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Testing Homogeneity of Ordered Variances

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Summary: Fujino (1979) studied several tests for homogeneity of nondecreasing variances and concluded that the modification of Bartlett's (1937) test, first proposed by Boswell & Brunk (1969), is generally superior to its competitors in terms of power. A weakness of this test, however, is that the null distribution of the test statistic has not been adequately determined for cases other than when the group sample sizes are equal. In this article a class of simple tests for equality of nondecreasing variances is proposed which can be used without special tables for arbitrary sample sizes. Some of these tests have operating characteristics which compare favorably to those of the modification of Bartlett's test. A prescription is also given for applying the tests in cases where the population variances are constrained by more general partial orders.

Key Words: Bartlett's test; Combining independent tests; Fisher's Combination method; Logit Combination method; Order-restricted inference.

1 Introduction

The assumption of equal error variances underlies many inferential procedures in the normal theory analysis of linear models. The well-known homoscedasticity tests of Bartlett (1937), Cochran (1941), and Hartley (1950) provide appropriate type I error control even if the population variances are *a priori* order-restricted, however their powers would be expected to be smaller than those of procedures which utilize the prior information about the ordering. In this spirit Fujino (1979) suggested modifying the classical tests by using the order-constrained maximum likelihood estimates of the variances in the test statistics in place of the sample variances. He considered the case where the variances are known to be in nondecreasing order and the sample sizes are equal. In an empirical study of the classical tests and their modifications, as well as a regression-type test due to Vincent (1961) and its modification, Fujino found that the modified tests have a substantial power advantage over the classical tests, and that the modification of Bartlett's test is preferable in this regard.

In this paper we employ the general strategy for order-constrained hypothesis testing proposed in Mudholkar & McDermott (1989) for constructing simple alternatives to the modified tests investigated by Fujino. In this approach the null hypothesis is decomposed into several nested component hypotheses, each of which can be tested using simple, well-known statistics. It can be easily shown that these statistics, and therefore the associated p-values, are mutually independent. These p-values are then pooled using such classical devices as Fisher's combination method to provide an overall test of the null hypothesis.

The modified versions of the classical tests of homoscedasticity proposed by Fujino (1979) are outlined in Section 2. In Section 3 the new tests are described for the case where the population variances are known to be in nondecreasing order. These tests are easy to implement for arbitrary sample sizes. A power study is undertaken in Section 4, using both exact computation and simulation, which compares the new tests with the modification of Bartlett's test. The new tests for homoscedasticity are extended in Section 5 to cases where the population variances are constrained by more general partial orders. The conclusions are summarized in Section 6.

2 Modifications of the Classical Tests

Let $\hat{\sigma}_i^2 = S_i^2$ $(i = 1, \ldots, k)$ be the unbiased estimates of the variances σ_i^2 of k normal populations, where the $\nu_i S_i^2/\sigma_i^2$ are independently distributed as χ^2 with ν_i degrees of freedom. It is of interest to test H_0: $\sigma_1^2 = \cdots = \sigma_k^2$ against the simple order alternative H_1: $\sigma_1^2 \leq \cdots \leq \sigma_k^2$, with at least one inequality strict.

Fujino (1979) assumed $\nu_1 = \cdots = \nu_k = \nu$ and developed the modifications M^*, $F^*_{\max}$, and G^*, G_* of the classical tests M due to Bartlett (1937), Hartley's (1950) $F_{\max}$, and G proposed by Cochran (1941) respectively for testing H_0 against H_1. These modifications were obtained by replacing the unrestricted maximum likelihood estimates of σ_i^2 in the classical statistics by the order-constrained maximum likelihood estimates $\hat{\sigma}_i^{*2}$:

$$M^* = k\nu \log \hat{\sigma}^2 - \nu \sum_{i=1}^{k} \log \hat{\sigma}_i^{*2} , \tag{2.1}$$

where $\hat{\sigma}^2$ is the maximum likelihood estimate of the common value σ^2 under H_0,

$$F^*_{\max} = \max_{1 \leq i \leq k} \hat{\sigma}_i^{*2} \Big/ \min_{1 \leq i \leq k} \hat{\sigma}_i^{*2} , \tag{2.2}$$

$$G^* = \max_{1 \leq i \leq k} \hat{\sigma}_i^{*2} \Big/ \sum_{i=1}^{k} \hat{\sigma}_i^{*2} , \qquad G_* = \min_{1 \leq i \leq k} \hat{\sigma}_i^{*2} \Big/ \sum_{i=1}^{k} \hat{\sigma}_i^{*2} . \tag{2.3}$$

Note that several algorithms are available for computing the constrained estimates $\hat{\sigma}_i^{*2}$ $(i = 1, \ldots, k)$; see Robertson, Wright & Dykstra (1988, Ch. 1) for details.

