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A Note on Three-Stage Confidence Intervals for the Difference of Locations: The Exponential Case

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Abstract: Fixed-width confidence intervals for the difference of location parameters of two independent negative exponential distributions are constructed via triple sampling when the scale parameters are unknown and unequal. The present three-stage estimation methodology is put forth because (i) it is operationally more convenient than the existing purely sequential counterpart, and (ii) the three-stage and the purely sequential estimation techniques have fairly similar asymptotic second-order characteristics.

Key Words and Phrases: Fixed-width intervals; Behrens-Fisher situation; two-parameter exponentials; triple sampling; second-order expansions.

1 Introduction

Let $\{X_{i1}, X_{i2}, \dots\}$, $i = 1, 2$ be two independent sequences of random variables where we assume that $X_{i1}, X_{i2}, \dots$ are independent and identically distributed (i.i.d.) having the probability density function $f(x; \mu_i, \sigma_i) = \sigma_i^{-1} \exp\{-(x - \mu_i)/\sigma_i\}$ $I(x > \mu_i)$, $0 < \sigma_i < \infty$, $-\infty < \mu_i < \infty$, $i = 1, 2$. Here and elsewhere, $I(\cdot)$ stands for the indicator function of $(\cdot)$. The particular type of distribution considered here has been used widely in many reliability and life testing experiments to describe the failure times of complex equipment, vacuum tubes etc. This distribution has also been suggested as a statistical model in several clinical trials, such as studies of behavior of tumor systems in animals and analysis of survival data in cancer research. One is referred to Zelen (1966). One is also referred to Mukhopadhyay (1988) for other citations.

The parameters μ_1, μ_2, when positive, may be interpreted as the minimum guarantee times or the thresholds of the distributions. The parameters σ_1, σ_2 are known as the scales of the distributions. We assume that all four parameters are

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unknown and our goal is to construct fixed-width confidence intervals for the parameter $\delta = \mu_1 - \mu_2$.

Mukhopadhyay and Hamdy (1984) proposed a two-stage procedure that came with the guarantee of the nominal level for the probability of coverage of δ by means of the constructed fixed-width confidence interval. On the average, however, the two-stage procedures have a tendency to oversample. Hence, in the same paper of Mukhopadhyay and Hamdy (1984), they had also proposed purely sequential procedures in order to achieve "efficiency". All these techniques were proposed and studied when σ_1, σ_2 are unknown and unequal. In the case when $\sigma_1 = \sigma_2 = \sigma$, say, but σ is unknown, Mukhopadhyay and Mauromoustakos (1987) had proposed a three-stage fixed-width confidence interval for δ, motivated by the works of Hall (1981).

In the present note, we design three-stage sampling procedures for constructing fixed-width confidence intervals for δ when σ_1, σ_2 are unknown and unequal. Section 2 gives some of the preliminaries and the main result (Theorem 1) that sets the rate of convergence of the difference between the achieved confidence level and the target value. The proof of Theorem 1 is included in Section 3.

2 Formulation and Preliminaries

Having recorded $X_{i1}, \ldots, X_{in_i}$ we write $X_{in_i(1)} = \min\{X_{i1}, \ldots, X_{in_i}\}$ and $U_{in_i} = (n_i - 1)^{-1} \sum_{j=1}^{n_i} (X_{ij} - X_{in_i(1)})$ for $n_i \geq 2$ which respectively estimates μ_i and σ_i, $i = 1, 2$. Let $\underset{\sim}{n} = (n_1, n_2)$ and given a preassigned number $d(> 0)$ we propose the fixed-width confidence interval

$$J(\underset{\sim}{n}) = [X_{1n_1(1)} - X_{2n_2(1)} \pm d] \tag{2.1}$$

for the parameter $\delta(= \mu_1 - \mu_2)$. We are also given a preassigned number $\alpha \in (0, 1)$ and we wish to conclude that $P\{\delta \in J(\underset{\sim}{n})\} \geq 1 - \alpha$ for all σ_1, σ_2. Now, with $a = \ln(1/\alpha)$, $n_i \geq a\sigma_i/d = C_i$, say, $P\{\delta \in J(\underset{\sim}{n})\}$ can be shown to be at least $(1 - \alpha)$. But C_1, C_2 are unknown and hence certain multistage procedures are in order. Mukhopadhyay and Hamdy (1984) considered "efficient" purely sequential sampling techniques. At present, we consider proposing three-stage estimation procedures so that these will be operationally more convenient than the purely sequential counterpart, and yet these three-stage methodologies and the purely sequential ones will have very similar asymptotic characteristics. Throughout, asymptotics are carried out as $d \to 0$.

