}$ is an enumeration of all rational numbers, and $r_p = 1/l^2$ if $x_p = k/l$, where $k$ and $l$ are coprime integers and $l \neq 0$, in which case the condition in the Corollary is indeed satisfied, we recover the corresponding result of Schmidt; viz.

3.6. COROLLARY. *The set of badly approximable (real) numbers is an $(\alpha, \beta)$-winning set in $\mathbb{R}$, for any $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$.*

To prove Theorem 3.2 we need the following lemma.

3.7. LEMMA. *Let $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$ be given. Let $\theta = \frac{1}{2}(1 - 2\alpha + \alpha\beta) \in (0, 1)$. Let $h$ be a positive integer such that $(\alpha\beta)^h < \theta$. Let $k \geq 0$ be arbitrary and let $B_k$ be the closed ball chosen by $\mathscr{B}$ at the kth stage. Let $V$ be a hyperplane and let $I$ be a closed subinterval of $V^\perp$, the orthocomplement of $V$ in $\mathbb{R}^m$. Let $l(I)$ be the length of $I$, $a$ be the mid-point of $I$ and suppose that*

$$l(I) < d(z(B_k), a + V) + \theta r(B_k)$$

*Then $\mathscr{A}$ can play in such a way that $B_{k+h}$ is disjoint from $I + V$.*

*Proof.* Let $L$ be the diameter of $B_k$ which is parallel to $V^\perp$ and let $x_1$ and $x_2$ be the endpoints of $L$. Without loss of generality we may assume that $d(x_1, a + V) \geq d(x_2, a + V)$. $\mathscr{A}$ shall choose $A_{k+1}$ to be the closed ball of radius $\alpha r(B_k)$ which is contained in $B_k$ and is tangential to the boundary of $B_k$ at $x_1$. Let $y_1$ be the point of $a + V$ nearest to $x_1$. Then evidently,

$$\|z(A_{k+1}) - y_1\| = \|z(B_k) - y_1\| + (1 - \alpha)r(B_k)$$

Let $B_{k+1}$ be the ball of radius $\beta r(A_{k+1})$ contained in $A_{k+1}$ chosen by $\mathscr{B}$. Let $y_2$ be the point of intersection of $z(B_{k+1}) + V$ and $L$. Then it is easy to see that

$$\begin{aligned}\|y_2 - y_1\| &\geq \|z(A_{k+1}) - y_1\| - (r(A_{k+1}) - r(B_{k+1}))\\ &= \|z(B_k) - y_1\| + (1 - \alpha)r(B_k) - (r(A_{k+1}) - r(B_{k+1}))\\ &= \|z(B_k) - y_1\| + 2\theta r(B_k) = d(z(B_k), a + V) + 2\theta r(B_k).\end{aligned}$$

In particular we get that $d(z(B_{k+1}), a + V) = \|y_2 - y_1\| \geq d(z(B_k), a + V) + 2\theta r(B_k) > l(I) + \theta r(B_k) > l(I) - \theta r(B_{k+1})$. Then the hypothesis of the Lemma is satisfied for $B_{k+1}$ in the place of $B_k$. Now $A_{k+2}$ may be chosen by the same procedure used for choosing $A_{k+1}$ within $B_k$; and the process may be continued indefinitely.

Suppose $A_{k+1}, A_{k+2}, \ldots, A_{k+h}$ are chosen by the above procedure, letting alternately $\mathscr{B}$ choose the ball according to the rules of the game. We show that $B_{k+h} \cap (I+V)$ is empty. From the construction we have $d(z(B_{k+h}), I+V) \geq d(z(B_{k+h-1}), I+V) \geq \cdots \geq d(z(B_{k+1}), I+V) \geq l(I) + \theta r(B_k) > l(I) + r(B_{k+h})$, since $(\alpha\beta)^h < \theta$. Hence $B_{k+h}$ and $I+V$ must be disjoint.

*Proof of Theorem* 3.2. Let $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$ be given. Let $B_0$ be the closed ball chosen by $\mathscr{B}$ to start the $(\alpha, \beta)$-game. As before let $\theta = \frac{1}{2}(1 - 2\alpha + \alpha\beta) \in (0, 1)$ and let $h$ be a positive integer such that $(\alpha\beta)^h < \theta$. Let $\mu = (\alpha\beta)^h$. Let $M \geq 1$, $\varepsilon \in (0, 1)$ and $\{\tau_p\}$ be such that conditions a) and b) are satisfied for $C = B_0$ and $\mu$ as above. Let $k_0$ be an integer such that $\mu^{k_0} < \min\{\varepsilon\mu r(B_0)^{-1}, M^{-1}\}$. We then choose

