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ON L^1-VORTICITY FOR 2-D INCOMPRESSIBLE FLOW

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We prove the existence of a classical weak solution for the 2-D incompressible Euler equations with initial vorticity $w_0 = w_0' + w_0''$, where w_0' is in $L^1(R^2) \cap H^{-1}(R^2)$, compactly supported, and w_0'' is a compactly supported positive Radon measure in $H^{-1}(R^2)$.

1. Introduction

The Euler equations for an inviscid incompressible 2-D fluid flow are given by

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p \qquad x \in R^2,\ t > 0$$

$$\operatorname{div} v = 0$$

$$v(x,0) = v_0(x)$$

where $v = (v_1, v_2)^t$ is the fluid velocity, p is the scalar pressure, and v_0 is an initial incompressible velocity field, i.e. $\operatorname{div} v_0 = 0$.

In Di Perna-Majda's fundamental paper [3], the following problems are studied, (we refer to [3] for notation and terminology.)

Problem 1. Assume that the initial incompressible velocity field v_0 vanishes as $|x| \to \infty$ and that the vorticity $w_0 = \operatorname{curl} v_0$ is a Radon measure contained in Sobolev space $H^{-1}_{loc}(R^2)$. Is there a classical weak solution $v(x,t)$ of 2-D Euler equation with initial velocity field v_0?

Problem 2. Given an initial velocity field v_0 with the same properties as in Problem 1, assume that $v^\epsilon(x,t)$ is an approximate solution sequence for 2-D Euler equations such that at time zero

$$v^\epsilon(x,0) = \tilde{v}_0^\epsilon(x) + \bar{v}^\epsilon(x)$$

and

$$\|\tilde{v}^\epsilon(0) - \tilde{v}_0\| + \|\bar{v}^\epsilon(0) - \bar{v}_0\| \longrightarrow 0$$

as $\epsilon \to 0$. Does $v^\epsilon(x,t)$ converge to a classical weak solution of the Euler equation with initial data $v_0(x)$ as $\epsilon \to 0$?

The definition of a classical weak solution for the 2-D Euler equations is given in [3] as follows.

Definition. *A velocity field $v(x,t) \in L^\infty([0,T], L^2(R^2;R^2))$ for any $T > 0$ and vanishing as $|x| \to \infty$ is a classical weak solution of the 2-D Euler equations with initial data $v_0(x)$ provided that*

(1) for all test functions $\Phi \in C_0^\infty(R^2 \times R^+; R^2)$ with $\operatorname{div}\Phi = 0$,

$$\int\int (\Phi_t \cdot v + \nabla\Phi : v \otimes v)\, dx\, dt = 0;$$

(2) the velocity is incompressible in the weak sense, i.e., for all $\phi \in C_0^\infty(R^2 \times R^+)$,

$$\int\int \nabla\phi \cdot v\, dx\, dt = 0;$$

(3) $v(x,t) \in Lip([0,T], H_{loc}^{-L}(R^2))$ for some $L > 0$ and $v_0(x,0) = v_0(x)$ in H_{loc}^{-L}.

Here $v \otimes v = (v_i v_j)$, $\nabla\Phi = (\partial\Phi_i/\partial x_j)$, and $A : B$ denotes the matrix product $\sum_{i,j} a_{ij} b_{ij}$.

In [3], Di Perna and Majda construct approximate solution sequences $v^\epsilon(x,t)$ in different ways and study the related concentration-cancellation phenomena. There are many other works devoted to these problems. See, among others, [4], [8].

