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Rosso's form and quantized Kac Moody algebras

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1 Introduction

When $\mathfrak{g}$ is semisimple, Rosso [R1, Sect. A.II] defined a remarkable bilinear form on $U = U_q(\mathfrak{g})$ which is ad-invariant, compatible with triangular decomposition, and when restricted to the "torus" recovers the "Cartan" inner product. In a later paper, Rosso [R2] connects this form to Drinfeld's bilinear pairing. We observe that Rosso's form can be defined for any quantized Kac–Moody algebra. Moreover we relate it to Kashiwara's form and in particular obtain a more precise uniqueness property (Theorem 4.8).

Section 5 of this paper is devoted to computing the quantum Shapovalov determinant. As in the Kac–Moody case, one can determine the possible factors using a quantum Casimir operator (see [L2, Sect.6.1]). This gives a fairly complete description of the determinant's factorization (Conjecture 5.6, Conjecture 5.7, Lemma 5.10). It leads to a new family of Verma submodules associated to roots of unity (even though q is an indeterminate) which have no analog either for $\mathfrak{g}$ semisimple or $q = 1$.

Our motivation for factoring the Shapovalov determinant was the following question of Drinfeld [D1, Question 8.1]. Assume that L is a simple highest weight U module with a specializable highest weight $q^\lambda : \lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ (Sect. 6). Does L specialize to the corresponding simple highest weight $U(\mathfrak{g})$ module? For $\mathfrak{g}$ semisimple, Lusztig [L1, 4.12] and Rosso [R1, Sect. C] showed that this is true when the highest weight of L is integrable, whilst [D1, Sect. 8] resolved the nonintegrable case when $\mathfrak{g}$ is semisimple. Drinfeld [D1, Question 8.1] conjectured that this also holds when $\mathfrak{g}$ is any Kac–Moody algebra. Using the Shapovalov determinant, one can compute the sum character formula which is exactly the same as the classical formula for specializable weights.

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We conjecture that the quantum Jantzen filtration specializes to the classical one which would establish Drinfeld's conjecture.

When $\mathfrak{g}$ is semisimple, U admits a separation of variables theorem [J-L2]. Rosso's form plays an important role in describing the $\operatorname{ad} U$ module structure of a certain associated graded algebra. By exploiting a deep orthonormality property of the Kashiwara form, relative to the crystal basis and the positivity of the Cartan matrix, we show that R restricted to certain subspaces of U is nondegenerate (Corollary 7.4). This result allows us to identify subspaces of U which map isomorphically to the endomorphism rings of the simple highest weight modules (Corollary 8.2). We also use (Lemma 8.5) the non-degeneracy of the restricted form to define the "harmonic elements" which are analogs to the harmonic elements for the classical enveloping algebra.

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2 Background and definitions

Our notation is based on [J-L1-3]; *but will be redefined as necessary.*

Let k denote a field of characteristic zero and set $K = k(q)$ where q is an indeterminate. Let C denote a symmetrizable generalized Cartan matrix, and let $\pi = \{\alpha_1, \ldots, \alpha_l\}$ denote the corresponding set of simple roots. We denote by $\mathfrak{g}$ the Kac–Moody algebra associated to C with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \otimes \mathfrak{h} \otimes \mathfrak{n}^+$. Let Δ^+ be its set of positive roots, $Q(\pi) = \sum_i \mathbb{Z}\alpha_i$ the root lattice, and $Q^+(\pi) = \sum_i \mathbb{N}\alpha_i$. Order $\mathfrak{h}^*$ by $\lambda \geqq \gamma$ if $\lambda - \gamma$ can be written as a sum of simple roots with non-negative coefficients. Note that $\dim \mathfrak{h} = 2l - n$ where $n = \operatorname{rank} C$. Complete π to a basis $\{\alpha_1, \ldots, \alpha_{2l-n}\}$ of $\mathfrak{h}^*$ as in [K, Sect. 1]. By [K, 2.1], there is a nondegenerate inner product (,) on $\mathfrak{h}^*$, so that $C_{ij} = (\alpha_i^\vee, \alpha_j)$ where $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Choose $\omega_i : i = 1, 2, \ldots, \ell$, in $\mathfrak{h}^*$ so that $(\alpha_i^\vee, \omega_j) = \delta_{ij}$ and let ρ denote their sum. Let $P(\pi) = \sum_i \mathbb{Z}\omega_i$ be the weight lattice.

The Hopf algebra $U = U_q(\mathfrak{g})$ is generated over K by elements $x_i, y_i, 1 \leqq i \leqq l$ and t_i, t_i^{-1}, $1 \leqq i \leqq 2l - n$ subject to

(2.1) $$t_i x_j t_i^{-1} = q^{(\alpha_i, \alpha_j)} x_j; \qquad t_i y_j t_i^{-1} = q^{-(\alpha_i, \alpha_j)} y_j$$

(2.2) $$x_i y_j - y_j x_i = \delta_{ij} \frac{(t_i^2 - t_i^{-2})}{q^{(\alpha_i, \alpha_i)} - q^{-(\alpha_i, \alpha_i)}}$$

(2.3) *The* t_i *commute.*

(2.4) *The* $x_i, 1 \leqq i \leqq l$ *(resp.* $y_i, 1 \leqq i \leqq l$*) satisfy the quantized Serre relations* (*see for example* [J-L1, Sect. 4].)

The free abelian multiplicative group of rank l generated by $t_1, \ldots, t_{2l-n}$ is denoted by T. The algebra U is a Hopf algebra with comultiplication Δ, counit ε, and antipode σ given by

(2.5) $$\Delta(t_i) = t_i \otimes t_i; \quad \Delta(t_i^{-1}) = t_i^{-1} \otimes t_i^{-1}$$

(2.6) $$\Delta(x_i) = x_i \otimes t_i^{-1} + t_i \otimes x_i; \quad \Delta(y_i) = y_i \otimes t_i^{-1} + t_i \otimes y_i$$

(2.7) $$\varepsilon(x_i) = \varepsilon(y_i) = 0; \quad \varepsilon(t_i) = \varepsilon(t_i^{-1}) = 1$$

(2.8) $$\sigma(t_i) = t_i^{-1}; \quad \sigma(t_i^{-1}) = t_i; \quad \sigma(x_i) = -q^{-(\alpha_i,\alpha_i)}x_i; \quad \sigma(y_i) = -q^{(\alpha_i,\alpha_i)}y_i$$

For a typical element $b \in U$, we use the Sweedler notation $\Delta(b) = b_{(1)} \otimes b_{(2)}$. Using the Hopf algebra structure, one can define an adjoint action, denoted by ad. (See for example [J-L1, Sect. 2].) The adjoint action of the generators on a typical element $b \in U$ is given by

(2.9) $$(\operatorname{ad} x_i)b = x_i b t_i - q^{-(\alpha_i,\alpha_i)} t_i b x_i$$

(2.10) $$(\operatorname{ad} y_i)b = y_i b t_i - q^{(\alpha_i,\alpha_i)} t_i b y_i$$

(2.11) $$(\operatorname{ad} t_i)b = t_i b t_i^{-1}; \qquad (ad t_i^{-1})b = t_i^{-1} b t_i$$

Let M be a U-module. A weight Λ of M is an element of the character group T^* of T for which there exists some non-zero $m \in M$ with $t_i m = \Lambda(t_i)m$ for $1 \leqq i \leqq 2l - n$. The corresponding space of weight vectors (or weight space, for short) is $M_\Lambda = \{m \in M | t_i m = \Lambda(t_i)m, 1 \leqq i \leqq 2l - n\}$. A weight Λ is said to be linear if there exists $\lambda \in \mathfrak{h}^*$ such that $(\lambda, \alpha_i) \in \mathbb{Z}$ and $\Lambda(t_i) = q^{(\lambda,\alpha_i)}$ for each i. In this case, we write $\Lambda = q^\lambda$ and set $M_\Lambda = M_\lambda$. Let U_λ denote the λ weight space of U considered as a U module via the adjoint action. If $a \in U$ is a weight vector for the adjoint action, then we write *wt a* for the weight of a.

Define a group isomorphism from the additive group $Q(\pi)$ to the multiplicative group T by $\tau(\alpha_i) = t_i$. Note that if $a \in U$ is a weight vector of weight q^λ for the adjoint action, then $\tau(\beta)a\tau(\beta)^{-1} = q^{(\lambda,\beta)}a$. It is sometimes necessary to replace K by its algebraic closure $\bar{K}$ as in Sect. 5 and to extend τ to $Q(\pi) + P(\pi)$ as in Sect. 8. This only modifies earlier sections in a trivial fashion.

Let κ denote the Chevellay antiautomorphism of U over K defined by

(2.12) $$\kappa(x_i) = y_i, \qquad \kappa(y_i) = x_i, \qquad \kappa(t_i) = t_i\,.$$

3 Kashiwara's bilinear form and related forms

Set $e_i = x_i t_i$ (resp. $f_i = t_i y_i$) and let G^+ (resp. G^-) be the subalgebra of U over K generated by e_i (resp. f_i), $1 \leqq i \leqq l$. In this section, we consider the bilinear form on G^- introduced by Kashiwara, and a modification of it.

Recall (2.9), the action of $\operatorname{ad} x_i$ on an element $b \in U$. One checks from relation (2.2) that

$$(\operatorname{ad} x_i) f_j = \delta_{ij} \left(\frac{t_i^4 - 1}{q^{2(\alpha_i,\alpha_i)} - 1} \right)$$

By induction on weight, it follows that $(\operatorname{ad} x_i)G^- \subseteq G^- t_i^4 + G^-$. Then following Kashiwara [Ka, Sect. 3] we may define linear maps x_i' and x_i'' on G^- by

$$(\operatorname{ad} x_i t_i^{-1})b = \frac{(t_i^2 x_i''(b) - t_i^{-2} x_i'(b))t_i^2}{q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}}$$

Note that $x_i''(f_j) = \delta_{ij} = x_i'(f_j)$. (We use "$x_i t_i^{-1}$" instead of "$x_i$" because Kashiwara uses a different set of generators for U.) By [Ka, 3.4.5], they satisfy

(3.1) $$x_i' x_j'' = q^{2(\alpha_i,\alpha_j)} x_j'' x_i' .$$

Kashiwara's form [Ka, 3.4.4] is the unique nondegenerate symmetric bilinear form $\langle , \rangle$ on G^- defined by

(3.2) $$\langle f_i f, g\rangle = \langle f, x_i'(g)\rangle$$

$$\langle 1, 1\rangle = 1$$

for $f, g \in G^-$ and $1 \leqq i \leqq l$. Note that if a and b are weight vectors in G^-, then

(3.3) $$\langle a, b\rangle = 0 \quad \text{unless } \operatorname{wt} a = \operatorname{wt} b .$$

After Kashiwara [Ka, 5.2.2 (i)] one has

Lemma 3.1 *If f is a vector in G^- of weight $-\beta$, and $g \in G^-$, then*

$$\langle f f_i, g\rangle = q^{-2(\beta,\alpha_i)} \langle f, x_i''(g)\rangle .$$

For us x_i' and x_i'' are not the most convenient maps. Thus, we define new endomorphisms e_i'' and e_i' of G^- by

(3.4) $$(\operatorname{ad} x_i)b = e_i''(b)t_i^4 - e_i'(b)$$

for $b \in G^-$.

Now assume that b is a vector in G^- of weight $-\beta$. Then

$$(\operatorname{ad} x_i)b = q^{-(\alpha_i,\beta)}(\operatorname{ad} x_i t_i^{-1})b = q^{-(\alpha_i,\beta)} \frac{(t_i^2 x_i''(b) - t_i^{-2} x_i'(b))t_i^2}{q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}}$$

Hence

(3.5) $$e_i'(b) = \frac{q^{(\alpha_i,\beta-\alpha_i)} x_i'(b)}{q^{2(\alpha_i,\alpha_i)} - 1} .$$

One can similarly relate e_i'' and x_i''.

Set $A(\beta) = \prod_i (1 - q^{2(\alpha_i,\alpha_i)})^{k_i}$ where $\beta = \sum_i k_i \alpha_i \in Q^+(\pi)$. We introduce a new bilinear form which we call the adjusted Kashiwara form on G^- by setting

(3.6) $$\langle a, b\rangle' = \frac{q^{(\rho,\beta)}}{A(\beta)} \langle a, b\rangle$$

where $b \in G^-$ is a vector of weight $-\beta$. The following lemma describes some of this form's properties.

Lemma 3.2 *Let b be a vector of weight $-\beta$ in G^- and a, a vector of weight $-\beta-\alpha$. Then for each $1 \leqq i \leqq l$*

(i) $q^{(\alpha_i-\beta,\alpha_i)}\langle bf_i, a\rangle' + \langle b, e_i''(a)\rangle' = 0$

(ii) $q^{(\alpha_i+\beta,\alpha_i)}\langle f_i b, a\rangle' + \langle b, e_i'(a)\rangle' = 0$

Proof. Fix i and set $\alpha = \alpha_i$, $f = f_i$, $e'' = e_i''$, and $e' = e_i'$. Using (3.6), we have

$$q^{(\alpha+\beta,\alpha)}\langle fb, a\rangle' = \frac{q^{(\alpha+\beta,\alpha)+(\rho,\beta+\alpha)}}{A(\beta+\alpha)}\langle fb, a\rangle .$$

By (3.2) and (3.5), the second expression above equals

$$\begin{aligned}\frac{q^{(\alpha+\beta,\alpha)+(\rho,\beta+\alpha)}}{A(\beta+\alpha)}\langle b, x'(a)\rangle &= q^{(\alpha+\beta,\alpha)+(\rho,\beta+\alpha)-(\alpha,\beta-\alpha)}\frac{(-A(\alpha))}{A(\beta+\alpha)}\langle b, e'(a)\rangle \\ &= \frac{-q^{(\rho,\beta)}}{A(\beta)}\langle b, e'(a)\rangle .\end{aligned}$$

Assertion (ii) now follows from (3.6). Assertion (i) is proved in a similar manner using Lemma 3.1. □

Remark. Clearly either (i) or (ii) and the normalization $\langle 1,1\rangle' = 1$ determine this form uniquely.