$$\delta = \mu^{k_0+1} r(B_0)$$

Then $0 < \delta < \varepsilon < 1$. We shall show that $\mathscr{A}$ can play in such a way that the point of intersection does not belong to $S(p, \delta)$ for any $p$. To that end, we shall show inductively that he can play so that for any $k \geq 0$, $B_{kh}$ does not intersect $S(p, \delta)$ for any $p$ such that $\tau_p \geq \mu^{-k_0+k}$. For $k = 0$ this holds because by condition a) and the choice of $k_0$ we have $\tau_p \leq M < \mu^{-k_0}$ for all $p$ for which $B_0 \cap S(p, \delta)$ is nonempty. Now let $k$ be any positive integer and suppose that $\mathscr{A}$ has played upto $(k-1)h$th stage so that $B_{(k-1)h}$ does not intersect $S(p, \delta)$ for any $p$ such that $\tau_p \geq \mu^{-k_0+k-1}$. To complete the inductive argument we only need to make sure that $\mathscr{A}$ can play (further) upto $kh$th stage in such a way that $B_{kh}$ does not intersect $S(p, \delta)$ for any $p$ such that $\mu^{-k_0+k} \leq \tau_p < \mu^{-k_0+k-1}$ We first show that there is at most one index $p$ such that $\mu^{-k_0+k} \leq \tau_p < \mu^{-k_0+k-1}$ and $B_{(k-1)h} \cap S(p, \delta)$ is nonempty. If $p$ and $q$ are two such indices then we get $d(S(p, \delta), S(q, \delta)) \leq 2r(B_{(k-1)h}) = 2\mu^{k-1} r(B_0) < 2\varepsilon\mu^{-k_0+k} \leq \varepsilon(\tau_p + \tau_q)$, because of our choices; but since $\tau_p^{-1}\tau_q \in (\mu, \mu^{-1})$, the combined inequality together with condition b) in the hypothesis imply that $p = q$.

If there is no index $p$ such that $\mu^{-k_0+k} \leq \tau_p \leq \mu^{-k_0+k-1}$ and $B_{(k-1)h} \cap S(p, \delta)$ is nonempty then $\mathscr{A}$ can play at random until the $kh$th stage, since the inductive assertion already holds. Otherwise let $q$ be the unique index for which the conditions hold. Then we have $\tau(S(q, \delta)) \leq \delta\tau_q = \mu^{k_0+1} r(B_0)\tau_q < \mu^k r(B_0) < \theta\mu^{k-1} r(B_0) = \theta r(B_{(k-1)h})$. Hence there exists a hyperplane $V$ such that $S(q, \delta)$ is contained in a set of the form $I + V$, where $I$ is an interval in $V^\perp$ of length less than $\theta r(B_{(k-1)h})$. Hence by Lemma 3.7 $\mathscr{A}$ can play the next $h$ turns in such a way that $B_{kh}$ does not intersect $S(q, \delta)$, whatever be the choices made by $\mathscr{B}$ within the rules of the game. Together with the inductive hypothesis and the uniqueness of $q$

as above this means that $B_{kh}$ does not intersect $S(p, \delta)$ for any $p$ such that $\tau_p \geq \mu^{-k_0+k}$, thus completing the inductive argument.

It is evident that if $\mathscr{A}$ plays the game as above then the point of intersection does not belong to $S(p, \delta)$ for any $p$. Hence $F$ is an $(\alpha, \beta)$-winning set.

3.8. COROLLARY. *Let the hypothesis and notation be as in Theorem 3.2. Then for any nonempty open subset $\Omega$ of $\mathbb{R}^m$, $F \cap \Omega$ is of Hausdorff dimension $m$.*

*Proof.* The theorem in particular implies that for any $\alpha \in (0, \frac{1}{2})$, $F$ is an $\alpha$-winning set. The Corollary therefore follows from Proposition 3.1.

## §4. Bounded trajectories and Hausdorff dimension

We shall now apply Theorem 3.2 to compute the Hausdorff dimensions of sets of bounded trajectories of flows as in §§1 and 2.

Let the notation be as in §§1 and 2. We equip $G$ with a Riemannian metric which is invariant under the left action of $G$ on itself. Any Lie subgroup of $G$ is equipped with the induced metric. The space $G/P$ is canonically identified with $K/M$ and is equipped with the metric obtained by projecting the metric on $K$. All Riemannian manifolds are considered as metric spaces canonically via the distance function corresponding to the Riemannian metric.

4.1. THEOREM. *For any nonempty open subset $\Omega$ of $G/P$, the Hausdorff dimension of $E^+(\Gamma) \cap \Omega$* (cf. §2 *for definition of $E^+(\Gamma)$*) *coincides with the dimension of $G/P$ as a manifold.*

*Proof.* Let $\lambda \in A$ and consider the map $\psi: N \to G/P$ defined by $\psi(n) = \lambda nwP$. Recall that $\psi$ is a diffeomorphism of $N$ onto the open submanifold $\lambda(G-P)/P$ (cf. Proposition 1.1). In particular, $\psi$ and the map $\psi^{-1}$ defined on $\lambda(G-P)/P$ are locally Lipschitz maps. Since Hausdorff dimension is (obviously) unchanged under bilipschitz maps, in view of Proposition 2.5, it is enough to prove that if (in the notation of Proposition 2.5)

$$X = \bigcup_{a \in A} \left( N - \bigcup_{i=1}^{\infty} \pi(n_i a_i^{-1} a^{-1} N^- A^-) \right) \tag{4.2}$$

then for all bounded open subsets $\Omega$ of $N$, $X \cap \Omega$ has Hausdorff dimension $m$ = dimension of $N$ = dimension of $G/P$.

Let $\mathfrak{n}$ be the Lie algebra of $N$ and let $\exp: \mathfrak{n} \to N$ be the usual exponential map. $N$ is a simply connected nilpotent Lie group and exp is a diffeomorphism of $\mathfrak{n}$ onto $N$. Let $\log: N \to \mathfrak{n}$ be the inverse map. We view $\mathfrak{n}$ as $\mathbb{R}^m$ and equip it with the usual Hilbert norm with respect to a basis consisting of eigenvectors of $\mathrm{Ad}(\exp Y)$. Recall that all eigenvalues of $\mathrm{Ad}(\exp Y)$ are real and positive. Let $\eta$ be the largest eigenvalue of $\mathrm{Ad}(\exp Y)$ and let

$$V = \{\xi \in \mathfrak{n} \mid \mathrm{Ad}(\exp Y)(\xi) = \eta\xi\}$$

Then $V$ is an abelian Lie subalgebra of $\mathfrak{n}$ invariant under $\mathrm{Ad}(\exp Y)$. We shall show that for all $x_0 \in N$