2.1 A Three-Stage Methodology

One starts with $X_{i1}, \ldots, X_{im}$ where the starting sample size $m(\geq 2)$ is such that $m = O(d^{-1/r})$ for some $r > 1$. We also choose and fix two numbers, $0 < \rho_1, \rho_2 < 1$. Now, let

$$T_i = T_i(d) = \max\{m, \langle a\rho_i U_{im} d^{-1}\rangle + 1\} , \tag{2.2}$$

$$N_i = N_i(d) = \max\{T_i, \langle aU_{iT_i} d^{-1} + \varepsilon_i\rangle + 1\} , \tag{2.3}$$

where $\langle x\rangle$ stands for the largest integer $< x$ and ε_1, ε_2 are *known* real numbers which are to be defined precisely in the sequel, $i = 1, 2$. Let us write $\underset{\sim}{N} = (N_1, N_2)$.

The three-stage methodology (2.2)–(2.3) is implemented as follows. Based on $X_{i1}, \ldots, X_{im}$, one determines T_i which estimates $\rho_i C_i$, a fraction of C_i. If $T_i = m$, then one does not take any more samples from X_i in the second stage, however if $T_i > m$, then one samples the difference $(T_i - m)$ from X_i in the second stage arriving at $X_{i1}, \ldots, X_{iT_i}$. Now, one determines N_i based on $X_{i1}, \ldots, X_{iT_i}$. Notice that N_i estimates C_i. If $N_i = T_i$, then one does not take any more samples from X_i in the third stage, however if $N_i > T_i$, then one samples the difference $(N_i - T_i)$ from X_i in the third stage, finally arriving at $X_{i1}, \ldots, X_{im}, \ldots, X_{iT_i}, \ldots, X_{iN_i}$. We then propose the fixed-width confidence interval $J(\underset{\sim}{N})$, defined in (2.1), for δ.

2.2 Preliminary Derivations

Notice that $I(\underset{\sim}{N} = \underset{\sim}{n})$ is independent of $(X_{1n_1(1)}, X_{2n_2(1)})$ for all fixed $\underset{\sim}{n}$, and hence

$$P\{\delta \in J(\underset{\sim}{N})\} = E[g(N_1/C_1, N_2/C_2)] \tag{2.4}$$

where

$$g(x, y) = 1 - [x^{-1} + y^{-1}]^{-1}[x^{-1}e^{-ax} + y^{-1}e^{-ay}] , \tag{2.5}$$

for $x > 0$, $y > 0$. It follows easily that $N_i \to \infty$ a.s., $N_i/C_i \to 1$ a.s. and hence $P\{\delta \in J(\underset{\sim}{N})\} \to 1 - e^{-a} = 1 - \alpha$ as $d \to 0$, that is, the three-stage fixed-width confidence interval procedure is asymptotically consistent in the Chow-Robbins (1965) sense. Our aim is to show that the difference, $P\{\delta \in J(\underset{\sim}{N})\} - (1 - \alpha)$, can be made $o(d)$ if one chooses the "fine-tuning" factors ε_1 and ε_2 in (2.3) appropriately.

The three-stage procedure (2.2)–(2.3), without the "fine-tuning" factors ε_i, is of the form considered in general in (2.1)–(2.2) of Mukopadhyay (1990) where $\gamma = \rho_i$, $\lambda = a/d$, $\theta = \sigma_i$ and Y_{ij}'s are i.i.d. $\frac{1}{2}\sigma_i\chi_2^2$. Hence Mukhopadhyay's (1990) Theorem 2 implies

$$E(N_i) = C_i + (\eta_i + \varepsilon_i) + o(d) \ , \tag{2.6}$$

where $\eta_i = \frac{1}{2} - \rho_i^{-1}$, $i = 1, 2$. Also, Theorem 3(i) of Mukhopadhyay (1990) shows that