The key point in proving that a weak limit $v(x,t)$ of an approximate solution sequence $v^\epsilon(x,t)$ is a classical weak solution of the 2-D Euler equations is to prove that $v_1^\epsilon v_2^\epsilon \rightharpoonup v_1 v_2$ and $v_1^{\epsilon 2} - v_2^{\epsilon 2} \rightharpoonup v_1^2 - v_2^2$. Here "$\rightharpoonup$" denotes weak convergence in the sense of distributions. By assuming that initial vorticity $w_0 = \operatorname{curl} v_0 \in L^p_{comp}(R^2)$, with $p > 1$, Di Perna and Majda [3] prove that the weak limit $v(x,t)$ is a classical weak solution for 2-D Euler equation by using Sobolev embedding theorem, whereas J.M. Delort [1] proves it for initial vorticity $w_0 = w_0' + w_0''$, where w_0' is a compactly supported positive Radon measure in $H^{-1}(R^2)$, and w_0'' is in L^p, $p > 1$, and compactly supported.

In this note, we prove the existence of a classical weak solution $v(x,t)$ for the 2-D incompressible Euler equations with initial vorticity $w_0 = w_0' + w_0''$, where w_0' is in $L^1(R^2) \cap H^{-1}(R^2)$, compactly supported, and w_0'' is a compactly supported positive Radon measure in $H^{-1}(R^2)$. Moreover, we prove that the vorticity $w(\cdot,t) = \operatorname{curl} v(\cdot,t)$ stays in $L^1(R^2)$ for each time $t > 0$ if the initial vorticity w_0 is in $L^1(R^2)$. The new element in the present paper is the idea of using Dunford-Pettis theorem to show that weak L^1_{loc}-convergence is preserved by the flow. Again by Dunford-Pettis theorem, this is sufficient to apply Delort's argument.

Note. After completion of this paper J.M. Delort kindly informed us that that P.L. Lions had commented to him on the possibility of proving the result described here. Actually such remark is referred to in a footnote on page 9 of [2]. However, to our knowledge, not even the hint of a proof has ever been published. We would like to thank Prof. Delort for his remarks and for making reference [2] available to us. Moreover, we would like to thank Bob Kohn for his support.

2. Main Results

Notations. Let $x = (x_1, x_2)$ be the coordinates on R^2, $\mathcal{D}'(R^2)$ denotes the space of distributions on R^2 and $M(R^2)$ is the space of Radon measures on R^2. If $\varphi \in \mathcal{D}'(R^2)$, we denote

$$\partial_j \varphi = \frac{\partial \varphi}{\partial x_j}, \qquad j = 1,2, \qquad \nabla \varphi = (\partial_1 \varphi, \partial_2 \varphi)^t,$$
$$\nabla^\perp \varphi = (-\partial_2 \varphi, \partial_1 \varphi)^t, \qquad \Delta \varphi = (\partial_1^2 + \partial_2^2)\varphi.$$

If $v = (v_1, v_2)^t$ and $w = \operatorname{curl} v$ is compactly supported, we have

$$v = \nabla^\perp \Delta^{-1} w,$$

where

$$u = \Delta^{-1} w = \int \log |x-y| w(y)\, dy.$$

Moreover we recall that if w is a radially symmetric function in $C_0^\infty(R^2)$, then $v = \nabla^\perp \Delta^{-1} w$ is a stationary solution of 2-D Euler equation.

Theorem 1. *Suppose w_0 is a compactly supported function on $L^1 \cap H^{-1}(R^2)$ and let $v_0 = \nabla^\perp \Delta^{-1} w_0$. Then there exists a function $v \in L^\infty_{loc}(R; L^2_{loc}(R^2; R^2))$, and a function $p \in L^\infty_{loc}(R; \mathcal{D}'(R^2))$, s.t. (v,p) is a classical weak solution of the 2-D Euler equation:*

$$\begin{cases} \dfrac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p & x \in R^2,\ t > 0 \\ div\, v = 0 \\ v(x,0) = v_0(x) \end{cases} \tag{2.1}$$