$$\bigcup_{a\in A}\left(V - \bigcup_{i=1}^{\infty} \log x_0\pi(n_i a_i^{-1} a^{-1} N^- A^-)\right) \tag{4.3}$$

is an $(\alpha, \beta)$-winning set for all $\alpha, \beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$. Since exp is a diffeomorphism, by Proposition 3.1 this implies that for all $x_0 \in N$, $(x_0^{-1} \exp V) \cap X \cap \Omega$, where $X$ is the set as in (4.2) and $\Omega$ is any bounded nonempty open subset of $N$, has Hausdorff dimension $l$, provided it is nonempty. Since the natural quotient map of $N$ onto $N/\exp V$ is differentiable, by Theorem 2.10.25 of [5] it follows that for all $s < m$ the $s$-dimensional Hausdorff measure of $X \cap \Omega$ is $\infty$; thus the Hausdorff dimension is at least $m$. But since $\Omega$ is also of Hausdorff dimension $m$ this implies that $X \cap \Omega$ is of Hausdorff dimension $m$, thus proving the theorem.

It remains to prove the assertion about the set in (4.3), for which we proceed as follows. For $t > 0$ let $\varphi_t = \exp(-\log t/\log \eta)Y$. Then $(\mathrm{Ad}\,\varphi_t^{-1})v = tv$ for all $v \in V$ and $t > 0$. We fix $x_0 \in N$ and for $i \in \mathbb{N}$ and $0 < t \le 1$ put

$$S(i, t) = \log x_0\pi(n_i a_i^{-1}\varphi_t^{-1} N^- A^-) \cap V.$$

We would like to estimate $\tau(S(i, t))$ (cf. §3 for definition). Fix $i \in \mathbb{N}$ and $0 < t \le 1$ and let $v_1, v_2 \in S(i, t)$. Then there exist $y_1, y_2 \in \pi(N^- A^-)$ such that $\exp v_j = x_0 n_i a_i^{-1}\varphi_t^{-1} y_j \varphi_t a_i$ for $j = 1$ and 2. Then $\exp(v_2 - v_1) = a_i^{-1}\varphi_t^{-1} y_1^{-1} y_2 \varphi_t a_i$. Therefore $v_2 - v_1 = \log(a_i^{-1}\varphi_t^{-1} y_1^{-1} y_2 \varphi_t a_i) = \mathrm{Ad}\, a_i^{-1}\varphi_t^{-1}(\log y_1^{-1} y_2)$. Thus $(\mathrm{Ad}\, a_i\varphi_t) \times (v_1 - v_2) = \log y_1^{-1} y_2$. Let

$$\Delta \doteq \sup\{\|\log y_1^{-1} y_2\| \mid y_1, y_2 \in \pi(N^- A^-)\} \tag{4.4}$$

which is finite since $\pi(N^- A^-)$ is a bounded subset of $N$ (cf. Corollary 1.7). Also we have $(\mathrm{Ad}\, a_i\varphi_t)(v_1 - v_2) = t^{-1}\eta^{-t_i}(v_1 - v_2)$ where, as in §2, $t_i \in \mathbb{R}$ are such that $a_i^{-1} = \exp t_i Y$. Thus we get that $\|v_1 - v_2\| \le \Delta t \eta^{t_i}$ for all $v_1, v_2 \in S(i, t)$. Hence

$\tau(S(i, t)) \leq \Delta t \eta^{t_i}$ for all $i \in \mathbb{N}$ and $0 < t < 1$. For all $i \in \mathbb{N}$, put $\tau_i = \Delta \eta^{t_i}$. We have noted earlier (see the proof of Proposition 2.5) that $t_i \leq 2\sigma$, where $\sigma$ is the constant as in Remark 2.3. Thus we get that $\tau_i \leq \Delta \eta^{2\sigma} = M$, say, and $\tau(S(i, t)) \leq t\tau_i$ for all $i \in \mathbb{N}$, which shows that condition a) of Theorem 3.2 is satisfied for the sets $S(i, t)$ (with constants independent of the compact set $C$ involved in the condition).

Now let $\mu \in (0, 1)$ be given and let $i$ and $j$ be such that $\mu\tau_i \leq \tau_j \leq \mu^{-1}\tau_i$. Since $\tau_i = \Delta\eta^{t_i}$ and $\tau_j = \Delta\eta^{t_j}$, we have $(\log \eta)|t_i - t_j| \leq -\log \mu$. Put $t_0 = -\log \mu/\log \eta > 0$. Then by Lemma 1.8 there exists a neighbourhood $\Omega$ of the identity in $N$ such that $nN^-A^-K \cap (\exp tY)N^-A^-K$ is nonempty for all $n \in \Omega$ and $t \in [-t_0, t_0]$. Recall also that by Remark 2.3 the sets $n_p w a_p N A_\sigma K$, $p \in \mathbb{N}$ are pairwise disjoint. In particular, $n_i a_i^{-1}(\exp -\sigma Y)N^-A^-K \cap n_j a_j^{-1}(\exp -\sigma Y)N^-A^-K$ is empty and hence so is $a_i^{-1}a_j N^-A^-K \cap a_j(\exp \sigma Y)(n_i^{-1}n_j)(\exp -\sigma Y)a_j^{-1}N^-A^-K$. Since $a_i^{-1}a_j = \exp(t_j - t_i)Y$ and $(t_i - t_j) \in [-t_0, t_0]$ we conclude that $a_j(\exp \sigma Y)(n_i^{-1}n_j)(\exp -\sigma Y)a_j^{-1}$ does not belong to $\Omega$. Let $r > 0$ be such that $\exp B(0, r) \subset \Omega$. Then $n_i^{-1}n_j$ does not belong to $a_j^{-1}\exp -\sigma Y \exp B(0, r)(\exp \sigma Y)a_j$ and hence $\log n_i^{-1}n_j$ does not belong to $\mathrm{Ad}(a_j^{-1}\exp -\sigma Y)(B(0, r))$.

