manuscripta math. 78, 417 - 443 (1993)

manuscripta mathematica

ON THE SINGULAR SET OF STATIONARY HARMONIC MAPS

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Let M and N be compact riemannian manifolds, and u a stationary harmonic map from M to N. We prove that $H^{n-2}(\Sigma) = 0$, where $n = \dim M$ and Σ is the singular set of u. This is a generalization of a result of C. Evans [7], where this is proved in the special case N is a sphere. We also prove that, if u is a weakly harmonic map in $W^{1,n}(M,N)$, then u is smooth. This extends results of F. Hélein for the case $n = 2$, or the case N is a sphere ([9], [10]).

I Introduction

Let (M,g) and (N,h) be two compact riemannian manifolds of dimension n and k respectively. We assume furthermore that $\partial N = \varnothing$. We may also assume (using Nash-Moser Theorem) that N is isometrically imbedded in some euclidean space $\mathbb{R}^K$. We consider the Sobolev space $H^1(M,N)$ defined by

$$H^1(M,N) = \{u \in H^1(M,\mathbb{R}^K),\ u(x) \in N \ \ a.e\},$$

and for a map u in $H^1(M,N)$, the energy functional $E(u) = \int_M |\nabla u|^2$.

We say that a map u in $H^1(M,N)$ is a weakly harmonic map if u is a critical point of $E(u)$ in the following sense

$$\forall \varphi \in C_c^\infty(M,\mathbb{R}^K),\ \frac{d}{dt}E(\Pi(u+t\varphi)) = 0, \tag{I.1}$$

where Π denotes the nearest point projection onto N. It is easy to verify that (I.1) is equivalent to the following system

$$-\Delta_g u = A(u)(\nabla u, \nabla u), \tag{I.2}$$

where $A(u)$ is the second fundamental form of N, and where we have used the notation
$A(u)(\nabla u, \nabla u) = g^{\alpha\beta} A(u) \left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right)$. Note that

$$\Delta u \perp T_{u(x)} N, \tag{I.3}$$

which is actually equivalent to (I.2).

In [9], F. Hélein has proved that any weakly harmonic map from M to N is smooth in the case $n = 2$. For higher dimensions ($n \geqslant 3$), this is no longer true however, as the following example shows : the map $\frac{x}{|x|}$ is a harmonic map from B^3 to S^2, and is singular at the origin. More generally, one can produce weakly harmonic maps having a singular set of dimension $n-3$ quite easily. Very recently, T. Rivière [16] has constructed weakly harmonic maps from B^n into the sphere S^k which are singular on the whole domain B^n, and therefore no partial regularity theory can be derived for "general" weakly harmonic maps. For that reason, we are going to turn our attention to a more restrictive class of weakly harmonic maps, namely stationary harmonic maps.

Definition : A map $u \in H^1(M, N)$ is a stationary harmonic map, if u satisfies (I.2) and if, for any smooth one-parameter family of diffeomorphisms Φ_t of M, satisfying $\Phi_{t|\partial M} = Id_{|\partial M}$, we have

$$\frac{d}{dt} E(u \circ \Phi_t) = 0. \tag{I.4}$$

(I.2) expresses the fact that u is critical for $E(u)$ with respect to variations on the target N, whereas (I.4) expresses the fact that u is critical for E with respect to variations of the domain.

Our main result is the following.

Theorem I.1 *Let $u \in H^1(M, N)$ be a stationary harmonic map. There exists a closed subset Σ of M such that*

$$H^{n-2}(\Sigma) = 0 \tag{I.5}$$

and

$$u \in C^\infty(M \backslash \Sigma, N). \tag{I.6}$$

Here H^{n-2} denotes the $(n-2)$-dimensional Hausdorff measure.

Remark I-1 : Minimizing harmonic maps are weakly stationary harmonic maps. A minimizing harmonic map minimizes the energy functional $E(u)$ on $H^1(M, N)$ among all maps having the same boundary value. For a minimizing harmonic map, R. Schoen and K. Uhlenbeck [18] have proved that the $(n-3)$-dimensional Hausdorff measure of the singular set is locally finite (compare with Theorem I-1 above).

Remark I-2 : In the special case where N is a standard sphere, Theorem I-1 has already been proved by C. Evans [7]. Our proof of Theorem I-1 will strongly rely on ideas introduced in [9] and [7].

As a by-product of our methods, we are also going to prove the following.

Theorem I.2 *Let u be a weakly harmonic map in $W^{1,n}(M,N)$. Then u is smooth in M.*

Remark I-3 : This result has already been proved by F. Hélein in the special case N is an homogeneous space ([10]).

For sake of simplicity, we will assume throughout the paper that M is flat, that is a bounded domain Ω in $\mathbb{R}^n$ (the proof in the general case M is any compact manifold is essentially the same, but technically a little more complicated).

The following monotonicity formula for stationary harmonic maps is crucial in our proof of Theorem I-1.

Theorem I.3 *Let x be a point in Ω. We have*

$$\frac{d}{d\rho}\left(\frac{1}{\rho^{n-2}}\int_{B(x,\rho)}|\nabla u|^2\right) \geqslant 0. \tag{I.7}$$

That is

$$\frac{1}{r_1^{n-2}}\int_{B(x,r_1)}|\nabla u|^2 \leqslant \frac{1}{r_2^{n-2}}\int_{B(x,r_2)}|\nabla u|^2, \tag{I.8}$$

provided $r_1 \leqslant r_2$ and $B(x,r_2) \subset \Omega$.

Theorem I-3 was proved by R. Schoen and K. Uhlenbeck (see [18]) in the case of minimizing harmonic maps, and by P. Price [15] for stationary harmonic maps.

Remark I-4 : The conclusion of Theorem I-1 would still hold, if instead of considering a stationary harmonic map, we would consider a weakly harmonic map satisfying (I.8).

The proof of Theorem I-1 is organized as follows. The main point is to prove the following ε-regularity theorem.

Theorem I.4 *There exists some $\varepsilon_0 > 0$ such that if $B(x_0,r_0) \subset \Omega$ and u satisfies*

$$\frac{1}{r_0^{n-2}}\int_{B(x_0,r_0)}|\nabla u|^2 < \varepsilon_0, \tag{I.9}$$

then u is smooth on $B(x_0,\frac{r_0}{4})$.

It is a standard procedure, based on a covering argument (see e.g [8]) to show that Theorem I.4 and the monotonicity formula (I.8) imply Theorem I-1 (see Section VII). In order to prove Theorem I-4 we are going to make use of Morrey-type inequalities (see e.g. [8]) for some suitable norm. For that purpose, we introduce the following notations.

Set, for $x_0 \in \Omega$, and for r such that $B(x_0, 2r) \subset \Omega$,

$$M(x_0, r) = \sup\left\{\frac{1}{\rho^{n-1}}\int_{B(x,\rho)} |\nabla u|, B(x,\rho) \subset B(x_0, r)\right\}, \tag{I.10}$$

(the supremum is taken all balls $B(x,\rho)$ included in $B(x_0, r)$). We first observe that $M(x_0, r)$ is bounded : this follows from the monotonicity formula and the assumption $B(x_0, 2r) \subset \Omega$. Indeed, we have, by the Cauchy-Schwarz inequality

$$\frac{1}{\rho^{n-1}}\int_{B(x,\rho)} |\nabla u| \leqslant \left(\frac{C}{\rho^{n-2}}\int_{B(x,\rho)} |\nabla u|^2\right)^{1/2}. \tag{I.11}$$

On the other hand Theorem I-3 yields

$$\frac{1}{\rho^{n-2}}\int_{B(x,\rho)} |\nabla u|^2 \leqslant \frac{1}{(\rho+r)^{n-2}}\int_{B(x,\rho+r)} |\nabla u|^2 \leqslant \frac{C}{r^{n-2}}\int_{B(x_0,2r)} |\nabla u|^2, \tag{I.12}$$

and hence combining (I.11) and (I.12) and taking the supremum for all balls $B(x,\rho) \subset B(x_0, \frac{3r}{2})$, we obtain

$$M(x_0, r) \leqslant \left(\frac{C}{r^{n-2}}\int_{B(x_0,2r)} |\nabla u|^2\right)^{1/2} < +\infty. \tag{I.13}$$

We also remark that

$$M(x_0, r_1) \leqslant M(x_0, r_2) \quad \text{if } r_1 \leqslant r_2.$$

We are going to prove the following.