Moreover, $v(x,t) = \bar{v}(x) + \tilde{v}(x,t)$, where $\bar{v} = \nabla^\perp \Delta^{-1} \bar{w}$, with $\bar{w} \in C_0^\infty(R^2)$ radially symmetric and $\int \bar{w}(x)\, dx = \int w_0$, $\tilde{v} \in L^\infty_{loc}(R; L^2(R^2; R^2))$. The vorticity $w(\cdot,t) = curl\, v(\cdot,t)$ is in $L^1(R^2)$ for every $t > 0$ and

$$\|w(\cdot,t)\|_{L^1(R^2)} \le \|w_0\|_{L^1}.$$

By combinning with the result of J.M. Delort [1], we have **Theorem 2.** *Suppose $w_0 = w_0' + w_0''$, where $w_0' \in L^1_{comp} \cap H^{-1}(R^2)$, w_0'' is a compactly supported positive Radon measure in Sobolev space $H^{-1}(R^2)$. Let $v_0 = \nabla^\perp \Delta^{-1} w_0$. Then there exists a function $v \in L^\infty_{loc}(R; L^2_{loc}(R^2; R^2))$, and a function $p \in L^\infty_{loc}(R; \mathcal{D}'(R^2))$, s.t. (v,p) is a classical weak solution of the 2-D Euler equation (2.1). Moreover, $v(x,t) = \bar{v}(x) + \tilde{v}(x,t)$, where $\bar{v} = \nabla^\perp \Delta^{-1} \bar{w}$, with $\bar{w} \in C_0^\infty(R^2)$ radially symmetric and $\int \bar{w}(x)\, dx = \int w_0$, $\tilde{v} \in L^\infty_{loc}(R; L^2(R^2; R^2))$. The vorticity $w(\cdot,t) = curl\, v(\cdot,t)$ can be decomposed as $w(x,t) = w'(x,t) + w''(x,t)$, where $w'(x,t) \in L^\infty(R; L^1(R^2))$ with*

$$\|w'(\cdot,t)\|_{L^1(R^2)} \le \|w_0'\|_{L^1},$$

for all $t > 0$, and $w''(\cdot,t)$ is a positive Radon measure for each $t > 0$ with total mass bounded by the total mass of w_0''.

3. Proof of the theorems

In [3] Di Perna and Majda construct approximate solution sequences for the 2-D Euler equation by smoothing the initial data. For completeness we repeat their construction here.

We consider an initial velocity field v_0 such that

$$w_0 = \operatorname{curl} v_0 \in H^{-1}_{comp}(R^2)\bigcap M(R^2). \tag{3.1}$$

Choose $\rho \in C_0^\infty(R^2)$, $\rho \geq 0$, $\int \rho = 1$, and define $v_0^\epsilon = \rho_\epsilon * v_0$, with $\rho_\epsilon(x) = \frac{1}{\epsilon^2}\rho(\frac{x}{\epsilon})$, then $v_0^\epsilon \in C^\infty(R^2)$ and $w_0^\epsilon = \operatorname{curl} v_0^\epsilon = \rho_\epsilon * w_0 \in C_0^\infty(R^2)$. It is well known (see [6]) that the 2-D Euler equations have a smooth global solution $v^\epsilon(x,t)$ so that $v^\epsilon(x,0) = v_0^\epsilon(x)$. Setting $w^\epsilon(\cdot,t) = \operatorname{curl} v^\epsilon(\cdot,t)$, we have that $w^\epsilon(x,t) \in C(R; C_0^\infty(R^2))$, and

$$\begin{cases} \dfrac{\partial w^\epsilon}{\partial t} + (v^\epsilon \cdot \nabla) w^\epsilon = 0 \qquad x \in R^2,\ t > 0 \\ w^\epsilon(x,0) = w_0^\epsilon(x). \end{cases} \tag{3.2}$$

So $w^\epsilon(x,t) = w_0^\epsilon(U_{0,t}^\epsilon(x))$, where $U_{0,t}^\epsilon(x)$ is a solution of the ODE

$$\begin{cases} \dfrac{d}{ds} U_{s,t}^\epsilon(x) = v^\epsilon(U_{s,t}^\epsilon(x), s) \\ U_{t,t}^\epsilon(x) = x \end{cases} \tag{3.3}$$