4 Bilinear forms on U

Rosso [R, Sect. A.II] introduced an ad-invariant nondegenerate bilinear form on U for $\mathfrak{g}$ semisimple compatible with the triangular decomposition of U. We investigate necessary and sufficient conditions for a bilinear form on U for any Kac Moody algebra $\mathfrak{g}$ to satisfy these properties. More precisely, recall the definition of G^- and G^+ (Sect. 3). There is an isomorphism of vector spaces

$$U \cong G^- \otimes_K KT \otimes_K G^+$$

via the multiplication map $a\otimes b\otimes c \mapsto abc$ by [R, Sect. A.II Remarque following Définition 5]. Let B be a non-zero bilinear form on U. We say that B is *triangular* provided

$$(4.1) \qquad B(a^-\tau(\lambda)a^+, b^-\tau(\gamma)b^+) = B(a^+, b^-)B(\tau(\lambda), \tau(\gamma))B(a^-, b^+)$$

for all $a^-, b^- \in G^-$; $a^+, b^+ \in G^+$; and $\tau(\lambda), \tau(\gamma) \in T$.

The form B is *ad-invariant* with respect to a set S if

$$(4.2) \qquad B((\operatorname{ad} a_{(1)})b, (\operatorname{ad} a_{(2)})c) = \varepsilon(a)B(b,c)$$

for all $a \in S$ with $\Delta(a) = a_{(1)} \otimes a_{(2)}$, and all $b, c \in U$.

Let U^- (resp. U^+) be the subalgebra of U generated by the $y_i, 1 \leqq i \leqq l$ (resp. $x_i, 1 \leqq i \leqq l$). If B is ad-invariant with respect to T (resp. U^-, U^+, a one element set $\{b\}$), we say that B is $\operatorname{ad} T$ (resp. $\operatorname{ad} U^-$, $\operatorname{ad} U^+$, $\operatorname{ad} b$) invariant.

Note that 1 is an element of G^-, T, and G^+. Hence if B is a triangular, then

$$B(1,1) = \pm 1 . \tag{4.3}$$

We shall take $B(1,1) = 1$. If B is $\operatorname{ad} t$ invariant, then

$$B((\operatorname{ad} t)b, (\operatorname{ad} t)c) = B(b,c) . \tag{4.4}$$

In particular, if b and c are weight vectors, then (4.4) implies that $B(b,c) = 0$ unless $\operatorname{wt} b = -\operatorname{wt} c$. If in addition, B is nondegenerate, then it must be nondegenerate on $U_{-\beta} \times U_\beta$ for all $\beta \in Q^+(\pi)$. Combining this observation with property (4.1) yields:

Lemma 4.1 *Let B be a triangular form on U nondegenerate on $G^- \times G^+$ and $G^+ \times G^-$. Then B is* $\operatorname{ad} T$ *invariant if and only if*

(i) $B(G^-_{-\mu}, G^+_\gamma) = B(G^+_\gamma, G^-_{-\mu}) = 0$ *for* $\mu \neq \gamma$

(ii) *B is nondegenerate when restricted to $T \times T$, $G^-_{-\beta} \times G^+_\beta$, and $G^+_\beta \times G^-_{-\beta}$ for all β in $Q^+(\pi)$.*

For Lemmas 4.2–4.7, assume that B is a triangular nondegenerate bilinear form on $G^- \times G^+$ and $G^+ \times G^-$ which is $\operatorname{ad} T$ invariant. We investigate necessary and sufficient conditions for B to be both $\operatorname{ad} U^+$ and $\operatorname{ad} U^-$ invariant. Most of the arguments involve manipulations of x_i, y_i, t_i, etc. To make it easier to read, we set $x = x_i$, $\alpha = \alpha_i$, $t = t_i$, $e = e_i$, $f = f_i$, $e' = e'_i$, and $e'' = e''_i$, with the understanding that the arguments work for any choice of i between 1 and l.

Lemma 4.2 *If B is* $\operatorname{ad} x$ *invariant, then*

$$B(t^{-4m}, t^{4s}) = q^{4ms(\alpha,\alpha)} = B(t^{4s}, t^{-4m})$$

where $m, s \in \mathbb{Z}$ and $m \geqq 0$.

Proof. Assume B is $\operatorname{ad} x$ invariant. Fix a nonnegative integer m. By [J-LI, 3.11], $t^{-4m}e^{2m}$ is a highest weight vector for the $\operatorname{ad} x$ action. Recall (2.7) that $\varepsilon(x) = 0$. By the definition of $\operatorname{ad} x$ (2.9), and ad-invariance (4.2), we have

$$B((\operatorname{ad} x)t^{-4m}e^{2m}, (\operatorname{ad} t^{-1})f^{2m+1}t^{4s}) + B((\operatorname{ad} t)t^{-4m}e^{2m}, (ad\, x)f^{2m+1}t^{4s}) = 0 .$$

Since $(\operatorname{ad} x)t^{-4m}e^{2m} = 0$, the first summand is zero. Now taking $q_{(\alpha,\alpha)} = q^{(\alpha,\alpha)/2}$ one has

$$(\operatorname{ad} x)f^{2m+1}t^{4s} = a_1 f^{2m}(t^4 - q^{8m}_{(\alpha,\alpha)})t^{4s} + a_2 f^{2m+1}t^{4s}e$$

where a_1 and a_2 are nonzero elements of K. Hence, using (4.1),

$$B(e^{2m}, f^{2m})B(t^{-4m}, (t^4 - q^{8m}_{(\alpha,\alpha)})t^{4s}) = 0 .$$

By Lemma 4.1, $B(e^{2m}, f^{2m}) \neq 0$. Thus $B(t^{-4m}, (t^4 - q^{8m}_{(\alpha,\alpha)})t^{4s}) = 0$, so

$$B(t^{-4m}, t^{4s+4}) = -q^{8m}_{(\alpha,\alpha)} B(t^{-4m}, t^{4s}) \tag{4.5}$$

for all nonnegative integer m. A similar argument shows that

$$B(t^{4s+4}, t^{-4m}) = -q^{8m}_{(\alpha,\alpha)} B(t^{4s}, t^{-4m}) \tag{4.6}$$

for all integer $m \geqq 0$. Recall that $B(1,1) = 1$. By using induction on s and $-s$ in (4.5) and (4.6) the lemma follows. □

In order to understand $\operatorname{ad} U^+$ invariance, we need some information about the action of $\operatorname{ad} U^+$ on U. If b is a weight vector in G^+ of weight β and $\tau(\lambda) \in T$, then

$$(*) \quad (\operatorname{ad} x)\tau(\lambda)b = x\tau(\lambda)bt - q^{-(\alpha,\alpha)} t\tau(\lambda)bx = \tau(\lambda)[q^{-(\lambda,\alpha)-(\beta,\alpha)}eb - q^{(\beta,\alpha)}be] .$$

If a is a vector in G^-, then recall (3.4):

$$(\operatorname{ad} x)a = e''(a)t^4 - e'(a) .$$

Lemma 4.3 *If B is* $\operatorname{ad} x$ *invariant, then*

(i) $q^{(\alpha-\beta,\alpha)} B(eb^+, b^-) + B(b^+, e''(b^-)) = 0$

(ii) $q^{(\alpha+\beta,\alpha)} B(b^+e, b^-) + B(b^+, e'(b^-)) = 0$

(iii) $q^{-(\alpha+\beta,\alpha)} B(b^-, eb^+) + B(e''(b^-), b^+) = 0$

(iv) $q^{(\beta-\alpha,\alpha)} B(b^-, b^+e) + B(e'(b^-), b^+) = 0$ *where* $b^+ \in G^+, b^- \in G^-$, $\operatorname{wt} b^+ = \beta$, $\operatorname{wt} b^- = -\beta - \alpha$.

Proof. Fix $m < 0$. The $\operatorname{ad} x$ invariance of B implies that

$$B((\operatorname{ad} x)t^{-4m}b^+, (\operatorname{ad} t^{-1})b^-) + B((\operatorname{ad} t)t^{-4m}b^+, (\operatorname{ad} x)b^-) = 0 .$$

By $(*)$ and triangular decomposition, we have

$$\begin{aligned} &B(q^{(4m\alpha-\beta,\alpha)}eb^+, q^{(\alpha,\beta+\alpha)}b^-)B(t^{-4m}, 1) - B(q^{(\beta,\alpha)}b^+e, q^{(\alpha,\beta+\alpha)}b^-)B(t^{-4m}, 1) \\ &\quad + B(q^{(\beta,\alpha)}b^+, e''(b^-))B(t^{-4m}, t^4) - B(q^{(\beta,\alpha)}b^+, e'(b^-))B(t^{-4m}, 1) = 0 . \end{aligned}$$

By Lemma 4.1, the expression above simplifies to

$$\begin{aligned} &q^{4m(\alpha,\alpha)+(\alpha,\alpha)} B(eb^+, b^-) - q^{2(\beta,\alpha)+(\alpha,\alpha)} B(b^+e, b^-) \\ &\quad + q^{4m(\alpha,\alpha)+(\beta,\alpha)} B(b^+, e''(b^-)) - q^{(\beta,\alpha)} B(b^+, e'(b^-)) = 0 \end{aligned} \tag{4.7}$$

Note that (4.7) holds for every $m > 0$. Hence, letting m vary in (4.7) yields (i) and (ii). A similar argument in which the order of $t^{-4m}b^+$ and b^- in $B(\ ,\ )$ is switched proves (iii) and (iv). □

Note the similarity in Lemma 4.3 between (i) and (ii) and also between (ii) and (iv). This suggests that B is almost symmetric. More precisely, we have

Lemma 4.4 *If B is* ad U^+ *invariant, then*

$$B(b^-, b^+) = q^{(\rho,\beta)} B(b^+, b^-)$$

where $b^+ \in G^+$, $b^- \in G^-$, wt $b^+ = \beta = -$wt b^-.

Proof. The lemma follows using induction on β as in the proof of the similar statement in [R1, Sect. 2 Theorem 6]. □

Let T_4 be the subgroup of T generated by the $t_i^4 : i = 1, 2, \ldots, \ell$.

Lemma 4.5 *If B is* ad U^+ *invariant, then*

$$B(\tau(\lambda), \tau(\gamma)) = q^{-1/4(\lambda,\gamma)} = B(\tau(\gamma), \tau(\lambda))$$

for all $\tau(\lambda)$, $\tau(\gamma) \in T_4$.

Proof. By Lemma 4.1, there exist $b^+ \in G^+$ and $b^- \in G^-$ with wt $b^+ = \beta$ and wt $b^- = -\beta - \alpha$ such that $B(eb^+, b^-) \neq 0$. By ad x invariance

$$B((\operatorname{ad} x)\tau(\lambda)b^+, (\operatorname{ad} t^{-1})b^-\tau(\gamma)) + B((\operatorname{ad} t)\tau(\lambda)b^+, (\operatorname{ad} x)b^-\tau(\gamma)) = 0\,.$$

Note that $B(\tau(\lambda)b^+, b^-\tau(\gamma)e) = 0$ by triangular decomposition and Lemma 4.1. Hence using (4.1) as in Lemma 4.3, it follows that

$$\begin{aligned} &q^{(-\lambda+\alpha,\alpha)} B(eb^+, b^-) B(\tau(\lambda), \tau(\gamma)) - q^{(2\beta+\alpha,\alpha)} B(b^+e, b^-) B(\tau(\lambda), \tau(\gamma)) \\ &\quad + q^{(\beta,\alpha)} B(b^+, e''(b^-)) B(\tau(\lambda), t^4\tau(\gamma)) - q^{(\beta,\alpha)} B(b^+, e'(b^-)) B(\tau(\lambda), \tau(\gamma)) = 0 \end{aligned}$$

Lemma 4.3 (i) and (ii) imply that

$$B(eb^+, b^-)[q^{-(\lambda,\alpha)} B(\tau(\lambda), \tau(\gamma)) - B(\tau(\lambda), t^4\tau(\gamma))] = 0\,.$$

By assumption, $B(eb^+, b^-) \neq 0$ so the square bracket term vanishes. A similar argument which switches the order of $\tau(\lambda)b^+$ and $b^-\tau(\gamma)$ in $B(\,,\,)$ and uses Lemma 4.3 (iii) and (iv) proves the corresponding result for the first entry. An easy induction argument completes the proof. □

We are now ready to give necessary and sufficient conditions for B to be ad U^+ invariant.