Theorem I.5 : *There exist* $\varepsilon_1 > 0$ *and* $0 < \theta_1 < 1/2$ *such that, if* $B(x_0, 2r) \subset \Omega$ *and if*

$$\frac{1}{(2r)^{n-2}}\int_{B(x_0,2r)} |\nabla u|^2 \leqslant \varepsilon_1, \tag{I.14}$$

then

$$M(x_0, \theta_1 r) \leqslant \frac{1}{4}M(x_0, r). \tag{I.15}$$

Remark I-5 : Related estimates (based on Campanato Spaces) have been used recently by F. Pacard [14].

Combining Theorem I-5 and the monotonicity formula, we deduce by routine arguments Theorem I-4 (see Section VI). Therefore, the main part of this paper is devoted to the proof of Theorem I-5.

The three main ingredients in the proof are the following :

- Hodge de Rham decompositions of forms,
- Compensation properties related to Jacobians and maps satisfying (I.8),
- An appropriate choice of an orthonormal frame on $T_{u(x)}N$ (as in Hélein [9]).

This paper is organized as follows. In section II we recall some useful facts concerning differential forms, with an emphasis on the Hodge de Rham decomposition, which has already been used in a similar context in [1] and by P. Choné [4]. In Section III we present some compensation properties related to Jacobians. This was discovered by R. Coifman, P.L. Lions, Y. Meyer and S. Semmes, who proved that certain non linear quantities, which are of interest for us, belong to the Hardy space $\mathcal{H}^1$ (a space a little smaller than L^1). We include some very useful observations due to C. Evans, in particular the fact that maps satisfying the monotonicity property (I.7) belong to BMO (a space a little larger than L^∞ and which is in duality with $\mathcal{H}^1$). In Section IV, we briefly describe Hélein's method for finding an appropriate tangent frame. In Section V we give the proof of Theorem I-5. In Section VI we show how Theorem I-4 can be derived from Theorem I-5, and we complete the proof of Theorem I-1, in Section VII. In Section VIII we prove Theorem I-2.

Acknowledgements : The author would like to thank F. Hélein and P. Haszlasz for interesting discussions. Part of this work was completed while the author was visiting Rutgers University. He thanks the Math. Department for its support and hospitality.

II Differential forms and Hodge de Rham decomposition

Since we shall use quite extensively the formalism of differentials forms, we recall next some useful facts and notations. For $n \geqslant 1$, we consider the euclidean space $I\!R^n$, and denote by $e_1, \cdots, e_n$ the standard dual basis. For $0 \leqslant l \leqslant 1$, $\wedge^l(I\!R)$ is the set of l-forms on $I\!R^n$ and consists of linear combinations of exterior products

$$e_I = e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_l},$$

where $I = (i_1, \cdots i_l)$ is on ordered l-tuple, i.e $1 \leqslant i_1 < i_2 \cdots < i_l$; $[e_I]$ is the standard basis of $\wedge^l$. For $\alpha = \sum \alpha^I e_I$ and $\beta = \sum \beta^I e_I$ we define the inner product of α and β by

$$< \alpha, \beta >= \sum \alpha^I \beta^I. \tag{II.1}$$

By convention, we set $\Lambda^0 = I\!R$, and if $l \notin \{0, 1, \cdots n\}, \Lambda^l = \{0\}$. The Hodge star operator $\star$ is a linear operator from $\Lambda^l(I\!R)$ to $\Lambda^{n-l}(I\!R^n)$ given by the rules

$$\star 1 = e_1 \wedge e_2 \wedge .. \wedge e_n \tag{II.2}$$

and

$$\alpha \wedge \star\beta = \beta \wedge \star\alpha =< \alpha, \beta > e_1 \wedge .. \wedge e_n, \tag{II.3}$$

for all forms α, β in $\Lambda^l(I\!R)$. We have the identity

$$\star\star = (-I)^{n(n-l)}$$

where I denotes the identity map from Λ^l.

We consider now a domain $\Omega \subset I\!R^n$, and differential l-forms on Ω, that is distributions with values in $\Lambda^l(I\!R)$, and we set $\Lambda^l(\Omega) = \mathcal{D}'(\Omega, \Lambda^l)$ (by convention $\Lambda^0(\Omega) = \mathcal{D}'(\Omega; I\!R)$ and $\Lambda^l = \{0\}$, if $l \notin \{0, 1 \cdots n\}$). The exterior derivative operator

$$d : \Lambda^{l-1}(\Omega) \to \Lambda^l(\Omega)$$

satisfies

$$d(w_1 \wedge w_2) = w_1 \wedge dw_2 + dw_1 \wedge w_2, \tag{II.4}$$

for any two differential forms w_1, w_2. Moreover, if $(x^1, \cdots, x^n)$ are cartesian coordinates on $I\!R^n$, we have

$$dx^i = e_i, \ \ \forall i \in \{1, \cdots, n\}. \tag{II.5}$$

Similarily, we write $dx^I = e_I$, and for every differential form w, we have the decomposition

$$w = \sum_I w_I \ dx^I. \tag{II.6}$$

The conjugate operator of the exterior differential is the Hodge operator

$$d^* : \Lambda^l(\Omega) \to \Lambda^{l-1}(\Omega) \tag{II.7}$$

given by

$$d^* = \star d \star. \tag{II.8}$$

For every α in $C_0^\infty(\Omega, \wedge^l)$, and β in $C_0^\infty(\Omega, \wedge^{l+1})$, we see indeed that

$$\int_\Omega < \alpha, d^*\beta > = (-1)^{nl+1} \int_\Omega < d\alpha, \beta > . \tag{II.9}$$

We have the following important property

$$dd = 0, \;\; d^*d^* = 0. \tag{II.10}$$

Remark that, for functions d plays the role of the familiar grad operator ($df = \sum_i \frac{\partial f}{\partial x^i} dx^i$) whereas d^* from $\wedge^1(\Omega) \to \wedge^0(\Omega)$ plays the role of the divergence operator, in the sense that $d^*(u_1\, dx^1 + u_2\, dx^2 + \cdots) = (-1)^{nl}[div(u)]$. Similarly the curl operator corresponds to $d : \wedge^1(\Omega) \to \wedge^2(\Omega)$.

The Laplace operator for differential forms is given by

$$\Delta = dd^* + d^*d \qquad \Delta : \wedge^l(\Omega) \to \wedge^l(\Omega).$$

It acts only on the coefficients of the form, since

$$\Delta w = (-1)^{nl+1} \sum_I (\Delta w_I) dx^I, \tag{II.11}$$

where Δw_I denotes the usual Laplacian for functions. For a function $f, d^*f = 0$, and therefore $\Delta f = d^*df$, and we recover the classical formula $\Delta = div(grad)$.

Hodge-de Rham decomposition : The following result, due to Iwaniec and Martin, is an extension of the classical Hodge de Rham decomposition (HDR) to forms in $L^p, 1 < p < +\infty$.