Boswell & Brunk (1969) first proposed M^* as a statistic for this problem and also obtained its large sample null distribution. Fujino (1979) tabulated the null distributions of M^* and $F^*_{\max}$. Fujino also performed a simulation study in order to evaluate the power properties of the above tests as well as a "regression type" test to due Vincent (1961), which is based on the statistic

$$V = \sum_{i=1}^{k} \{i - \tfrac{1}{2}(k+1)\}\hat{\sigma}_i^2 \Big/ \sum_{i=1}^{k} \hat{\sigma}_i^2 , \tag{2.4}$$

and a modified version based on the statistic

$$V' = \sum_{i=1}^{k} \{i - \tfrac{1}{2}(k+1)\} \log \hat{\sigma}_i^2 . \tag{2.5}$$

The test based on M^* was found to have the best overall performance in this investigation.

3 A New Class of Tests

Following the approach given in Mudholkar & McDermott (1989), one may view the problem of testing H_0: $\sigma_1^2 = \cdots = \sigma_k^2$ subject to the simple order constraint $\sigma_1^2 \leq \cdots \leq \sigma_k^2$ as the conjunction of $k-1$ nested problems of testing $H_{0(i)}$: $\sigma_1^2 = \cdots = \sigma_{i-1}^2 = \sigma_i^2$ against the alternative $H_{1(i)}$: $\sigma_1^2 = \cdots = \sigma_{i-1}^2 < \sigma_i^2$ for $i = 2, \ldots, k$. To ease notation, let

$$S_{[i-1]}^2 = \sum_{j=1}^{i-1} \nu_j S_j^2 \Big/ \sum_{j=1}^{i-1} \nu_j ,$$

$$\nu_{[i-1]} = \sum_{j=1}^{i-1} \nu_j \qquad (i = 2, \ldots, k) . \tag{3.1}$$

It is well-known that the uniformly most powerful unbiased test of $H_{0(i)}$ against $H_{1(i)}$ is based on the statistic $F_i = S_i^2/S_{[i-1]}^2$. The construction of the new tests of H_0 against H_1 depends on the following result:

Theorem 1: Under the null hypothesis the $(k-1)$ test statistics F_i are mutually independently distributed as F with ν_i and $\nu_{[i-1]}$ degrees of freedom $(i = 2, \ldots, k)$.

The theorem follows immediately from the following lemma:

Lemma 1: Let $V_1, \ldots, V_n$ be independent gamma random variables with the same scale parameter, i.e.

$$f_{V_i}(v_i) = \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} v_i^{\alpha_i - 1} e^{-\beta v_i} , \qquad v_i > 0$$

for $i = 1, \ldots, n$. Then the random variables

$$W_2 = \frac{V_2}{V_1} , \qquad W_3 = \frac{V_3}{V_1 + V_2} , \ldots, W_n = \frac{V_n}{V_1 + V_2 + \cdots + V_{n-1}}$$

are mutually independent.

Proof: Let $V_1 = W_1$. It follows that

$$V_1 = W_1 , \qquad V_2 = W_1 W_2 , \qquad V_3 = W_1(1 + W_2) W_3, \ldots,$$
$$V_n = W_1(1 + W_2) \ldots (1 + W_{n-1}) W_n .$$

The Jacobian matrix

$$\mathbf{J} = \left(\frac{\partial V_j}{\partial W_i} \right) , \qquad i = 1, \ldots, n, \quad j = 1, \ldots, n ,$$

is upper-triangular, therefore

$$|\mathbf{J}| = \prod_{i=1}^{n} \frac{\partial V_i}{\partial W_i}$$
$$= W_1 (W_1 (1 + W_2)) \cdots (W_1 (1 + W_2) \cdots (1 + W_{n-1}))$$
$$= W_1^{n-1} \prod_{i=2}^{n-1} (1 + W_i)^{n-i} .$$