Lemma 4.6 *B is* ad U^+ *invariant if and only if*

(i) $q^{(\alpha_i-\beta,\alpha_i)} B(e_i(b^+), b^-) + B(b^+, e_i''(b^-)) = 0$,

(ii) $q^{(\alpha_i+\beta,\alpha_i)} B(b^+(e_i), b^-) + B(b^+, e_i'(b^-)) = 0$,

(iii) $B(a^-, b^+) = q^{(\rho,\beta)} B(b^+, a^-)$,

(iv) $B(\tau(\lambda), \tau(\gamma)) = q^{-\frac{1}{4}(\lambda,\gamma)}$ *for all* $1 \leqq i \leqq l$, $b^+ \in G^+, a^-, b^- \in G^-$ *with* wt $b^+ = -$wt $a^- = -$wt $b^- - \alpha = \beta$, *and* $\tau(\lambda), \tau(\gamma) \in T_4$.

Proof. In view of Lemmas 4.3, 4.4, and 4.5 it remains to show that (i)–(iv) imply that

$$B((\mathrm{ad}\,x)c^-\tau(\lambda)b^+,(\mathrm{ad}\,t^{-1})b^-\tau(\gamma)c^+) + B((\mathrm{ad}\,t)c^-\tau(\lambda)b^+,(\mathrm{ad}\,x)b^-\tau(\gamma)c^+) = 0 \tag{4.8}$$

equals zero. In (4.8) we may assume that c^-, c^+, b^-, and b^+ are weight vectors. By Lemma 4.1, and triangular decomposition, both summands are zero unless

$$-\mathrm{wt}\,c^- = \mathrm{wt}\,c^+ \quad \text{and} \quad \mathrm{wt}\,b^+ = -\mathrm{wt}\,b^- - \alpha \tag{4.9}$$

or

$$-\mathrm{wt}\,c^- + \alpha = \mathrm{wt}\,c^+ \quad \text{and} \quad \mathrm{wt}\,b^+ = -\mathrm{wt}\,b^- . \tag{4.10}$$

Assume that (4.9) holds with $\mathrm{wt}\,b^+ = \beta$ and $\mathrm{wt}\,b^- = -\beta - \alpha$. Then (4.8) equals

$$\begin{aligned}[(q^{(\alpha-\lambda,\alpha)}&B(eb^+,b^-) - q^{(\alpha+2\beta,\alpha)}B(b^+e,b^-))B(\tau(\lambda),\tau(\gamma))\\ &+ q^{(\beta,\alpha)}B(b^+,e''(b^-))B(\tau(\lambda),t^4\tau(\gamma))\\ &- q^{(\beta,\alpha)}B(b^+,e'(b^-))B(\tau(\lambda),\tau(\gamma))]B((\mathrm{ad}\,t)c^-,(\mathrm{ad}\,t^{-1})c^+)\end{aligned}$$

which vanishes by (i), (ii), and (iv). A similar argument can be made under assumption (4.10). □

Define $f', f'' \in \mathrm{End}\,G^+$ through

$$(\mathrm{ad}\,y)b = t^4 f''(b) - f'(b)$$

for $b \in G^+$.

A straightforward calculation shows that for all $b \in U$,

$$\kappa((\mathrm{ad}\,x)b) = -q^{-(\alpha,\alpha)}(\mathrm{ad}\,y)\kappa(b) \tag{4.11}$$

Hence for $b \in G^-$ we have

$$f''(\kappa(b)) = -q^{-(\alpha,\alpha)}\kappa(e''(b))$$

and

$$f'(\kappa(b)) = -q^{-(\alpha,\alpha)}\kappa(e'(b))$$

Observe that $U_4 := G^-T_4G^+$ is an ad U invariant subalgebra of U.

Lemma 4.7 *Assume B is* ad U^+ *invariant. Then B is* ad U^- *invariant if*

$$B(\kappa(a),\kappa(b)) = B(b,a) \tag{4.12}$$

for all a, $b \in U_4$.

Proof. By (4.11) and (4.12) the condition for ad U^+ invariance translates to the condition for ad U^- invariance. □

Theorem 4.8 *Up to a sign there is a unique triangular* ad U^+T *invariant bilinear form* B *on* U_4 *which is nondegenerate on* $G^+ \times G^-$ *and* $G^- \times G^+$. *Moreover*

(4.14) $$B(a^-\tau(\lambda)a^+, b^-\tau(\gamma)b^+) = q^{(\rho, \mathrm{wt}\; b^+)}\langle\kappa(a^+), b^-\rangle' q^{-1/4(\lambda,\gamma)}\langle a^-, \kappa(b^+)\rangle'$$

for weight vectors $a^-, b^- \in G^+$ *and* $a^+, b^+ \in G^+$, *and* $\tau(\lambda), \tau(\gamma) \in T_4$. *Moreover* B *satisfies* (4.12) *and is* ad-*invariant.*

Proof. For uniqueness it suffices to show that B is unique when restricted to $G^+ \times G^-$, $G^- \times G^+$, and $T_4 \times T_4$. By Lemma 4.5:

(4.15) $$B(\tau(\lambda), \tau(\gamma)) = q^{-1/4(\lambda,\gamma)} = B(\tau(\gamma), \tau(\lambda))$$

for all $\tau(\lambda), \tau(\gamma) \in T_4$.

By Lemma 4.6(iii), B restricted to $G^- \times G^+$ is the same as B evaluated on $G^+ \times G^-$ up to a power of q. Thus it remains to note that Lemma 4.6(i) completely determines the evaluation of B on $G^+ \times G^-$ by an easy induction on weights.

Now consider the form B defined in (4.14). This form is triangular, and is nondegenerate on $G^+ \times G^-$ and $G^- \times G^+$ since the adjusted Kashiwara form $\langle\,,\,\rangle'$ is nondegenerate. Moreover it is clear that B is ad T invariant.

To show that B is ad U^+ invariant, we verify conditions (i)–(iv) of Lemma 4.6. Condition (iv) is immediate. By (4.14) and the symmetry of the adjusted Kashiwara form,

$$\begin{aligned} B(a^-, b^+) &= q^{(\rho, \mathrm{wt}\, b^+)}\langle a^-, \kappa(b^+)\rangle' = q^{(\rho, \mathrm{wt}\, b^+)}\langle\kappa(b^+), a^-\rangle' \\ &= q^{(\rho, \mathrm{wt}\, b^+)} B(b^+, a^-)\,. \end{aligned}$$

Hence (iii). By Lemma 3.2(i), we have

$$\begin{aligned} B(e_i b^+, b^-) &= \langle\kappa(e_i b^+), b^-\rangle' = \langle\kappa(b^+) f_i, b^-\rangle' \\ &= -q^{-(\alpha_i - \beta, \alpha_i)}\langle\kappa(b^+), e_i''(b^-)\rangle' = -q^{-(\alpha_i - \beta, \alpha_i)} B(b^+, e_i''(b^-))\,. \end{aligned}$$

Hence (i). Similarly Lemma 3.2(ii) gives (ii). ·

Now

$$\begin{aligned} B(\kappa(b^+), \kappa(a^-)) &= q^{(\rho, \mathrm{wt}\, \kappa(a^-))}\langle\kappa(b^+), a^-\rangle' \\ &= q^{(\rho, \mathrm{wt}\, \kappa(a^-))}\langle a^-, \kappa(b^+)\rangle' = B(a^-, b^+)\,. \end{aligned}$$

Similarly $B(\kappa(a^-), \kappa(b^+)) = \langle a^-, \kappa(b^+)\rangle' = \langle\kappa(b^+), a^-\rangle' = B(b^+, a^-)$. Obviously B satisfies (4.15) and then by triangularity (4.12) results. Hence by Lemma 4.7, B is ad U^- invariant. We have thus shown that B is ad U invariant which completes the proof. □

Remark. One can check that the bilinear form $a, b \to \langle K(a), b\rangle'$ corresponds to the Hopf algebra pairing between G^+ and G^- up to suitable normalization.

Hence formula (4.14) in Theorem 4.8 extends Rosso's result connecting R and the Hopf pairing in the semisimple case ([R2, Theorem 7]).

If we assume that

$$B(\tau(\lambda), \tau(\gamma)) = q^{-1/4(\lambda,\gamma)} \tag{4.16}$$

for all $\tau(\lambda), \tau(\gamma) \in T$, then we obtain a unique form on U that generalizes Rosso's form to the Kac–Moody case. Furthermore, the nondegeneracy of $(\ ,\ )$ implies that this form is nondegenerate on T and hence on U. In general, there is more than one choice of bilinear form B on all of U which satisfies the conditions of Theorem 4.8. Let $\bar{K}$ denote the algebraic closure of K, and let $s_1 = 1, s_2, \ldots, s_r$ be a set of coset representatives for $K[t_1, \ldots, t_l]/T_4$. Since each $G^- s_i T_4 G^+$ is ad-invariant it follows that B may be modified on each such summand. Hence the

Theorem 4.9 *For each* $\{a_{ij} | 1 \leqq i, j \leqq r\}$ *with* $a_{ij} \in \bar{K}$ *and* $a_{11} = 1$, *there exists a unique triangular* ad-*invariant bilinear form* B *on* $G^- K[t_1, \ldots, t_l] G^+$ *nondegenerate on* $G^- \times G^+$ *and* $G^+ \times G^-$ *such that* $B(s_i, s_j) = a_{ij}$ *for all* $1 \leqq i, j \leqq r$.

For the remainder of this paper, we denote by R the unique nondegenerate ad-invariant bilinear form on U which satisfies (4.16). As in Theorem 4.8, we have

$$R(a^- \tau(\lambda) a^+, b^- \tau(\gamma) b^+) = q^{(\rho, \mathrm{wt}\, b^+)} \langle \kappa(a^+), b^- \rangle' q^{-1/4(\lambda,\gamma)} \langle a^-, \kappa(b^+) \rangle' \tag{4.17}$$

for weight vectors $a^-, b^- \in G^-$, and $a^+, b^+ \in G^+, \tau(\lambda), \tau(\gamma) \in T$.

5 The quantum Shapovalov determinant

Shapovalov introduced a remarkable determinant for semisimple $\mathfrak{g}$, whose factorization was completely described in [S]. The factorization was extended to the Kac–Moody case in [K-K] and to U for $\mathfrak{g}$ semisimple, in [D-K]. In this section, we examine the factorization of the quantum Shapovalov determinant associated to U. The argument uses ideas from both [D-K] and [K-K], but there are some differences. We use Rosso's form, instead of a PBW type basis, to compute the highest and lowest degree terms. This is a significant simplification over [D-K]. A new difficulty in the quantum Kac–Moody case is that not all the obvious factors are irreducible. Here we are forced to use specialization.

Set $U_> = \sum_i y_i U + \sum_i U x_i$. As in [J-L1], we have a direct sum decomposition

$$U = K[T] \oplus U_> \, .$$

Let φ denote the projection of U onto the first summand $K[T]$. Given a weight η, choose bases $\{a^i_\eta\}$ (resp. $\{a^i_{-\eta}\}$) for the η (resp. $-\eta$) weight space of U^+ (resp. U^-). Define the quantum Shapovalov determinant corresponding to η by

$$\det_\eta := \det[\varphi(a^i_\eta a^j_{-\eta})] \, .$$

Up to a nonzero scalar in K, it is independent of the choice of basis.

The next three lemmas compute the highest and lowest degree terms of $\det_\eta$. Let $P(\eta)$ denote the generalized Kostant partition function defined by

$$\sum_{\eta\in Q^+(\pi)} P(\eta)e^\eta = \prod_{\beta\in\Delta^+_{\text{mult}}} (1-e^\beta)^{-1} =: D$$

where Δ^+_{mult} denotes the set of positive roots counted with multiplicities.

Lemma 5.1 $P(\eta)\eta = \sum_{i=1}^\infty \sum_{\beta\in\Delta^+_{\text{mult}}} P(\eta - i\beta)\beta$.

Proof. This is an elementary combinatorial result and we indicate the proof for completion. It is clear that we may write

$$\sum_{\eta\in Q^+(\pi)} P(\eta)e^{t\eta} = \prod_{\beta\in\Delta^+_{\text{mult}}} (1-e^{t\beta})^{-1}$$

where t may be considered as a free parameter. Differentiation with respect to t and setting $t=1$ gives

$$\sum_{\eta\in Q^+(\pi)} \eta P(\eta)e^\eta = \sum_{\beta\in\Delta^+_{\text{mult}}} \frac{\beta e^\beta}{(1-e^\beta)}D = \sum_{\beta\in\Delta^+_{\text{mult}}} \sum_{\eta\in Q^+(\pi)} \sum_{i=1}^{\infty} \beta P(\eta)e^{i\beta+\eta}\,.$$

Then the assertion results on equating coefficients of e^η.

The next lemma is about a property of Rosso's form which will be used to determine the highest and lowest degree terms of $\det_\eta$. Given $r\in K[t_1^4,\dots,t_l^4]$, let $r\|_0$ denote the evaluation of r at $t_1^4=\dots=t_l^4=0$. The defining relation (2.2) implies that $\varphi(e_if_j)\in K[t_1^4,\dots,t_l^4]$ for each $1\leqq i,j\leqq l$. By induction on weight, $\varphi(ab)\in K[t_1^4,\dots,t_l^4]$ for each $a\in G^+$ and $b\in G^-$ and hence it makes sense to consider $\varphi(ab)\|_0$.