Theorem II.1 *([11]) Let w be in $L^p(I\!R^n, \wedge^l)$. Then there is a $(l-1)$-form α and a $(l+1)$-form β such that*

$$w = d\alpha + d^*\beta \tag{II.12}$$

$$d^*\alpha = d\beta = 0. \tag{II.13}$$

The differential forms α and β belong to $W^{1,p}$ and

$$||\alpha||_{W^{1,p}(I\!R^n)} + ||\beta||_{W^{1,p}(I\!R^n)} \leqslant C(p,k,n)||w||_{L^p}. \tag{II.14}$$

Moreover α and β are unique. If $dw = 0$ (resp. $d^*w = 0$) then

$$\beta = 0 \;\; (resp. \;\; \alpha = 0). \tag{II.15}$$

Sketch of the proof : Let G be the fundamental solution of the Laplacian in R^n and set

$$\varphi = G \star w.$$

Then φ is in $W^{2,p}(I\!R^n, \wedge^l)$ and satisfies

$$\Delta\varphi = (dd^\star + d^\star d)\varphi = d(d^\star\varphi) + d^\star(d\varphi) = w.$$

We take $\alpha = d^\star\varphi, \beta = d\varphi$ and easily verify that (II.13), (II.14) and (II.15) are satisfied. In the case $dw = 0$ (*resp* $d^\star w = 0$), $\Delta\beta = 0$ (*resp* $\Delta\alpha = 0$) and therefore $\beta = 0$ (*resp.* $\alpha = 0$).

We will also use the following (classical) result.

Proposition II.1 *Let Ω be a smooth bounded domain in $I\!R^n$. Let w be a form in $L^p(\Omega, \wedge^l)$ $(1 < p < +\infty)$ such that*

$$dw = 0 \ (resp.\ d^\star w = 0) \tag{II.16}$$

and

$$w = 0 \ on\ \partial\Omega \ (resp.\ \star w = 0). \tag{II.17}$$

There is some $(l-1)$-form α in $W^{1,p}(\Omega, \wedge^{l-1})$ (resp. $(l+1)$-form β in $W^{1,p}(\Omega, \wedge^{l+1})$) such that

$$w = d\alpha \ (resp.\ w = d^\star\beta) \tag{II.18}$$

and

$$||\alpha||_{W^{1,p}} \leqslant C_p||w||_{L^p} \ (resp.\ ||\beta||_{W^{1,p}} \leqslant C_p||w||_{L^p}). \tag{II.19}$$

Proof : Extend w by $w = 0$ on $I\!R^n\backslash\Omega$, and note that $dw = 0$ on $I\!R^n$. The conclusion than follows directly from Theorem II-1.

Next, we are going to apply the HDR-decomposition to the following special case.

HDR-Decomposition of the product of a gradient by a function : Let $B(\rho)$ be a ball in $I\!R^n$, and let f be in $H^1(B(\rho), I\!R) \cap L^\infty$ and g be in $H^1(B(\rho), I\!R)$. We consider the 1-form $w = f \wedge dg$, which can be written in coordinates

$$w = f \wedge dg = f\frac{\partial g}{\partial x_1}dx^1 + f\frac{\partial g}{\partial x_2}dx^2 + \cdots + f\frac{\partial g}{\partial x^n}dx^n. \tag{II.20}$$

Let ζ be a smooth function from $I\!R^n$ to $I\!R^+$ such that

$$\zeta \equiv 1 \text{ on } B(\rho/2), \tag{II.21}$$

$$\zeta \equiv 0 \text{ on } \mathbb{R}^n \backslash B(\rho), \tag{II.22}$$

$$|\nabla \zeta| \leqslant \frac{4}{\rho}. \tag{II.23}$$

Let $\tilde{w}$ be the 1-form defined on $\mathbb{R}^n$ by

$$\tilde{w} = f \wedge d(\zeta g). \tag{II.24}$$

Note that $\tilde{w} = w$ on $B(\rho/2)$.

Applying Theorem II-1 to $\tilde{w}$ we find a function α in $H^1(\mathbb{R}^n, \mathbb{R})$ and a 2-form β in $H^1(\mathbb{R}^n; \wedge^2)$ such that

$$\tilde{w} = f \wedge d(\zeta g) = d\alpha + d^*\beta \text{ on } \mathbb{R}^n, \tag{II.25}$$

$$d\alpha = 0, \;\; d^*\beta = 0 \text{ on } \mathbb{R}^n, \tag{II.26}$$

$$||\alpha||_{H^1(\mathbb{R}^n)} + ||\beta||_{H^1(\mathbb{R}^n)} \leqslant C||f||_{L^\infty}||\zeta g||_{H^1}. \tag{II.27}$$

In particular, we have

$$w = f \wedge dg = d\alpha + d^*\beta \text{ on } B(\rho/2). \tag{II.28}$$

Taking the exterior derivative of (II.25), we obtain

$$dd^*\beta = df \wedge d(\zeta g) \text{ on } \mathbb{R}^n. \tag{II.29}$$

Since $d\beta = 0$, (II.29) reads as

$$\Delta\beta = df \wedge d(\zeta g) \text{ on } \mathbb{R}^n. \tag{II.30}$$

Expressing β in the standard basis of $\wedge^2$, that is

$$\beta = \sum_{i<j} \beta^{i,j} dx^i \wedge dx^j,$$

we are led to the equation

$$\Delta\beta^{i,j} = \{f, \zeta g\}_{i,j} \text{ on } \mathbb{R}^n. \tag{II.31}$$

We have used the notation

$$\{f, g\}_{i,j} = f_{x_i} g_{x_j} - g_{x_i} f_{x_j}, \tag{II.32}$$

where subscripts stand for partial derivatives.

A Remark on notation : In the sequel, we will often have to deal with vector-valued forms, that is forms whose coefficients take values in $\mathbb{R}^K$ instead of $\mathbb{R}$. We will denote by $\wedge^l(\mathbb{R}^n; \mathbb{R}^K)$ the set of these forms. Let $(f^1, \cdots, f^k)$ be the standard basis on $\mathbb{R}^K$. A form α in $\wedge^l(\mathbb{R}^n; \mathbb{R}^K)$ writes in coordinates as

$$\alpha = \sum_{j=1}^{k} \alpha_j f^j \tag{II.33}$$

where α_j is a real-valued form $\alpha_j \in \wedge^l(\mathbb{R}^n; \mathbb{R})$. We will use the notation, for α in $\wedge^l(\mathbb{R}^n; \mathbb{R}^K)$ and β in $\wedge^q(\mathbb{R}^n; \mathbb{R}^K)$

$$\alpha \wedge \beta = \sum_{j=1}^{k} \alpha_j \wedge \beta_j \tag{II.34}$$

and hence $\alpha \wedge \beta$ will represent a real-valued form. If α is a zero-form, we will write indistinctibly $\alpha \wedge \beta$ or $\alpha.\beta$, where the point stands for the scalar product.

Most of the previous results extend in a straightforward way to forms in $\wedge^l(\mathbb{R}^n; \mathbb{R}^K)$.

III Compensation phenomena

For f and g in H^1 the right hand side of (II.31) (that is $\{f, g\}_{i,j}$) is clearly in L^1. Thanks to its divergence stucture, namely

$$\{f, g\}_{i,j} = (f g_{x_j})_{x_i} - (f g_{x_i})_{x_j} \tag{III.1}$$

R. Coifman, P.L. Lions, Y. Meyer and S. Semmes were able to prove the following.

Theorem III.1 *([5]) : Assume that f and g are in $H^1(\mathbb{R}^n)$. Then $\{f, g\}_{i,j}$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|\{f, g\}_{i,j}\|_{\mathcal{H}^1(\mathbb{R}^n)} \leqslant C \|\nabla f\|_{L^2} \|\nabla g\|_{L^2}, \tag{III.2}$$

where C is an absolute constant.