$$P(N_i \le \xi C_i) = O(C_i^{-Lm}) \ , \tag{2.7}$$

for arbitrarily large, but fixed, $L(>0)$. Let us write $N_i^* = C_i^{-1/2}(N_i - C_i)$, and then Mukhopadhyay's (1990) Theorem 3(ii) shows that

$$N_i^* \xrightarrow{\mathscr{L}} N(0, p_i) \ , \tag{2.8}$$

where $p_i = \rho_i^{-1}$. Also, one will observe that

$$|N_i^*|^s \text{ is uniformly integrable, } s = 1, 2, 3 \ . \tag{2.9}$$

2.3 *Main Result*

Theorem 1: In the three-stage estimation methodology (2.2)–(2.3), *suppose that one chooses* $\varepsilon_i = \frac{1}{2}(a + 3 - \rho_i)/\rho_i$, $i = 1, 2$. *Then, we have as* $d \to 0$,

$$P\{\delta \in J(\underset{\sim}{N})\} = (1 - \alpha) + o(d) \ .$$

In other words, $P\{\delta \in J(\underset{\sim}{N})\}$ is approximately $(1 - \alpha)$ up to the order $o(d)$. Note that (2.6) already gives the asymptotic second-order expansion of $E(N_i - C_i)$.

3 Proof of Theorem 1

We expand $g(x, y)$ given in (2.5) around $(x, y) = (1, 1)$ and hence for some random variables W_i between 1 and N_i/C_i, $i = 1, 2$, one obtains from (2.4),

$$P\{\delta \in J(\underset{\sim}{N})\} = (1 - e^{-a}) + \frac{1}{2}de^{-a} \sum_{i=1}^{2} \sigma_i^{-1} E(N_i - C_i)$$

$$- \frac{1}{4}de^{-a}(1 + a) \sum_{i=1}^{2} \sigma_i^{-1} E(N_i^{*2}) + \frac{1}{2}de^{-a}(\sigma_1\sigma_2)^{-1} E(N_1^* N_2^*)$$

$$+ E\left[\frac{1}{6}d^{3/2}a^{-3/2} \sum_{i=1}^{2} \sigma_i^{-3/2} e^{-aW_i}(a^{-1}V^{-4}W_i^{-4} + 4a^{-2}V^{-4}W_i^{-5}\right.$$

$$\left. - 4a^{-3}V^{-5}W_i^{-6})N_i^{*3}\right] + E\left[\frac{1}{6}d^{3/2}a^{-3/2} \sum_{1 \le i \ne j \le 2} \sigma_i^{-3/2} e^{-aW_i}\right.$$

$$\{V^{-3}(2a^{-2}W_i^{-3}W_j^{-2} + 6a^{-3}W_i^{-4}W_j^{-2} - 6a^{-3}W_i^{-2}W_j^{-4})$$

$$+ V^{-4}(3a^{-1}W_i^{-3}W_j^{-1} + 3a^{-1}W_i^{-2}W_j^{-2} + a^{-1}W_i^{-1}W_j^{-3}$$

$$+ 11a^{-2}W_i^{-4}W_j^{-1} + 3a^{-2}W_i^{-2}W_j^{-3} + 10a^{-2}W_i^{-3}W_j^{-2}$$

$$+ 8a^{-3}W_i^{-5}W_j^{-1} + 4a^{-3}W_i^{-3}W_j^{-3} + 12a^{-1}W_i^{-4}W_j^{-2}$$

$$+ 6a^{-4}W_i^{-2}W_j^{-5} - 6a^{-4}W_i^{-5}W_j^{-2}) + V^{-5}(12a^{-3}W_i^{-5}W_j^{-1}$$

$$+ 12a^{-3}W_i^{-4}W_j^{-2} + 4a^{-3}W_i^{-3}W_j^{-3} + 8a^{-4}W_i^{-6}W_j^{-1}$$

$$\left. + 8a^{-4}W_i^{-4}W_j^{-3} + 16a^{-4}W_i^{-5}W_j^{-2})\}N_i^{*3}\right]$$

$$+ E\left[\frac{1}{6}d^{3/2}a^{-3/2} \sum_{1 \le i \ne j \le 2} \sigma_i^{-1}\sigma_j^{-1/2} e^{-aW_j}(6a^{-4}V^{-4}W_i^{-3}W_j^{-4}\right.$$