Lemma 5.2 *For all $a\in G^+$ and $b\in G^-$ we have*

$$\varphi(ab)\|_0 = R(a,b)\,. \tag{5.1}$$

Proof. By definition $R(a,b)=\langle\kappa(a),b\rangle$ for all $a\in G^+$, $b\in G^-$, so by the remark following Lemma 3.2, it is enough to show that $\varphi(ab)\|_0 = -q^{-(\gamma,\alpha)}\varphi(ce'(b))$ with $ce=a$ and $\gamma=\alpha+\text{wt}(c)$.

Recall that $e=xt$ and $(\operatorname{ad}x)b = xbt - q^{-(\alpha,\alpha)}tbx$. It follows that

$$\varphi(ceb) = q^{-(\alpha,\gamma)}\varphi(c(\operatorname{ad}x)b)$$

which equals $q^{-(\gamma,\alpha)}\varphi(ce''(b)t^4 - ce'(b))$. Since $\varphi(ce''(b)t^4)\|_0 = \varphi(ce''(b)\|_0 t^4\|_0 = 0$, we have

$$\varphi(ab)\|_0 = -q^{-(\alpha,\gamma)}\varphi(ce'(b))\|_0$$

as required. □

We now compute the highest and lowest degree terms of $\det_\eta$. Here, we take the usual degree function on $K[T]$: the degree of t_i is 1 for each $1\leqq i\leqq l$.

For each $\mu \in Q^+(\pi)$ set

$$h_{i,\mu} = \frac{t_i^2 q^{-2(\mu,\alpha_i)} - t_i^{-2} q^{2(\mu,\alpha_i)}}{q^{(\alpha_i,\alpha_i)} - q^{-(\alpha_i,\alpha_i)}} .$$

Set $h_{i,0} = h_i$. Take $h_{i,\mu} = 1$ if $\mu < 0$.

Lemma 5.3 *The highest degree term of* $\det_\eta$ *is*

(i) $\prod_{i \geqq 1} \prod_{\beta \in \Delta^+} \tau(2\beta)^{P(\eta - i\beta)}$ *and the lowest degree term is*

(ii) $\prod_{i \geqq 1} \prod_{\beta \in \Delta^+} \tau(-2\beta)^{P(\eta - i\beta)}$.

(iii) $\tau(P(\eta)2\eta)\det_\eta \in KT_4$ *and has a nonzero constant term.*

(iv) $\det_\eta$ *is a polynomial in the* $h_{i,\mu} : \mu \in Q^+(\pi),\ \mu < \eta$.

Proof. By [J-L1], the dimension of the η (resp. $-\eta$) weight space of U^+ (resp. U^-) is the same as that of the η (resp. $-\eta$) weight space of $U(\mathfrak{n}^+)$ (resp. $U(\mathfrak{n}^-$.) In particular, $\dim_K U^+_\eta = P(\eta) = \dim_K U^-_{-\eta}$. Therefore, $\det_\eta \tau(P(\eta)2\eta) = \det[\varphi(a^i_\eta a^j_{-\eta})\tau(2\eta)]$.

Note that $a^i_\eta \tau(\eta) \in G^+$ and $a^j_{-\eta}\tau(\eta) \in G^-$, hence $\varphi(a^i_\eta a^j_{-\eta})\tau(2\eta) \in KT_4$ for all i,j. This establishes the first part of (iii).

By Lemma 5.2,

$$\begin{aligned}\det_\eta \tau(P(\eta)2\eta)\|_0 &= \det[\varphi(a^i_\eta a^j_{-\eta})\tau(2\eta)]\|_0 = \det[q^{(\eta,\eta)}\varphi(a^i_\eta \tau(\eta) a^j_{-\eta}\tau(\eta))\|_0] \\ &= \det[q^{(\eta,\eta)} R(a^i_\eta \tau(\eta), a^j_{-\eta}\tau(\eta))] .\end{aligned}$$

which is non-zero by the non-degeneracy of R. Hence (iii).

For (iv) it is enough to establish the corresponding assertion for $\varphi(a_\eta a_{-\eta})$ where a_η (resp. $a_{-\eta}$) is monomial in the x_i (resp. y_i). We can write $a_\eta = a_{\eta-\alpha_i} x_i$ and $a_{-\eta} = y_{-(\eta-\mu_s-\alpha_i)} y_i y_{-\mu_s}$ for each factor of y_i in $a_{-\eta}$. Then

$$\varphi(a_\eta a_{-\eta}) = \sum_s \varphi(a_{\eta-\alpha_i} a_{-(\eta-\mu_s-\alpha_i)}[x_i, y_i] a_{-\mu_s}) .$$

Each term in the above sum takes the form

$$\varphi(a_{\eta-\alpha_i} a_{-\eta-\mu-\alpha_i} h_i a_{-\mu}) = h_{i,\mu}\varphi(a_{\eta-\alpha_i} a_{-(\eta-\alpha_i)})$$

from which the required assertion follows by induction. Finally (i), (ii) follows from (iii), (iv) and Lemma 5.1. $\square$

We now turn our attention to the factors of the Shapovalov determinant. Drinfeld constructed a Casimir operator in [D2]; but this operator is in (the completion of) a larger algebra and was not applicable to non-integrable weights (see Lemma 5.4 below). Let $\hat{A}$ be the completion of U defined as follows. Using the standard partial ordering on $Q^+(\pi)$, set A_η equal to the direct sum of the subspaces $U^-_\gamma K[T] U^+_\gamma, \gamma \geqq \eta$. Define $\hat{A}$ to be the inverse limit of the U/A_γ taken over the partially ordered set $Q^+(\pi)$. We modified Drinfeld's

Casimir to a Casimir semi-invariant Ω in $\hat{A}$ which satisfies

$$t_i^{-2}x_i\Omega = t_i^2\Omega x_i, \qquad \Omega y_i = y_i\Omega t_i^4, \qquad \Omega\tau = \tau\Omega$$

for $1 \leqq i \leqq l$ and for all $\tau \in T$. Since our paper was first communicated (July 1993) Lusztig's book appeared giving this Casimir and so we refer to his book for details [L2, Sect. 6.1]. Whereas we had obtained Ω by a direct calculation, Lusztig obtains it by adapting the following method of Drinfeld [D2, Sect. 2]. Write the R matrix as $\sum r_i'\otimes r_i''$ and set $u = \sum \sigma(r_i'')r_i'$ which is more properly the element of the dual of the Hopf dual of U obtained by applying $\mu(\sigma\otimes 1)$ to the conjugate of R by the flip $\mathbf{t}$. (Here μ is just the multiplication map from $U\otimes U$ to U.) Then [D2, Proposition 2.1] R being invertible implies that u is invertible. Also, $\mathbf{t}R$ commuting with all $\Delta(a)$, $a \in U$, implies that conjugation by u is the square of the antipode. This holds for any Hopf algebra with an R matrix; but in the present case the square of the antipode is also conjugation by $\tau(-4\rho)$. This gives a central element, namely $\tau(4\rho)u$. In order to stay in the completion $\hat{A}$ one is obliged to use a quasi R matrix ([L2, Chapter 4] rather than the true R matrix as in Drinfeld [D2] and this leads only to a semi-invariant. For details see [L2, Sect. 6.1]. It is straightforward to check that

$$\Omega y_{-\beta} = q^{-2(\beta,\beta)+4(\rho,\beta)}y_{-\beta}\Omega\tau(4\beta)$$

for each $y_{-\beta} \in U^-$ of weight $-\beta$ where $\beta \in Q^+(\pi)$.

Fix $\Lambda \in T^*$ and let v be the highest weight generating vector of $M(\Lambda)$. Since $\Omega \in 1 + \sum_{v\in Q(\pi)} U^-_{-v}U^+_v$, it follows that $\Omega w = w$ for any highest weight vector w. Suppose that $b \in U^-$ with weight $-\beta$ is such that bv is another highest weight vector in $M(\Lambda)$. The following lemma is immediate and provides a condition which β must satisfy.

Lemma 5.4 $\Lambda(\tau(-4\beta))q^{-4(\rho,\beta)+2(\beta,\beta)} = 1$. *Furthermore, suppose that each* $\beta \in Q^+(\pi)$ *has the property that* $\Lambda(\tau(-4\beta)) \neq q^{4(\rho,\beta)-2(\beta,\beta)}$. *Then* $M(\Lambda)$ *is simple.*

Given a weight $\Lambda \in T^*$, define a symmetric bilinear form φ_Λ on U^- as in say [J-L1, 5.4]. One has $\varphi_\Lambda(a,b) = \Lambda(\varphi(\kappa(a)b))$. By [J-L1, 5.4], $\ker\varphi_\Lambda$ is the maximal proper submodule of $M(\Lambda)$. Thus $\Lambda(\det_\eta) = 0$ if and only if $M(\Lambda)$ has a highest weight vector of weight $\Lambda q^{-\beta}$ with $\beta \leqq \eta$.

Set $Q^+_{\text{irr}}(\pi) = \{\gamma \in Q^+(\pi) | \gamma = r\beta \,:\, \beta \in Q^+(\pi),\ r \in \mathbb{N}^+ \text{ implies } r = 1\}$. Take $\gamma \in Q^+_{\text{irr}}(\pi)$, $m \in \mathbb{N}^+$ and ζ a $4m^{th}$ root of unity. Then $\tau(\gamma) - \zeta q^{-(\rho,\gamma)+\frac{m}{2}(\gamma,\gamma)}$ is irreducible in $\bar{K}T$ and every irreducible factor of $\tau(4\beta) - q^{-4(\rho,\beta)+2(\beta,\beta)}$: $\beta \in Q^+(\pi)$ takes this form. Let $H_{\gamma,m,\zeta}$ be the hypersurface defined by the above irreducible (Laurent) polynomial.

By Lemma 5.3(iii), $\det_\eta \neq 0$. Let H be an irreducible component of its zero variety. By Lemma 5.4 taking each $\Lambda(t_i) \in \bar{K}^*$ arbitrary and the previous remarks it follows that H lies in the union of the $H_{\gamma,m,\zeta}$ with $m\gamma \leqq \eta$. However this is a finite set and so H must equal one of them. This determines the possible irreducible factors of $\det_\eta$.

By Lemma 5.3(iv), $\det_\eta \in KT_4$ up to units. Since U is defined over $\mathbb{Q}(q)$ we can further assume that $\det_\eta \in \mathbb{Q}(q)T_4$ (up to units). For each $s \in \mathbb{N}^+$, let $\mathscr{P}_s$ denote the set of primitive s roots of unity. We conclude that

Proposition 5.5 *Up to units in $\bar{K}T$ every factor of* $\det_\eta$ *has the form*

$$\prod_{\zeta \in \mathscr{P}_s} (\tau(4\beta) - \zeta q^{-4(\rho,\beta)+2m(\beta,\beta)})$$

for some $\beta \in Q^+_{\rm irr}(\pi)$, $m \in \mathbb{N}^+$ with $m\beta \leqq \eta$ and s a divisor of m.

We can describe the factors in $\det_\eta$ corresponding to $\zeta = 1$ through specialization. Let A denote the localization of $k[q,q^{-1}]$ at the prime ideal $\langle q-1\rangle$. Let $\hat{U}$ denote the A subalgebra of U generated by $x_i, y_i, t_i^{\pm 1}, 1 \leqq i \leqq l$. As in say [J-L2, 6.11],

$$\hat{U} \otimes_A / \langle q-1\rangle \cong U(\mathfrak{g}) \otimes_k k[T] \tag{5.3}$$

where the images of $x_i, y_i, h_i \ : \ i = 1,2,\ldots,\ell$, in the left hand side form the canonical generators of $\mathfrak{g}$ and where the elements in T are central after specialization and satisfy $t_i^4 = 1$.

Set $U_1 = \hat{U} \otimes_A A/\langle q-1\rangle$. We tensor again:

$$U_1 \otimes_{k[T]} k[T]/\langle t_i - 1 | 1 \leqq i \leqq l\rangle \cong U(\mathfrak{g})\,. \tag{5.4}$$

We call the process of tensoring twice using (5.3) and (5.4) the specialization of $\hat{U}$ at $q = 1$.

By Lemma 5.3(iv), $\det_\eta$ specializes to a polynomial $\det_\eta^1$ in the $h_i \ : \ i = 1,2,\ldots,\ell$ (which could be zero). However it is also clearly the corresponding Shapovalov determinant for $U(\mathfrak{g})$ determined in [K-K, Theorem 1] to be given by

$$(\det_\eta^1)(\lambda) = \prod_{m \geqq 1} \prod_{\beta \in \Delta^+_{\rm mult}} (2(\lambda+\rho,\beta) - m(\beta,\beta))^{P(\eta - m\beta)}$$

for all $\lambda \in \mathfrak{h}^*$, up to a non-zero scalar.

For $\zeta \neq 1$, the factors in the conclusion of Proposition 5.5 specialize at $q = 1$ to non-zero scalars. When $\zeta = 1$, these factors become zero and since $\det_\eta^1 \neq 0$, it was in fact appropriate to first divide each by $q-1$. Then each such factor evaluated on $\lambda \in \mathfrak{h}^*$ gives the term $4(\lambda+\rho,\beta) - 2m(\beta,\beta)$. Let mult γ denote the multiplicity of $\gamma \in Q^+(\pi)$ in Δ^+ (with the convention that mult $\gamma = 0$ if $\gamma \notin \Delta^+$).

Set $\Delta^+_{\rm irr} = \Delta^+ \cap Q^+(\pi)$. By [K, Prop. 5.5] one has $\Delta^+ = \mathbb{N}\Delta^+_{\rm irr} \cap \Delta^+$. We have proved the following conjecture for the case $\zeta = 1$.