We recall that the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is the set of functions ψ in L^1 such that the maximal function

$$\psi^\star(x) = \sup_{r>0} \left| \frac{1}{r^n} \int g(y)\phi\left(\frac{(x-y)}{r}\right) dr \right| \tag{III.3}$$

is also in L^1. In (III.3), ϕ represents any smooth function with support in the unit ball, such that $\int \phi = 1$. A norm on $\mathcal{H}^1$ is

$$||\psi||_{\mathcal{H}^1} = ||\psi||_{L^1} + |\psi^\star|_{L^1}.$$

For equivalent definitions, and more information we refer the reader to Meyer [13], Stein [17], or Torchinsky [19].

A fundamental theorem of Fefferman asserts that the dual space of $\mathcal{H}^1$ is BMO, the space of function ζ such that

$$||\zeta||_{BMO} \equiv \sup \left\{ \fint_{B(x,r)} |\zeta - (\zeta)_{x,r}|, x \in \mathbb{R}^n, r > 0 \right\} < +\infty,$$

where we have set

$$(\zeta)_{x,r} = \fint_{B(x,r)} \zeta = \frac{1}{|B(x,r)|} \int_{B(x,r)} \zeta \ .$$

The duality $\mathcal{H}^1 - BMO$ can therefore been expressed as

$$\left| \int_{\mathbb{R}^n} \psi\zeta \right| \leqslant C||\zeta||_{BMO}||\psi||_{\mathcal{H}^1}. \tag{III.4}$$

In particular, for f, g in $H^1(\mathbb{R}^n)$ and h in BMO we see that

$$\left| \int_{\mathbb{R}^n} \{f, g\}_{i,j} .h \right| \leqslant C||\nabla f||_{L^2}||\nabla g||_{L^2}||h||_{BMO}. \tag{III.5}$$

This will be very useful in our context, in view of the following observation (due to C. Evans).

Proposition III.1 *Let $1 \leqslant p < +\infty$, and h be a map in $W^{1,p}(\mathbb{R}^n)$ such that*

$$\bar{M} \equiv \sup \left\{ \left(\frac{1}{r^{n-p}} \int_{B(x,r)} |\nabla h|^p \right)^{1/p}, \ \ x \in \mathbb{R}^n, r > 0 \right\} < +\infty.$$

Then u belongs to $BMO(\mathbb{R}^n)$ and

$$||h||_{BMO(\mathbb{R}^n)} \leqslant C\bar{M}. \tag{III.6}$$

Proof : By the Poincaré inequality we have

$$\frac{1}{r^n}\int_{B(x,r)} |h-(h)_{x,r}| \leqslant C\left(\frac{1}{r^{n-p}}\int_{B(x,r)}|\nabla h|^p\right)^{1/p},$$

which yields (III.6).

Combining (III.5) and (III.6), we are led to

$$\left|\int_{\mathbb{R}^n}\{f,g\}_{i,j}\,h\right| \leqslant C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2}\bar{M}. \tag{III.7}$$

Remark : In the case $p \neq 1$, S. Chanillo ([2] for $p \geqslant 2$) and S. Chanillo and Y.Y. Li ([3] for $p > 1$) have given a more elementary proof of (III.7), which does not rely on the $\mathcal{H}^1 - BMO$ duality theorem.

We notice that in the definitions above, we have taken the domain to be $\mathbb{R}^n$ whereas we mainly need estimates for bounded domains. Using a tronchation argument (as in [7]) we may prove the following local version of inequality (III.7).

Proposition III.2 *Let f and g be in $H^1(B(r))$, and assume that*

$$f = C^{te} \text{ on } \partial B(r) \text{ or } g = C^{te} \text{ on } \partial B(r). \tag{III.8}$$

Let h be in $W^{1,p}(B(2r))$ $(1 \leqslant p < +\infty)$ such that

$$\bar{M}(2r) = \sup\left\{\left(\frac{1}{\rho^{n-p}}\int_{B(x,\rho)}|\nabla h|^p\right)^{1/p}, B(x,\rho)\subset B(2r)\right\} < +\infty.$$

We have

$$\left|\int_{B(r)}\{f,g\}_{i,j}h\right| \leqslant C\|\nabla f\|_{L^2}\|\nabla g\|_{L^2}\bar{M}(2r), \tag{III.9}$$

where C is an absolute constant.

Proposition III.2 is due to Evans [7]. For sake of completeness we will give a proof in the Appendix.

We end this section with an elementary, yet crucial remark.

Proposition III.3 *Let f, g and h be functions in $H^1(B(r))\cap L^\infty$ such that*

$$f = 0 \text{ on } \partial B(r) \text{ or } g = 0 \text{ on } \partial B(r) \text{ or } h = 0 \text{ on } \partial B(r). \tag{III.10}$$

We have

$$\int_{B(r)}\{f,g\}_{i,j}\,h = \int_{B(r)}\{g,h\}_{i,j}\,f = \int_{B(r)}\{h,f\}_{i,j}\,g. \tag{III.11}$$

Proof : use (III.1) and integrate by parts.

IV Construction of a tangent frame and rewriting of the equation

We first briefly recall some of the main arguments in Hélein's paper [9] which are going to be useful for us.

The first observation is that we may always assume that the image of Ω by u lies in an open subset of N where there exists a smooth orthonormal tangent frame $\tilde{e} = (\tilde{e}_1, \cdots, \tilde{e}_k)$; that is $\tilde{e}_i.\tilde{e}_j = \delta_{ij}$ and $\tilde{e}_i(y) \in T_yN$. Indeed, if that were not the case, one may construct a larger manifold $\bar{N}$, such that $N \subset \bar{N}$, such that u is still harmonic as a map into $\bar{N}$, and such that the assumption holds. Therefore we might replace N by $\bar{N}$.

The second key idea is to construct a frame which is more adapted to u. Consider a ball $B(\rho) \subset \Omega$, the gauge transformation of $\tilde{e}$ given by

$$e_i(x) = R_{ij}(x)[\tilde{e}_j u(x)], \tag{IV.1}$$

where $R = R_{ij}$ is a rotation in $H^1(B(\rho), SO(k))$ and the functional

$$F(R) = \int_{B(\rho)} \sum_{i,j} < \nabla e_i, e_j >^2 . \tag{IV.2}$$

By a result of Dell' Antonio and Zwangiger [6], it can be proved that

$$\bar{\mu} = \inf\{F(R), R \in H^1(B(\rho); SO(k))\}$$

is achieved. We denote by R_0 a minimizer, and $e = (e_1, \cdots, e_k)$ the corresponding tangent frame. Clearly $e_i(x) \in T_{u(x)}N$ and $e_i.e_j = \delta_{ij}$.

The Euler-Lagrange equation for e is

$$\sum_i \frac{-\partial}{\partial x^i} < e_l, \frac{\partial e_m}{\partial x^i} >= 0, \ \forall l, m \text{ in } \{1, \cdots k\}, \tag{IV.3}$$

with a Neumann type boundary condition

$$< e_l, \frac{\partial e_m}{\partial \nu} >= 0, \tag{IV.4}$$

where ν is the exterior normal to $\partial B(\rho)$ (see [9]). From (IV.2) and some geometric arguments, we deduce that

$$\int_{B(\rho)} |\nabla e_l|^2 \leqslant C \int_{B(\rho)} |\nabla u|^2, \ \forall l \in \{1, \cdots, k\}. \tag{IV.5}$$

Equation (IV.3) can be written in a slightly form different using the formalism of forms, namely

$$d^* < e_l \wedge de_m >= 0. \tag{IV.6}$$

Hence by Proposition (II.1), (IV.6) and (IV.4) we see that there is some 2-form $D_{l,m}$ in $H^1(B(r), \wedge^2)$ such that

$$d^* D_{l,m} = e_l \wedge de_m = e_l.de_m \tag{IV.7}$$

and

$$\int_{B(\rho)} |\nabla D_{l,m}|^2 \leqslant C \int_{B(\rho)} |\nabla u|^2, \tag{IV.8}$$

(for (IV.8) we have used (IV.5)).