The Jacobian of $x \mapsto U_{0,t}^\epsilon(x)$ is identically equal to 1 for all $t > 0$, $\epsilon > 0$, since $\operatorname{div} v^\epsilon = 0$, (see [6]). Consequently,

$$\int w^\epsilon(x,t)\,dx = \int w_0^\epsilon(x)\,dx = \int w_0 \qquad \text{for all } t, \epsilon > 0, \tag{3.4}$$

$$\int |w^\epsilon(x,t)|\,dx = \int |w_0^\epsilon(x)|\,dx \leq \int |w_0| \qquad \text{for all } t, \epsilon > 0. \tag{3.5}$$

It is shown in [3] that $v^\epsilon(x,t) = \tilde{v}^\epsilon(x,t) + \bar{v}(x)$, where $\tilde{v}^\epsilon \in L^\infty_{loc}(R; L^2(R^2; R^2))$ uniformly in ϵ, i.e. for all $T > 0$, there is a constant C_T, such that

$$\sup_{0\leq t\leq T}\ \sup_{0<\epsilon\leq 1} \|\tilde{v}^\epsilon(\cdot,t)\|_{L^2(R^2;R^2)} \leq C_T. \tag{3.6}$$

Moreover $\bar{v}(x) = \nabla^{\perp}\Delta^{-1}\bar{w}(x)$, with $\bar{w} \in C_0^\infty(R^2)$ chosen s.t. $\bar{w}$ is radially symmetric and $\int \bar{w}(x)\,dx = \int w_0$. The velocity field $\bar{v}(x)$ is a stationary solution of the 2-D Euler equation and satisfies

$$\max_{0\le t\le T}(\|(1+|x|)\bar{v}\|_{L^\infty} + \|(1+|x|)^2\nabla\bar{v}\|_{L^\infty}) \le C_T, \tag{3.7}$$

for all $T > 0$, where C_T is a constant depends only on T. Therefore for all $T > 0$, $v^\epsilon \in L^\infty([0,T], L^2_{loc}(R^2; R^2))$ uniformly in ϵ. Moreover $v^\epsilon \in \mathrm{Lip}([0,T], H^{-L}_{loc}(R^2))$ and v^ϵ is uniformly Lipschitz [3, Lemma 1.1]. So there is a subsequence of v^ϵ, which we do not relabel , such that for all $t > 0$ fixed, there is a function $v(\cdot,t) \in L^2_{loc}(R^2)$, which is the weak limit of $v^\epsilon(\cdot,t)$ in $L^2_{loc}(R^2)$, so that

$$v^\epsilon(\cdot,t) \rightharpoonup v(\cdot,t), \qquad \text{as } \epsilon \to 0. \tag{3.8}$$

For the corresponding vorticity sequence $w^\epsilon(x,t)$ one has

$$w^\epsilon(\cdot,t) \rightharpoonup w(\cdot,t) \qquad \text{weak} * \text{ in } M(R^2). \tag{3.9}$$

with $w(\cdot,t) = \operatorname{curl} v(\cdot,t) \in M(R^2)\bigcap H^{-1}(R^2)$, since for all $t > 0$, $w^\epsilon(x,t) \in L^1(R^2)$ uniformly in ϵ (3.5). It is easy to see that $v(x,t)$ is also a weak limit of $v^\epsilon(x,t)$ under the topology $L^2_{loc}(R^2 \times R^+)$ (by Lebesgue's convergence theorem). Therefore $v(x,t)$ satisfies all the properties in Theorem 1.1 of [3], and moreover, $v(x) = \tilde{v}(v,t) + \bar{v}(x,t)$, where $\tilde{v}(x,t) \in L^\infty_{loc}(R; L^2(R^2 : R^2))$. In order to show that $v(x,t)$ is a weak classical solution of Euler equation, we only need to show that for any $\phi \in C_0^\infty(R^2 \times R^+)$,

$$\int\int v_1^\epsilon(x,t)v_2^\epsilon(x,t)\phi(x,t)\,dx\,dt \longrightarrow \int\int v_1(x,t)v_2(x,t)\phi(x,t)\,dx\,dt \tag{3.10}$$

and

(3.11)
$$\int\int(v_1^{\epsilon 2}(x,t) - v_2^{\epsilon 2}(x,t))\phi(x,t)\,dx\,dt \longrightarrow \int\int(v_1^2(x,t) - v_2^2(x,t))\phi(x,t)\,dx\,dt.$$