Conjecture 5.6 *The factor $\tau(4\beta) - \zeta q^{-4(\rho,\beta)+2m(\beta,\beta)}$, $\beta \in \Delta^+_{\rm irr}$, $m \in \mathbb{N}^+$ in* $\det_\eta$ *occurs with multiplicity* $(\sum_{r|m,\zeta^r=1} \text{mult}(r\beta))\ P(\eta - m\beta)$.

Remark. It can happen that $(\beta,\beta)=0$. Then the above result should be taken to mean that the resulting factor occurs with multiplicity

$$\sum_{m=1}^{\infty}\left(\sum_{r|m,\zeta^r=1}\text{mult}(r\beta)\right)P(\eta-m\beta)\,.$$

Taking account of Lemma 5.3 we would have the following result, which we state as

Conjecture 5.7 *Up to a non-zero scalar*

$$\det_\eta=\prod_{m=1}^{\infty}\prod_{\beta\in\Delta^+_{\text{mult}}}(\tau(2\beta)-q^{-4(\rho,\beta)+2m(\beta,\beta)}\tau(-2\beta))^{P(\eta-m\beta)}\,.$$

By careful reworking of the argument in [K-K] we can establish an averaged version of this conjecture (Lemma 5.10) which, in particular, asserts the presence of factors from Conjecture 5.6 at $\zeta\neq 1$. When one of these remaining factors vanishes on Λ, we obtain Verma submodules of $M(\Lambda)$ which are unexpected since they do not occur in the quantum semisimple case, and have no analog for the classical Kac–Moody algebras. The analysis is sketched in the lemmas below.

Extending k we may assume it to contain $\mathbb{C}$. Fix positive real numbers u_i; $i=1,2,\dots,\ell$ linearly independent over $\mathbb{Q}$.

Lemma 5.8 *Fix $m\in\mathbb{N}^+$, $\zeta\in\mathscr{P}_s$. Then for each $\alpha\in Q^+(\pi)$ isotropic (resp. non-isotropic) there exists $\Lambda\in T^*$ such that $\Lambda(\tau(4\beta))=\zeta q^{-4(\rho,\beta)+2m(\beta,\beta)}$ implies $\beta\in\mathbb{Q}\alpha$ (resp. $\beta=\alpha$).*

Proof. The general case will be easily deduced from the case $m=1$, $\zeta=1$. Up to a renumbering of the simple roots we may write

$$\alpha=\sum_{i=1}^{s}r_i\alpha_i\;:\;r_i\in\mathbb{N}^+,\quad s\leqq\ell\,.$$

Choose $\lambda\in P(\pi)$ such that

$$(\lambda+\rho,\alpha_i^\vee)=r_i^2+\sum_{j=i+1}^{s}(\alpha_i^\vee,\alpha_j)r_ir_j\in\mathbb{Z}\,.$$

Then $2(\lambda+\rho,\alpha)=(\alpha,\alpha)$. Defining $\Lambda\in T^*$ through

$$\Lambda(t_i)=\begin{cases}q^{(\lambda,\alpha_i)}\exp\frac{1}{r_i}\,(u_i-u_{i-1}) & 1\leqq i\leqq s\\ q^{(\lambda,\alpha_i)}\exp u_i & i>s\end{cases}$$

with $u_0=u_s$, gives the required result. $\square$

For each $\alpha\in Q^+_{\text{irr}}(\pi)$, let $d_{\eta,\zeta,m}(\alpha)$ denote the multiplicity of the factor $\tau(2\alpha)-\zeta q^{-4(\rho,\alpha)+2m(\alpha,\alpha)}\tau(-2\alpha)$ occurring in $\det_\eta$.

Lemma 5.9 *Fix* $m \in \mathbb{N}^+$, $\zeta \in \mathscr{P}_s$, $\alpha \in Q^+_{\text{irr}}(\pi)$. *There exist integers* $c_{\zeta,m}(\alpha)$ *independent of* η *such that*

$$\text{(i)} \qquad \sum_{m=1}^{\infty} d_{\eta,\zeta,m}(\alpha) = \sum_{m=1}^{\infty} c_{\zeta,m}(\alpha) P(\eta - m\alpha)$$

if α *is isotropic,*

$$\text{(ii)} \qquad d_{\eta,\zeta,m}(\alpha) = c_{\zeta,m}(\alpha) P(\eta - m\alpha)$$

if α *is non-isotropic.*

Before proving Lemma 5.9, we need to define the quantum Jantzen filtration. The definition is very similar to the classical, and we will take a closer look at this filtration in Sect. 6.

Definition 5.1 *Let* Λ *be an element in* T^* *and let* S *be an indeterminant. Denote by* $M(\Lambda S^\rho)$ *the* $U \otimes_K K(S)$ *Verma module with highest weight* ΛS^ρ *and highest weight vector* v. *As in say* [J-L1, 5.4], *there is a contravariant form* $\mathscr{F} = \mathscr{F}_{\Lambda S^\rho}$ *on* $M(\Lambda S^\rho)$ *uniquely determined by the normalization* $\mathscr{F}(v,v) = 1$. (*In fact* $\mathscr{F}(av,bv) = \varphi_{\Lambda S^\rho}(a,b) = \Lambda S^\rho(\varphi(\kappa(a)b))$ *for* $a,b \in U^-$.) *Proposition* 5.5 *implies that* $M(\Lambda S^\rho)$ *is simple and so* $\mathscr{F}$ *is nondegenerate. Set* $\hat{M}(\Lambda S^\rho) = U \otimes_K K[S]v$ *and* $M^i(\Lambda S^\rho) = \{m \in \hat{M}(\Lambda S^\rho) \mid \mathscr{F}(a,m) \in (S-1)^i K[S]$ *for all* $a \in \hat{M}(\Lambda S^\rho)$. *Let* $M^i(\Lambda)$ *be the image of* $M^i(\Lambda S^\rho)$ *in the* U *Verma module* $M(\Lambda)$ *using the identification of* $\hat{M}(\Lambda S^\rho) \otimes_K K[S]$ *with* $M(\Lambda)$. *The modules* $M^i(\Lambda)$ *are called the* **Jantzen filtration** *for* $M(\Lambda)$.

Proof of Lemma 5.9 Choose Λ through the conclusion of Lemma 5.8. Then as in the classical case [J, Satz 5.2] one obtains

$$\text{(5.5)} \qquad \sum_{m=1}^{\infty} d_{\eta,\zeta,m}(\alpha) = \sum_{i=1}^{\infty} \dim M^i(\Lambda)_{\Lambda q^{-\eta}}$$

if α is isotropic and

$$\text{(5.6)} \qquad d_{\eta,\zeta,m}(\alpha) = \sum_{i=1}^{\infty} \dim M^i(\Lambda)_{\Lambda q^{-\eta}}$$

if α is non-isotropic.

Suppose α is non-isotropic. It follows from Lemma 5.4 and Lemma 5.8 that either $M(\Lambda)$ is simple and the right hand side of (5.6) is zero, or $M(\Lambda q^{-m\alpha})$ is a simple submodule of $M(\Lambda)$ and every simple submodule takes this form. Hence $M^i(\Lambda)$ is $c_{\zeta,m,i}(\alpha)$ copies of $M(\Lambda q^{-m\alpha})$. Setting $c_{\zeta,m}(\alpha) = \sum_{i \geqq 1} c_{\zeta,m,i}(\alpha)$ and noting that $\dim M(\Lambda q^{-m\alpha})_{\Lambda q^{-\eta}} = P(\eta - m\alpha)$, (ii) results from (5.6).

Suppose α is isotropic. Although $M(\Lambda)$ need not have finite length, it follows by weight space decomposition [K, 9.6, 9.7] that the "Jordan–Hölder" factors $[M(\Lambda') : L(\Lambda'')]$ are well-defined and form a triangular matrix with ones on the diagonal. Take $m \in \mathbb{N}$. By Lemma 5.4 and Lemma 5.8 it follows that $[M(\Lambda q^{-m\alpha}) : L(\Lambda')] = 0$: *unless* $\Lambda' = \Lambda q^{-n\alpha}$: $n \geqq m$. In particular using [·] to denote Grothendieck group representatives we can write

$$[M^i(\Lambda)] = \sum_{n=1}^{\infty} b_{\zeta,n,i}(\alpha)[L(\Lambda q^{-n\alpha})]\,, \quad \text{for some } b_{\zeta,n,i}(\alpha) \in \mathbb{N}\,,$$

$$= \sum_{s=1}^{\infty} c_{\zeta,s,i}(\alpha)[M(\Lambda q^{-s\alpha})]\,, \text{ for some } c_{\zeta,s,i}(\alpha) \in \mathbb{Z}\,.$$

Setting $c_{\zeta,s}(\alpha) = \sum_{i \geqq 1} c_{\zeta,s,i}(\alpha)$, we obtain (i) from (5.5). □

Let $\mathscr{P}_{(m)}$ denote the set of m^{th} roots of unity. By Lemma 5.5 one has $d_{\eta,\zeta,m}(\alpha) = 0$ unless $\zeta \in \mathscr{P}_{(m)}$.

Lemma 5.10 *For all* $m \in \mathbb{N}^+$, $\alpha \in Q^+_{\text{irr}}(\pi)$ *one has*

$$\sum_{\zeta \in \mathscr{P}_{(m)}} c_{\zeta,m}(\alpha) = \sum_{r|m} r(\text{mult } r\alpha)\,.$$

Proof. Comparison of top order terms in $\det_\eta$ gives by Lemma 5.9, Lemma 5.3(i) and Lemma 5.1, that

$$\sum_{m=1}^{\infty} \sum_{\zeta \in \mathscr{P}_{(m)}} \sum_{\alpha \in Q^+_{\text{irr}}(\pi)} 2\alpha\, c_{\zeta,m}(\alpha) P(\eta - m\alpha) = 2\eta\, P(\eta) = \sum_{m=1}^{\infty} \sum_{\alpha \in \Delta^+_{\text{mult}}} 2\alpha P(\eta - m\alpha)\,.$$

The linear independence of the functions $\eta \mapsto P(\eta - \beta) \,:\, \beta \in \mathbb{Q}(\pi)$ completes the proof. □

Conjecture 5.6 is equivalent to $c_{\zeta,m}(\alpha) = \sum_{r|m, \zeta^r = 1} \text{mult } r\alpha$. Since we already know that this holds for $\zeta = 1$, it follows from Lemma 5.10 that it also holds if α is a real root (since then mult $r\alpha = 0$ for $r > 1$). This recovers in particular the result in [D-K, Proposition 1.9] for $\mathfrak{g}$ semisimple. (Of course the real root case can be obtained directly from the case $\zeta = 1$.) More interestingly, Lemmas 5.3 and 5.5 imply that Conjecture 5.6 also holds for imaginary roots with respect to factors corresponding to m being prime. We remark that to prove the conjectures it is enough to show that $c_{\zeta,m}(\alpha) \leqq c_{\xi,m}(\alpha)$, when the multiplicative order of ζ is greater than the order of ξ.

6 The Jantzen filtration and specialization

In this section, we consider the connection between the Jantzen filtration and Drinfeld's problem discussed in the introduction. In this it is convenient to assume that $\lambda \in \mathfrak{h}^*_{\mathbb{Z}} := \{\lambda \in \mathfrak{h}^* \,|\, (\lambda, \alpha^\vee) \in \mathbb{Z},\ \forall \alpha \in \Delta\}$. The more general case $\lambda \in \mathfrak{h}^*_{\mathbb{Q}} := \{\lambda \in \mathfrak{h}^* \,|\, (\lambda, \alpha^\vee) \in \mathbb{Q},\ \forall \alpha \in \Delta\}$ can be similarly handled by including appropriate fractional powers of q.

The quantum sum character formula can be computed as in the classical case using the factorization of the Shapovalov determinant. Recall the definition of the Jantzen filtration given in Definition 5.1. With r an indeterminate, a similar construction using $\lambda + r\rho$ gives the Jantzen filtration

$m^n(\lambda)$ of the $U(\mathfrak{g})$ Verma module $m(\lambda)$ (See [J, Sect. 5] and [K-K].) Set $\Delta^+_{\text{mult}}(\lambda) = \{\alpha \in \Delta^+_{\text{mult}}(\pi) \mid (\lambda + \rho, \check{\alpha}) \in \mathbb{N}^+\}$ where we are counting the appropriate positive roots with multiplicities. Recall that $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^*$ is its dual. One has the

Proposition 6.1 *For all* $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$

(i) $\sum_{n>0} \operatorname{ch} m^n(\lambda) = \sum_{\alpha \in \Delta^+_{\text{mult}}(\lambda)} \operatorname{ch} m(s_\alpha \cdot \lambda)$

(ii) $\sum_{n>0} \operatorname{ch} M^n(q^\lambda) = \sum_{\alpha \in \Delta^+_{\text{mult}}(\lambda)} \operatorname{ch} M(q^{s_\alpha \cdot \lambda})$.