The probability density function of $\mathbf{W} = (W_1, \ldots, W_n)'$ is then easily shown to be

$$g_{\mathbf{W}}(\mathbf{w}) \propto w_1^{\sum_{i=1}^{n} \alpha_i - 1} \exp\left\{-\beta w_1 \prod_{i=2}^{n} (1 + w_i)\right\} \prod_{i=2}^{n} w_i^{\alpha_i - 1}(1 + w_i)^{\sum_{j=i+1}^{n} \alpha_j} .$$

Hence the joint density of $\mathbf{W}^* = (W_2, \ldots, W_n)'$, obtained by integrating out W_1, is

$$h_{\mathbf{W}^*}(\mathbf{w}^*) \propto \prod_{i=2}^{n} w_i^{\alpha_i - 1}(1 + w_i)^{\sum_{j=i+1}^{n} \alpha_j} \int_0^\infty w_1^{\sum_{i=1}^{n} \alpha_i - 1} \exp\left\{-\beta w_1 \prod_{i=2}^{n} (1 + w_i)\right\} dw_1$$

$$\propto \prod_{i=2}^{n} w_i^{\alpha_i - 1}(1 + w_i)^{-\sum_{j=1}^{i} \alpha_j} .$$

This is clearly the product of the individual densities of $W_2, \ldots, W_n$. Therefore $W_2, \ldots, W_n$ are mutually independent.

Now let P_i be the p-values associated with the test statistics F_i, which by Theorem 1 are mutually independent. The new tests of H_0 subject to the simple order constraint are based on various methods of combining independent p-values. In this article we study four such tests based on the combination statistics $\Psi_T = \min(P_i)$ due to Tippett, $\Psi_F = -2\sum \log P_i$ introduced by Fisher, Liptak's $\Psi_N = \sum \Phi^{-1}(1 - P_i)$, and the logit statistic $\Psi_L = -A^{-1/2}\sum \log\{P_i/(1 - P_i)\}$ proposed by Mudholkar & George (1979), where $A = \pi^2 m(5m + 2)/(15m + 12)$ and $m = k - 1$ is the number of p-values being combined. Small values of Ψ_T and large values of Ψ_F, Ψ_N, and Ψ_L are seen as evidence against the null hypothesis. Under H_0 Ψ_T is distributed as the minimum of m uniform variates, Ψ_F has a χ^2 distribution with $2m$ degrees of freedom, Ψ_N has a $N(0, m)$ distribution, and Ψ_L has a distribution that is very well approximated by Student's t with $5m + 4$ degrees of freedom. Note that Ψ_F and Ψ_L are asymptotically equivalent and optimal among all monotone combination methods, and in some cases among all tests based on the data, in terms of Bahadur's exact slopes; for example see Berk & Cohen (1979).

4 Power Comparisons

Consider the simple case where $k = 3$ and the degrees of freedom are all equal, i.e. $\nu_1 = \nu_2 = \nu_3 = \nu$. To calculate the power of the new tests for H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2$ against the alternative $\sigma_1^2 \leq \sigma_2^2 \leq \sigma_3^2$, with at least one inequality strict, it is necessary to examine the distributions of the component p-values under the alternative hypothesis.

Consider the random variables

$$F_2' = \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2}\ , \qquad F_3' = \frac{2S_3^2/\sigma_3^2}{(S_1^2/\sigma_1^2) + (S_2^2/\sigma_2^2)}\ ,$$

where σ_1^2, σ_2^2, and σ_3^2 cancel under the null hypothesis. Clearly these random variables may be expressed as $F_2' = \mathscr{F}_{\nu,\nu}^{-1}(U_1)$ and $F_3' = \mathscr{F}_{\nu,2\nu}^{-1}(U_2)$, where $\mathscr{F}_{\nu_1,\nu_2}(\cdot)$ is the cumulative distribution function of the F-distribution with ν_1 and ν_2 degrees of freedom and U_1 and U_2 are independent $U(0, 1)$ random variables. Hence, after some manipulation, it is easily seen that the random variables F_2 and F_3 satisfy

$$F_2 \doteq \frac{\sigma_2^2}{\sigma_1^2}\mathscr{F}_{\nu,\nu}^{-1}(U_1)\ , \qquad F_3 \doteq \frac{\sigma_3^2\{1 + \mathscr{F}_{\nu,\nu}^{-1}(U_1)\}}{\sigma_1^2 + \sigma_2^2\mathscr{F}_{\nu,\nu}^{-1}(U_1)}\mathscr{F}_{\nu,2\nu}^{-1}(U_2)\ ,$$

where $\doteq$ denotes equivalence in law. Therefore the component p-values satisfy