We only prove that (3.10) is true under the conditions of Theorem 1 or Theorem 2. The proof of (3.11) is similar. By Lebesgue's convergence theorem, we only need to show that for any $\phi \in C_0^\infty(R^2)$,

$$\int v_1^\epsilon(x,t)v_2^\epsilon(x,t)\phi(x)\,dx \longrightarrow \int v_1(x,t)v_2(x,t)\phi(x)\,dx, \tag{3.12}$$

for a.e. $t > 0$. We proceed as in [1].

Fix $t > 0$, take $\phi \in C_0^\infty(R^2)$ and assume that $\operatorname{supp}\phi \in B_R(0)$. Taking $\psi \in C_0^\infty(R^2)$, s.t. $\psi(x) \equiv 1$, for $|x| < R$, since $v^\epsilon = \nabla^\perp \Delta^{-1} w^\epsilon$, we have

$$\int v_1^\epsilon(x,t) v_2^\epsilon(x,t)\phi(x)\,dx = \int w^\epsilon(x,t) w^\epsilon(y,t) H_\phi(x,y)\,dx\,dy, \tag{3.13}$$

where

$$H_\phi(x,y) = c\,p.v. \int \frac{x_1 - z_1}{|x-z|^2}\frac{y_2 - z_2}{|y-z|^2}\phi(z)\,dz, \tag{3.14}$$

c being a universal constant. (In the following, we use the letter c to indicate a universal constant, which may be different in each case.) We know that (see [7])

$$p.v. \int \frac{x_1 - z_1}{|x-z|^2}\frac{y_2 - z_2}{|y-z|^2}\,dz = c\,h(x-y), \tag{3.15}$$

where

$$h(x) = \frac{x_1 x_2}{|x|^2}. \tag{3.16}$$

Now setting

$$r(x,y) = c\,p.v. \int \frac{x_1 - z_1}{|x-z|^2}\frac{y_2 - z_2}{|y-z|^2}(\phi(z)\psi(z) - \phi(x)\psi(y))\,dz, \tag{3.17}$$

we have

$$H_\phi(x,y) = c\,h(x-y)\phi(x)\psi(y) + r(x,y). \tag{3.18}$$

It is easy to check that $r(x,y) \in \mathrm{Lip}^{1/2}(R^2)$ and

$$r(x,y) \le \frac{c}{|x| + |y|}, \qquad \text{when } |x| + |y| \gg 1. \tag{3.19}$$

Therefore by the fact that

$$w^\epsilon(\cdot,t) \rightharpoonup w(\cdot,t) \qquad \text{weak} * \text{ in } M(R^2),$$

and by Ascoli-Arzela's Theorem,

(3.20)
$$\int w^\epsilon(x,t) w^\epsilon(y,t)((1-\chi(x-y))h(x-y)\phi(x)\psi(y) + r(x,y))\,dx\,dy$$
$$\longrightarrow \int w(x,t) w(y,t)((1-\chi(x-y))h(x-y)\phi(x)\psi(y) + r(x,y))\,dx\,dy,$$

where χ is any C_0^∞ function satisfies that $\chi(x) \equiv 1$, when $|x| < \delta$, for some $\delta > 0$. As in [1], the following lemma implies (3.12). Lemma 1 Assume the conditions as in Theorem 1 or Theorem 2. Then for any integer j, there is δ_j, ϵ_j, s.t.

