manuscripta math. 78, 223 - 243 (1993)

manuscripta mathematica

The Space of Surface Group Representations

Jun Li

In this note we prove that the number of irreducible components of $\mathrm{Hom}(\pi, G)$ is the same as $\pi_1(G)$, where π is a surface group and G is complex semisimple. This is established by studying the flat bundles on Riemann surfaces.

0. Introduction

Let X be a closed oriented Riemann surface of genus $g > 1$ and let π be its fundamental group. For any connected Lie group G, we denote by $\mathrm{Hom}(\pi, G)$ the analytic space of all homomorphisms from π to G. In this paper, we calculate the number of connected components of $\mathrm{Hom}(\pi, G)$ when G is complex semisimple. We prove

Theorem 0.1: *Let G be a connected complex semi-simple Lie group. Then $\pi_0(\mathrm{Hom}(\pi, G))$ is isomorphic to $\pi_1(G)$.*

For any homomorphism $\rho : \pi \to G$, there is a canonical flat connection on the marked principal G-bundle $P = \widetilde{X} \times G/\pi$ and vice verse, where $\widetilde{X}$ is the universal covering space of X. We fix such a topological principal G-bundle P. According to [GM1], if we denote by $\mathrm{Hom}(\pi, G)_P$ the subset of $\mathrm{Hom}(\pi, G)$ consisting of all homomorphisms ρ whose associated flat bundles P_ρ is topologically equivalent to P and denote by $F(P)$ the space of flat connections on P, then $F(P)$ is a principal bundle over $\mathrm{Hom}(\pi, G)_P$. Theorem 0.1 will follow if we prove

Theorem 0.2: *Let G be a connected complex semi-simple Lie group and P be*

The present work is partially supported by NSF grant DMS89-04922

an arbitrary principal G-bundle over X such that F(P) is non-empty. Then F(P) is an irreducible and simply connected infinite-dimensional complex variety.

Theorem 0.3: *Let G be any connected complex semi-simple Lie group. Then there are exactly $\pi_1(G)$ many distinct topological principal G-bundles and for each of such bundle P, $F(P)$ is non-empty.*

When G is simply connected, we calculate the fundamental group of $\mathrm{Hom}(\pi, G)$,

Theorem 0.4: *Assume that G is a connected, simply connected complex semi-simple Lie group, then $\pi_1(\mathrm{Hom}(\pi, G)) = \{e\}$.*

We now turn to the situation when G is a compact semisimple Lie group. Observe that G acts on $\mathrm{Hom}(\pi, G)$ by conjugation. In case that G is compact, the quotient space $\mathrm{Hom}(\pi, G)/G$ is a Hausdorff space carrying rich geometric structures. It has been extensively studied by [Ra] and by [AB]. Though they haven't stated explicitly, a combination of their argument shows:

Theorem 0.5: *Let G be a compact, connected semi-simple Lie group. Then $\pi_0(\mathrm{Hom}(\pi, G)/G)$ is isomorphic to $\pi_1(G)$.*

Theorem 0.1 was conjectured by W.Goldman. He showed that theorem 0.1 is true when G is $\mathrm{SL}(2, \mathbb{C})$ [Go].

We now outline the proof of theorem 0.2. Clearly, every flat structure on P induces a holomorphic structure on the same bundle. Let $\eta : F(P) \to \mathcal{C}_P$ be such a correspondence, where $\mathcal{C}_P$ is the set of all holomorphic structures on P. If we let $\mathcal{C}_P^0$ be the set

$$\{\bar\partial \in \mathcal{C}_P \mid H^0(X, adP_{\bar\partial}) = \{0\}\},$$

then $\eta : \eta^{-1}(\mathcal{C}_P^0) \to \mathcal{C}_P^0$ is a fiber bundle with affine fibers. Now using the fact that $\mathcal{C}_P^0$ is zariski open in $\mathcal{C}_P$ and $\mathcal{C}_P$ is affine, $\eta^{-1}(\mathcal{C}_P^0)$ is connected (irreducible). Theorem 0.2 will be proved if we can show that $\eta^{-1}(\mathcal{C}_P^0)$ is dense in $F(P)$. We will prove this by showing that any flat structure on P can be deformed to flat structures in $\eta^{-1}(\mathcal{C}_P^0)$.