We turn now to equation (I-2) (or its equivalent (I-3)). Since ∇u lies in $T_{u(x)}N$ which is spanned by the orthonormal frame $(e_1(x), \cdots, e_n(x))$ we may write

$$\nabla u = \sum_l < \nabla u, e_l > e_l. \tag{IV.9}$$

We have for $l \in \{1, \cdots, k\}$

$$\text{div } (< \nabla u, e_l >) = \Delta u.e_l + \nabla u.\nabla e_l. \tag{IV.10}$$

By (I.3) we see that

$$\Delta u.e_l = 0, \tag{IV.11}$$

since $e_l \in T_{u(x)}N$ and $\Delta u \perp T_{u(x)}N$. On the other hand, we may write according to (IV.9)

$$\begin{aligned} \nabla u.\nabla e_l &= \sum_m < \nabla u.e_m, \nabla e_l.e_m > \\ &= \sum_m < du.e_m, d^* D_{l,m} >= \sum_m < d^* D_{l,m}, du > .e_m, \end{aligned}$$

where we have used (IV.7). By (II.3) we may write the vector $< d^* D_{l,m}, du >$ as

$$\begin{aligned} < d^* D_{l,m}, du > &= -\star < d(\star D_{l,m}) \wedge du > \\ &= (-1)^{(n+1)} \sum_{i<j} \left\{ D_{l,m}^{i,j}, u \right\}_{i,j}. \end{aligned} \tag{IV.12}$$

Finally, combining (IV.10), (IV.11) and (IV.12) we obtain

$$\operatorname{div}\,(<\nabla u, e_l>) = (-1)^{(n+1)} \sum_m \sum_{i<j} \left\{D^{i,j}_{l,m}, u\right\}_{i,j} .e_m \tag{IV.13}$$

Rewriting (IV.13) as an elliptic system : Let ζ be a cut-off function satisfying (II.21), (II.22) and (II.23). For l in $\{1, \cdots, k\}$ let $w_l(u)$ be the 1-form defined on $B(\rho)$ by

$$w_l(u) = e_l \wedge du,$$

and $\tilde{w}_l(u)$ be the 1-form on $\mathbb{R}^n$ defined by

$$\tilde{w}_l(u) = e_l \wedge d(\zeta(u - u_\rho)),$$

where

$$u_\rho = \fint_{B(\rho)} u \ .$$

Note that

$$w_l = \tilde{w}_l \text{ in } B(\rho/2). \tag{IV.14}$$

We use Proposition II-2 and write the HDR - decomposition of $\tilde{w}_l(u)$ on $\mathbb{R}^n$,

$$\tilde{w}_l(u) = d\alpha_l + d^*\beta_l, \tag{IV.15}$$

where α_l is a function in $H^1(\mathbb{R}^n; \mathbb{R}^K)$ and β_l a 2-form in $H^1(\mathbb{R}^n; \mathbb{R}^K)$ such that

$$||\alpha||_{H^1(\mathbb{R}^n)} + ||\beta||_{H^1(\mathbb{R}^n)} \leqslant C||\nabla u||_{L^2}, \tag{IV.16}$$

and

$$d^*\alpha = 0, \quad d\beta = 0. \tag{IV.17}$$

Clearly, we have by (IV.9) and (IV.14) and (IV.15)

$$|\nabla u| \leqslant C \sum_{l=1}^{k} (|\nabla \alpha_l| + |\nabla \beta_l|) \quad \text{in } B(\rho/2). \tag{IV.18}$$

By (II.31), the coeffficients $\beta_l^{i,j}$ of β_l in the standard basis satisfy the equation

$$\Delta \beta_l^{i,j} = \{e_l, \zeta(u - u_0)\}_{i,j} \quad \text{in } \mathbb{R}^n. \tag{IV.19}$$

On the other hand, applying the Hodge operator d^* to (IV.15), we obtain

$$d^*\tilde{\omega}_l(u) = d^*d\alpha_l = \Delta\alpha_l$$

(since $d^*\alpha_l = 0$). This yields in view of the equation (IV.13) and of (IV.14)

$$\Delta\alpha_l = (-1)^{n+1}\sum_m\sum_{i<j}\left\{D^{i,j}_{l,m}, u\right\}_{i,j}.e_m \text{ in } B(\rho/2). \tag{IV.20}$$

Hence, we have written equation (IV.13) in the form of an elliptic system (IV.19) and (IV.20) for α_l and β_l.

V Proof of Theorem I-5

Let $B(x_0, r)$ be a ball in Ω such that $B(x_0, 2r) \subset \Omega$ and such that the smallness assumption (I.14) holds, for some ε_1 to be determined later. Let $B(x,\rho)$ be a ball included in $B(x_0, \frac{r}{2})$ (in particular $\rho < \frac{r}{2}$). We deduce from (I.12) and (I.14) that

$$M(x, 2\rho) \leqslant M(x_0, r) \leqslant C\varepsilon_1^{1/2}. \tag{V.1}$$

On the other hand, the monotonicity formula (I.8) and (I.14) yield

$$\frac{1}{\rho^{n-2}}\int_{B(x,\rho)} |\nabla u|^2 \leqslant C\varepsilon_1. \tag{V.2}$$

We construct the tangent frame $e = (e_1, \cdots, e_k)$ as in section IV, on the ball $B(x,\rho)$. We are going to estimate $|\nabla\alpha_l|$ and $|\nabla\beta_l|$, which will give us by (IV.15) an estimate for $|\nabla u|$.

Estimates for β_l : Multiplying (IV.18) by $\beta_l^{i,j}$ and integrating on $I\!R^n$, we obtain

$$\int_{I\!R^n} |\nabla\beta_l^{i,j}|^2 = \int_{I\!R^n} \{e_l, \zeta(u-(u)_{x,\rho})\}_{i,j}\, \beta_l^{i,j}, \tag{V.3}$$

for all indexes, i, j and l. By Proposition III-3 we have

$$\int_{I\!R^n} \{e_l, \zeta(u-(u)_{x,\rho})\}_{i,j}.\beta_l^{i,j} = -\int_{I\!R^n} \left\{\beta_l^{i,j}, e_l\right\}_{i,j}.\zeta(u-(u)_{x,\rho}). \tag{V.4}$$

By Lemma A-1 of the Appendix, we have

$$||\zeta(u-(u)_{x,\rho})||_{BMO(I\!R^n)} \leqslant M(x, 2\rho). \tag{V.5}$$

Applying Proposition II-2, we see that

$$\begin{aligned}\left|\int_{B(x,\rho)}\left\{\beta_l^{i,j}, e_l\right\}_{i,j}.\zeta(u-(u)_{x,\rho})\right| &\leqslant C||\nabla e_l||_{L^2}||\nabla\beta_l^{i,j}||_{L^2}M(x,2\rho)\\ &\leqslant C||\nabla u||_{L^2}||\nabla\beta_l^{i,j}||_{L^2}M(x,\rho)\\ &\leqslant C\rho^{\frac{n-2}{2}}\varepsilon_1^{1/2}M(x,\rho)||\nabla\beta_l^{i,j}||_{L^2},\end{aligned} \tag{V.6}$$

where we have used (IV.5) and (V.2) for the last inequalities. Going back to (V.3) we are led to

$$\int_{\mathbb{R}^n} |\nabla\beta_l^{i,j}|^2 \leqslant C\rho^{n-2}\varepsilon_1 M^2(x,\rho),$$

which gives, by Cauchy - Schwarz inequality,

$$\frac{1}{\rho^{n-1}}\int_{B(x_0,\rho)} |\nabla\beta_l^{i,j}| \leqslant C\varepsilon_1^{1/2}M(x,\rho). \tag{V.7}$$