Proof. Assertion (i) is just [K-K] in the Kac–Moody case and [J, 5.3] in the semisimple case. The proof of (ii) is exactly the same as in [J, 5.3] except one uses the quantum Shapovalov determinant. Here we remark that the principal ideal domain $k[r]$ is replaced by the principal ideal domain $K[S]$ and p is the ideal generated by $S - 1$. So the factor in the quantum Shapovalov form $(q^\lambda S^\rho)\tau(4\beta) - \zeta q^{-4(\rho,\beta)+2m(\beta,\beta)}$ has nonzero p-adic evaluation if and only if $\beta \in \Delta^+_{\text{mult}}(\lambda)$ and $\zeta = 1$ and in this case has p-adic evaluation equal to one. Note that since the factors with $\zeta \neq 1$ have zero p-adic evaluation, the information given in Conjecture 5.6 for $\zeta = 1$ is enough to compute the sum character formula. In particular the exponent of the above factor for $\beta \in \Delta^+_{\text{mult}}(\lambda)$ and $\zeta = 1$ matches up with $\operatorname{ch} M(s_\beta \cdot \lambda)$ taking multiplicities into account. Thus (ii) follows as in the classical case [J, 5.2, 5.3]. □

Intuitively one expects the Jantzen filtration of $M^n(q^\lambda)$ to be coarser than that of $m^n(\lambda)$. Then the above sum rule would force equality throughout and imply that $L(\lambda) = M(q^\lambda)/M^1(q^\lambda)$ specializes to $\ell(\lambda) = m(\lambda)/m^1(\lambda)$. Unfortunately we cannot directly compare $M^n(q^\lambda)$ and $m^n(\lambda)$ since $M^n(q^\lambda)$ is defined over $k(q)$. In order to proceed it is necessary to deal simultaneously with the two indeterminates q, S.

Let A (resp. B) denote the localization of the polynomial algebra $k[q - 1]$ (resp. $k[q - 1, (S - 1)/(q - 1)]$) at its augmentation ideal. Of course A is just the algebra defined in Sect. 5 and identifying r with $(S - 1)/(q - 1)$ it follows that $B/(q - 1)B$ is just the localization C of $k[r]$ at its augmentation ideal. Let U_A (resp. U_B) be the A (resp. B) subalgebra of $U \otimes_{k(q)} k(q, S)$ generated by $x_i, y_i, t_i^{\pm 1} : 1 \leqq i \leqq \ell$ over A (resp. B). Then U_A identifies with $\hat{U}$ of Sect. 5. Let $M_A(q^\lambda)$ (resp. $M_B(q^\lambda S^\rho)$) denote the U_A (resp. U_B) Verma module with highest weight λ (resp. $q^\lambda S^\rho$). The localization of the polynomial algebra $K[S - 1]$ at its augmentation identifies with $B[(q - 1)^{-1}]$ and then $M_B(q^\lambda S^\rho) \otimes_B K[S - 1]$ identifies with the Verma module $\hat{M}(q^\lambda S^\rho)$ which is used (Sect. 5) in constructing the Jantzen filtration of $M(q^\lambda)$. Now let $U(\mathfrak{g})_C$ denote the C-subalgebra of $U(\mathfrak{g}) \otimes_k k(r)$ generated by a Chevalley basis for $\mathfrak{g}$ and $m_C(\lambda + r\rho)$ the Verma module for $U(\mathfrak{g})_C$ with highest weight $\lambda + r\rho$ which is used [J, Sect. 5] in constructing the Jantzen filtration of $m(\lambda)$. One checks that $M_B(q^\lambda S^\rho) \otimes B/(q - 1)B$ identifies with $m_C(\lambda + r\rho)$. This gives us the required means for comparing Jantzen filtrations.

Consider the Shapovalov forms defined on the $\mathfrak{h}$ weight submodules of $M_B(q^\lambda S^\rho)$. These are free finite rank B modules and the forms take values in B. Consequently they can be viewed as being given by an $n \times n$ matrix with values in B. For each such form the corresponding form on $\hat{M}(q^\lambda S^\rho)$ is obtained by inverting $q-1$ and the Jantzen filtration on $M(q^\lambda)$ restricted to that $\mathfrak{h}$ weight subspace is obtained by diagonalizing the form (see below) and extracting powers of $(S-1)$ exponents. On the other hand the corresponding form on $m_C(\lambda + r\rho)$ is obtained by specialization at $q = 1$ and the Jantzen filtration on $m(\lambda)$ on that $\mathfrak{h}$ weight subspace is obtained by diagonalizing the form (see below) and extracting powers of r exponents. It is clear that the Jantzen filtrations coincide if and only if all such exponents coincide. A simple lemma of commutative algebra given below shows that this is further equivalent to the forms *already diagonalizing over B.*

It is convenient to replace $q-1$ by t. Then $A = k[t]_0$, $B = k[r,t]_0$ where the zero subscript means localization at the augmentation ideal. Given a non-degenerate A-bilinear form φ on A^n one may find A bases $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$ of A^n such that $\varphi(e_i, f_j) = 0$ if $i \neq j$ and $\varphi(e_i, f_i) = t^{\ell_i}$ for some $\ell_i \in \mathbb{N}$. One says that φ is diagonalized and the ℓ_i are the exponents of the form which can be assumed to be increasing. Then their i^{th} partial sum $\ell_1 + \ell_2 + \cdots + \ell_i$ is just the largest t power dividing all the $i \times i$ minors of φ viewed as a matrix with respect to any pair of bases. This result for the more difficult case of an arbitrary principal ideal domain can for example be read off from [La1, XV, Sect. 2, Theorem 5 and Lemma 3].

The corresponding result for B fails. On the other hand the diagonalization in the above sense of such a form over any commutative domain can always be achieved if at each step there is after dividing out rows or columns by common elements a matrix entry which is a unit. This observation makes the case of a local principal ideal domain, for example A particularly trivial.

Let φ be a non-degenerate bilinear form on B^n. One can recover a local principal ideal domain from B in two different ways. First, invert t that is form $B[t^{-1}] = k(t)[r]_0$. Let $k_1 \leqq k_2 \leqq \cdots \leqq k_n$ be the corresponding (generic) exponents of φ over the above localization. Second, form B/tB and let $\ell_1 \leqq \ell_2 \leqq \cdots \leqq \ell_n$ be the corresponding (special) exponents of φ. From the previous characterization of exponents it is immediate that we have inequalities on the partial sums $k_1 + k_2 + \cdots + k_i \leqq \ell_1 + \ell_2 + \cdots + \ell_i$ rather than the stronger assertions $k_i \leqq \ell_i$ which seemed more intuitive and when inserted into the Jantzen sum rules of 6.1, namely

$$(*) \qquad k_1 + k_2 + \cdots + k_n = \ell_1 + \ell_2 + \cdots + \ell_n$$

give the required equalities $k_i = \ell_i$. In fact the correct result is the following.

Lemma 6.2 *Assume* $(*)$ *holds. Then the following are equivalent*

(i) $k_i = \ell_i$ *for all* i.
(ii) φ *can be diagonalized over* B.

Clearly (ii) implies $k_i \leqq \ell_i$ for all i and hence (i) by $(*)$. Conversely suppose (i) holds. Fixing bases we can view φ as given by an $n \times n$ matrix with entries in B. By the characterization of exponents it follows that each entry is divisable by r^{k_1} in $B[(q-1)^{-1}]$ and hence in B. Dividing out say columns by this factor one can assume $k_1 = 0$ without loss of generality and one further obtains at least one matrix entry, say a, not divisable by r. Equivalently a has a non-zero image in B/rB. Set $t = 0$ in all such matrix entries. If they all become divisable by r then one must have $\ell_1 > 0$ contradicting the hypothesis of (i). Consequently there is at least one matrix entry not contained in the maximal ideal of the local ring B and which is hence a unit. By our previous observation this process eventually gives (ii). □

It is easy to give an example of a form φ not satisfying the equivalent conditions of the lemma. For example take $n = 2$ and φ to be given by the matrix $\begin{pmatrix} r^2t & r \\ r & t \end{pmatrix}$. One easily calculates its generic exponents to be $0, 2$ and its special exponents to be $1, 1$. This example shows that one cannot immediately use 6.1 to deduce a positive answer to Drinfeld's problem. Moreover this example is serious from the point of view of trying to gain some extra information on the Shapovalov forms to show that they could be diagonalized over B. Identifying $\mathrm{Hom}_B(B^n, B)$ with B^n identifies a B-linear form with an element of $\mathrm{End}_B B^n$. One can expect using $\mathfrak{sl}(2)$ theory to show that each such form can be factorized as a product of B-linear transformations depending either on r (coming from $\mathfrak{sl}(2)$ theory) or only on t (coming from relating different $\mathfrak{sl}(2)$ subalgebras). However the above example admits such a factorization, namely

$$\begin{pmatrix} r^2t & r \\ r & t \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-t^2 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally we note below that for $\mathfrak{g}$ semisimple Drinfeld's isomorphism gives a Jantzen filtration for $M_A(q^\lambda) : \lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ which specializes to that of $m(\lambda)$.

Let u be an indeterminate and identify $A = k[q-1]_0$ with a subalgebra of $k[[u]]$ by taking $q = \exp u$. For each k subspace (resp. A module) V let $V[[u]]$ denote the u-adic completion of $V \otimes_k k[[u]]$ (resp. $V \otimes_A k[[u]]$) called simply the completion of V. Consider the algebra $\mathbb{U}$ over $k[[u]]$ generated as an algebra complete in the u-adic topology by the x_i, y_i, and the space $\mathfrak{h}$ in which the torus element t_i is identified with $\exp uh_i$ and the h_i in $\mathfrak{h}$ corresponds to α_i via the Killing form. After Drinfeld [D2, Prop. 4.3] one has an isomorphism $\mathbb{U} \xrightarrow{\sim} U(\mathfrak{g})[[u]]$ which gives rise to an embedding $U_A[[u]] \hookrightarrow U(\mathfrak{g})[[u]]$ where t_i becomes $\exp uh_i$. Now set $S = q^r$ and $X = (S-1)/(q-1)$. It is straightforward to show that B is a subring of $C[[u]]$. Realizing the generators $x_i, y_i, t_i^{\pm 1}$ of U in $U(\mathfrak{g})[[u]]$ gives an A-algebra embedding $\psi : U_A \hookrightarrow U(\mathfrak{g})[[u]]$ which extends to a map Ψ of $U_B = U_A \otimes_A B$ into $(U(\mathfrak{g}) \otimes_k C)[[u]]$ by $\Psi(\sum u_i \otimes b_i) = \sum \psi(u_i) b_i$. To see that Ψ is also an embedding, we may "clear denominators" and assume that b_i is equal to X^i. Injectivity follows because $U(\mathfrak{g})[[u]] \cap k[X] = k$ and this implies that the X^i are linearly independent over

$U(\mathfrak{g})[[u]]$. Let v_λ be a highest weight vector for $m_C(\lambda+r\rho)$. Then as an element of $m_C(\lambda+r\rho)[[u]]$ it is also a highest weight vector with respect to U_B and hence generates a highest weight module whose completion is $m_C(\lambda+r\rho)[[u]]$ and hence must itself coincide with the Verma module $M_B(q^\lambda S^\rho)$ for U_B. Let $m_C^i(\lambda+r\rho)$ denote the filtration of $m_C(\lambda+r\rho)$ defined by r-adic evaluation [J, Sect. 5] of the Shapovalov form. Note that the form φ is preserved under Drinfeld's embedding. Recalling the notation of definition 5.1 one has, as above, that $\hat{M}(q^\lambda S^\rho)((u)) = m_C(\lambda+r\rho)((u))$ and so $\hat{M}(q^\lambda S^\rho) \cap m_C^i(\lambda+r\rho)((u)) = M^i(q^\lambda S^\rho)$.

Since $k(u)[S-1]_0$ is a principal ideal domain over which the T weight submodules of $\hat{M}(q^\lambda S^\rho)$ are torsion-free and finitely generated (as in say [J-L1, 5.10(i)]), it follows that $\hat{M}(q^\lambda S^\rho)$ is a direct sum of T weight submodules which are free and of finite rank. Consequently $m_C(\lambda+r\rho)((u))$ is simply the direct product of these weight submodules extended over $k((u))[S-1]_0$. Since the contravariant form respects weight decomposition $M^i(q^\lambda S^\rho)$ and $m_C^i(\lambda+r\rho)((u))$ are similarly related and hence so are their specializations at $S-1$. In particular $M(q^\lambda) \cap m^i(\lambda)((u)) = M^i(q^\lambda)$. Now define $M_A(q^\lambda)$ to be the U_A submodule of $m(\lambda)[[u]]$ generated by the image of v_λ and $M_A^i(q^\lambda) = M_A(q^\lambda) \cap m^i(\lambda)[[u]]$. Trivially $M_A^i(q^\lambda)$ specializes at $u=0$ to $m^i(\lambda)$, whilst $M_A^i(q^\lambda)[t^{-1}] = M^i(q^\lambda)$ by the previous result. Thus our stated aim is achieved.

Either directly or through the lemma we conclude that the Shapovalov forms on $M_B(q^\lambda S^\rho)$ can be diagonalized over B in the semisimple case and we suggest that this must also hold in the Kac–Moody case. It motivates the study of adapted bases effecting this diagonalization. In the semisimple case these adapted bases are prescribed by the Drinfeld isomorphism which separates out the r and q (or u) variable.

7 A nondegeneracy theorem

In this section, we assume that U is the quantized enveloping algebra associated to a semisimple Lie algebra $\mathfrak{g}$. Here $\mathbb{C}$ denotes the complex field; but may also be taken to be any subfield stable by complex conjugation and then an extension of the latter.