Now let $\Delta$, $\sigma$ and $r$ be as above and let $\varepsilon_1 > 0$ be such that $\mathrm{Ad}\, \varphi_{\varepsilon_1}^{-1}(B(0, \Delta)) \subset \mathrm{Ad}(\exp -\sigma Y)(B(0, r))$. Note that such $\varepsilon_1$ exists since all eigenvalues of $\mathrm{Ad}\, \varphi_t^{-1}$ tend to 0 as $t \to 0$. Further, we get that for all $t \in (0, \varepsilon_1)$, $\mathrm{Ad}\, \varphi_t^{-1}(B(0, \Delta)) \subset \mathrm{Ad}(\exp -\sigma Y)(B(0, r))$. Now we claim that if $i, j \in \mathbb{N}$ are such that $\mu\tau_i \leq \tau_j \leq \mu^{-1}\tau_i$ and $t \in (0, \varepsilon_1)$ then $S(i, t)$ and $S(j, t)$ are disjoint: Let, if possible, the intersection be nonempty. Then there exist $z_1 \in a_i^{-1}\varphi_t^{-1}\pi(N^-A^-)\varphi_t a_i$ and $z_2 \in a_j^{-1}\varphi_t^{-1}\pi(N^-A^-)\varphi_t a_j$ such that $x_0 n_i z_1 = x_0 n_j z_2$. Then $n_i^{-1}n_j = z_1 z_2^{-1} \in (a_i^{-1}\varphi_t^{-1}\pi(N^-A^-)\varphi_t a_i)(a_j^{-1}\varphi_t^{-1}\pi(N^-A^-)^{-1}\varphi_t a_j)$. Without loss of generality we may assume $t_i \geq t_j$; then $a_i^{-1}\pi(N^-A^-)a_i = \pi(a_i^{-1}N^-A^-) = \pi(a_j^{-1}(a_i^{-1}a_j)N^-A^-) = \pi(a_j^{-1}N^-(a_j^{-1}a_i)A^-) \subset \pi(a_j^{-1}N^-A^-) = a_j^{-1}\pi(N^-A^-)a_j$. Thus we get that

$$\begin{aligned} n_i^{-1}n_j &\in (a_j^{-1}\varphi_t^{-1}\pi(N^-A^-)\varphi_t a_j)(a_j^{-1}\varphi_t^{-1}\pi(N^-A^-)^{-1}\varphi_t a_j) \\ &= a_j^{-1}\varphi_t^{-1}\pi(N^-A^-)\pi(N^-A^-)^{-1}\varphi_t a_j \\ &\subset \exp \mathrm{Ad}\, a_j^{-1}\varphi_t^{-1}(B(0, \Delta)) \subset \exp \mathrm{Ad}\, a_j^{-1}(\exp -\sigma Y)\,(B(0, r)). \end{aligned}$$

Thus we find that $\log\, n_i^{-1}n_j \in \mathrm{Ad}\, a_j^{-1}(\exp -\sigma Y)\ (B(0, r))$, which however contradicts the choice of $r$; hence the claim must hold.

To complete the proof we shall find $\varepsilon \in (0, \varepsilon_1)$ so that condition b) of Theorem 3.2 is satisfied. First we prove the following Lemmas.

Recall, that $\pi(N^-A^-)$ contains a neighbourhood of the identity in $N$. Hence there exists $\Delta' \in (0, \Delta)$, where $\Delta$ is as in (4.4), such that $\exp B(0, \Delta')$ is contained in $\pi(N^-A^-)$.

4.5. LEMMA. *Let $t, t' > 0$ be such that $4m^2\Delta^2(\Delta')^{-2}t \leq t' < \frac{1}{4}$. Let $x_1 \in (\operatorname{Ad} a_i^{-1}\varphi_t^{-1})(B(0, \Delta))$ and $x_2 \notin (\operatorname{Ad} a_i^{-1}\varphi_{t'}^{-1})\ (B(0, \Delta'))$. Then*

$$\|x_1 - x_2\| \geq (m^{-1}t'\Delta' - t\Delta)\eta^{t_i-\sigma}$$

*where, as before, m is the dimension of N.*

*Proof.* Let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathfrak{n}$ consisting of eigenvectors of Ad exp $Y$ and let $\eta_1, \ldots, \eta_m$ be the corresponding eigenvalues. We may assume $\eta_1 = \eta$. For $s_1, \ldots, s_m > 0$ let

$$R(s_1, \ldots, s_m) = \{\textstyle\sum \xi_i e_i \mid |\xi_i| < s_i \text{ for all } i\}$$