We estimate next α_l. We write, on $B(x,\rho/2)$

$$\alpha_l = \alpha_l^0 + \alpha_l^1, \tag{V.8}$$

where α_l^0 is the solution to

$$\Delta\alpha_l^0 = 0 \text{ in } B(x,\rho/2) \tag{V.9}$$

$$\alpha_l^0 = \alpha_l \text{ on } \partial B(x,\rho/2), \tag{V.10}$$

and α_l^1 is the solution to

$$\Delta\alpha_l^1 = (-1)^{(n+1)}\sum_m\sum_{i<j}\left\{D_{l,m}^{i,j},u\right\}_{i,j}.e_m \text{ on } B(x,\rho/2) \tag{V.11}$$

$$\alpha_l^1 = 0 \text{ on } \partial B(x,\rho/2). \tag{V.12}$$

Estimate for α_l^1 : Let ψ be the solution to

$$\Delta\psi = \operatorname{div}\left(\frac{\nabla\alpha_l^1}{|\nabla\alpha_l^1|}\right) \text{ in } B(x,\rho/2) \tag{V.13}$$

$$\psi = 0 \text{ on } \partial B(x,\rho/2). \tag{V.14}$$

Since $\left|\frac{\nabla\alpha_l^1}{|\nabla\alpha_l^1|}\right| = 1$, the r.h.s. of (V.13) is the divergence of a bounded function and therefore, by standard elliptic estimates ψ is bounded in $W^{1,q}(B(x,\rho/2))$ for every $1 < q < +\infty$. Hence by the Sobolev imbedding Theorem ψ is bounded in L^∞. We have

$$||\psi||_{L^\infty} \leqslant C\rho \tag{V.15}$$

and

$$||\nabla\psi||_{L^2} \leqslant C\rho^{n/2}, \tag{V.16}$$

where C is an absolute constant.

Since $\alpha_l^1 = \psi = 0$ on $\partial B(x,\rho)$, we see that, integrating by parts

$$\begin{aligned}\int_{B(x,\rho/2)} \Delta\alpha_l^1.\psi &= \int_{B(x,\rho/2)} \Delta\psi.\alpha_l^1 = \int_{B(x,\rho/2)} \operatorname{div}\left(\frac{\nabla\alpha_l^1}{|\nabla\alpha_l^1|}\right).\alpha_l^1 \\ &= \int_{B(x,\rho/2)} |\nabla\alpha_l^1|.\end{aligned}$$

Therefore we obtain, using equation (V.11)

$$\int_{B(x,\rho/2)} |\nabla\alpha_l^1| = (-1)^{(n+1)} \sum_m \sum_{i<j} \int_{B(x,\rho/2)} \left\{D_{l,m}^{i,j}, u\right\}_{i,j}.\psi e_m \quad . \tag{V.17}$$

Since e_m is uniformly bounded ($|e_m| = 1$), and since ψ is bounded by (V.15), ψe_m belongs to $H^1(B(x,\rho/2); I\!R^K)$. More precisely

$$||\nabla(\psi e_m)||_{L^2} \leqslant ||\psi||_{L^\infty}||\nabla e_m||_{L^2} + ||\nabla\psi||_{L^2},$$

which yields, in view of (V.15), (IV.5) and (V.16)

$$||\nabla(\psi e_m)||_{L^2(B(x,\rho/2))} \leqslant C\rho^{n/2}. \tag{V.18}$$

By Proposition III-3, we see that

$$\int_{B(x,\rho/2)} \left\{D_{l,m}^{i,j}, u\right\}_{i,j}.\psi e_m = -\int_{B(x,\rho/2)} \left\{D_{l,m}^{i,j}, \psi e_m\right\}_{i,j}.u, \tag{V.19}$$

and by Proposition II-2 (with $p = 1$) we obtain

$$\begin{aligned}\left|\int_{B(x,\rho/2)} \left\{D_{l,m}^{i,j}.\psi e_m\right\}_{i,j}.u\right| &\leqslant C||\nabla(\psi e_m)||_{L^2}||\nabla D_{l,m}^{i,j}||_{L^2} M(x,\rho) \\ &\leqslant C\rho^{n-1}\varepsilon_1^{1/2} M(x,\rho).\end{aligned} \tag{V.20}$$

We have used (V.18), (IV.8) and (V.2) for the last inequality. Combining (V.20) and (V.17) we are led to

$$\frac{1}{\rho^{n-1}}\int_{B(x,\rho/2)}|\nabla\alpha_l^1|\leqslant C\varepsilon_1^{1/2}M(x_0,r),\quad \forall l\in\{1,\cdots,k\}. \tag{V.21}$$

Estimates for α_l^0 : By (V.8) and (V.15) we have

$$|\nabla\alpha_l^0|\leqslant|\nabla\alpha_l^1|+|\nabla\alpha_l|\leqslant|\nabla\alpha_l^1|+C(|\nabla u|+|\nabla\beta_l|).$$

Therefore by (V.7) and (V.21) we obtain

$$\frac{1}{\rho^{n-1}}\int_{B(x_0,\rho/2)}|\nabla\alpha_l^0|\leqslant CM(x_0,r). \tag{V.22}$$

Since α_l^0 is a harmonic function, it follows from standard elliptic estimates that for any $0\leqslant\theta\leqslant\frac{1}{2}$, we have

$$\frac{1}{(\theta\rho)^{n-1}}\int_{B(x,\theta\rho)}|\nabla\alpha_l^0|\leqslant C\theta\left(\frac{1}{\rho^{n-1}}\int_{B(x,\rho)}|\nabla\alpha_l^0|\right)$$

which yields, combined with (V.22)

$$\frac{1}{(\theta\rho)^{n-1}}\int_{B(x,\theta_\rho)}|\nabla\alpha_l^0|\leqslant C\theta M(x_0,r), \tag{V.23}$$

for $0\leqslant\theta<\frac{1}{2}$ and l in $\{1,\cdots,k\}$.

Choice of θ_0 and ε_1, and completion of the proof : Since

$$|\nabla u|\leqslant C\sum_{l=1}^{k}(|\nabla\alpha_l^0|+|\nabla\alpha_l^1|+|\nabla\beta_l|), \tag{V.24}$$

we see that

$$\frac{1}{(\theta\rho)^{n-1}}\int_{B(x,\theta\rho)}|\nabla u|\leqslant C\sum_{l=1}^{k}\frac{1}{(\theta\rho)^{n-1}}\left(\int_{B(x,\theta\rho)}|\nabla\alpha_l^0|+\int_{B(x,\rho/2)}(|\nabla\alpha_l^1|+|\nabla\beta_l|)\right).$$

Combining the previous inequality with estimates (V.8), (V.21) and (V.23) we are led to

$$\frac{1}{(\theta\rho)^{n-1}}\int_{B(x,\theta\rho)}|\nabla u|\leqslant C\left(\theta+\frac{\varepsilon_1^{1/2}}{\theta^{n-1}}\right)M(x_0,r). \tag{V.25}$$

We first choose some number θ_0 in $(0,\frac{1}{2})$ such that

$$C\theta_0 \leqslant \frac{1}{8}. \tag{V.26}$$

We then determine ε_1 such that

$$C^{1/2}\varepsilon_1 \leqslant \frac{1}{8}\theta_0^{n-1}, \tag{V.27}$$

and we set

$$\theta_1 = \frac{1}{2}\theta_0. \tag{V.28}$$

Going back to (V.25), we see that for any ball $B(x,\rho) \subset B(x_0, \frac{r}{2})$ we have

$$\frac{1}{(\theta_0\rho)^{n-1}} \int_{B(x,\theta_0\rho)} |\nabla u| \leqslant \frac{1}{4} M(x_0, r). \tag{V.29}$$

