ON THE NUMERICAL APPROACH TO A TWO-PHASE STEFAN PROBLEM WITH NON-LINEAR FLUX

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ABSTRACT - This paper is devoted to the numerical analysis of a multidimensional two-phase Stefan problem, with a non-linear flux condition on the fixed boundary; the enthalpy formulation is used. A numerical approach suggested by the theory of non-linear semigroup of contractions in $L^1(\Omega)$ is introduced; some converging algorithms based on the Crandall-Liggett formula and on the non-linear Chernoff formula are studied. The algebraic non-linear equations are solved by a modified Gauss-Seidel method. The results of several numerical tests are exhibited and discussed.

1. Introduction.

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set, with smooth boundary; we fix $T>0$ and set $Q=\Omega\times]0,T[$, $\Sigma=\Gamma\times]0,T[$. We introduce directly the enthalpy formulation of the two-phase Stefan problem with non-linear flux:

$$
(P)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta\beta(u) = 0 & \text{in } Q \\[2ex] \dfrac{\partial \beta(u)}{\partial \nu} + g(\beta(u)) = 0 & \text{on } \Sigma \\[2ex] u(0) = u_0 & \text{in } \Omega \end{cases}
$$

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where u denotes the enthalpy density; $\theta=\beta(u)$ is the temperature. The non-decreasing function β is characteristic of the material; β can be assumed of the form

$$\beta(\xi) = \begin{cases} \alpha_1(\xi-L) & \xi > L \\ 0 & 0 \leq \xi \leq L, \qquad \xi \in \mathbb{R} \\ \alpha_2\xi & \xi < 0 \end{cases} \tag{1}$$

L being the latent heat; thus u>L corresponds to the liquid phase, u<0 to the solid phase, and $\theta=0$ is the phase transition temperature. Finally

(2) $g : \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing, of class C^0 and $g(0) = 0$.

In the case of a single space variable, the classical formulation of this problem has already been extensively studied starting from [22]. In the case of several space variables, weak formulations in terms of enthalpy or of freezing index have been considered (see [33, 44] for references). When g is *non-linear*, weak variational formulations of the problem have been studied recently [46, 37, 12]. Many converging algorithms and error estimates are available for the numerical approximation of the problem with a *linear* g or Dirichlet data: the approximation for the enthalpy formulation can be found in [27, 26, 38, 39, 20, 3, 14, 35, 47, 48, 49]; the approximation for the freezing index formulation is studied in [7, 43]. For a *non-linear* g, some numerical algorithms have been proposed, with no theoretical justification and no error estimates (see [33] for references). As far as the author knows, convergence of an algorithm has been proved for the first time in [45] (see also [33]). Recently error estimates have been given (see [40]).

A weak approach based on the theory of non-linear semigroups of contractions in $L^1(\Omega)$ has been introduced by Brézis for Dirichlet conditions; the numerical approximation has been studied in this framework in [6]. A similar approach for the problem with *non-linear* g is given in [34, 51, 52], following ideas and techniques used for instance in [5, 6, 9, 16, 17]. The non-linear operator $A : \varphi \rightarrow -\Delta\beta(\varphi)$ with domain

$$\begin{aligned} D(A) = \{&\varphi \in L^1(\Omega) : \beta(\varphi) \in W^{1,1}(\Omega), \Delta\beta(\varphi) \in L^1(\Omega), g(\beta(\varphi)) \in L^1(\Gamma), \\ &\frac{\partial\beta(\varphi)}{\partial\nu} + g(\beta(\varphi)) = 0 \text{ on } \Gamma \text{ in the «weak» sense}\} \end{aligned} \tag{3}$$

is m-accretive in $L^1(\Omega)$. Hence $-A : D(A) \to L^1(\Omega)$ generates a non-linear semigroup of contractions denoted by $S(t)$, that can be defined by Crandall-Liggett's formula:

$$(4) \qquad \forall\, u_0 \in L^1(\Omega), \quad S(t)u_0 = \lim_{k\to\infty} \left(I + \frac{t}{k} A\right)^{-k} u_0 \quad \text{uniformly in } [0,T].$$

Then $u(t) = S(t)u_0 \in C^0([0,T];L^1(\Omega))$ and $u(t)$ is the generalized solution of problem (P) in the sense of Crandall-Liggett [15] and Bénilan [4].

This approach seems to be expecially useful also for numerical purposes; infact Crandall-Liggett's formula and other formulae of the type of Chernoff suggest several converging algorithms, as we shall see in this paper following the suggestion of [6].

2. **Algorithms based on the theory of non-linear semigroups of contractions.**

We first present the algorithms without spatial discretizations.

Let $\lambda = \frac{T}{n}$, $n \geq 1$, be the time-step. Let $\sigma_\lambda :]0,\infty[\to]0,\infty[$ be a function s.t.

$$(5) \qquad \lim_{\lambda\to 0} \sigma_\lambda = 0 \quad \text{and} \quad \sigma_\lambda \geq \alpha\lambda \quad \text{for each } \lambda > 0$$

where $\alpha = \max(\alpha_1, \alpha_2)$ is the Lipschitz constant of β.

$G(f^i)$ denotes the piecewise constant function

$$(6) \qquad G(f^i)(\cdot,t) = \begin{cases} f^0(\cdot) & \text{for } t = 0 \\ f^i(\cdot) & \text{for } t \in](i-1)\lambda, i\lambda],\ i = 1,\dots,n. \end{cases}$$

Let u_0 be in $L^1(\Omega)$.

2.1. *Algorithm (S1) based on Crandall-Liggett's formula.*

We consider the following algorithm

$$w_\lambda^0 = u_0$$

$$(7) \qquad \begin{cases} w_\lambda^{i+1} - \lambda\Delta\beta(w_\lambda^{i+1}) = w_\lambda^i & \text{in } \Omega \\ \dfrac{\partial\beta(w_\lambda^{i+1})}{\partial\nu} + g(\beta(w_\lambda^{i+1})) = 0 & \text{on } \Gamma \end{cases} \qquad i = 0,1,\dots,n-1$$

that is, if we define $J(t)\varphi=(I+tA)^{-1}\varphi$, $\varphi\in L^1(\Omega)$, then $w_\lambda^{i+1}=J^{i+1}(\lambda)u_0$.

Remark 1. One can prove that, if $u_0\in L^\infty(\Omega)$, then

(8) $$\sup_i \|w_\lambda^i\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}.$$

Sketch of the *Proof.* Let be $M_2\leq u_0\leq M_1$ a.e. in Ω, $M_2\leq 0$, $M_1\geq L$. Define the positive function:

$$\Phi(\xi) = \begin{cases} (\xi-M_1)^2 & \text{if } \xi\geq M_1 \\ 0 & \text{if } M_2\leq\xi\leq M_1 \\ (\xi-M_2)^2 & \text{if } \xi\leq M_2 \end{cases}.$$

Multiplying (7) by $v_\lambda^{i+1}=(\beta(w_\lambda^{i+1})-\beta(M_1))^+-(\beta(w_\lambda^{i+1})-\beta(M_2))^-$, integrating on Ω and summing on i, we get:

$$\frac{1}{\lambda}\sum_{i=0}^{m-1}\int_\Omega (w_\lambda^{i+1}-w_\lambda^i)v_\lambda^{i+1}dx \geq \frac{1}{2\lambda}\|\Phi(w_\lambda^m)\|^2_{L^2(\Omega)}$$

$$\sum_{i=0}^{m-1}\int_\Omega \nabla\beta(w_\lambda^{i+1})\cdot\nabla v_\lambda^{i+1}dx = \sum_{i=0}^{m-1}\int_\Omega |\nabla v_\lambda^{i+1}|^2dx \geq 0$$

$$\sum_{i=0}^{m-1}\int_\Gamma g(\beta(w_\lambda^{i+1}))v_\lambda^{i+1}d\sigma \geq 0$$

for each $m\leq n$, whence $\Phi(w_\lambda^m)=0$ a.e. in Ω, i.e. $M_2\leq w_\lambda^m\leq M_1$ in Ω, i.e. (8). □

Remark 2. If $u_0\in D(A)$, then the convergence result (4) can be improved as follows:

(9) $$\sup_i \|w_\lambda^i(\cdot)-u(\cdot,i\lambda)\|_{L^1(\Omega)} \leq C\sqrt{\lambda},\quad \text{C depending upon } u_0.$$

In fact, we have ([15]):

$$\left\|\left(I+\frac{t}{i}A\right)^{-i}u_0-u(\cdot,t)\right\|_{L^1(\Omega)} \leq \frac{2t}{\sqrt{i}}\|Au_0\|_{L^1(\Omega)},\ \forall\, t\in[0,T];$$

setting $t=i\lambda$, $i=1,\dots,n$, (9) follows with $C=2\sqrt{T}\|Au_0\|_{L^1(\Omega)}$.