Then $x_1 \in (\operatorname{Ad} a_i^{-1}\varphi_t^{-1})R(\Delta, \ldots, \Delta) = R(\Delta_1, \ldots, \Delta_m)$ where $\Delta_k$, $k = 1, \ldots, m$, are $\Delta$ times the eigenvalues of $\operatorname{Ad} a_i^{-1}\varphi_t^{-1}$ corresponding to $e_k$. On the other hand $x_2$ does not belong to $(\operatorname{Ad} a_i^{-1}\varphi_{t'}^{-1})(R(m^{-1}\Delta', \ldots, m^{-1}\Delta')) = R(m^{-1}\Delta'_1, \ldots, m^{-1}\Delta'_m)$, where $\Delta'_k$, $k = 1, 2, \ldots, m$ are $\Delta'$ times the eigenvalues of $\operatorname{Ad} a_i^{-1}\varphi_{t'}^{-1}$ corresponding to $e_k$. Hence for some $k = 1, \ldots, m$ the $e_k$ coordinate of $x_1 - x_2$ is at least $m^{-1}\Delta'_k - \Delta_k$ and hence $\|x_1 - x_2\| \geq m^{-1}\Delta'_k - \Delta_k$. Observe that if $\eta_k = \eta$ then $\Delta'_k = \eta^{t_i}t'\Delta'$ and $\Delta_k = \eta^{t_i}t\Delta$ and the lemma would hold. We shall now uphold it in general.

Since $G$ is of $\mathbb{R}$-rank 1, for any one-parameter subgroup whose adjoint action is diagonalisable over $\mathbb{R}$, the logarithms of the eigenvalues of the adjoint action of any nontrivial element of the subgroup, form a root system in $\mathbb{R}$. In particular, it follows that the only possible eigenvalue of Ad exp $Y$ other than $\eta$ is $\sqrt{\eta}$ (a version of the lemma can also be proved without using this fact, if the condition on $t'$ is modified suitably – but, for simplicity, we choose the present course). Thus if for some $k = 1, \ldots, m$, $\eta_k \neq \eta$ then $m^{-1}\Delta'_k - \Delta_k = \eta^{t_i/2}(m^{-1}\Delta'\sqrt{t'} - \Delta\sqrt{t}) \geq \eta^{t_i-\sigma}(m^{-1}\Delta'\sqrt{t'} - \Delta\sqrt{t})$, since $t_i \leq 2\sigma$ for all $i$. Observe that $m^{-1}\Delta'(\sqrt{t'} - t') = m^{-1}\Delta'\sqrt{t'}(1 - \sqrt{t'}) \geq 2\Delta\sqrt{t}(1 - \sqrt{t'})$; since $t' < \frac{1}{4}$, $(1 - \sqrt{t'}) \geq \frac{1}{2}(1 - \sqrt{t})$ and hence we get $m^{-1}\Delta'(\sqrt{t'} - t') \geq \Delta(\sqrt{t} - t)$ and consequently $m^{-1}\Delta'\sqrt{t'} - \Delta\sqrt{t} \geq m^{-1}\Delta't' - \Delta t$. Hence for $k$ as above we get $m^{-1}\Delta'_k - \Delta_k \geq \eta^{t_i-\sigma}(m^{-1}\Delta't' - \Delta t)$ as desired.

For any $i \in \mathbb{N}$ and $t > 0$ we put

$$S'(i, t) = \log \pi(n_i a_i^{-1}\varphi_t^{-1}N^- A^-) = n_i a_i^{-1}\varphi_t^{-1}\pi(N^- A^-)\varphi_t a_i.$$

4.6. LEMMA. *For any compact subset $C'$ of $\mathfrak{n}$ there exists a constant $L > 0$ such that if for some $i$, $S'(i, 1) \cap C'$ is nonempty then $S'(i, 1)$ is contained in $B(0, L)$.*

*Proof.* Recall that if $t_i \in \mathbb{R}$ are such that $a_i^{-1} = \exp t_i Y$ then $t_i \leq 2\sigma$ for all $i$, where $\sigma$ is the constant as in Remark 2.3. Hence by Lemma 1.2 $\bigcup_{i=1}^{\infty} \pi(a_i^{-1}N^-A^-)$ is contained in a compact subset, say $Q$. Let $L_1 \geq 0$ be such that $(\exp C')Q^{-1}$ is contained in $\exp B(0, L_1)$. Let $L > 0$ be such that $(\exp B(0, L_1))Q$ is contained in $\exp B(0, L)$. Let $i$ be such that $S'(i, 1) \cap C'$ is nonempty; then so is $\exp S'(i, 1) \cap \exp C'$. Hence there exists $y_0 \in \pi(a_i^{-1}N^-A^-)$ such that $n_i y_0 \in \exp C'$. Then $n_i \in (\exp C')y_0^{-1} \subset (\exp C')Q^{-1}$ and hence $\log n_i \in B(0, L_1)$. Then for any $y \in \pi(a_i^{-1}N^-A^-) \subset Q$ we have $\log n_i y \in \log(\exp B(0, L_1))Q \subset B(0, L)$, which proves the lemma.

We are now ready to verify condition b) of Theorem 3.2 for the family of sets $S(i, t)$. Let $\mu \in (0, 1)$ be as before and let a compact subset $C$ of $V$ be also given. Let $L > 0$ be such that the conclusion of Lemma 4.6 holds for $C' = \log x_0^{-1}(\exp C)$. Since exp is a diffeomorphism and the metric on $N$ is translation-invariant there exists $c \in (0, 1)$ such that

$$c^{-1}\|\log y_1 - \log y_2\| \geq \|\log y_1^{-1}y_2\| \geqq c\,\|\log y_1 - \log y_2\| \tag{4.7}$$

for all $y_1, y_2 \in x_0 \exp B(0, L) \cup Q^{-1}Q$, where, as in Lemma 4.6, $Q$ is a compact set containing $\pi(a_i^{-1}N^-A^-)$ for all $i$. Let $\Delta$, $\Delta'$, $\eta$, $\sigma$, $m$ and $\varepsilon_1$ be as before. Without loss of generality, we may assume $\varepsilon_1 < \frac{1}{2}$, $\Delta' < 1$ and $\Delta > 1$. Now put

$$\varepsilon = \tfrac{1}{5}c^2(\Delta^{-1}\Delta' m^{-1})^2 \eta^{-\sigma}(1 + \mu^{-1})^{-1}\varepsilon_1$$