We consider the sets

$$\mathcal{A}_0 = \{B(x,\tilde{\rho}), \text{ such that } B(x,\tilde{\rho}) \subset B(x_0, \theta_1 r)\}$$

and

$$\mathcal{A}_1 = \{B(x,\theta_0\rho), \text{ such that } B(x,\rho) \subset B(x_0, \frac{r}{2})\},$$

and we easily verify that

$$\mathcal{A}_0 \subset \mathcal{A}_1. \tag{V.30}$$

Indeed, let $B(x,\tilde{\rho})$ be such that $B(x,\tilde{\rho}) \subset B(x_0,\theta_1 r)$. Since $\theta_0 = 2\theta_1$, we see that $B(x, \frac{\tilde{\rho}}{\theta_0}) \subset B(x_0, \frac{r}{2})$, and hence $B(x,\tilde{\rho}) \in \mathcal{A}_1$. From (V.29), we therefore deduce that

$$\frac{1}{\tilde{\rho}^{n-1}} \int_{B(x,\tilde{\rho})} |\nabla u| \leqslant \frac{1}{4} M(x_0, r). \tag{V.31}$$

If we take the supremum of the left-hand side of (V.31) for all balls $B(x,\tilde{\rho}) \in \mathcal{A}_1$ we are led to

$$M(x_0, \theta_1 r) \leqslant \frac{1}{4} M(x_0, r),$$

and this completes the proof of Theorem I-5.

VI Proof of Theorem I-4

We assume that (I.9) holds, for some ε_0 to be determined later. Let x_1 be a point on $B(x_0, \frac{r_0}{4})$. From the monotonicity formula (I.8) we deduce that for any $0 \leqslant r \leqslant \frac{r_0}{2}$ we have

$$\frac{1}{r^{n-2}}\int_{B(x_1,r)}|\nabla u|^2 \leqslant \frac{1}{(\frac{3}{4}r_0)^{n-2}}\int_{B(x_1,\frac{3}{4}r_0)}|\nabla u|^2 \leqslant (\frac{4}{3})^{n-2}\varepsilon_0, \quad \text{(VI.1)}$$

where we have used (I.9) for the last inequality. We determine the value of ε_0 by

$$\varepsilon_0 = (\frac{3}{4})^{n-2}\varepsilon_1. \quad \text{(VI.2)}$$

Going back to (VI.1) we see that for $0 \leqslant r \leqslant \frac{r_0}{2}$, and x_1 in $B(x_0, \frac{r_0}{4})$

$$\frac{1}{r^{n-2}}\int_{B(x_1,r)}|\nabla u|^2 \leqslant \varepsilon_1. \quad \text{(VI.3)}$$

On the other hand, we deduce from (I.13) that for every $0 \leqslant r \leqslant \frac{r_0}{4}$ and x_1 in $B(x_0, \frac{r_0}{4})$ we have

$$M(x_1, r) \leqslant 2^{\frac{n-2}{2}}\varepsilon_1^{1/2}. \quad \text{(VI.4)}$$

Since (VI.3) holds, we may apply Theorem I-6 to u restricted to $B(x_1, r)$ (for $0 \leqslant r \leqslant \frac{r_0}{4}$). This yields first for $r = \frac{r_0}{4}$,

$$M(x_1, \theta_1\frac{r_0}{4}) \leqslant \frac{1}{4}M(x_1, \frac{r_0}{4}).$$

Taking then successively $r = \theta_1\frac{r_0}{4}, r = \theta_1^2\frac{r_0}{4}, \cdots, r = \theta_1^m\frac{r_0}{4}$ and iterating inequality (I.15) we are led to, for any $m \in \mathbb{N}^*$

$$M(x_1, \theta_1^m\frac{r_0}{4}) \leqslant (\frac{1}{4})^m M(x_1, \frac{r_0}{4}) \leqslant (\frac{1}{4})^m 2^{\frac{n-2}{2}}\varepsilon_1^{1/2}. \quad \text{(VI.5)}$$

For $0 \leqslant r \leqslant \frac{r_0}{4}$, there is some $m \in \mathbb{N}^*$ such that

$$\theta_1^{m+1}\frac{r_0}{4} \leqslant r \leqslant \theta_1^m\frac{r_0}{4}. \quad \text{(VI.6)}$$

Clearly we have

$$m \leqslant \log(\frac{4r_0/r}{\theta_1}).$$

Hence, it follows from (VI.5) that

$$\begin{aligned} M(x_1, r) \leqslant M(x_1, \theta_1^m\frac{r_0}{4}) &\leqslant C(\frac{1}{4})^m \\ &\leqslant C \exp\ (m \log r(\frac{\log 1/4}{\log\theta_1})) \\ &\leqslant Cr^\mu, \end{aligned} \quad \text{(VI.7)}$$

where $\mu = \frac{\log 1/4}{\log \theta_1}$. On the other hand, from the definition of $M(x_1, r)$ we see that

$$\frac{1}{r^{n-1}} \int_{B(x_1,r)} |\nabla u| \leqslant M(x_1, r),$$

and hence, by (VI.7)

$$\int_{B(x_1,r)} |\nabla u| \leqslant C r^{\mu+(n-1)}, \tag{VI.8}$$

for every x_1 in $B(x_0, \frac{r_0}{4})$ and $r \leqslant \frac{r_0}{4}$. By a classical result due to Morrey (see e.g. [8], Theorem 1.1 p 64) that u is hölder continuous in $B(x_0, \frac{r_0}{4})$. Higher regularity can thereafter be derived by standard arguments.

VII Proof of Theorem I-1 completed

We use a standard covering argument (see e.g. [18], Corollary 2-7). Let $B(\lambda)$ be a ball in Ω of radius λ. Let $\delta > 0$ be small, and $B(x_1, \delta), B(x_2, \delta), \cdots, B(x_l, \delta)$ be a maximal family of l balls covering $B(\lambda)$. By maximality of this family, we have

$$l \leqslant C\delta^{-n}, \tag{VII.1}$$

where C is some absolute constant. We also see that

$$B(\frac{\lambda}{2}) \subset \cup_{i=1}^{l} B(x_i, 2\delta). \tag{VII.2}$$

Relabelling if necessary the points x_i, let $x_1, x_2, \cdots, x_p$ be the points such that

$$\frac{1}{(8\delta)^{n-2}} \int_{B(x_i,8\delta)} |\nabla u|^2 \geqslant \varepsilon_0. \tag{VII.3}$$

It follows from Theorem I-4 (ε- regularity) that

$$\Sigma \cap B(\frac{\lambda}{2}) \subset \cup_{i=1}^{p} B(x_i, 2\delta). \tag{VII.4}$$

Integrating $|\nabla u|^2$ on $\cup_{i=1}^{p} B(x_i, 8\delta)$, we obtain, in view of (VII.3) (by maximality of the family $B(x_i, \delta)$)

$$p\delta^{n-2} \leqslant C\varepsilon_0^{-1} \int_{\cup_{i=1}^{p} B(x_i,8\delta)} |\nabla u|^2 \leqslant C\varepsilon_0^{-1} \int |\nabla u|^2, \tag{VII.5}$$

and therefore $\mathcal{H}^{n-2}(\sum \cap B(\frac{\lambda}{2})) \leqslant CE(u)$ (letting $\delta \to 0$); hence $\mathcal{H}^{n}(\cup_{i=1}^{p} B(x_i, 8\delta))$ tends to zero as $\delta \to 0$. It follows that

$$\lim_{\delta \to 0} \int_{\cup_{i=1}^{p} B(x_i,8\delta)} |\nabla u|^2 = 0.$$

Going back to (VII.5), this yields

$$\mathcal{H}^{n-2}(\Sigma \cap B(\frac{\lambda}{2})) = 0,$$

and completes the proof of Theorem I-1.