Remark 3. To solve the problem (7), we can approach the functions β and g by sequences $\{\beta_\varepsilon\}$, $\{g_\varepsilon\}$, $\varepsilon>0$, respectively, as follows:

$$(10) \qquad \beta_\varepsilon \in C^2(\mathbb{R}),\ \varepsilon \leqslant \beta_\varepsilon' \leqslant \alpha,\ \beta_\varepsilon(0)=0,\ \beta_\varepsilon \to \beta \text{ uniformly in } \mathbb{R}$$

$$(11) \qquad g_\varepsilon(\xi) = \begin{cases} g(\xi) \text{ if } |g(\xi)| \leqslant 1/\varepsilon \\ 1/\varepsilon \quad \text{if } |g(\xi)| > 1/\varepsilon \end{cases}$$

and we consider the ε-regularized problem corresponding to (7)

$$w_{\lambda,\varepsilon}^{0} = u_{0,\varepsilon}$$

$$(12) \qquad \begin{cases} w_{\lambda,\varepsilon}^{i+1} - \lambda\Delta\beta_\varepsilon(w_{\lambda,\varepsilon}^{i+1}) = w_{\lambda,\varepsilon}^{i} & \text{in } \Omega \\ \dfrac{\partial\beta_\varepsilon(w_{\lambda,\varepsilon}^{i+1})}{\partial\nu} + g_\varepsilon(\beta_\varepsilon(w_{\lambda,\varepsilon}^{i+1})) = 0 & \text{on } \Gamma \end{cases} \qquad \text{for } i=0,1,\dots,n-1$$

where $u_{0,\varepsilon}$ belongs to $C^\infty(\bar{\Omega})$. By [9] this problem has one and only one solution $w_{\lambda,\varepsilon}^{i+1}$ belonging, in particular, to $H^1(\Omega) \cap C^0(\bar{\Omega})$; if $u_0 \in L^2(\Omega)$, we can show that:

$$(13) \qquad \begin{aligned} & w_{\lambda,\varepsilon}^{i} \to w_\lambda^{i} \quad \text{weakly in } L^2(\Omega) \\ & \beta_\varepsilon(w_{\lambda,\varepsilon}^{i}) \to \beta(w_\lambda^{i}) \quad \text{strongly in } L^2(\Omega) \end{aligned} \qquad \text{for } i=1,\dots,n$$

if $u_{0,\varepsilon} \to u_0$ in $L^2(\Omega)$, as $\varepsilon \to 0$.

In fact, by using standard monotonicity and compatness techniques (see, e.g., [9, 34, 52]) we can prove that ($w_{\lambda,\varepsilon}^{i} \to w_\lambda^{i}$ weakly in $L^2(\Omega)$) entails:

$$w_{\lambda,\varepsilon}^{i+1} \to w_\lambda^{i+1} \qquad \text{weakly in } L^2(\Omega)$$

$$\beta_\varepsilon(w_{\lambda,\varepsilon}^{i+1}) \to \beta(w_\lambda^{i+1}) \qquad \text{strongly in } L^2(\Omega).$$

2.2. *Algorithms based on non-linear Chernoff's formula.*

Algorithm (S2). It is known that the operator $\tilde{A} : \psi \to -\Delta\psi$ with domain $D(\tilde{A}) = \{\psi \in W^{1,1}(\Omega), \Delta\psi \in L^1(\Omega), g(\psi) \in L^1(\Gamma), \dfrac{\partial\psi}{\partial\nu} + g(\psi) = 0$ on Γ in the weak sense} is m-accretive in $L^1(\Omega)$ [21], hence it generates a non-linear semigroup of contractions $\tilde{S}(t)$. We define

$$(14)\qquad F(t)\varphi = (I - \frac{t}{\sigma_t}(\beta - \tilde{S}(\sigma_t)\beta))\varphi, \quad \varphi \in L^1(\Omega)$$

and we shall prove now that $F(t) : D(A) \to L^1(\Omega)$ fulfils the hypothesis i), ii) of theorem 3.2 of [8] (the non-linear Chernoff's formula):
i) $F(t)$ is a contraction in $L^1(\Omega)$, i.e.

$$(15)\qquad \|F(t)\varphi_1 - F(t)\varphi_2\|_{L^1(\Omega)} \leq \|\varphi_1 - \varphi_2\|_{L^1(\Omega)} \ \forall t > 0, \ \forall \varphi_1, \varphi_2 \in L^1(\Omega).$$

Recalling that $\tilde{S}(t)$ is a non-linear semigroup of contractions, from (14) we get

$$(16)\qquad \begin{aligned} \|F(t)\varphi_1 - F(t)\varphi_2\|_{L^1(\Omega)} &\leq \frac{t}{\sigma_t}\|\beta(\varphi_1) - \beta(\varphi_2)\|_{L^1(\Omega)} + \\ &+ \|(\varphi_1 - \varphi_2) - \frac{t}{\sigma_t}(\beta(\varphi_1) - \beta(\varphi_2))\|_{L^1(\Omega)}. \end{aligned}$$

As $\beta(\xi)$ and $\xi - \frac{t}{\sigma_t}\beta(\xi)$ are non-decreasing functions in $\mathbb{R}$, we have

$$(17)\qquad \frac{t}{\sigma_t}|\beta(\xi_1) - \beta(\xi_2)| + |(\xi_1 - \xi_2) - \frac{t}{\sigma_t}(\beta(\xi_1) - \beta(\xi_2))| = |\xi_1 - \xi_2| \quad \text{for } \xi_1, \xi_2 \in \mathbb{R},$$

therefore (15) follows.

ii)

$$(18)\qquad \lim_{t \to 0} (I + \frac{\lambda}{t}(I - F(t)))^{-1}\varphi = (I + \lambda A)^{-1}\varphi \qquad \text{in } L^1(\Omega), \ \forall \lambda > 0, \ \forall \varphi \in L^1(\Omega)$$

or equivalently

$$(19)\qquad \lim_{t \to 0} \frac{\varphi - F(t)\varphi}{t} = A\varphi \qquad \text{in } L^1(\Omega), \ \forall \varphi \in D(A).$$

Using (14), this is true if

$$(20)\qquad \lim_{t \to 0} \left(\frac{I - \tilde{S}(\sigma_t)}{\sigma_t}\right)\beta(\varphi) = -\Delta\beta(\varphi) = A\varphi \qquad \text{in } L^1(\Omega), \ \forall \varphi \in D(A)$$

or, as $\varphi \in D(A) \Longrightarrow \beta(\varphi) \in D(\tilde{A})$, if

$$(21) \qquad \lim_{t\to 0} \left(\frac{I-\tilde{S}(\sigma_t)}{\sigma_t}\right) \psi = -\Delta\psi = \tilde{A}\psi \qquad \text{in } L^1(\Omega),\ \forall \psi \in D(\tilde{A}).$$

Since $\tilde{A}$ is m-accretive in $L^1(\Omega)$ and accretive in $L^\infty(\Omega)$ [21], $-\tilde{A}$ is a strong infinitesimal generator of the semigroup $\tilde{S}(t)$ (theorem 1 of [30]), i.e. (21) is true.

Then, by means of theorem 3.2 of [8], we have:

$$(22) \qquad \forall u_0 \in L^1(\Omega), \quad \lim_{k\to\infty} F^k\left(\frac{t}{k}\right) u_0 = u(t) \qquad \text{uniformly in } [0,T].$$

This suggest the following algorithm

$$(23) \qquad \begin{aligned} & w_\lambda^0 = u_0 \\ & w_\lambda^{i+1} = F(\lambda) w_\lambda^i \quad i=0,1,\dots,n-1. \end{aligned}$$

Consequently for $i=0,1,\dots,n-1$ we solve the problem

$$(24) \qquad \begin{cases} v^i(0) = \beta(w_\lambda^i) & \text{in } \Omega \\ \dfrac{\partial v^i}{\partial t} - \Delta v^i = 0 & \text{in } \Omega \times]0,\sigma_\lambda[\\ \dfrac{\partial v^i}{\partial \nu} + g(v^i) = 0 & \text{on } \Gamma \times]0,\sigma_\lambda[\end{cases}$$

and set

$$(25) \qquad w_\lambda^{i+1} = w_\lambda^i - \frac{\lambda}{\sigma_\lambda}[\beta(w_\lambda^i) - v^i(\sigma_\lambda)].$$

Algorithm (S3). We define

$$(26) \qquad F(t)\varphi = \left(I - \frac{t}{\sigma_t}(\beta - (I+\sigma_t\tilde{A})^{-1}\beta)\right)\varphi = \left(I - \frac{t}{\sigma_t}(\beta - \tilde{J}(\sigma_t)\beta)\right)\varphi, \quad \varphi \in L^1(\Omega)$$

and we still apply theorem 3.2 of [8].

i) From (26) we have

$$\|F(t)\varphi_1 - F(t)\varphi_2\|_{L^1(\Omega)} \leq \frac{t}{\sigma_t} \|\tilde{J}(\sigma_t)\beta(\varphi_1) - \tilde{J}(\sigma_t)\beta(\varphi_2)\|_{L^1(\Omega)} + \tag{27}$$
$$+ \left\|(\varphi_1 - \varphi_2) - \frac{t}{\sigma_t}(\beta(\varphi_1) - \beta(\varphi_2))\right\|_{L^1(\Omega)}.$$

As $\tilde{A}$ is m-accretive in $L^1(\Omega)$, $\tilde{J}(\sigma_t)$ is a contraction in $L^1(\Omega)$, whence (15) follows.

ii) Thanks to (26) we must prove

$$\lim_{t\to 0}\left(\frac{I - \tilde{J}(\sigma_t)}{\sigma_t}\right)\beta(\varphi) = A\varphi \qquad \text{in } L^1(\Omega),\ \forall \varphi \in D(A) \tag{28}$$

i.e.

$$\lim_{t\to 0}\left(\frac{I - \tilde{J}(\sigma_t)}{\sigma_t}\right)\psi = \tilde{A}\psi \qquad \text{in } L^1(\Omega),\ \forall \psi \in D(\tilde{A}) \tag{29}$$

and this is true also for [30].