Acknowledgment

The author likes to take this opportunity to express his gratitude to S.T. Yau, the author benefitted greatly from conversations with him. He also thanks D. Gieseker for introducing him Ramanathan's work. The author also likes *to express his gratitude to the hospitality of the Mathematics department in UCLA.*

1. Connections

Let X be a complex manifold, G be a complex Lie group and P be a principal G-bundle over X. The goal of this section is to understand the space of complex structures and the space of flat structures on P. We refer to standard text [AB][Ko] for the definition and basic properties of connections on principal bundles.

We first introduce two relevant vector bundles associated to P. Let $autP$ be the twisted product $autP = P \times_G G$, where G acts on G via conjugation. Clearly, associated to every G-invariant fiber preserving map $\rho: P \to P$ there is a global section of the bundle $autP$. We call $\mathcal{G} = C^\infty(autP)$ the gauge group of P. The adjoint bundle adP is the vector bundle $adP = P \times_G g$, where g is the Lie algebra of G and G acts on g via the adjoint representation. Let D be a connection on P. D is given by a connection form ω which is a g-valued 1-form on P. Equivalently, D is defined by a G-equivariant splitting of the following exact sequence of vector bundles over P,

$$0 \longrightarrow T^\vee P \longrightarrow TP \overset{i_D}{\leftrightarrows} p_X^* TX \longrightarrow 0, \tag{1.1}$$

where $T^\vee P$ is the vertical tangent bundle and $p_X: P \to X$ is the projection.

If we denote by J_G the complex structure on g and by J_X the complex structure on TX, we can define an almost complex structure J_P on TP which is the direct sum $J_X \oplus J_G$ induced by the splitting i_D. We have the following

Lemma 1.1: [Ko] *Let D be a connection on P and ω be its connection form. Then there is a unique almost complex structure J_P on the manifold P such that for any tangent vector $v \in TP$, we have*
(1) $\omega(J_P v) = J_G\, \omega(v)$,
(2) $p_{X*}(J_P v) = J_X(p_{X*} v)$.
Moreover, J_P is integrable if and only if the (0,2) part of the curvature form $\Theta(D) = \mathrm{d}\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(X, adP)$ is identically zero. □

When X is a Riemann surface, there is no non-trivial (0,2) forms on X. So we have

Corollary 1.2: *If* $\dim X = 1$, *then any connection D on P induces a holomorphic structure on P such that both $p_X : P \to X$ and the multiplication* *map $P \times G \to P$ are holomorphic. Moreover, the connection form ω of D is a g-valued (1,0)-form on P.* □

A principal bundle P with such a holomorphic structure is called a holomorphic principal bundle. A connection on the holomorphic principal bundle whose connection form is of (1,0) type is called a compatible connection. Since the difference of two connection forms is in $\Omega^1(X, adP)$, the space of compatible connections on P is an affine space isomorphic to $\Omega^{1,0}(X, adP)$ and the space of holomorphic structures on P is an affine space isomorphic to $\Omega^{0,1}(X, adP)$.

Since we intend to study the relation between the flat structures and holomorphic structures on P, it is convenient if we can find a canonical compatible connection on P. Let G be a semi-simple complex Lie group and let K be a maximal compact subgroup of G. If we denote by g_0 and $g^{\mathbb{R}}$ the (real) Lie algebra of K and G respectively, then $g^{\mathbb{R}} = g_0 + Jg_0$, where J is the complex structure of $g^{\mathbb{R}}$. g_0 is called a compact real form of g. We fix a compact real form $g_0 \subset g^{\mathbb{R}}$ once and for all. Then we can canonically express any element $Z \in g$ as $Z = X + JY$, $X, Y \in g_0$. Consequently, g can be written as $g_0 \otimes_{\mathbb{R}} \mathbb{C}$. We define the conjugation $\sigma : g \to g$ by $\sigma(X + JY) = X - JY$. If $B(\cdot,\cdot)$ is the killing form of g, the hermitian form $< \cdot,\cdot >_K$ on $g \times g$ defined by $< u, v >_K = -B(u, \sigma v)$ is positive definite. One notes that both the conjugation σ and the hermitian form $< \cdot,\cdot >_K$ are invariant under the adjoint action of K. We first reduce the structure group of P to K.