Recall that we have (already) chosen $\tau_i = \Delta\eta^{t_i}$ for all $i$. Now let $t \in (0, \varepsilon)$ and let $i, j \in \mathbb{N}$ be such that $\mu\tau_i \leq \tau_j \leq \mu^{-1}\tau_i$ and $S(i, t) \cap C$ and $S(j, t) \cap C$ are nonempty. By our earlier discussion the first condition implies that $S(i, t')$ and $S(j, t')$ are disjoint for $t' \in (0, \varepsilon_1)$. In particular this implies that

$$d(S(i, t), S(j, t)) \geq d(S(i, t), \partial S(i, t')) \tag{4.8}$$

for all $t' \in (t, \varepsilon_1)$, where $\partial$ denotes the boundary of the set in question. Now let $t' \in (\varepsilon_1 - \varepsilon, \varepsilon_1)$. Note that in particular $t' > 4m^2\Delta^2(\Delta')^{-2}\varepsilon$. Let $v_1 \in S(i, t)$ and $v_2 \in \partial S(i, t')$. Then there exist $y_1 \in \pi(a_i^{-1}\varphi_t^{-1}N^-A^-)$ and $y_2 \in \partial\pi(a_i^{-1}\varphi_{t'}^{-1}N^-A^-)$ such that $\exp v_k = x_0 n_i y_k$ for $k = 1$ and 2. Then by (4.7) we have $\|v_1 - v_2\| \geq c\,\|\log(x_0 n_i y_1)^{-1}(x_0 n_i y_2)\| = c\,\|\log y_1^{-1}y_2\| \geq c^2\|\log y_1 - \log y_2\|$, since $y_1, y_2 \in Q$. Since $y_2 \in \partial\pi(a_i^{-1}\varphi_{t'}^{-1}N^-A^-)$, $\log y_2$ does not belong to $(\operatorname{Ad} a_i^{-1}\varphi_{t'}^{-1})B(0, \Delta'))$. On the other hand $\log y_1$ is contained in $(\operatorname{Ad} a_i^{-1}\varphi_t^{-1})(B(0, \Delta))$. Hence, by Lemma 4.5 $\|\log y_1 - \log y_2\| \geq (m^{-1}t'\Delta' - t\Delta)\eta^{t_i - \sigma}$. Hence $\|v_1 - v_2\| \geq c^2(m^{-1}t'\Delta' - t\Delta)\eta^{t_i-\sigma} = c^2\Delta^{-1}\tau_i(m^{-1}t'\Delta' - t\Delta)\eta^{-\sigma}$. Since $t \in (0, \varepsilon)$ and $t' > \varepsilon_1 - \varepsilon$, we have $m^{-1}t'\Delta' - t\Delta \geq m^{-1}(\varepsilon_1 - \varepsilon)\Delta' - \varepsilon\Delta \geq m^{-1}\varepsilon_1\Delta' - 2\varepsilon\Delta \geq 5c^{-2}(1 +$

$\mu^{-1})\varepsilon\Delta\eta^\sigma - 2\varepsilon\Delta \geq c^{-2}(1+\mu^{-1})\varepsilon\Delta\eta^\sigma$. Hence $\|v_1 - v_2\| \geq \varepsilon(1+\mu^{-1})\tau_i \geq \varepsilon(\tau_i + \tau_j)$. Since $v_1$ and $v_2$ were arbitrary elements of $S(i, t)$ and $\partial S(i, t')$ we conclude that $d(S(i, t).\ \partial S(i, t')) \geq \varepsilon(\tau_i + \tau_j)$. Hence by (4.8), $d(S(i, t), S(j, t)) \geq \varepsilon(\tau_i + \tau_j)$ for any $t \in (0, \varepsilon)$, which shows that condition b) of Theorem 3.2 is satisfied for the sets $S(i, t)$, $i \in \mathbb{N}$, $0 < t < \varepsilon$. Hence by that theorem, for every $x_0 \in N$, the set in (4.3) is an $(\alpha,\beta)$-winning set for all $\alpha,\beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$. As noted before this implies the theorem.

4.9. *Remark.* If $G = SO(m, 1)$, the special orthogonal group of a quadratic form of signature $(m, 1)$ and $A = (\exp tY)$ is a one-parameter subgroup such that $\mathrm{Ad}\, a$, $a \in A$, is diagonalisable over $\mathbb{R}$ then $\mathrm{Ad}(\exp Y)$ has only one eigenvalue on the Lie subalgebra $\mathfrak{n}$ of the Lie subgroup $N$ as defined in §1. In this case, $N$ is canonically isomorphic to $\mathbb{R}^{m-1}$ (via the exponential map) and the above proof actually shows that the set $X$ defined by (4.2) is itself an $(\alpha,\beta)$-winning set for any $\alpha,\beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$. It turns out that in this case each $\pi(n_i a_i^{-1} a^{-1} N^- A^-)$ is an open ball and the condition as in Corollary 3.4 is satisfied. We refer the reader to [4] for a discussion in this regard (see also Corollary 5.2 below).