Then the convergence result (22) is still valid.

We perform the algorithm (23) solving the following problem

$$\begin{cases} z^i - \sigma_\lambda \Delta z^i = \beta(w_\lambda^i) & \text{in } \Omega \\ \dfrac{\partial z^i}{\partial \nu} + g(z^i) = 0 & \text{on } \Gamma \end{cases} \tag{30}$$

for $i=0,1,\ldots,n-1$ and setting

$$w_\lambda^{i+1} = w_\lambda^i - \frac{\lambda}{\sigma_\lambda}[\beta(w_\lambda^i) - z^i]. \tag{31}$$

Remark 4. For all $u_0 \in L^1(\Omega)$, we have $\sup_i \|w_\lambda^i\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}$, where w_λ^i is given by (7), or (25), or (31).

In fact, as $(I+\lambda A)^{-1}$ (resp. $F(\lambda)$) is a contraction in $L^1(\Omega)$, and $(I+\lambda A)^{-1}0=0$ (resp. $F(\lambda)0=0$), then:

$$\|w_\lambda^{i+1}\|_{L^1(\Omega)} \leq \|w_\lambda^i\|_{L^1(\Omega)} \leq \ldots \leq \|u_0\|_{L^1(\Omega)}.$$

We shall prove now the following convergence result:

Proposition 1. For all $u_0 \in L^1(\Omega)$, we have $\lim_{n\to\infty} G(w_\lambda^i)=u$ in $L^1(Q)$, where $\{w_\lambda^i\}_{i=0}^n$ is given by algorithm (S1), or (S2), or (S3).

Proof. Let be $u_0 \in L^1(\Omega)$. Set $u^i=u(t_i)$, $t_i=i\lambda$; as $u \in C^0([0,T];L^1(\Omega))$, we have $\lim_{n\to\infty} G(u^i)=u$ in $L^1(Q)$, therefore it is enough to prove: $\lim_{n\to\infty} G(w_\lambda^i)=G(u^i)$ in $L^1(Q)$. Setting $w_{t/k}^k=(I+\frac{t}{k}A)^{-k}u_0$ (resp. $w_{t/k}^k=F^k\left(\frac{t}{k}\right)u_0$), Crandall-Liggett's formula (resp. Chernoff's formula) implies that

$$
(32) \quad \begin{aligned} &\lim_{k\to\infty} w_{t/k}^k = u(t) \qquad \text{in } L^1(\Omega), \text{ uniformly in } [0,T], \\ &\text{i.e. } \forall\varepsilon>0,\ \exists\ k_\varepsilon\colon \forall k\geq k_\varepsilon\colon \|w_{t/k}^k-u(t)\|_{L^1(\Omega)} \leq \varepsilon,\ \forall t \in [0,T]. \end{aligned}
$$

For any $\varepsilon>0$, let k_ε be as in (32). For every $n>k_\varepsilon$, we get

$$
\|G(w_\lambda^i)-G(u^i)\|_{L^1(Q)} = \lambda \sum_{i=1}^{k_\varepsilon} \|w_\lambda^i-u^i\|_{L^1(\Omega)}+\lambda \sum_{i=k_\varepsilon+1}^{n} \|w_\lambda^i-u^i\|_{L^1(\Omega)} \leq
$$

$$
(\text{as } w_\lambda^i = w_{t_i/i}^i) \leq \lambda k_\varepsilon \sup_{1\leq i\leq k_\varepsilon} \|w_\lambda^i-u^i\|_{L^1(\Omega)}+\lambda(n-k_\varepsilon)\varepsilon.
$$

As $\sup_i \|w_\lambda^i-u^i\|_{L^1(\Omega)}\leq C$ (C independent of λ), it is sufficient to take n such that $\frac{k_\varepsilon}{n}<\varepsilon$ in order to obtain

$$
\|G(w_\lambda^i)-G(u^i)\|_{L^1(Q)} \leq T(1+C)\varepsilon. \qquad \square
$$

2.3. *Numerical implementation of the algorithms.*

For the sake of simplicity, in this section 2.3 we shall require the following assumption:

$$
(33) \qquad \exists\ C_1,C_2 > 0 \text{ such that } |g(\xi)| \leq C_1|\xi|+C_2 \quad \forall\xi \in \mathbb{R}.
$$

We note that this growth condition is not restrictive; indeed if it did not hold, then it is not difficult to introduce a further approximation, replacing g by a sequence $\{g_\varepsilon\}$ as in (11), and finally to take the limit as $\varepsilon\to 0$.

We always use linear finite elements for the spatial discretization. Therefore we must employ convergence results in L^2 together with aforesaid convergences in L^1. Let us introduce some notations. Let $\{\Delta_h\}$ be a family of triangulations of Ω. We suppose that $\bar{\Omega}=\Omega_h=\bigcup_{\tau\in\Delta_h}\tau$, hence $\Gamma_h=\partial\Omega_h=\Gamma$. We assume that the family $\{\Delta_h\}$ is regular (see [13]) and of acute type (see [50]). Let

$$M_h = \{\chi\in C^0(\bar{\Omega}) : x_{|\tau} \text{ linear } \forall\tau\in\Delta_h\},$$

$\{\chi_j\}_{j=1}^r$ be the canonical base of M_h, $\{x^j\}_{j=p+1}^r$ be the set of nodes from Γ.

We will use only piecewise linear functions because the regularity of the solution of Stefan problem does not justify the use of higher order elements. We choose the following quadrature formula for a N-dimensional simplex τ of vertices x^j: $I_\tau(f)=\dfrac{1}{N+1}\text{ meas }(\tau)\sum_{j=1}^{N+1} f(x^j)$ and we set the following approximations:

$$(34)\qquad \begin{aligned} (v,w) &= \int_\Omega vwdx \cong (v,w)_h = \sum_{\tau\in\Delta_h} I_\tau(vw)\\ <v,w> &= \int_\Gamma vwd\sigma \cong <v,w>_h = \sum_{\tau'\in\Gamma_h} I_{\tau'}(vw), \qquad \tau' \text{ face of } \Gamma. \end{aligned}$$

At last we define:

$$(35)\qquad \begin{aligned} M &= ((\chi_i,\chi_j)_h)_{i,j=1}^r = (m_{ij})\\ K &= ((\nabla\chi_i,\nabla\chi_j))_{i,j=1}^r = (k_{ij})\\ B &= (<\chi_i,\chi_j>_h)_{i,j=1}^r = (b_{ij}). \end{aligned}$$

We have (see [50]):

$$(36)\qquad \begin{aligned} &m_{ii} > 0, \quad m_{ij} = 0 \quad \text{for } i\neq j\\ &b_{ii} \geqslant 0, \quad b_{ij} = 0 \quad \text{for } i\neq j\\ &\sum_{j=1}^r k_{ij} = 0, \quad k_{ii} > 0, \quad k_{ij} \leqslant 0 \quad \text{for } i\neq j, \quad k_{ij}=k_{ji}. \end{aligned}$$

Let u_0 be in $L^2(\Omega)$.

Algorithm (S1). In order to solve problem (7) numerically, we can discretize by linear finite elements either the problem itself or the ε-regularized problem (12), with $g_\varepsilon=g$ thanks to (33). Here we make a unique treatment by setting $w^i_\lambda=w^i_{\lambda,0}$, $\beta=\beta_0$. To obtain the approximation $w^i_{\lambda,\varepsilon,h}\in M_h$ to $w^i_{\lambda,\varepsilon}$ we introduce the following problem:

$$(37)\qquad w^0_{\lambda,\varepsilon,h}= P_h(u_{0,\varepsilon}) \ (P_h \text{ projection of } L^2(\Omega) \text{ onto } M_h)$$

$$(38)\qquad \begin{aligned}&(w^{i+1}_{\lambda,\varepsilon,h},\chi)_h+\lambda(\nabla I_h\beta_\varepsilon(w^{i+1}_{\lambda,\varepsilon,h}),\nabla\chi)+\lambda<g(\beta_\varepsilon(w^{i+1}_{\lambda,\varepsilon,h})),\chi>_h =\\ &= (w^i_{\lambda,\varepsilon,h},\chi)_h \qquad \forall\chi\in M_h, \text{ for } i=0,1,\dots,n-1 \ (I_h \text{ interpolation in } M_h).\end{aligned}$$