$$P_2 \doteq 1 - \mathscr{F}_{\nu,\nu}\left\{\frac{\sigma_2^2}{\sigma_1^2}\mathscr{F}_{\nu,\nu}^{-1}(U_1)\right\}\ , \tag{4.1}$$

$$P_3 \doteq 1 - \mathscr{F}_{\nu,2\nu}\left[\frac{\sigma_3^2\{1 + \mathscr{F}_{\nu,\nu}^{-1}(U_1)\}}{\sigma_1^2 + \sigma_2^2\mathscr{F}_{\nu,\nu}^{-1}(U_1)}\mathscr{F}_{\nu,2\nu}^{-1}(U_2)\right]\ . \tag{4.2}$$

The power function of the new test based on $\Psi_F = -2\log P_2 + -2\log P_3$ may be found directly using (4.1) and (4.2). Note however that this power function is not a simple convolution of $-2\log P_2$ and $-2\log P_3$ because, in general, the component p-values are not mutually independent under the alternative hypothesis. However because P_2 is a function of U_1 alone, the power function may be obtained by conditioning on U_1:

$$\begin{aligned} pr(\Psi_F \geq C) &= pr(-2\log P_2 + -2\log P_3 \geq C) \\ &= pr(P_3 \leq e^{-C/2}/P_2) \\ &= E\{pr(P_3 \leq e^{-C/2}/P_2)|U_1\}\ , \end{aligned}$$

where C is the upper $100\alpha\%$ point of the χ_4^2 distribution. Expressions (4.1) and (4.2) may be substituted above for P_2 and P_3 respectively, leading to the power function

$$1 - \int_0^y \mathscr{F}_{\nu,2\nu}\left(\frac{\sigma_1^2 + \sigma_2^2\mathscr{F}_{\nu,\nu}^{-1}(u)}{\sigma_3^2\{1 + \mathscr{F}_{\nu,\nu}^{-1}(u)\}}\mathscr{F}_{\nu,2\nu}^{-1}\left[1 - \frac{e^{-C/2}}{1 - \mathscr{F}_{\nu,\nu}\{\mathscr{F}_{\nu,\nu}^{-1}(u)\sigma_2^2/\sigma_1^2\}}\right]\right)du\ , \tag{4.3}$$

where

$$y = \mathscr{F}_{\nu,\nu}\{\mathscr{F}_{\nu,\nu}^{-1}(1 - e^{-C/2})\sigma_1^2/\sigma_2^2\} \ .$$

The power functions of Ψ_T, Ψ_N, and Ψ_L may be obtained with some effort in a similar fashion.

The power function (4.3) of Ψ_F and analogous expressions for Ψ_T, Ψ_N, and Ψ_L were numerically evaluated using NAG (1981) fortran library subroutines D01AJF and D01AHF for quadrature. The significance level was taken to be 5% and values of $\nu = 5$, 10 were used. The power function of M^* was estimated using Monte-Carlo simulation. NAG fortran library subroutines G05CCF and G05DDF were used to generate $n_i = \nu + 1$ independent normal random variables with mean 0 and variance σ_i^2, $i = 1, 2, 3$, and using these M^* was calculated. The powers were estimated by calculating the percentage of times out of 100,000 repetitions that M^* exceeded the appropriate critical value. The results of the comparisons of these five power functions are presented in Table 1(a). Various configurations $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ of the variances were used and these are listed in Table 1(b).

Table 1(a). Powers at the 5% level of five competing tests for H_0: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2$ in the model $\sigma_1^2 \le \sigma_2^2 \le \sigma_3^2$

Configuration	ν	M^*	Ψ_F	Ψ_L	Ψ_N	Ψ_T
Null	5	0.050	0.050	0.050	0.050	0.050
	10	0.050	0.050	0.050	0.050	0.050
Step (1, 2)	5	0.404	0.416	0.411	0.396	0.369
	10	0.710	0.729	0.720	0.691	0.656
Step (2, 3)	5	0.454	0.434	0.391	0.348	0.453
	10	0.737	0.702	0.640	0.565	0.728
Linear	5	0.378	0.392	0.402	0.397	0.325
	10	0.644	0.664	0.679	0.674	0.556
Quadratic	5	0.378	0.392	0.401	0.395	0.328
	10	0.640	0.658	0.672	0.666	0.556
Logarithm	5	0.380	0.395	0.404	0.399	0.329
	10	0.654	0.677	0.689	0.682	0.570