4.10. *Remark.* Though for simplicity in the proof of Theorem 4.1 we chose $V$ to be the eigenspace corresponding to the largest eigenvalue of $\mathrm{Ad}(\exp Y)$, it is not difficult to modify the proof to show the following: If $V$ is an abelian $\mathrm{Ad}(\exp Y)$-invariant Lie subalgebra of $\mathfrak{n}$ such that the largest eigenvalue of $\mathrm{Ad}(\exp Y)$ on $V$ is also the largest among all eigenvalues, then for any $x_0 \in N$ the set defined in (4.3) is an $(\alpha,\beta)$-winning set for all $\alpha,\beta \in (0, 1)$ such that $1 - 2\alpha + \alpha\beta > 0$. This has to do with the fact that the conditions in Theorem 3.2 involve thicknesses of sets and not diameters. It, however, does seem necessary to assume $V$ to be abelian, since otherwise, in the computation for verifying condition a), we get various terms that cannot be controlled. We shall however not go into the details regarding these observations.

## §5. Bounded orbits of flows

We can deduce the following conclusion about the set of bounded orbits of flows, rather than trajectories.

5.1. THEOREM. *Let $G$ be a connected semisimple Lie group of $\mathbb{R}$-rank 1 and let $\Gamma$ be a lattice in $G$. Let $G$ be equipped with a metric obtained as a quotient of a left-invariant metric on $G$. Let $(g_t)$ be a one-parameter subgroup such that $\mathrm{Ad}\, g_1$ has an eigenvalue (possibly complex) of absolute value other than one. Then* *for any nonempty open subset $\Omega$ of $\Gamma\backslash G$ the set*

$$\{\Gamma g \in \Omega \mid \text{the } (g_t)\text{-orbit of } \Gamma g \text{ is bounded}\}$$

*is of Hausdorff dimension equal to the dimension of $G$.*

*Proof.* Let $(g_t)$ be a one-parameter subgroup as in the hypothesis. Let $g_t = s_t u_t$, $t \in \mathbb{R}$, be the Jordan decomposition; here $(s_t)$ and $(u_t)$ are one-parameter subgroups consisting of semisimple and unipotent elements respectively, (that is, the matrix for the adjoint action is semisimple or unipotent respectively) commuting with each other. Since $\operatorname{Ad} g_1$ has an eigenvalue of absolute value other than 1, $\operatorname{Ad} s_t$ is non-trivial for all $t \neq 0$. Since $G$ is of $\mathbb{R}$-rank 1 such an element $s_t$ does not commute with any non-central unipotent element. Since the center of $G$ is discrete it follows that the one-parameter subgroup $(u_t)$ is trivial. Hence $(g_t)$ consists of semisimple elements. Then $g_t$ may be expressed as $g_t = c_t d_t$, $t \in \mathbb{R}$, where $(c_t)$ and $(d_t)$ are one-parameter subgroups commuting with each other and such that $(c_t)$ is contained in a compact subgroup of $G$ and $(d_t)$ consists of semisimple elements such that all the eigenvalues of $\operatorname{Ad} d_t$, $t \in \mathbb{R}$ are real. Evidently, for any $g \in G$, $\{\Gamma g g_t \mid t \in \mathbb{R}\}$ is bounded if and only if $\{\Gamma g d_t \mid t \in \mathbb{R}\}$ is bounded. Hence, in proving the theorem, we may without loss of generality also assume that all eigenvalues $\operatorname{Ad} g_t$ are real. By conjugating by a suitable element we may assume $(g_t) = (\exp tY)$, the one-parameter subgroup as in the earlier sections.

We now use the notation as in §1. Also $\Phi: G \to G/P \times G/P$ be the map defined by $\Phi(g) = (gP, gP^-)$. Let $E^+(\Gamma)$ be the subset of $G/P$ defined in §2 and let $E^-(\Gamma)$ be the subset of $G/P^-$ defined analogously by

$$E^-(\Gamma) = \{xP^- \in G/P^- \mid \{\Gamma x(\exp -tY) | \, t \geq 0\} \text{ is bounded}\}$$

It is obvious that for $g \in G$, $\{\Gamma g g_t \mid t \in \mathbb{R}\}$ is bounded if and only if $\Phi(g) \in E^+(\Gamma) \times E^-(\Gamma)$. By Theorem 4.1, for any nonempty open subset $\Omega^+$ of $G/P$, the Hausdorff dimension of $E^+(\Gamma) \cap \Omega^+$ equals the dimension of $G/P$. Similarly applying that result to $(\exp -tY)$ in the place of $(\exp tY)$ we conclude that for any nonempty open subset $\Omega^-$ of $G/P^-$, $E^-(\Gamma) \cap \Omega^-$ has Hausdorff dimension equal to the dimension of $G/P^-$. Since the Hausdorff dimension of the Cartesian product of two metric spaces is at least as much as the sum of the Hausdorff dimensions of the components (this follows from Theorem 2.10.27 of [5], for instance) we can conclude from the above that for any nonempty open subset $\Omega'$ of $G/P \times G/P^-$ the Hausdorff dimension of $E^+(\Gamma) \times E^-(\Gamma) \cap \Omega'$ equals the dimension of $G/P \times G/P^-$. Observe that $P \cap P^-(= MA$ in the notation of §1) is of codimension equal to the dimension of $G/P \times G/P^-$. Hence by the rank theorem it follows that the map $\Phi$ defined above is an open map. Now let $\Omega$ be a nonempty open set and $\Omega' = \Phi(\Omega)$. Then $\Omega'$ is a nonempty open set and by our earlier observation $E^+(\Gamma) \times E^-(\Gamma) \cap \Omega'$ has Hausdorff dimension equal to the dimension of $G/P \times G/P^-$. Since $\Phi$ is differentiable, by Theorem 2.10.25 of [5] this implies that $\Phi^{-1}(E^+(\Gamma) \times E^-(\Gamma)) \cap \Omega$ has Hausdorff dimension equal to the dimension of $G$. Since for any $g \in G$, $\{\Gamma g g_t \mid t \in \mathbb{R}\}$ is bounded if and only if $\Phi(g) \in E^+(\Gamma) \times E^-(\Gamma)$ and $\Gamma\backslash G$ is equipped with a metric obtained as a quotient of the metric on $G$, the last assertion implies the theorem.