We set $W^i_j=w^i_{\lambda,\varepsilon,h}(x^j)$, $j=1,\dots,r$; choosing $\chi=\chi_j$, $j=1,\dots,r$ in (38) we see that the computation of $w^{i+1}_{\lambda,\varepsilon,h}$ is equivalent to the solution of the non-linear algebraic system

$$(39)\qquad MW^{i+1}+\lambda K\beta(W^{i+1})+\lambda Bg(\beta(W^{i+1})) = MW^i$$

for $i=0,1,\dots,n-1$, where

$$(40)\qquad \begin{aligned}&W^i = (W^i_1,\dots,W^i_r)^T\\ &\beta(W^i) = (\beta_\varepsilon(W^i_1),\dots,\beta_\varepsilon(W^i_r))^T\\ &g(\beta(W^i)) = (0,\dots,0,g(\beta_\varepsilon(W^i_{p+1})),\dots,g(\beta_\varepsilon(W^i_r)))^T.\end{aligned}$$

It is easy to show that, for every $\varepsilon\geq 0$, the system (39) has one and only one solution, which can be approximated by the following non-linear Gauss-Seidel method (see [42, 45, 47]). We set $A=\lambda M^{-1}K$, $C=\lambda M^{-1}B$, and we define the function $R:\mathbb{R}\to\mathbb{R}^r$

$$(41)\qquad \xi\in\mathbb{R}: R_j(\xi) = \xi+a_{jj}\beta_\varepsilon(\xi)+c_{jj}g(\beta_\varepsilon(\xi)) \quad j=1,2,\dots,r.$$

If $A=D-A_1-A_2$ is the usual splitting of the matrix A, we set $\Sigma^i_0=W^i$ and construct the sequence $\Sigma^{i+1}_k=(\xi^{i+1}_{1,k},\dots,\xi^{i+1}_{r,k})^T$, $k=1,2,\dots$, by solving the following non-linear equations:

$$(42)\qquad \xi^{i+1}_{j,k+1} = R_j^{-1}([A_1\beta(\Sigma^{i+1}_{k+1})+A_2\beta(\Sigma^{i+1}_k)]_j+W^i_j) \qquad j=1,\dots,r.$$

As the function $\xi \rightarrow R_j(\xi)$ is strictly increasing, we can easily evaluate $\xi_{j,k+1}^{i+1}$.

For every $\varepsilon>0$, one can show that the sequence $\beta(\Sigma_k^{i+1})$ converges at least as fast as the linear Gauss-Seidel method for the positive definite matrix $1/\alpha M+\lambda K$ (see [50]). An analogous convergence result is true also for $\varepsilon=0$, that is:

Proposition 2. If $\varepsilon=0$, the sequence Σ_k^{i+1} converges to the solution of the system (39) at a rate not lower than that of the linear Gauss-Seidel method corresponding to the positive definite matrix $1/\alpha M+\lambda K$.

Proof. (For g linear, see [47]). As β and g are monotone, we have:

$$(43)\qquad |R_j(\xi_1)-R_j(\xi_2)| = |\xi_1-\xi_2|+a_{jj}|\beta(\xi_1)-\beta(\xi_2)|+c_{jj}|g(\beta(\xi_1))-g(\beta(\xi_2))|$$

$$\forall \xi_1,\xi_2 \in \mathbb{R},\ j=1,2,\dots,r$$

whence:

$$(44)\qquad |\xi_1-\xi_2| \leqslant |R_j(\xi_1)-R_j(\xi_2)|$$

and, as β is also Lipschitz-continuous (here we suppose with constant 1):

$$(45)\qquad |\beta(\xi_1)-\beta(\xi_2)| \leqslant \frac{1}{1+a_{jj}}|R_j(\xi_1)-R_j(\xi_2)|.$$

Therefore $(R_j(\xi_{j,k}^i) \rightarrow R_j(W_j^i)\ j=1,\dots,r)$ entails $\Sigma_k^i \rightarrow W^i$. We have

$$(46)\qquad R_j(W_j^i) = W_j^i+a_{jj}\beta(W_j^i)+c_{jj}g(\beta(W_j^i)) = W_j^{i-1}+[(A_1+A_2)\beta(W^i)]_j.$$

Taking the difference between (46) and (42), we get:

$$(47)\qquad R_j(W_j^i)-R_j(\xi_{j,k+1}^i) = [A_1(\beta(W^i)-\beta(\Sigma_{k+1}^i))]_j+[A_2(\beta(W^i)-\beta(\Sigma_k^i))]_j;$$

setting

$$(48)\qquad \alpha_{j,k}^i = |R_j(W_j^i)-R_j(\xi_{j,k}^i)|, \text{ and } \alpha_k^i = (\alpha_{1,k}^i,\dots,\alpha_{r,k}^i)^T,$$

as A_1, A_2 are positive, by (48) and (45) we have

(49) $\alpha^i_{j,k+1} \leqslant [A_1(D+I)^{-1}\alpha^i_{k+1}]_j+[A_2(D+I)^{-1}\alpha^i_k]_j$

that is

(50) $(I-A_1(D+I)^{-1})\alpha^i_{k+1} \leqslant A_2(D+I)^{-1}\alpha^i_k.$

As the inverse of $I-A_1(D+I)^{-1}$ is positive, (50) yields:

(51) $\alpha^i_{k+1} \leqslant (I-A_1(D+I)^{-1})^{-1}A_2(D+I)^{-1}\alpha^i_k.$

Therefore $R_j(\xi^i_{j,k}) \to R_j(W^i_j)$ $j=1,\dots,r$, as the spectral radius of $X=(I-A_1(D+I)^{-1})^{-1}$ $A_2(D+I)^{-1}$ is less than 1; in fact X is the iteration matrix of the linear Gauss-Seidel method corresponding to the non-singular M-matrix $(A-D)(D+I)^{-1}+I=$ $=M^{-1}(M+\lambda K)(D+I)^{-1}$. □

Finally we shall prove the convergence of the solution of system (39) to the solution of problem (P). Setting $\chi=I_h\beta_\varepsilon(w^{i+1}_{\lambda,\varepsilon,h})$ in (38), by using standard monotonicity and compatness techniques, one can prove that:

$$w^i_{\lambda,\varepsilon,h} \to w^i_{\lambda,\varepsilon} \qquad \text{weakly in } L^2(\Omega)$$

(52) $$\beta_\varepsilon(w^i_{\lambda,\varepsilon,h}) \to \beta_\varepsilon(w^i_{\lambda,\varepsilon}) \qquad \text{strongly in } L^2(\Omega)$$

$$\text{as } h \to 0, \text{ for every } \varepsilon \geqslant 0, \qquad \text{for } i=1,\dots,n.$$

Then, by means of (52), proposition 1 (and also (13) if $\varepsilon>0$), we have that:

$$G(w^i_{\lambda,\varepsilon,h}) \to u \qquad \text{weakly in } L^1(Q)$$

(53) $$G(\beta_\varepsilon(w^i_{\lambda,\varepsilon,h})) \to \beta(u) \qquad \text{strongly in } L^1(Q)$$

$$\text{as } h \to 0,\ \varepsilon \to 0,\ \lambda \to 0 \text{ (in this order)},$$

in the sense that: $\lim_{\lambda\to 0} (\lim_{\varepsilon\to 0} (\lim_{h\to 0} G(w^i_{\lambda,\varepsilon,h})))=u$ weakly in $L^1(Q)$.

We observe that ($w^i_{\lambda,\varepsilon,h} \to w^i_{\lambda,\varepsilon}$ weakly in $L^1(\Omega)$, as $h\to 0$, for $i=1,\dots,n$) entails ($G(w^i_{\lambda,\varepsilon,h}) \to G(w^i_{\lambda,\varepsilon})$ weakly in $L^1(Q)$, as $h\to 0$).

Algorithm (S2). Let $w_{\lambda,\Delta t,h}^{0} = P_h u_0$. We discretize the heat problem (24) by backward Euler method in time and linear finite elements in space. Hence, setting $\Delta t = \frac{\sigma_\lambda}{m}$ (where $m \geq 1$ is an integer), for $i=0,1,\dots,n-1$, we solve the following non-linear system:

$$
(54)\qquad
\begin{aligned}
&V^{i,0} = \beta(W^i)\\
&MV^{i,k+1} + \Delta t K V^{i,k+1} + \Delta t B g(V^{i,k+1}) = MV^{i,k} \quad \text{for } k=0,1,\dots,m-1
\end{aligned}
$$

where:

$$
(55)\qquad
\begin{aligned}
&\beta(W^i) = (\beta(W_1^i),\dots,\beta(W_r^i))^T, \qquad W_j^i = w_{\lambda,\Delta t,h}^{i}(x^j)\\
&V^{i,k} = (V_1^{i,k},\dots,V_r^{i,k})^T\\
&g(V^{i,k}) = (0,\dots,0,g(V_{p+1}^{i,k}),\dots,g(V_r^{i,k}))^T
\end{aligned}
$$

and we define

$$
(56)\qquad w_{\lambda,\Delta t,h}^{i+1} = w_{\lambda,\Delta t,h}^{i} - \frac{\lambda}{\sigma_\lambda} I_h \beta(w_{\lambda,\Delta t,h}^{i}) + \frac{\lambda}{\sigma_\lambda} \sum_{j=1}^{r} V_j^{i,m} \chi_j .
$$