Powers are based on 100,000 simulated replications for M^* and are exact for Ψ_F, Ψ_L, Ψ_N, and Ψ_T

Table 1(b). Configurations $(\sigma_1^2, \sigma_2^2, \sigma_3^2)$

Null	(1.00, 1.00, 1.00)
Step (1, 2)	(1.00, 4.00, 4.00)
Step (2, 3)	(1.00, 1.00, 4.00)
Linear	(1.00, 2.50, 4.00)
Quadratic	(1.00, 2.25, 4.00)
Logarithm	(1.00, 2.89, 4.00)

Table 2(a). Estimated powers at the 5% level of five competing tests for H_0: $\sigma_1^2 = \cdots = \sigma_6^2$ in the model $\sigma_1^2 \leq \cdots \leq \sigma_6^2$

Configuration	v	M^*	Ψ_F	Ψ_L	Ψ_N	Ψ_T
Null	5	0.050	0.050	0.051	0.050	0.051
	10	0.050	0.050	0.050	0.050	0.051
Step (1, 2)	5	0.367	0.375	0.359	0.344	0.282
	10	0.694	0.706	0.661	0.616	0.540
Step (3, 4)	5	0.707	0.699	0.628	0.571	0.609
	10	0.949	0.943	0.900	0.847	0.882
Step (5, 6)	5	0.501	0.415	0.331	0.259	0.481
	10	0.776	0.673	0.553	0.411	0.752
Linear	5	0.460	0.490	0.494	0.483	0.315
	10	0.742	0.773	0.790	0.781	0.508
Quadratic	5	0.497	0.512	0.504	0.485	0.357
	10	0.777	0.782	0.782	0.764	0.574
Logarithm	5	0.418	0.462	0.471	0.460	0.282
	10	0.707	0.758	0.774	0.767	0.470

Table 2(b). Configurations $(\sigma_1^2, \ldots, \sigma_6^2)$

Null	(1.00, 1.00, 1.00, 1.00, 1.00, 1.00)
Step (1, 2)	(1.00, 4.00, 4.00, 4.00, 4.00, 4.00)
Step (2, 3)	(1.00, 1.00, 1.00, 4.00, 4.00, 4.00)
Step (5, 6)	(1.00, 1.00, 1.00, 1.00, 1.00, 4.00)
Linear	(1.00, 1.60, 2.20, 2.80, 3.40, 4.00)
Quadratic	(1.00, 1.26, 1.69, 2.29, 3.06, 4.00)
Logarithm	(1.00, 2.16, 2.84, 3.32, 3.69, 4.00)

A second simulation study was done in order to compare the five power functions for $k = 6$, again at the 5% level of significance and for $v = 5$, 10. The simulation was performed as described above, except that $N = 50{,}000$ repetitions were used for this study. The results and the configurations $(\sigma_1^2, \ldots, \sigma_6^2)$ of the variances employed are listed in Tables 2(a) and 2(b) respectively. These configurations are taken to be the same as in Fujino (1979) to facilitate comparison with his results.

For most of the configurations considered, the test based on Fisher's combination method is superior to the modification of Bartlett's test. As seen in Mudholkar & McDermott (1989) the tests based on the Logit and Liptak combination methods perform quite well for some configurations but quite poorly for others. The test based on Tippett's combination method is generally unsatisfactory.

It should be noted that the application of these combination methods in this setting is not invariant. Indeed one could proceed by first testing equality of any two adjacent variances, then one could test equality of their common value with an adjacent variance, and continue in this fashion. Clearly the power of the overall test will depend on the way in which the procedure is carried out. This in part explains why the tests based on combination methods outperform the modification of Bartlett's test for some configurations, but not for others.

5 Applications to Other Partial Orders

The new tests proposed in this article can be extended to cases where the variances are constrained by partial orders other than simple order. As an illustration of the necessary notation, consider the constraints

$$\sigma_1^2 \leq \sigma_2^2 \ ; \qquad \sigma_3^2 \leq [\sigma_4^2, \sigma_5^2, \sigma_6^2] \leq [\sigma_7^2, \sigma_8^2] \ .$$

Here the eight variances are divided into two *blocks*, where the separation is indicated by a semicolon, and it is understood that there are no order restrictions among the variances in different blocks. The inclusion of variances within a bracket, such as $[\sigma_4^2, \sigma_5^2, \sigma_6^2]$, implies a lack of any order restrictions among them. More generally, the k variances will be divided into blocks such that there are no order restrictions among the variances in different blocks, but the variances within each block will be restricted in some manner.