5.2. COROLLARY. *Let $M$ be a Riemannian manifold of constant negative curvature and finite Riemannian volume. Let $S(M)$ be the unit tangent bundle of $M$; that is,*

$$S(M) = \{(x, \xi) \mid x \in M,\ \xi \text{ a tangent vector at } x \text{ such that } \|\xi\| = 1\}$$

*For $(x, \xi) \in S(M)$ let $\gamma(x, \xi)$ be the geodesic on $M$ through the point $x$ in the direction of $\xi$. Let $S(M)$ be equipped with the canonical structure of a Riemannian manifold. Then for any nonempty open subset $\Omega$ of $S(M)$ the set*

$$\{(x, \xi) \in \Omega \mid \gamma(x, \xi) \text{ is bounded in } M\}$$

*has Hausdorff dimension equal to the dimension of $S(M)$, viz. $2m - 1$ where $m$ is the dimension of $M$.*

*Proof.* It is well-known (cf. [11], for instance) that $S(M)$ as above can be realised as a double coset space $\Gamma\backslash G/C$, where $G = SO(m, 1)$, the orthogonal group corresponding to a quadratic form of signature $(m, 1)$, $\Gamma$ is a lattice in $G$ and $C$ is a compact subgroup of $G$; further, there exists a maximal compact subgroup $K$ of $G$ containing $C$ such that $M$ may be identified with $\Gamma\backslash G/K$, so that the canonical quotient maps of $\Gamma\backslash G/C$ onto $\Gamma\backslash G/K$ and of $S(M)$ onto $M$ correspond to each other. The geodesics on $M$ correspond to the images of orbits of a one-parameter subgroup $(g_t)$ (viz. $\{\Gamma g g_t K \mid t \in \mathbb{R}\}$, $g \in G$) such that $(\operatorname{Ad} g_t)$ is diagonalisable over $\mathbb{R}$. Since $G = SO(m, 1)$ is a simple Lie group of $\mathbb{R}$-rank 1, $\Gamma$ is a lattice in $G$ and $K$ is a compact subgroup Theorem 5.1 implies that for any nonempty open subset $\Omega'$ of $\Gamma\backslash G$, $\{\Gamma g \in \Omega' \mid \{\Gamma g g_t K \mid t \in \mathbb{R}\}$ is bounded in $\Gamma\backslash G\}$ is of Hausdorff dimension equal to the dimension of $\Gamma\backslash G$; hence the image of that set in $\Gamma\backslash G/C$ has Hausdorff dimension equal to the dimension of $S(M)$. But, by the above comments, for any nonempty open subset $\Omega$ of $S(M)$ $\{(x, \xi) \in \Omega \mid \gamma(x, \xi)$ is bounded$\}$ is the image in $\Gamma\backslash G/C = S(M)$ of a set as above, for a suitable open set $\Omega'$ in $\Gamma\backslash G$. Hence the Corollary.

COMMENTS AND QUESTIONS. i) It would be interesting to know whether the analogue of Theorem 5.1 holds for any Lie group $G$, closed subgroup $H$ such that $H\backslash G$ admits a finite $G$-invariant measure and $(g_t)$ such that $\mathrm{Ad}\, g_t$ is semisimple for all $t$. Using the same ideas as in §4 of [2] and a recent result of D. Witte (cf. [19] Corollary 4.13) it is easy to reduce the question to the case of a semisimple factor group $G'$ of $G$ and a lattice $\Gamma'$ in $G'$ in the place of $G$ and $\Gamma$ respectively. If $G'$ is of $\mathbb{R}$-rank $\leq 1$ (for instance if $G$ has no factor group of $\mathbb{R}$-rank $\geq 2$) then Theorem 5.1 applies and we get the desired analogue. Also, as noted in the introduction, by a Theorem of W. M. Schmidt [15] and the correspondence established in [2] for the flows on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ induced by one-parameter subgroups of the form $\mathrm{diag}\,(e^{-t}, \ldots, e^{-t}, e^{\lambda t}, \ldots, e^{\lambda t})$ the set of points on bounded trajectories is of Hausdorff dimension equal to the dimension of the homogeneous space. For $n \geq 3$, this is, of course, not covered by Theorem 5.1, and suggests that the analogue sought after might indeed hold.

ii) Let $G$ and $\Gamma$ be as in Theorem 5.1. But now suppose that $\mathrm{Ad}\, g_t$ is unipotent for all $t \in \mathbb{R}$. Then $(g_t)$ is contained in a horospherical subgroup, say $N$, corresponding to a one-parameter subgroup $(a_t)$ such that $\mathrm{Ad}\, a_t$ is diagonalisable over $\mathbb{R}$. In [2] we proved that for $g \in C$, $Ng\Gamma$ is compact or dense in $G/\Gamma$ according to whether $\{a_t g\Gamma \mid t \geq 0\}$ is divergent or not. In the former case the orbit $\{g_t g\Gamma \mid t \in \mathbb{R}\}$ is evidently bounded. It seems reasonable to conjecture that in the latter case also $\{g_t g\Gamma \mid t \in \mathbb{R}\}$ is bounded only if it is contained in a compact orbit of a proper subgroup $H$ of $G$.

iii) Corollary 5.2 suggests the question whether analogous assertion would hold for manifolds of variable negative curvature.

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