Since the operator $\tilde{A}$ with domain $\{\psi \in D(\tilde{A}) \cap L^2(\Omega), \Delta\psi \in L^2(\Omega)\}$ is m-accretive in $L^2(\Omega)$ ([21]), the Crandall-Liggett's formula implies

$$
(57)\qquad \lim_{m\to\infty} (I + \frac{\lambda}{m}\tilde{A})^{-m} \beta(w_\lambda^i) = v^i(\lambda) \qquad \text{in } L^2(\Omega), \text{ for } i=0,1,\dots,n-1
$$

from which, using results on finite element method (see, e.g., [10, 24]) one obtaines

$$
(58)\qquad \lim_{\Delta t\to 0} (\lim_{h\to 0} \sum_{j=1}^{r} V_j^{i,m} \chi_j) = v^i(\lambda) \qquad \text{in } L^2(\Omega), \text{ for } i=0,1,\dots,n-1.
$$

Finally, by means of proposition 1 and of (58), we have that:

$$G(w^i_{\lambda,\Delta t,h}) \to u \qquad \text{strongly in } L^1(Q)$$

(59) $$G(\beta(w^i_{\lambda,\Delta t,h})) \to \beta(u) \qquad \text{strongly in } L^1(Q)$$

as $h \to 0$, $\Delta t \to 0$, $\lambda \to 0$ (in this order).

Algorithm (S3). Let $w^0_{\lambda,h}=P_h u_0$. Using linear finite elements for approximating the elliptic problem (30), for $i=0,1,\ldots,n-1$, we solve the following non-linear system

(60) $$MZ^{i+1}+\sigma_\lambda KZ^{i+1}+\sigma_\lambda Bg(Z^{i+1}) = M\beta(W^i)$$

where:

$$Z^i = (Z^i_1,\ldots,Z^i_r)^T$$

(61) $$g(Z^i) = (0,\ldots,0,g(Z^i_{p+1}),\ldots,g(Z^i_r))^T$$

$$\beta(W^i) = (\beta(W^i_1),\ldots,\beta(W^i_r))^T, \qquad W^i_j = w^i_{\lambda,h}(x^j)$$

and we define

(62) $$w^{i+1}_{\lambda,h} = w^i_{\lambda,h}-\frac{\lambda}{\sigma_\lambda} I_h\beta(w^i_{\lambda,h})+\frac{\lambda}{\sigma_\lambda}\sum_{j=1}^{r} Z^i_j\chi_j.$$

By means of proposition 1 and of results such as [10, 24], one has that

$$G(w^i_{\lambda,h}) \to u \qquad \text{weakly in } L^1(Q)$$

$$G(\beta(w^i_{\lambda,h})) \to \beta(u) \qquad \text{strongly in } L^1(Q)$$

as $h \to 0$, $\lambda \to 0$ (in this order).

Remark 5. The problem (P) is well-posed with respect to u_0 ([5]). If $u_0 \in L^1(\Omega)$, let $u_0^k \in L^2(\Omega)$, $k=1,2,\ldots$, and denote by $u^k(t)$ the function constructed by means of Crandall-Liggett formula: if $u_0^k \to u_0$ in $L^1(\Omega)$ then $u^k \to u$ in $C^0([0,T]; L^1(\Omega))$. Whence, the convergence results of algorithms (S1), (S2), (S3) are also true for $u_0 \in L^1(\Omega)$.

3. **Remarks on the variational formulation.**

The usual formulation of the problem (P) is the weak variational formulation in $L^2(Q)$, and is essentially due to [28, 41, 23]; it can be set as follows:

Given $u_0 \in L^2(\Omega)$, find u such that

$$(P_v) \quad \begin{aligned} & u \in L^2(Q),\ \beta(u) \in L^2(0,T;H^1(\Omega)),\ g(\beta(u)) \in L^2(\Sigma) \\ & -\int_Q u \frac{\partial v}{\partial t} dxdt + \int_Q \nabla\beta(u)\cdot\nabla v dxdt + \int_\Sigma g(\beta(u)) v dxdt + \int_\Omega u_0 v(0) dx = 0 \\ & \forall v \in K = \{v \in H^1(Q),\ v(\cdot,T)=0\}. \end{aligned}$$

It has been proved that if $u_0 \in L^\infty(\Omega)$, then (P_v) has one and only one solution such that $u \in L^\infty(Q)$, $\beta(u) \in L^\infty(Q)$, see [46, 37, 12]. Moreover if $\beta(u_0) \in C^0(\bar\Omega)$, then $\beta(u) \in C^0(\bar Q)$, see [11, 18].

It is easy to show that if $u_0 \in L^\infty(\Omega)$, then the generalized solution (4) and the variational one coincide. We just sketch the a priori estimates and the limit procedure which are similar to those of [46]. As the solutions fulfil a maximum principle, it is not restrictive to assume that g fulfils the order of growth assumption (33). If u_λ denotes the piecewise linear interpolate of $\{w_\lambda^i\}_{i=0}^n$, setting $\hat u_\lambda = G(w_\lambda^i)$, we get:

$$(63) \quad \begin{cases} \dfrac{\partial u_\lambda}{\partial t} - \Delta\beta(\hat u_\lambda) = 0 & \text{in } Q \\ \dfrac{\partial\beta(\hat u_\lambda)}{\partial\nu} + g(\beta(\hat u_\lambda)) = 0 & \text{on } \Sigma \\ u_\lambda(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Multiplying (63) by $\beta(\hat u_\lambda)$ and integrating on Q, we have:

$$(64) \quad \|\beta(\hat u_\lambda)\|_{L^2(0,T;H^1(\Omega))} \leq C \text{ (constant C independent of } \lambda)$$

whence $\|\beta(\hat u_\lambda)\|_{L^2(\Sigma)} \leq C$, and by assumptions on g:

$$(65) \quad \|g(\beta(\hat u_\lambda))\|_{L^2(\Sigma)} \leq C;$$

by assumption on β, (64) entails also:

$$\|\hat{u}_\lambda\|_{L^2(Q)} \leq C. \tag{66}$$

Then, there exist u,θ,η such that, possibly taking subsequences

$$\begin{aligned} &\hat{u}_\lambda \to u && \text{weakly in } L^2(Q) \\ &\beta(\hat{u}_\lambda) \to \theta && \text{weakly in } L^2(0,T;H^1(\Omega)) \\ &g(\beta(\hat{u}_\lambda)) \to \eta && \text{weakly in } L^2(\Sigma) \end{aligned} \tag{67}$$

as $\lambda \to 0$. Using standard monotonicity techniques, taking $\lambda \to 0$ in (63), one can show that u is the solution of problem (P_v). Moreover, by the Crandall-Liggett's formula we have $\hat{u}_\lambda \to u = S(t)u_0$ in $L^1(Q)$.

The numerical approximation of (P_v) is based on the numerical solution of the non-linear heat equation:

$$(P_\varepsilon) \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \Delta\beta_\varepsilon(u_\varepsilon) = 0 & \text{in } Q \\ \dfrac{\partial\beta_\varepsilon(u_\varepsilon)}{\partial v} + g(\beta_\varepsilon(u_\varepsilon)) = 0 & \text{on } \Sigma \\ u_\varepsilon(0) = u_{0,\varepsilon} & \text{in } \Omega. \end{cases}$$

Indeed, one can show (see, e.g., [46]) that if $u_{0,\varepsilon} \to u_0$ strongly in $L^2(\Omega)$, then as $\varepsilon \to 0$

$$\begin{aligned} &u_\varepsilon \to u && \text{weakly in } L^2(Q) \\ &\beta_\varepsilon(u_\varepsilon) \to \beta(u) && \text{strongly in } L^2(Q). \end{aligned} \tag{68}$$

The discretization of problem (P_ε) by means of the implicit Euler method with respect to time and by linear finite elements with respect to space would yield the same algorithm (S1). The following convergence results are deduced by using (68) and [50]:

$$G(w^i_{\varepsilon,\lambda,h}) \to u \qquad \text{weakly in } L^2(Q)$$

(69) $$G(\beta_\varepsilon(w^i_{\varepsilon,\lambda,h})) \to \beta(u) \qquad \text{strongly in } L^2(Q)$$

$$\text{as } (h,\lambda) \to 0,\ \varepsilon \to 0 \qquad \text{(in this order).}$$

All methods existing in literature for the approximation of the non-linear heat equation with non-linear boundary condition can be used; they generally require a higher regularity on the non-linear therm of the equation. The problem (P_ε) is certainly regular enough, but such a regularity is not preserved in the limit problem (P_v). Whence these methods are generally less efficient than the previous one (see, e.g., [32] where an extrapolated Crank-Nicholson method is used; see also the final remark of the present work).

4. **Numerical results.**

Remark 6. In some of our tests there is a second member f(x,t) in the equation. If $f \in C^0([0,T];L^1(\Omega))$, then $\forall u_0 \in L^1(\Omega)$

$$U_f(t)u_0 = \lim_{k\to\infty} \prod_{i=1}^{k} \left(I+\frac{t}{k}\left(A-f\left(i\,\frac{t}{k}\right)\right)\right)^{-1} u_0 \text{ (uniformly in } [0,T])$$

is the generalized solution (see [16]) of the problem (P) with the equation $\frac{\partial u}{\partial t} - \Delta\beta(u) = f$. The aforesaid numerical algorithms can be easily extended.