$$(3.21) \quad \int_{|x-y|<\delta_j} |w^\epsilon(x,t)||w^\epsilon(y,t)||\phi(x,y)|\,dx\,dy < 1/j, \qquad \text{for all } 0 < \epsilon < \epsilon_j.$$

Proof. Suppose that $w_0 = w_0' + w_0''$, where $w_0' \in L^1(R^2)$ and w_0'' is a positive Radon measure. Then

$$(3.22) \qquad w^\epsilon(x,t) = w_0^\epsilon(U_{0,t}^\epsilon(x)) = w'^\epsilon_0(U_{0,t}^\epsilon(x)) + w''^\epsilon_0(U_{0,t}^\epsilon(x)),$$

where $w'^\epsilon_0 = w_0' * \rho_\epsilon$, and $w''^\epsilon_0 = w_0'' * \rho_\epsilon$, therefore we have that $w'^\epsilon_0 \to w_0'$ in $L^1(R^2)$. We write $w'^\epsilon(x,t) = w'^\epsilon_0(U_{0,t}^\epsilon(x))$, $w''^\epsilon(x,t) = w''^\epsilon_0(U_{0,t}^\epsilon(x))$ where $w''^\epsilon(x,t)$ is positive.

As in [1], it is enough to prove that there exist sequences δ_j, and ϵ_j, s.t.

$$(3.23) \qquad \sup_{x\in B_R(0)} \int_{|x-y|<\delta_j} |w^\epsilon(y,t)|\,dy \le 1/j, \qquad \text{for all } 0 < \epsilon < \epsilon_j.$$

If (3.23) were not true, then for some $\delta > 0$, there would be a subsequence $\epsilon_n \to 0$, and a sequence $x_n \in B_R(0)$, $x_n \to x_0 \in B_R(0)$, s.t.

$$(3.24) \qquad \delta \le \int_{|y-x_n|<1/n} |w^{\epsilon_n}(y,t)|\,dy.$$

We recall that level sets of w^ϵ are preserved by the flow since the Jacobian of $x \mapsto U_{0,t}^\epsilon(x)$ is identically equal to 1, and so by Dunford-Pettis theorem (see [5, p.240]), since $w'^\epsilon_0 \to w_0'$ in $L^1(R^2)$, there is a subsequence of w'^{ϵ_n}, w''^{ϵ_n}, which we do not relabel, s.t.

$$(3.25) \qquad w'^{\epsilon_n}(\cdot,t) \rightharpoonup w'(\cdot,t) \qquad \text{weak in } L^1_{loc}(R^2),$$

and

$$(3.26) \qquad w''^{\epsilon_n}(\cdot,t) \rightharpoonup w''(\cdot,t) \qquad \text{weak} * \text{ in } M(R^2).$$

The same Dunford-Pettis theorem also implies that there is an $N > 0$ s.t., when $n > N$, we have

$$(3.27) \qquad \int_{|y-x_n|<1/n} |w'^{\epsilon_n}(y,t)|\,dy \le \delta/2.$$

So (3.24) and (3.27) implies that

$$\delta/2 \leq \int_{|y-x_n|<1/n} w''^{\epsilon_n}(y,t)\,dy, \tag{3.28}$$

for $n > N$. This is impossible since the measure $w_t = w(\cdot, t)$ is diffuse, and so is $w''_t = w''(\cdot, t)$, since $w'(\cdot, t) \in L^1$ (see [1]). This concludes the Lemma.

It is easy to see from (3.25), (3.26) that

$$w(x,t) = w'(x,t) + w''(x,t),$$

and $w'(\cdot, t) \in L^1(R^2)$,

$$\|w'(\cdot,t)\|_{L^1(R^2)} \leq \|w'_0\|_{L^1}. \tag{3.29}$$

Moreover $w''(\cdot, t)$ is positive with total measure bounded by the total measure of w''_0. This proves Theorem 2. Theorem 1 follows by assuming that $w''_0 = 0$.

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