One of the nice properties of Kashiwara's form is that his crystal basis is "practically" an orthonormal basis for it. More precisely, take the base field k to be $\mathbb{Q}$ and let $\mathbb{Q}[q]_{(q)}$ denote the localization of $\mathbb{Q}[q]$ at the prime ideal generated by q. Let (L, B) denote the crystal basis for G^-. (See [Ka, Sect. 2] for definitions.) In particular, L is a free $\mathbb{Q}[q]_{(q)}$ module such that $L \otimes_{\mathbb{Q}[q]_{(q)}} \mathbb{Q}(q) = G^-$ and B is a basis for L/qL. By [Ka, 5.1.2], the restriction of $\langle\, , \rangle$ to L takes values in $\mathbb{Q}[q]_{(q)}$, and hence we may obtain a $\mathbb{Q}$-valued from $\langle\, , \rangle|_0$ by evaluation at $q=0$. By [Ka, Sect. 7], we can lift B to a basis $\mathscr{B}$ of L such that the image of $\mathscr{B}$ in L/qL is B.

Theorem 7.1 [Ka, 5.1.2 (iii)] *B is an orthonormal basis for* $\langle\, , \rangle|_0$.

Now let the base field k be $\mathbb{C}$. Given an ordered pair $\xi = (\beta, \gamma) \in Q^+(\pi) \times Q^+(\pi)$ define

$$\mathscr{B}(\xi) = \{b\tau(\lambda)\kappa(b') | b, b' \in \mathscr{B},\ \mathrm{wt}(b) = -\beta,\ \mathrm{wt}(b') = -\gamma;\ \lambda \in Q(\pi)\}.$$

Set $\mathscr{B} = \bigcup_\xi \mathscr{B}(\xi)$ and note that $\mathscr{B}$ is a $\mathbb{C}(q)$ basis for U. Define a $\mathbb{R}(q)$ antiautomorphism of U by

$$\kappa^*\left(\sum_{v\in\mathscr{B}} a_v v\right) = \sum_{v\in\mathscr{B}} \bar{a}_v \kappa(v)$$

where "-" denote complex conjugation extended to $\mathbb{C}(q)$.

Consider two vectors v and w in $\mathscr{B}$ with $v = b\tau(\lambda)\kappa(b')$ and $w = c\tau(\eta)\kappa(c')$. We write $v \sim w$ if $b = c$ and $b' = c'$.

Let ν be the q-adic valuation on $\mathbb{C}(q)$. In particular, if $f = q^m g$ where g has no poles or zeros at $q = 0$, then $\nu(f) = m$. Equivalently, $q^{-\nu(f)} f|_0 \neq 0$ where "$|_0$" denotes evaluation at $q = 0$. Furthermore, if $s < \nu(f)$, then $q^{-s} f|_0 = 0$. Set $\nu(f) = \infty$, if $f = 0$.

Lemma 7.2

(i) *Assume $\xi_1 = (\beta_1, \gamma_1)$ and $\xi_2 = (\beta_2, \gamma_2)$ are ordered pairs in $Q^+(\pi) \times Q^+(\pi)$, $v \in \mathscr{B}(\xi_1)$ and $w \in \mathscr{B}(\xi_2)$. If $\xi_1 \neq \xi_2$, then $R(v, \kappa^*(w)) = 0$.*

(ii) *Suppose that $v, w \in \mathscr{B}(\xi)$ with $v = b\tau(\lambda)\kappa(b')$ and $w = c\tau(\eta)\kappa(c')$. Set $N(\xi) = (\rho, \beta) + (\rho, \beta + \gamma)$. Then*

$$q^{1/4(\lambda,\eta) - N(\xi)} R(v, \kappa^*(w))|_0 = \begin{cases} 1 & \textit{if } v \sim w \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since R is triangular, we have

$$(7.2) \qquad R(v, \kappa^*(w))) = R(\kappa(b'), c')R(\tau(\lambda), \tau(\eta))R(b, \kappa(c)).$$

By Lemma 4.1, (7.2) equals zero unless $\xi_1 = \xi_2$. This proves assertion (i).

Assume that $\xi_1 = \xi_2$. By (4.17), and the definition of the adjusted Kashiwara form (3.6), we have

$$R(v, \kappa^*(w)) = \frac{q^{-1/4(\lambda,\eta) + N(\xi)}}{A(\beta)A(\gamma)} \langle b', c'\rangle \langle b, c\rangle.$$

By definition, $A(\beta)|_0 = A(\gamma)|_0 = 1$. Hence by Theorem 7.1

$$q^{1/4(\lambda,\eta) - N(\xi)} R(v, \kappa^*(w))|_0 = \begin{cases} 1 & \text{if } v \sim w \\ 0 & \text{otherwise}. \end{cases} \qquad \square$$

We have the following nondegeneracy condition for R.

Theorem 7.3 *Assume that $\mathfrak{g}$ is semisimple and $k = \mathbb{C}$. For each $a \in U$ nonzero, there exists an integer s such that $q^{-s} R(a, \kappa^*(a))|_0$ is a strictly positive real number.*

Proof. Write $a = \sum a_\xi$ where $\xi \in Q^+(\pi) \times Q^+(\pi)$ and a_ξ lies in the span of $\mathscr{B}(\xi)$. According to Lemma 7.2(i),

$$R(a, \kappa^*(a)) = \sum_\xi R(a_\xi, \kappa^*(a_\xi))$$

Then in view of Lemma 7.2(ii) we may assume that only one ξ appears in the sum, so we may write $a = \sum a_v v$ where a_v lies in $\mathbb{C}(q)$ and $v \in \mathscr{B}(\xi)$. Set $\operatorname{Supp} a = \{v \in \mathscr{B}(\xi) | a_v \neq 0\}$.

Now

$$R(a, \kappa^*(a)) = \sum_{v,w} a_v \bar{a}_w R(v, \kappa^*(w))$$

Let $s = \min_{v,w \in \operatorname{Supp} a}\{v(a_v) + v(a_w) + v(R(v, \kappa^*(w)))\}$. We show that this minimal value s is only achieved on the diagonal contributions (i.e., when $v = w$.)

We already know by Lemma 7.2(ii) that

$$v(R(v, \kappa^*(w))) \geqq -1/4(\lambda, \eta) + N(\xi)\,,$$

with a strictly inequality exactly when $v \not\sim w$.

Since the Cartan inner product is positive definite on $Q(\pi)$, we have

$$(\lambda, \lambda) + (\eta, \eta) \geqq 2(\lambda, \eta)\,.$$

Thus if $v \not\sim w$ we have

$$\begin{aligned} 2[v(a_v) + v(a_w) + v(R(v, \kappa^*(w)))] &> 2[v(a_v) + v(a_w) - 1/4(\lambda, \eta) + N(\xi)] \\ &\geqq [v(a_v) + v(a_v) - 1/4(\lambda, \lambda) + N(\xi)] + [v(a_w) + v(a_w) - 1/4(\eta, \eta) + N(\xi)] \\ &\geqq 2s\,. \end{aligned}$$

Next suppose that $v \sim w$ but $v \neq w$. Then $\lambda \neq \eta$. In this case $(\lambda, \lambda) + (\eta, \eta) > 2(\lambda, \eta)$, whilst by Lemma 7.2(ii), $vR(v, \kappa^*(w)) = -1/4(\lambda, \eta) + N(\xi)$. Thus the above argument gives again the required inequality. Finally

$$\begin{aligned} q^{-s} R(a, \kappa^*(a))|_0 &= \sum_{v,w} q^{-s} a_v \bar{a}_w R(v, \kappa^*(w))|_0 \\ &= \sum_v q^{-s} a_v \bar{a}_v R(v, \kappa^*(v))|_0, \text{ by the above}\,. \end{aligned}$$

Yet by Lemma 7.2(ii) each non-zero term in the above sum equals $(q^{v(a_v)} a_v|_0)(q^{-v(a_v)} a_v|_0)$, which is a positive real number. □

Now let V be a subspace of U such that $\kappa^*(V) \subseteq V$. (Since κ^* is an antiautomorphism of U of order 2, this is equivalent to $\kappa^*(V) = V$.) If $a \in V$, then Theorem 7.3 implies that $R(a, \kappa^*(a)) \neq 0$. Thus we have

Corollary 7.4 *Assume that* $\mathfrak{g}$ *is semisimple and* $k = \mathbb{C}$. *Let* V *be a subspace of* U *such that* $\kappa^*(V) \subseteq V$. *Then* $R|_V$ *is nondegenerate.*

8 Applications to the locally finite part of U

In this section, we continue to assume that $\mathfrak{g}$ is semisimple and for the moment that $k = \mathbb{C}$. The structure of the locally finite part of U is investigated extensively in [J-L1] and [J-L2]. Here, we consider the additional information Corollary 7.4 provides.

Define the locally finite part of U to be $F = \{a \in U | \dim_K(\operatorname{ad} U)a < \infty\}$. The set F is an algebra ([J-L1, Corollary 2.3]) which is "large" inside U ([J-L1, Theorem 6.4]).

We can completely describe the set $T \cap F$ as follows. Let $P^+(\pi) = \sum_i \mathbb{N}\omega_i$ denote the set of dominant integral weights. Set $R^+(\pi) = -4P^+(\pi) \cap Q(\pi)$. By [J-L1, Lemma 6.1], $F \cap T = \tau(R^+(\pi))$.

We have the following direct sum decomposition of F([J-L2, Theorem4.12])

$$F = \bigoplus_{\lambda \in R^+(\pi)} (\operatorname{ad} U)\tau(\lambda)\,. \tag{8.1}$$

Moreover, we can completely describe the ad U module structure of each summand in the right hand side of (8.1). Let $L(\lambda)$ denote the simple U module with highest weight q^λ. Note that $\lambda \in R^+(\pi)$ implies $-1/4\lambda \in P^+(\pi)$. So $-1/4\lambda$ is a dominant integral weight and the corresponding simple module $L(-1/4\lambda)$ is finite dimensional. By [J-L2, Corollary 3.5], we have an isomorphism of ad U modules for each $\lambda \in R^+(\pi)$:

$$(\operatorname{ad} U)\tau(\lambda) \cong \operatorname{End}_K L(-1/4\lambda)\,. \tag{8.2}$$

The proof of (8.2) does not explicitly give the isomorphism map. The argument depends on the following observations. Given a subspace S of F, let $S^\perp = \{a \in F | R(a,s) = 0 \text{ for all } s \in S\}$. For all $\lambda \in R(\pi)$ we have ([J-L2, Corollary 3.3])

$$[(\operatorname{ad} U)\tau(\lambda)]^\perp = \operatorname{Ann}_F L(-1/4\lambda) \tag{8.3}$$

and ([J-L2], 3.5)

$$\dim_K(\operatorname{ad} U)\tau(\lambda) = \dim_K \operatorname{End} L(-1/4\lambda)\,. \tag{8.4}$$

On the other hand, there is a natural map from $(\operatorname{ad} U)\tau(\lambda)$ to $\operatorname{End} L(-1/4\lambda)$ by the action of U on $L(-1/4\lambda)$. Namely, an element $a \in (\operatorname{ad} U)\tau(\lambda)$ is sent to the map L_a where $L_a(v) = av$ for each $v \in L(-1/4\lambda)$. (In other words, L_a is left multiplication by a.) In [J-L2, Remark 3.5], it is conjectured that this map is an isomorphism from $(\operatorname{ad} U)\tau(\lambda)$ onto $\operatorname{End} L(-1/4\lambda)$. Using Sect. 7, we shall show this is true.

Let λ be any element in $Q(\pi)$. In order to apply Corollary 7.4 to $(\operatorname{ad} U)\tau(\lambda)$, we must show that it is invariant under κ^*. As in (4.11) we have for each

$1 \leqq i \leqq l$

$$\kappa^*((\operatorname{ad} x_i)a) = \kappa^*(x_i a t_i - q^{-(\alpha_i, \alpha_i)} t_i a x_i) \tag{8.5}$$
$$= t_i \kappa^*(a) y_i - q^{-(\alpha_i, \alpha_i)} y_i \kappa^*(a) t_i$$
$$= -q^{-(\alpha_i, \alpha_i)} (\operatorname{ad} y_i) \kappa^*(a) .$$

Each element in $(\operatorname{ad} U)\tau(\lambda)$ can be written as a finite linear combination of elements of the form $(\operatorname{ad} b)\tau(\lambda)$ where b is a product in the $x_i's$ and $y_i's$, $1 \leqq i \leqq l$. Since $\kappa^*(\tau(\lambda)) = \tau(\lambda)$, then by (8.5), it follows that $\kappa^*((\operatorname{ad} b)\tau(\lambda))$ is also contained in $(\operatorname{ad} U)\tau(\lambda)$. Hence $\kappa^*((\operatorname{ad} U)\tau(\lambda)) \subseteq (\operatorname{ad} U)\tau(\lambda)$. By Corollary 7.4, we now have (extending scalars to prove the general case).

Theorem 8.1 *Assume that* $\mathfrak{g}$ *is semisimple. The restriction of the Rosso form to* $(\operatorname{ad} U)\tau(\lambda)$ *is nondegenerate for any* $\lambda \in R^+(\pi)$.