For numerical purposes, we can choose a piecewise linear β_ε:

$$\beta_\varepsilon(\xi) = \begin{cases} \alpha_1(\xi - L) + \varepsilon & \xi > L \\ \dfrac{\varepsilon}{L}\xi & 0 \leq \xi \leq L, \\ \alpha_2 \xi & \xi < 0 \end{cases} \quad \text{with } \alpha_1 = k_1/c_1,\ \alpha_2 = k_2/c_2.$$

We stop the Gauss-Seidel iterations when the relative error is less than .0001, i.e. $|\xi^i_{j,k+1} - \xi^i_{j,k}| \leq .0001\ \|\Sigma^i_{k+1}\|_\infty \quad j=1,\dots,r.$

First, we describe the numerical experiences concerning tests for which the exact solution is known (exs. 1,2,3); then we show the evolution of a mushy region (ex. 4,5). Example 1 is for just one space variable, whereas the other ones are in two space variables with $\Omega =]0,a[\times]0,b[$. The mesh $\{\Delta_h\}$ is uniform; the sides of the triangles are parallel to the axes and to the diagonal $y = {}^b\!/\!_a x$ of Ω.

Notations.

n = number of time steps; n_x, n_y = number of grid points;
$e_1\%$ = discrete relative L^1-error in the temperature;
CPU = processing time.

Example 1 (see [6]). $\Omega =]0,2[$, $T=1$; $c_1=2$, $k_1=2$, $c_2=1$, $k_2=1$, $L=1$. The exact solution is

$$u(x,t) = \begin{cases} c_1(e^{-x+t+1}-1)+L & \Phi \geqslant 0 \\ c_2(e^{-x+t+1}-1) & \Phi < 0 \end{cases}$$

where $\Phi(x,t) = -x+t+1=0$ is the free boundary. A non-linear Neumann condition is assigned on the boundary.

	n	n_x	$e_1\%$	CPU
(S1)	20	20	1.56	4
	40	20	0.93	7
	80	40	0.38	32
	160	40	0.23	50
(S2)	20	20	4.00	0.8
	40	20	2.59	1.4
	80	40	1.58	3
	160	40	1.09	6

Table 1. Discrete relative L^1-error for several implementations of algorithms (S1) and (S2) for problem (1).

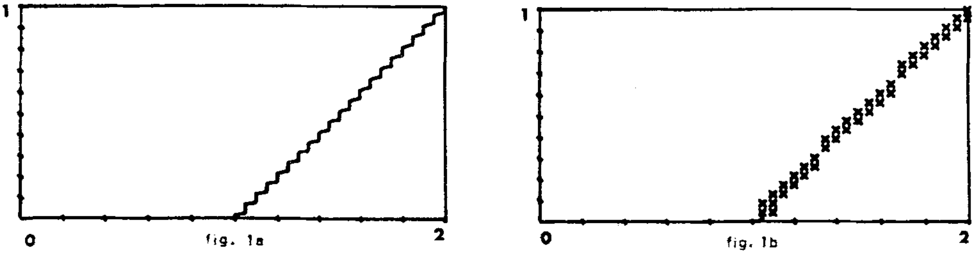


Figure 1. Discrete free boundary of problem (1), obtained using the algorithms:
(S1) with n=160, n_x=40, ε=0.001 (fig. 1a),
(S2) with n=160, n_x=40, m=2 (fig. 1b).

Example 2 (see [14]). $\Omega=]-\frac{1}{2},\frac{1}{2}[\times]-\frac{1}{2},\frac{1}{2}[$, T=0.1; $c_1=6$, $k_1=2$, $c_2=2$, $k_2=1$, L=1. The exact solution (enthalpy) is:

$$u(x,y,t) = \begin{cases} c_1\Phi(x,y,t)+L & \Phi \geqslant 0 \\ c_2\Phi(x,y,t) & \Phi < 0 \end{cases}$$

where $\Phi(x,y,t)=x^2+y^2-e^{-4t}/4=0$ is the free boundary. As the solution is symmetric, we can solve our problem in $]0,\frac{1}{2}[\times]0,\frac{1}{2}[$, requiring a vanishing flux condition on the sides x=0 and y=0. A non-linear Neumann condition is assigned on the sides x=½ and y=½.

	n	n_x	n_y	e_1%	CPU
(S1)	40 80	10 20	10 20	8.18 2.99	71 568
(S2)	40 80	10 20	10 20	13.60 8.16	21 166

Table 2: Discrete relative L^1-error for several implementations of algorithms (S1) and (S2) for problem (2).

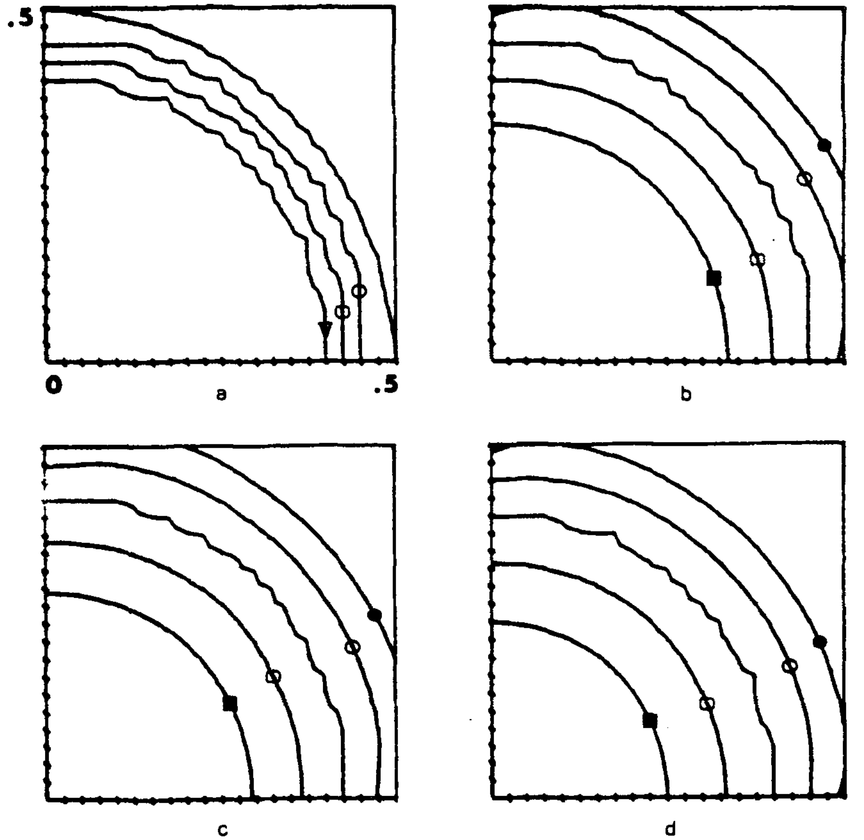


fig. 2'

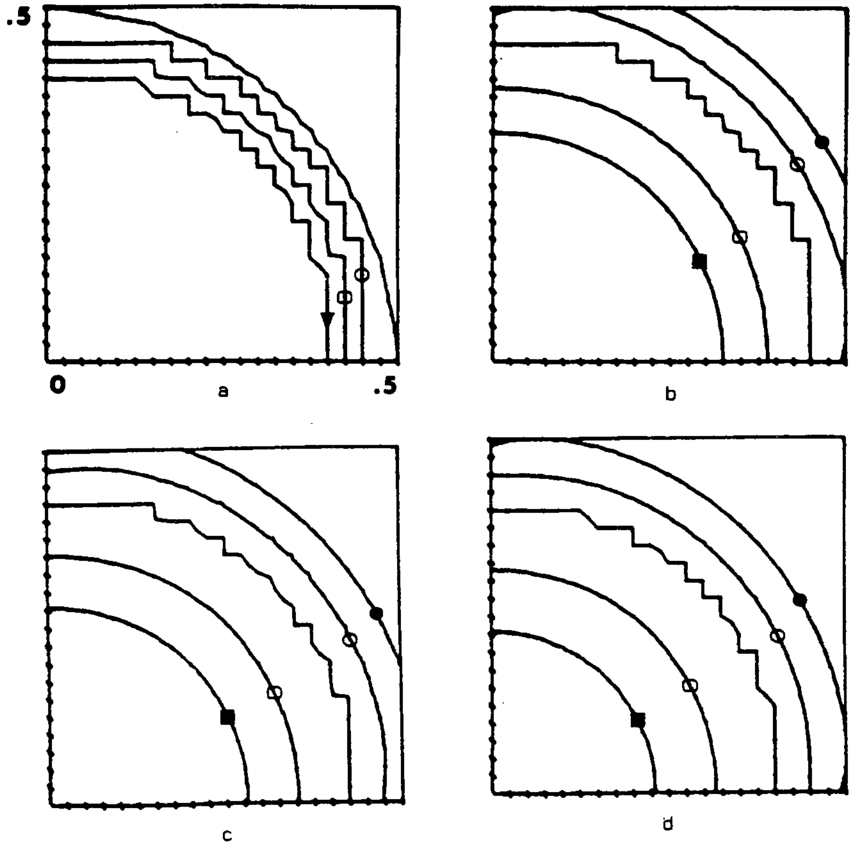


fig. 2''

Figure 2: Numerical solutions of problem (2), obtained using the algorithms:
(S1) with n=80, $n_x=n_y=20$, $\varepsilon=0.0005$ (fig. 2'),
(S2) with n=80, $n_x=n_y=20$, m=2 (fig. 2'').