To test the overall null hypothesis, first conduct the following steps for each block separately:

Step 1: Use Bartlett's test for testing equality of the variances included within a bracket. Obtain the corresponding significance probability for each bracket.
Step 2: Assume that all variances within each bracket are equal, thus yielding a simple order structure for the block. If there are, say, r inequalities in the block, obtain the $r-1$ significance probabilities using the method described above for the case of simple order.

Having treated each of the blocks in this manner, assume equality within each block and use Bartlett's statistic to test equality of the variances between blocks. The component p-values involved in the procedure can be easily shown to be mutually independent. They can therefore be combined as described above in order to test equality of the variances.

In the above example, the new testing procedure would result in six component p-values arising from the following tests: (1) Bartlett's test for H_0: $\sigma_4^2 = \sigma_5^2 = \sigma_6^2$; (2) Bartlett's test for H_0: $\sigma_7^2 = \sigma_8^2$; (3) F-test for H_0: $\sigma_1^2 = \sigma_2^2$; (4) F-test for equality of σ_3^2 and the assumed common value of σ_4^2, σ_5^2, and σ_6^2; (5) F-test for equality of the common value of $\sigma_3^2, \ldots, \sigma_6^2$ and the common value of σ_7^2 and σ_8^2; (6) Bartlett's test for equality of the common value of σ_1^2 and σ_2^2 and the common value of $\sigma_3^2, \ldots, \sigma_8^2$.

The independence of the component test statistics relies on the following result (e.g. see Johnson and Kotz (1970)): if $X_1, \ldots, X_k$ are independent gamma random variables having the same scale parameter, then the two random variables $\sum X_i$ and $X_j/\sum X_i$ are independent for each j ($j = 1, \ldots, k$). As an illustration of the use of this result in proving independence consider the case where the variances are subject to the simple tree order, $\sigma_1^2 \le [\sigma_2^2, \ldots, \sigma_k^2]$. The Bartlett statistic used for testing equality of $\sigma_2^2, \ldots, \sigma_k^2$ at the first stage is easily seen to be a simple function of

$$Q_1 = \prod_{i=2}^{k} \left\{ \sum_{j=2}^{k} \nu_j S_j^2 / S_i^2 \right\}^{\nu_i} .$$

The above property of gamma random variables can be applied directly to show that Q_1 is independent of the F-statistic used at the second stage, which is a simple function of

$$Q_2 = \sum_{i=2}^{k} \nu_i S_i^2 / S_1^2 .$$

A combination statistic Ψ may then be applied to the two associated component p-values in order to test equality of the k variances.

When testing equality of unrestricted variances in the above procedure, it is apparent that one may apply either Hartley's or Cochran's test instead of Bartlett's without sacrificing independence of the test statistics. This follows from the fact that, for independent gamma random variables having the same scale parameter, $\sum X_i$ will be independent of max $X_i/\sum X_i$, min $X_i/\sum X_i$, and their ratio max X_i/min X_i.

The new procedures may be extended further to more complex situations where the variances are subject to more general restrictions such as $\sigma_1^2 \le [\sigma_3^2; \sigma_7^2; \sigma_2^2 \le [\sigma_4^2, \sigma_5^2, \sigma_6^2]]$. Here groups of variances separated by semicolons are understood to have no order restrictions among them. In this case equality of the seven variances may be tested by first comparing $\sigma_2^2, \ldots, \sigma_7^2$ subject to the constraint to the right of σ_1^2, and then using an F-test to compare the common value of $\sigma_2^2, \ldots, \sigma_7^2$ with σ_1^2. An example of an ordering for which the new procedures may not be applied is $\sigma_1^2 \le [\sigma_2^2, \sigma_4^2]$; $\sigma_3^2 \le \sigma_4^2$, where σ_4^2 appears in both blocks. In general this new approach may be applied whenever a particular variance appears in one and only one block, not for general partial orders.

6 Conclusions

Until now satisfactory procedures for testing homogeneity of normal variances subject to order constraints have existed only for the simple order restriction and equal sample sizes. The new approach to the problem outlined in this paper is easy to implement for arbitrary sample sizes and may be applied to a wide variety of order restrictions. The power comparisons show that, in particular, the proposed test based on Fisher's combination method compares favorably with the best of the existing tests, namely the modification of Bartlett's test.

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