$$\left. - 2a^{-2}V^{-3}W_i^{-3}W_j^{-2} - 4a^{-3}V^{-3}W_i^{-3}W_j^{-3})N_i^{*2}N_j^*\right]$$

$$+ E\left[\frac{1}{6}d^{3/2}a^{-3/2} \sum_{1 \le i \ne j \le 2} \sigma_i^{-1}\sigma_j^{-1/2} e^{-aW_i}\{V^{-4}(3a^{-2}W_i^{-3}W_j^{-2}\right.$$

$$+ 6a^{-2}W_i^{-3}W_j^{-3} + 3a^{-2}W_i^{-1}W_j^{-4} + 2a^{-3}W_i^{-4}W_j^{-2} + 6a^{-3}W_i^{-1}W_j^{-4}$$

$$+ 8a^{-3}W_i^{-3}W_j^{-3} - 6a^{-4}W_i^{-3}W_j^{-4}) + 4a^{-1}V^{-3}W_i^{-2}W_j^{-3}$$

$$- V^{-5}(4a^{-3}W_i^{-4}W_j^{-2} + 12a^{-3}W_i^{-3}W_j^{-3} + 12a^{-3}W_i^{-2}W_j^{-4}$$

$$+ 4a^{-3}W_i^{-1}W_j^{-5} + 8a^{-4}W_i^{-4}W_j^{-3} + 8a^{-4}W_i^{-2}W_j^{-5}$$

$$\left. + 4a^{-4}W_i^{-3}W_j^{-4})\} N_i^{*2}N_j^{*} \right]$$

$$= (1-\alpha) + E(R_1 + R_2) - E(R_3 + R_4) + E(R_5) + E(R_6 + R_7)$$

$$+ E\left[\sum_{1 \le i \ne j \le 2} R_{8,i,j}\right] + E\left[\sum_{1 \le i \ne j \le 2} R_{9,i,j}\right] + E\left[\sum_{1 \le i \ne j \le 2} R_{10,i,j}\right], \quad \text{say,} \tag{3.1}$$

where

$$V = (W_1^{-1} + W_2^{-1})/a \ .$$

From (2.6), one gets

$$E(R_i) = \tfrac{1}{2} d e^{-a} \sigma_i^{-1}(\eta_i + \varepsilon_i) + o(d) \ , \qquad i = 1, 2 \tag{3.2}$$

and from (2.8)–(2.9) one has

$$E(R_i) = \tfrac{1}{4} d e^{-a}(1 + a)\sigma_i^{-1} p_i + o(d) \ , \qquad i = 3, 4 \ . \tag{3.3}$$

Then, note that

$$E(R_5) = \tfrac{1}{2} d e^{-a}(\sigma_1\sigma_2)^{-1} E(N_1^*)E(N_2^*)$$

$$= \tfrac{1}{2} d e^{-a}(\sigma_1\sigma_2)^{-1}(\eta_1 + \varepsilon_1 + o(1))(\eta_2 + \varepsilon_2 + o(1))(C_1 C_2)^{-1/2}$$

$$= O(d^2) + o(d^2) = o(d) \ , \tag{3.4}$$

in view of (2.6).

The terms R_6, R_7, $R_{8,i,j}$ are handled in similar fashions and note that each is a linear combination of expressions of which a typical one can be written as

$$L_{ij} = B d^{3/2} e^{-aW_i} V^{-i_1} W_i^{-i_2} W_j^{-i_3} N_i^{*3} \tag{3.5}$$

where i_1, i_2, i_3 are fixed non-negative integers. Here and elsewhere we write B as a positive generic constant, independent of d. The aim is to verify that $E(L_{ij}) = o(d)$ for all appropriate i, j. For brevity, we only show how to handle $E(L_{12})$. For $0 < \varepsilon < 1$, define $A_1 = [N_1 > \varepsilon C_1] \cap [N_2 > \varepsilon C_2]$, $A_2 = [N_1 > \varepsilon C_1] \cap [N_2 \le \varepsilon C_2]$, $A_3 = [N_1 \le \varepsilon C_1] \cap [N_2 > \varepsilon C_2]$, and $A_4 = [N_1 \le \varepsilon C_1] \cap [N_2 \le \varepsilon C_2]$. Now,

$$|L_{12}|I(A_1) \le Bd^{3/2}\varepsilon^{-(i_2+i_3)}|N_1^*|^3 \text{ and hence}$$