Figure 2a: Free boundary at times: t=0 (solid line), t=T/3 (circle), t=2T/3 (diamond), t=T (triangle).

Figure 2b,c,d: Temperature values at times t=T/3, 2T/3, T: free boundary (solid line), $\theta=0.1$ (open circle), $\theta=0.2$ (solid circle), $\theta=-0.05$ (open diamond), $\theta=-0.1$ (solid diamond).

Example 3. $\Omega=]0,2[\times]0,2[$, $T=1$; $c_1=2$, $k_1=2$, $c_2=1$, $k_2=1$, $L=1$. The exact solution (enthalpy) is

$$u(x,y,t) = \begin{cases} c_1(e^{-x-y+2t+1}-1)+L & \Phi \geq 0 \\ c_2(e^{-x-y+2t+1}-1) & \Phi < 0 \end{cases}$$

where $\Phi(x,y,t)=-x-y+2t+1=0$ is the free boundary. A non-linear Neumann condition is assigned on the boundary.

	n	n_x	n_y	$e_1\%$	CPU
(S1)	10	10	10	8.06	10
	25	10	10	4.24	42
	100	20	20	1.51	423
(S2)	10	10	10	12.84	1.5
	25	10	10	6.48	17
	100	20	20	2.25	168

Table 3: Discrete relative L^1-error for several implementations of algorithms (S1) and (S2) for problem (3).

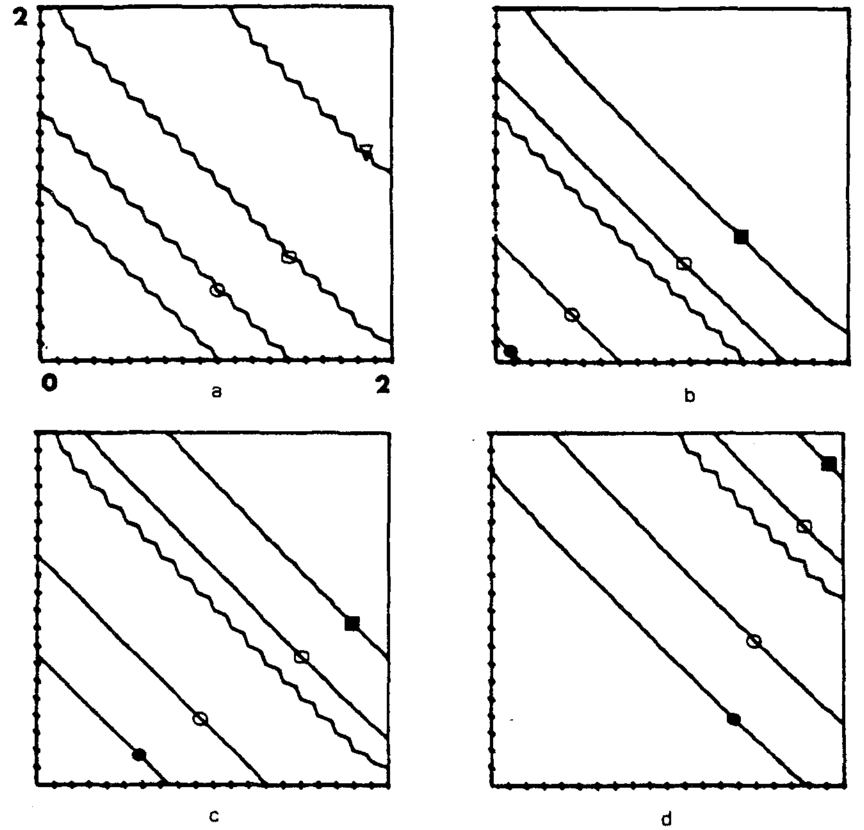


fig. 3[1]

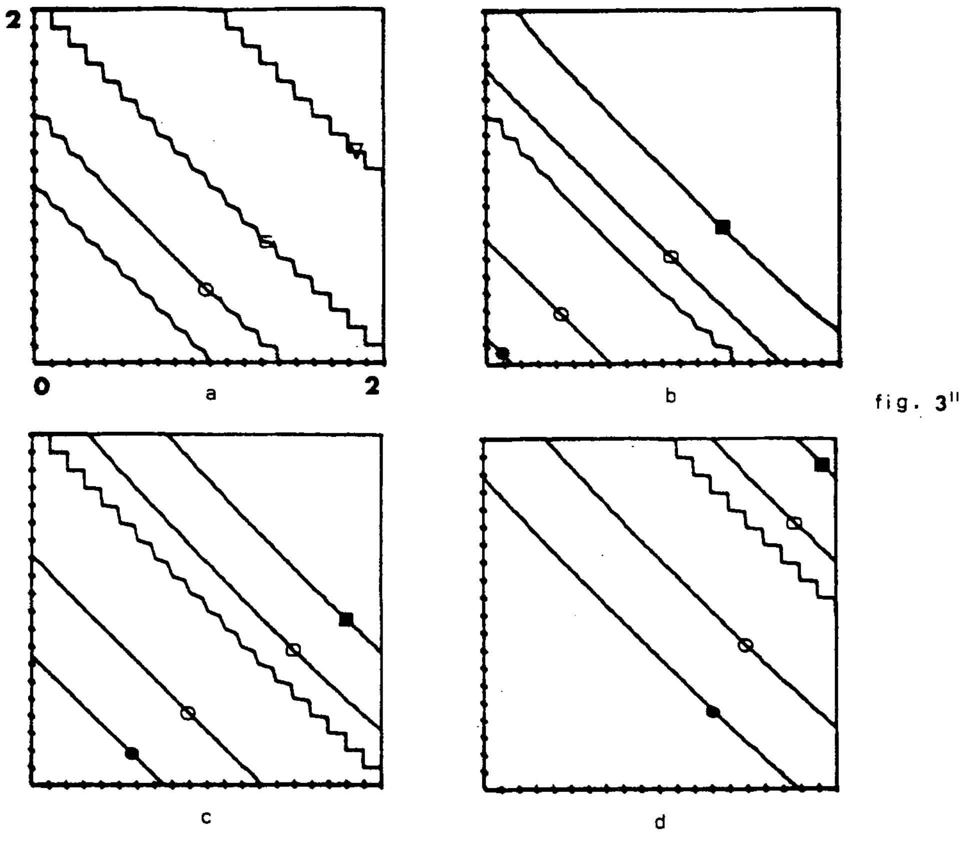
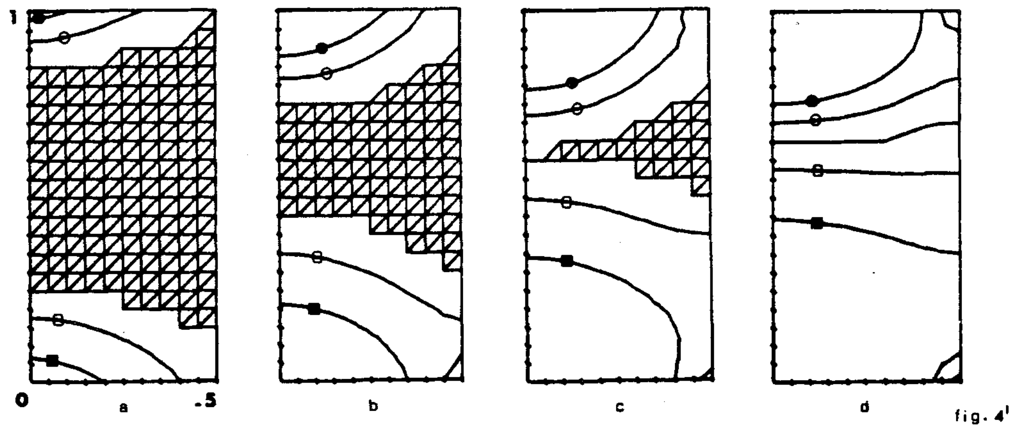


Figure 3: Numerical solutions of problem (3), obtained using the algorithms:
(S1) with n=80, $n_x=n_y=20$, $\varepsilon=0.001$ (fig. 3′),
(S2) with n=80, $n_x=n_y=20$, m=2 (fig. 3″).

Figure 3a: Free boundary at times: t=0 (solid line), t=2T/5 (circle), t=T/2 (diamond), t=T (triangle).

Figure 3b,c,d: Temperature values at times t=2T/5,T/2,T: free boundary (solid line), $\theta=2$ (open circle), $\theta=5$ (solid circle), $\theta=-0.2$ (open diamond), $\theta=-0.5$ (solid diamond).