VIII Proof of Theorem I-2

The proof is essentially the same as the proof of Theorem I-1. Since u is in $W^{1,n}(M, N)$ we have, by Hölder's inequality

$$\frac{1}{r^{n-2}} \int_{B(x,r)} |\nabla u|^2 \leqslant C \left(\int_{B(x,r)} |\nabla u|^n \right)^{2/n} < +\infty \tag{VIII.1}$$

and

$$\frac{1}{r^{n-1}} \int_{B(x,r)} |\nabla u| \leqslant C \left(\int |\nabla u|^n \right)^{1/n} < +\infty. \tag{VIII.2}$$

It follows from (VII.2), that for every ball $B(x, r) \subset \Omega$

$$M(x_0, r) \leqslant C E_n(u)^{1/n}, \tag{VIII.3}$$

where we have set $E_n(u) = \int_\Omega |\nabla u|^n$. In view of (VIII.1) and (VIII.3) we may easily adapt the proof of Theorem I-6, and obtain, the following.

Theorem VIII.1 : *There are constant $\varepsilon_2 > 0$ and $0 < \theta_2 < \frac{1}{2}$ such that if u in $W^{1,n}(\Omega, N)$ is weakly harmonic and if, for $B(x_0, r) \subset \Omega$*

$$\int_{B(x_0,r)} |\nabla u|^n \leqslant \varepsilon_2, \tag{VIII.4}$$

then

$$M(x_0, \theta_2 r) \leqslant \frac{1}{4} M(x_0, r). \tag{VIII.5}$$

Readily reproducing the arguments of Section VI, we may prove that there is some constant $\varepsilon_3 > 0$, such that if

$$\int_{B(x_0,r)} |\nabla u|^n \leqslant \varepsilon_3, \tag{VIII.6}$$

then, u is regular in $B(x_0, \frac{r}{4})$. Since one may always choose some r sufficiently small such that (VIII.6) is satisfied, this completes the proof of Theorem I-2.

APPENDIX

Lemma A.1 *Let h be a function in $W^{1,1}(B(x_0,2r))$ such that*

$$(h)_{x_0,2r} = \fint_{B(x_0,2r)} h = 0 \tag{A.1}$$

and

$$M(x_0,2r) < +\infty.$$

Let ζ be a smooth function from $\mathbb{R}^n \to \mathbb{R}^+$ such that

$$\zeta(x) \equiv 1 \text{ on } B(x_0,r), \tag{A.2}$$

$$\zeta(x) \equiv 0 \text{ on } \mathbb{R}^n \backslash B\left(x_0, \frac{3r}{2}\right), \tag{A.3}$$

$$|\nabla \zeta| \leqslant \frac{4}{r}. \tag{A.4}$$

Then ζh is in $BMO(\mathbb{R}^n)$ and

$$||\zeta h||_{BMO(\mathbb{R}^n)} \leqslant CM(x_0,2r), \tag{A.5}$$

where C is an absolute constant.

Proof : The argument is due to C. Evans [7]. For sake of completeness, we recall it. We have for any ball $B(x,\rho) \subset B(x_0,2r)$

$$\int_{B(x,\rho)} |h - (h)_{x,r}| \leqslant C\frac{1}{\rho^{n-1}} \int_{B(x,\rho)} |\nabla h| \leqslant CM(x_0,2r), \tag{A.6}$$

which implies by the John and Nirenberg inequality ([JN]) that h is bounded in L^p, for every $1 \leqslant p < +\infty$ and, since (A.1)) holds,

$$\left(\int_{B(x_0,2r)} |h|^p\right)^{1/p} \leqslant C_p r^{n/p} M(x_0,2r). \tag{A.7}$$

We see that for any ball $B(x,\rho) \subset B(x_0,2r)$ we have, for any y in $B(x,\rho)$

$$
\begin{aligned}
|(\zeta h)_{x,\rho} - \zeta(y)(h)_{x,\rho}| &= |\fint_{B(x,\rho)} \zeta h - \zeta(y) \fint_{B(x,\rho)} h\,| \\
&= |\fint_{B(x,\rho)} (\zeta - \zeta(x))h + (\zeta(x) - \zeta(y)) \fint_{B(x,\rho)} h\,| \\
&\leqslant C\frac{\rho}{r} \fint_{B(x,\rho)} |h|, \qquad \text{(A.8)}
\end{aligned}
$$

where we have used assumption (A.4) for the last inequality. It follows that

$$
\begin{aligned}
\fint_{B(x,\rho)} |\zeta h - (\zeta h)_{x,\rho}| &\leqslant \fint_{B(x,\rho)} |\zeta h - \zeta(h)_{x,\rho}| \fint_{B(x,\rho)} |\zeta(h)_{x,\rho} - (\zeta h)_{x,\rho}| \\
&\leqslant \fint_{B(x,\rho)} |h - (h)_{x,\rho}| + C\frac{\rho}{r} \fint_{B(x,\rho)} |h| \\
&\leqslant CM(2r) + C(\int |h|^n)^{1/n} \\
&\leqslant CM(2r),
\end{aligned}
$$

the last inequalities resulting form (A.6) and (A.7) with $p = n$. This completes the proof of Lemma A-1.

Proof of Proposition II-2 : We assume that, for instance $f = C^{te}$ on $\partial B(r)$. Substracting if necessary suitable constants, we may always assume that

$$
f = 0 \text{ on } \partial B(r) \tag{A.9}
$$

and

$$
\fint_{B(r)} g = 0. \tag{A.10}
$$

We may therefore find an extension $\tilde{g}$ of g to $\mathbb{R}^n$ such that $\tilde{g} = g$ on $B(r), \tilde{g} = 0$ on $\mathbb{R}^n \backslash B(2r)$ and

$$
||\tilde{g}||_{H^1(\mathbb{R}^n)} \leqslant C||g||_{H^1(\mathbb{R}^n)}. \tag{A.11}
$$

We extend f by $\tilde{f}$ to $\mathbb{R}^n$, such that $\tilde{f} = 0$ on $\mathbb{R}^n \backslash B(r)$. Set

$$
\bar{h} = h - \fint_{B(2r)} h = h - (h)_{0,2r},
$$

and let ζ by as in (A.2), (A.3) (A.5). By Lemma A-1 we have

$$
||\zeta \bar{h} + (h)_{0,2r}||_{BMO(\mathbb{R}^n)} = ||\zeta \bar{h}||_{BMO(\mathbb{R}^n)} \leqslant CM(2r). \tag{A.12}
$$

On the other hand

$$\int_{B(r)} \{f,g\}_{i,j}\, h = \int_{I\!R^n} \left\{\tilde{f},g\right\}_{i,j} .(\zeta\bar{h} + (h)_{o,2r}).$$

Hence

$$\begin{aligned} \left|\int_{B(r)} \{f,g\}_{i,j} .h\right| &\leqslant C\|\nabla\tilde{f}\|_{L^2(I\!R^n)}\|\nabla\tilde{g}\|_{L^2(I\!R^n)}\|\rho\bar{h} + (h)_{0,2r}\|_{BMO(I\!R^n)} \\ &\leqslant C\|\nabla f\|_{L^2(B(r))}\|\nabla g\|_{L^2B(r))}M(2r), \end{aligned}$$

where we have used (A.11) and (A.12). This completes the proof of Proposition II-2.

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(Received November 19, 1992)