Recall that $(\operatorname{ad} U)\tau(\lambda) \subseteq F$ for $\lambda \in R^+(\pi)$; the conclusion of Theorem 8.1 is equivalent to

$$(\operatorname{ad} U)\tau(\lambda) \cap ((\operatorname{ad} U)\tau(\lambda))^{\perp} = 0 . \tag{8.6}$$

By (8.3), it follows that $(\operatorname{ad} U)\tau(\lambda) \cap \operatorname{Ann}_F L(-1/4\lambda) = 0$ for all $\lambda \in R^+(\pi)$. Hence, the map $a \mapsto L_a$ is injective. The dimension equality in (8.4) now gives us

Corollary 8.2 *Assume that* $\mathfrak{g}$ *is semisimple. For* $\lambda \in R^+(\pi)$ *the map* $a \mapsto L_a$ *is an isomorphism of* $(\operatorname{ad} U)\tau(\lambda)$ *onto* $\operatorname{End} L(-1/4)$.

Remark 8.3 It is unclear what happens when we drop the assumption that $\mathfrak{g}$ is semisimple. The proof for Theorem 8.1 fails in general because the argument for Theorem 7.3 requires the bilinear form (,) on $Q(\pi)$ to be positive definite.

In [J-L2], an analog of Kostant's Separation of Variables Theorem is proved for the quantized enveloping algebra. The core of the classical theorem is that $S(\mathfrak{g})$ is isomorphic to a tensor product over k of the $\operatorname{ad} U(\mathfrak{g})$ invariant elements and an $\operatorname{ad} U$ module, usually referred to as the "harmonics", which is realized as the subspace of $S(\mathfrak{g})$ orthogonal to the $\operatorname{ad} U(\mathfrak{g})$ invariant elements of $S(\mathfrak{g}^*)$ under the bilinear pairing on $S(\mathfrak{g}^*) \times S(\mathfrak{g})$. (See [Ko, Sect. 1] for details.) In the quantum case, the "harmonics" are not completely determined. Using Rosso's form, we make a choice that parallels the classical one. This requires a base field to admit complex conjugation so that κ^* is defined.

We need to define two objects used in [J-L2]. The first is the simply connected quantized enveloping algebra $\check{U}$. Note that $Q(\pi) \subset P(\pi)$ and set $\check{T} = \tau(P(\pi)) \supset T$. The algebra $\check{U}$ is generated by U and $\check{T}$ such that $\check{U} \cong U^- \otimes K[\check{T}] \otimes U^+$ as vector spaces over K and

$$\tau(\lambda) x_i \tau(\lambda)^{-1} = q^{(\alpha_i, \lambda)} x_i; \qquad \tau(\lambda) y_i \tau(\lambda)^{-1} = q^{-(\alpha_i, \lambda)} y_i .$$

One can also make $\check{U}$ into a Hopf algebra by setting $\varepsilon(\tau(\lambda)) = 1$, $\Delta(\tau(\lambda)) = \tau(\lambda) \otimes \tau(\lambda)$, and $\sigma(\tau(\lambda)) = \tau(\lambda)^{-1}$ for all $\lambda \in P(\pi)$.

The reason we introduce $\check{U}$ is that the Quantized Separation of Variables Theorem [J-L2, Theorems 7.3 and 7.4] does not apply to the smaller algebra U (see [J-L2, Ex 5.5]). Note that all the results in the preceding sections of this paper also apply to $\check{U}$. In particular, we can define Rosso's form on $\check{U}$ where we allow $\tau(\lambda)$ and $\tau(\gamma)$ in (4.16) and (4.17) to be any elements in $\check{T}$. Of course the statements in this section also apply to $\check{U}$ where we replace $R^+(\pi)$ with $-4P^+(\pi)$ and F with $\check{F} = \{a \in \check{U} | \dim_K(\operatorname{ad} U)a < \infty\}$. Note that κ^* extends to an antiautomorphism of $\check{U}$ by setting $\kappa^*(\tau(\lambda)) = \tau(\lambda)$ for all $\tau(\lambda) \in T$.

The second object of interest is a graded algebra associated to $\check{U}$. Define an $\operatorname{ad} U$ invariant filtration using degree where

$$\deg x_i = \deg y_i = 1$$
$$\deg t_i = -\deg t_i^{-1} = -1$$

for $1 \leqq i \leqq l$. (See [J-L2, 2.2] for more precision.) Consider the associated graded algebra of $\check{U}$ for this filtration. We use the notation $\lceil a \rceil$ for the "top symbol" or "graded image" of an element $a \in \check{U}$. Similarly, this notation can be used for sets so that $\lceil \check{U} \rceil$ is just the associated graded algebra of $\check{U}$. Note that the adjoint action preserves degree and hence $\lceil \check{U} \rceil$ inherits an $\operatorname{ad} U$ action (see [J-L2, Sect. 2]).

Now κ^* preserves degree and hence induces an antiautomorphism on $\lceil \check{U} \rceil$ which we also denote by κ^*. Then

(8.7) $$\kappa^* \lceil a \rceil = \lceil \kappa^*(a) \rceil .$$

For each $\lambda \in -4P^+(\pi)$, we have the following equality and isomorphism of $\operatorname{ad} U$ modules ([J-L2, Lemma 4.5]).

(8.8) $$\lceil (\operatorname{ad} U)\tau(\lambda) \rceil = (\operatorname{ad} U)\lceil \tau(\lambda) \rceil \cong (\operatorname{ad} U)\tau(\lambda) .$$

Furthermore, by [J-L2, 4.10],

(8.9) $$\lceil \check{F} \rceil = \bigoplus_{\lambda \in -4P^+(\pi)} (\operatorname{ad} U)\lceil \tau(\lambda) \rceil .$$

Note that an immediate consequence of (8.8) is: for each $a \in \lceil (\operatorname{ad} U)\tau(\lambda) \rceil$, there exists a unique $b \in (\operatorname{ad} U)\tau(\lambda)$ such that $\lceil b \rceil = a$. Indeed if b_1 and b_2 are two such elements then $\deg(b_1 - b_2)$ is strictly less than $\deg b_1$. Yet $\lceil b_1 - b_2 \rceil \in \lceil (\operatorname{ad} U)\tau(\lambda) \rceil = (\operatorname{ad} U)\lceil \tau(\lambda) \rceil$, whose non-zero elements are homogeneous of degree equal to $\deg \tau(\lambda)$. This forces $\lceil b_1 - b_2 \rceil = 0$ and so $b_1 = b_2$. We conclude that R induces an ad-invariant bilinear form $\lceil R \rceil$ on $\lceil (\operatorname{ad} U)\tau(\lambda) \rceil$, defined by $\lceil R \rceil(\lceil a \rceil, \lceil b \rceil) = R(a,b)$. It is non-degenerate by (8.8). Extend $\lceil R \rceil$ to $\lceil \check{F} \rceil$ by making (8.9) an orthogonal direct sum. We have the following graded version of Corollary 7.4.

Lemma 8.4 *Let V be a subspace of $\lceil (\operatorname{ad} U)\tau(\lambda) \rceil$ such that $\kappa^*(V) \subseteq V$. Then $\lceil R \rceil |_V$ is nondegenerate.*

Proof. Recall that $\kappa^*((\mathrm{ad}\, U)\tau(\lambda)) = (\mathrm{ad}\, U)\tau(\lambda)$. Let W be the unique subspace of $(\mathrm{ad}\, U)\tau(\lambda)$ such that $\lceil W \rceil = V$. Then $\kappa^*(W) \subseteq (\mathrm{ad}\, U)\tau(\lambda)$. By (8.7)

$$\lceil \kappa^*(W) \rceil = \kappa^* \lceil W \rceil = \kappa^*(V) \subseteq V .$$

Hence, $\kappa^*(W) \subseteq W$. Since $\lceil R \rceil(\lceil a \rceil, \lceil b \rceil) = R(a,b)$ for all a, b in W, it follows from Corollary 7.4 that $\lceil R \rceil|_V$ is also nondegenerate. □

We use the graded Rosso's form to construct the "harmonic" elements of $\lceil \check{F} \rceil$. Let Z denote the set of $\mathrm{ad}\, U$ invariant elements of $\check{U}$. Then $\lceil Z \rceil$ is the set of $\mathrm{ad}\, U$ invariant elements of both $\lceil \check{U} \rceil$ and $\lceil \check{F} \rceil$. Let I be the ideal in $\lceil \check{F} \rceil$ generated by the homogeneous elements in $\lceil Z \rceil$ with positive degree. Set $I(\lambda) = I \cap \lceil (\mathrm{ad}\, U)\tau(\lambda) \rceil$. The harmonics can be any graded $\mathrm{ad}\, U$ module complementing I in $\lceil \check{F} \rceil$ (see [J-L2, 7.3]). Set

$$H(\lambda) = \{a \in \lceil (\mathrm{ad}\, U)\tau(\lambda) \rceil \,|\, \lceil R \rceil (a, I(\lambda)) = 0\}$$

for λ in $-4P^+(\pi)$ and

$$H = \bigoplus_{\lambda \in -4P^+(\pi)} H(\lambda) = \{a \in \lceil \check{F} \rceil \,|\, \lceil R \rceil (a, I) = 0\} .$$

Lemma 8.5 *For each* $\lambda \in -4P^+(\pi)$,

$$\lceil (\mathrm{ad}\, U)\tau(\lambda) \rceil \cong H(\lambda) \oplus I(\lambda) . \tag{8.10}$$

as ad *U modules.*

Proof. To show (8.10), it is sufficient to show that $\lceil R \rceil$ is nondegenerate when restricted to $I(\lambda)$.

Since κ^* is an antiautomorphism of $\check{U}$, it follows that $\kappa^*(Z) = Z$. Thus, by (8.7), $\kappa^* \lceil Z \rceil = \lceil Z \rceil$. Since κ^* preserves degree it follows easily that $\kappa^*(I(\lambda)) \subset I(\lambda)$. Then by Lemma 8.4, $\lceil R \rceil$ restricted to $I(\lambda)$ is nondegenerate. The proof follows. □

Let $\mathbb{H}(\lambda)$ be the $\mathrm{ad}\, U$ submodule of $(\mathrm{ad}\, U)\tau(\lambda)$ such that $\lceil \mathbb{H}(\lambda) \rceil = H(\lambda)$. Set

$$\mathbb{H} = \bigoplus_{\lambda \in -4P^+(\pi)} \mathbb{H}(\lambda) .$$

By Lemma 8.5 it follows that H is a complement to I in $\lceil \check{F} \rceil$. Since each $\lceil (\mathrm{ad}\, U)\tau(\lambda) \rceil$ is a homogeneous component of $\lceil \check{F} \rceil$, we have that H is a graded submodule of $\lceil \check{F} \rceil$. Thus by [J-L2, Proposition 7.3] and [J-L2, Theorem 7.4].

Theorem 8.5 *The map* $h \otimes y \mapsto hy$ *is an isomorphism of* $H \otimes_K \lceil Z \rceil$ *onto* $\lceil \check{F} \rceil$, *and the map* $b \otimes z \mapsto z$ *is an isomorphism of* $\mathbb{H} \otimes_K Z$ *onto* $\check{F}$.

References

[D-K] De Concini, C., Kac, V.G.: Representations of quantum groups at roots of 1. In: Connes, A., et al.: Operator algebras, unitary representations, enveloping algebras and invariant theory, (Progress in Mathematics, pp. 471–506) Boston: Birkhauser 1990

[D1] Drinfeld, V.G., Kulish, P.: On some unsolved problems in quantum group theory. In: Quantum Groups, (Lect. Notes Math., vol. 1510) Berlin: Springer-Verlag, 1992
[D2] Drinfeld, V.G..: On almost cocommutative Hopf algebras. Leningrad Math. J. **1**, 321–342 (1990)
[H] Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. New York: Springer-Verlag 1972
[J] Jantzen, J.C.: Moduln mit einem höchsten Gewicht (Lect. Notes Math., vol. 750) Berlin: Springer-Verlag 1979
[J-L1] Joseph, A., Letzter, G.: Local finiteness of the adjoint action for quantized enveloping algebras. J. Algebra **153**, 289–318 (1992)
[J-L2] Joseph, A., Letzter, G.: Separation of variables for quantized enveloping algebras. Amer. J. Math. **116** No. 1, 127–177 (1994)
[J-L3] Joseph, A., Letzter, G.: Verma module annihilators for quantized eveloping algebras. Ann. Ec. Norm. Sup. (in the press)
[K] Kac, V.G.: Infinite dimensional Lie algebras. Cambridge: Cambridge Univ. Press 1985
[K-K] Kac, V.G., Kazhdan, D.A.: Structure of representations with highest weight of infinite dimensional Lie algebras. Adv. Math. **34**, 97–108 (1979)
[Ka] Kashiwara, M.: On crystal bases of the q-analogue of universal enveloping algebra. Duke Math. J. **63**, 465–516 (1991)
[Ko] Kostant, B.: Lie group representations on polynomial rings. Amer. J. Math. **85**, 327–404 (1963)
[La1] Lang, S.: Algebra. London: Addison-Wesley 1965
[L1] Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. **70**, 237–249 (1988)
[L2] Lusztig, G.: Introduction to Quantum groups. Boston: Birkhäuser (1993)
[R1] Rosso, M.: Analogues de la forme de Killing et du théorème de Harish-Chandra pour les groupes quantiques. Ann. Ec. Norm. Sup. **23**, 445–467 (1990)
[R2] Rosso, M.: Certaines formes bilinéares sur les groupes quantiques et une conjecture de Schechtman et Varchenko. Comptes Rendus **314**, 5–8 (1992)
[S] Shapovalov, N.N.: On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra. Funct. Anal. Appl. **5**, 307–312 (1971)