Example 4 (see [7]). $\Omega =]0,\frac{1}{2}[\times]0,1[$, $T=0.04$; $c_1=1$, $k_1=2$, $c_2=1$, $k_2=1$, $L=1$. The exact solution is not known. We set $u_0(x,y)=\frac{1}{2}$, i.e. Ω is a mushy region for $t=0$. A non-zero Dirichlet data is assigned on the sides $y=0$ and $y=1$; a zero Neumann conditions is assigned on the sides $x=0$ and $x=\frac{1}{2}$.



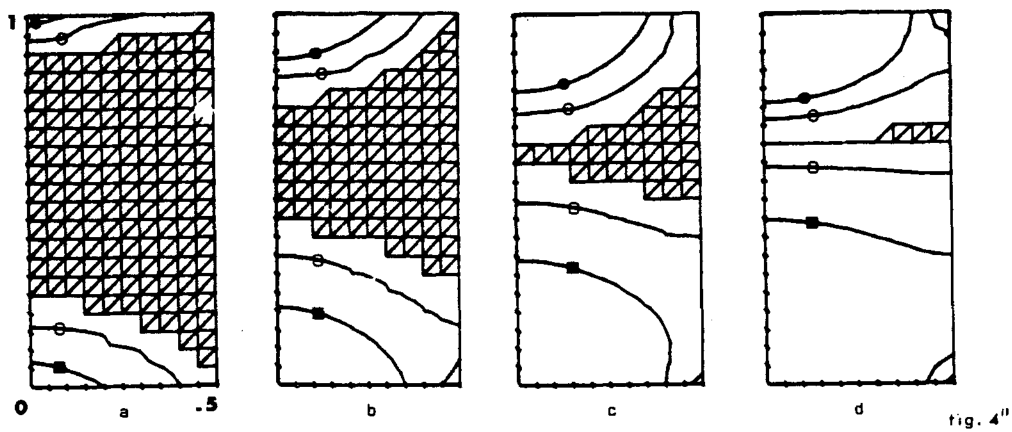


Figure 4: Numerical solutions of problem (4), obtained using the algorithms:
(S1) with $n=100$, $n_x=10$, $n_y=20$, $\varepsilon=0.001$ (fig. 4′),
(S2) with $n=100$, $n_x=10$, $n_y=20$, $m=2$ (fig. 4″).

Figure 4a,b,c,d: Temperature values at times $t=T/4, T/2, 3T/4, T$: mushy region (solid region), $\theta=0.1$ (open circle), $\theta=0.3$ (solid circle), $\theta=-0.05$ (open diamond), $\theta=-0.1$ (solid diamond).

Example 5. This is a classical counterexample due to Friedman [23] (see also [1]). $\Omega=]0,1[\times]0,3[$, $T=0.5$; $c_1=2$, $k_1=2$, $c_2=1$, $k_2=1$, $L=1$. The exact solution is not known. At the initial time we have a «bone-shaped» ice-block, which melts and then disconnects. On account of the symmetry, we can solve our problem in $]0,\frac{1}{2}[\times]0,\frac{3}{2}[$, requiring a vanishing flux condition on the sides $x=\frac{1}{2}$ and $y=\frac{3}{2}$.

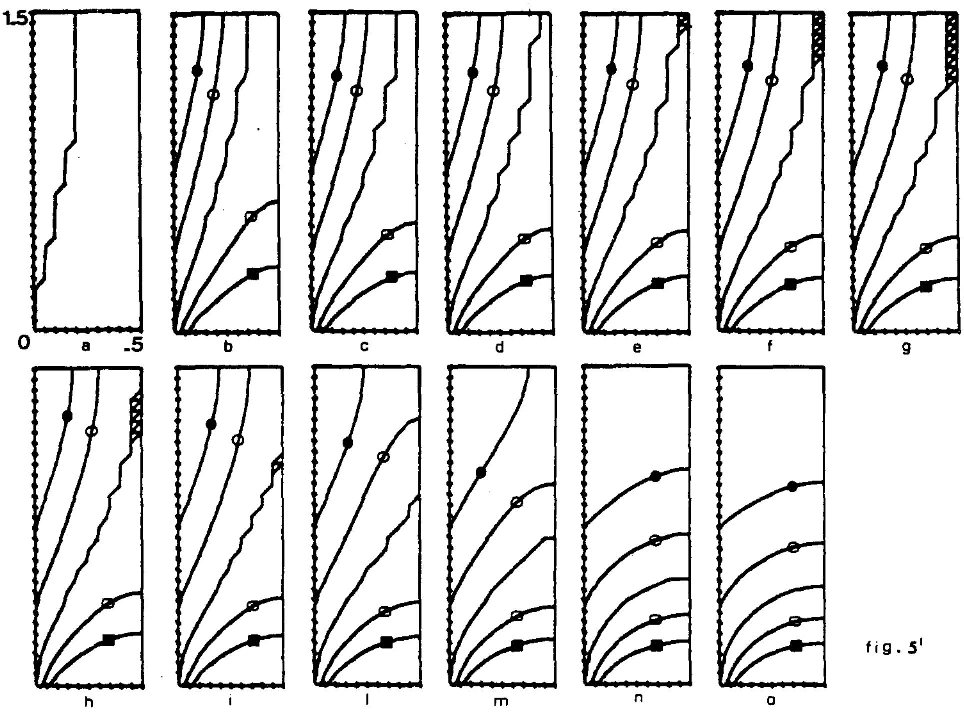


fig. 5'

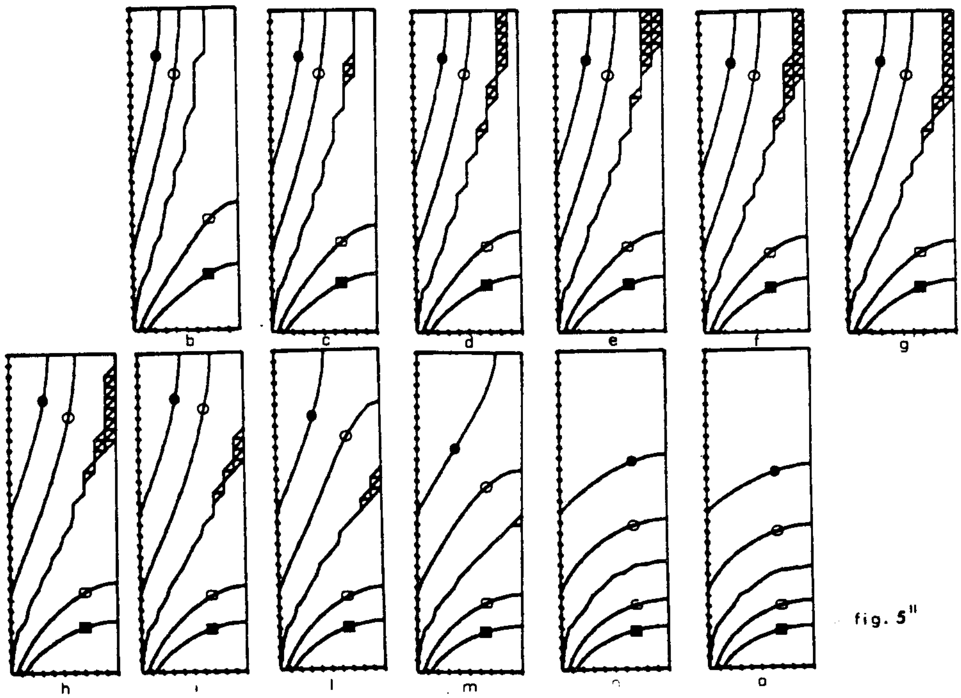


Figure 5: Numerical solutions of problem (5), obtained using the algorithms:
(S1) with n=100, n_x=10, n_y=30, ε=0.001 (fig. 5′),
(S2) with n=100, n_x=10, n_y=30, m=2 (fig. 5″).

Figure 5a′,b′,...,o′: Temperature values at times: t=0, 0.1, 0.12, 0.1275, 0.13, 0.14, 0.145, 0.1475, 0.17, 0.2, 0.25, 0.4, 0.5.

Figure 5b″,...,o″: Temperature values at times: t=0.1, 0.12, 0.1275, 0.13, 0.14, 0.15, 0.1525, 0.17, 0.2, 0.25, 0.4, 0.5.
Free boundary (solid line), θ=0.5 (open circle), θ=1 (solid circle), θ=–0.5 (open diamond), θ=–1 (solid diamond).

Final remarks. Further numerical results were obtained using both algorithm (S3) and the Crank-Nicholson extrapolated method already mentioned in Section 3; as expected, the latter was found less competitive (see [45]). Algorithm (S1) is a natural generalization of backward difference in time and linear finite element in space methods (see [27, 35, 47, 50]) to the case of non-linear flux condition. Algorithm (S1) seems to give good results when the exact solution is known as wellas when the mushy region is present. On the contrary, algorithms (S2) and

(S3), which have similar precision, are less usual; basically they are a linearization of problem (P). Of course in the literature one can find many other methods, but with an incomplete theoretical justification (see, e.g., «moving finite element method» [1, 36], or methods such as «front-tracking», see, e.g., [25]).

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