$$|E(L_{12}I(A_1))| = O(d^{3/2}) = o(d) \tag{3.6}$$

in view of (2.9). Also,

$$|L_{12}I(A_2)| \le Bd^{3/2}(C_2/N_2)^{i_3}|N_1^*|^3 I(A_2) \ , \qquad \text{and thus}$$

$$\begin{aligned} |E(L_{12}I(A_2))| &\le Bd^{(3/2)-i_3}E[|N_1^*|^3 I(N_1 > \varepsilon C_1)]P(N_2 \le \varepsilon C_2) \\ &= O(d^{Lm+(3/2)-i_3}) = o(d) \ , \end{aligned} \tag{3.7}$$

in view of (2.7) and since "L" can be chosen large. Again,

$$\begin{aligned} |E(L_{12}I(A_3))| &\le Bd^{3/2}E[|N_1^*|^3(C_1/N_1)^{i_3}I(A_3)] \\ &\le Bd^{-i_2}\int_{A_3}\left(1-\frac{N_1}{C_1}\right)^3 dP \le Bd^{-i_2}P(A_3) \\ &= O(d^{Lm-i_2}) = o(d) \ , \end{aligned} \tag{3.8}$$

along the lines of (3.7). Similarly,

$$\begin{aligned} |E(L_{12}I(A_4))| &\le Bd^{-i_2-i_3}P(N_1 \le \varepsilon C_1)P(N_2 \le \varepsilon C_2) \\ &= O(d^{2Lm-i_2-i_3}) = o(d) \ , \end{aligned} \tag{3.9}$$

along the lines of (3.7). Now, combining (3.6)–(3.9), we claim that $E(L_{12}) = o(d)$. In fact, the terms $R_{9,i,j}$ and $R_{10,i,j}$ also can be evaluated in similar fashions and shown to be $o(d)$, since N_i^* and N_j^* are indeed independent random variables.

Now, combining all these with (3.1)–(3.9), one obtains

$$P\{\delta \in J(\underset{\sim}{N})\} = (1-\alpha) + \frac{1}{2}de^{-a}\sum_{i=1}^{2}\left\{(\eta_i + \varepsilon_i) - \frac{1}{2}(1+a)p_i\right\}\sigma_i^{-1} + o(d)$$

which simplifies to $(1 - \alpha) + o(d)$ if one chooses

$$\varepsilon_i = \frac{1}{2}(1 + a)p_i - \eta_i = \frac{1}{2}(3 + a - \rho_i)/\rho_i \ , \qquad i = 1, 2 \ .$$

Remark 1: Suppose now one chooses ε_1, ε_2 as given in Theorem 1 and obtains $X_{i1}, \ldots, X_{iN_i}$, $i = 1, 2$ by implementing the three-stage methodology (2.2)–(2.3). Let us then consider separate confidence intervals for μ_1, μ_2 and propose the natural fixed-width confidence interval $J_i(N_i) = [X_{iN_i(1)} - d, X_{iN_i(1)}]$ for the location parameter μ_i, $i = 1, 2$. From Theorem 2 of Mukhopadhyay and Mauromoustakos (1987), we can immediately conclude that $P\{\mu_i \in J_i(N_i)\} \geq (1 - \alpha) + o(d)$.

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References

Chow YS, Robbins H (1965) On the asymptotic theory of fixed-width sequential confidence intervals for the mean. *Ann. Math. Statist.* 36:457–462

Hall P (1981) Asymptotic theory of triple sampling for sequential estimation of a mean. *Ann. Statist.* 9:1229–1238

Mukhopadhyay N (1988) Sequential estimation problems for negative exponential populations. *Commun. Statist., Theory and Methods (Reviews Section)* 17:2471–2506

Mukhopadhyay N (1990) Some properties of a three-stage procedure with applications in sequential anslysis. *Sankhya, Ser. A* 51:218–231

Mukhopadhyay N, Hamdy HI (1984) On estimating the difference of location parameters of two negative exponential distributions. *Canad. J. Statist.* 12:67–76

Mukhopadhyay N, Mauromoustakos A (1987) Three-stage estimation procedures for the negative exponential distributions. *Metrika* 34:83–93

Zelen M (1966) Application of exponential models to problems in cancer research. *J. Roy. Statist. Soc.* 129:368–398

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