Lemma 1.3: *Any principal G-bundle can be reduced to a principal K-bundle. That is, there is a principal K-bundle P_K such that $P = P_K \times_K G$.*

Proof: The proof follows from the fact that K is homotopic equivalent to G. See [Ra]. □

Lemma 1.4: *Let P_K be a principal K-bundle, $P = P_K \times_K G$. Let adP be the adjoint bundle. Then adP is a complex vector bundle and on adP, there is a hermitian metric $< \cdot,\cdot >$ such that at every point $x \in X$, $< \cdot,\cdot >_{|adP_x} = < \cdot,\cdot >_K$.*

Proof: Clearly, $g = g_0 \otimes_{\mathbb{R}} \mathbb{C}$ induces a complex structure on the vector bundle adP. We define a hermitian metric as follows: Since $P = P_K \times_K G$, $adP = P \times_G g = P_K \times_K g$, where K acts on g via the induced adjoint action. The hermitian metric $< \cdot,\cdot >_K$ on $g = g_0 \otimes_{\mathbb{R}} \mathbb{C}$ induces a hermitian metric H on $P_K \times g$. Since H is invariant under the adjoint action of K, H descends to a hermitian metric on $P_K \times_K g = adP$ with the desired property. □

Let $P = P_K \times_K G$ be a holomorphic G-bundle. A connection D on P is said to be unitary if D is compatible and if D is induced from a connection on P_K.

lemma 1.5: There is a unique unitary connection on any holomorphic principal G-bundle $P = P_K \times_K G$.

Proof: Since $adP = P_K \times_K g$, the conjugation σ on g extends to a conjugation $\sigma : adP \to adP$. Combined with the conjugation on $T^*_{\mathbb{C}}X$, we can define an involution $\theta : adP \otimes_{\mathbb{C}} T^*_{\mathbb{C}}X \to adP \otimes_{\mathbb{C}} T^*_{\mathbb{C}}X$. Let D be any compatible connection on P and D_1 be a connection induced from a connection on P_K. We can write $D = D_1 + \psi + \omega$, where $\omega \in \Omega^{0,1}(X, adP)$ and $\psi \in \Omega^{1,0}(X, adP)$. Define a new connection D' by

$$D' = D_1 + \omega + \theta\omega.$$

One checks directly that D' is a unitary connection. The uniqueness of the unitary connection is obvious and we leave it to the readers. □

2. Flat connections and their deformations

In the remainder sections, unless otherwise is stated, we assume that X is a Riemann surface of genus $g > 1$, that G is a connected complex semi-simple Lie group and that P is a principal G-bundle with a fixed reduction $P = P_K \times_K G$. Hence adP admits a canonical hermitian metric and any holomorphic structure on P defines a unique unitary connection. For the moment, we assume $F(P)$ is non-empty.

It is known that both the space $\mathcal{A}_P$ of connections on P and the space $\mathcal{C}_P$ of holomorphic structures on P are affine spaces. Further, if we fix a connection $D \in \mathcal{A}_P$, then there are identifications $\mathcal{A}_P \cong \Omega^1(X, adP)$ and $\mathcal{C}_P \cong \Omega^{0,1}(X, adP)$. Under these identifications, the projection $\Omega^1(X, adP) \to \Omega^{0,1}(X, adP)$ is compatible with the projection $\eta_D : \mathcal{A}_P \to \mathcal{C}_P$ introduced by corollary 1.2. If we endow $\mathcal{A}_P$ and $\mathcal{C}_P$ the complex structures induced by the affine structures, $\eta : \mathcal{A}_P \to \mathcal{C}_P$ is complex linear.

A connection D is said to be flat if its curvature $\Theta(D) \in \Omega^2(X, adP)$ is identical to zero. It is known that the parallel transform guided by a flat connection has vanishing local holonomy and its global holonomy induces a homomorphism $\rho : \pi_1(X) \to G$. In fact, if we fix a base point $x_0 \in X$ and let $\mathcal{G}_0 = \{\mathrm{h} \in \mathcal{G} \mid \mathrm{h}_{|P_{x_0}} = \mathrm{id}\}$, the global holonomy map $\mathcal{H}$ from the space of flat connections $F(P)$ to $\mathrm{Hom}(\pi, G)_P$ defines a principal bundle

$$\mathcal{H} : F(P) \to \mathrm{Hom}(\pi, G)_P \tag{2.1}$$

with structure group $\mathcal{G}_0$ [GM1]. In order to rigorously justify our argument, we need to introduce the Sobolev norms on the spaces of sections of the relevant bundles. We topologize the space $\Omega^{i,j}(X, adP)$ by using the sobolev L^p_k norm induced by a Kahler metric on X and the hermitian metric $< \cdot, \cdot >$ on adP with p large and $k = 3 - i - j$. Similarly, we use L^p_3 to topologize the space $\mathcal{G}_0$. A standard argument shows that both $\mathcal{A}_P$, $\mathcal{C}_P$ and $\mathcal{G}_0$ are smooth infinite-dimensional Banach manifolds and the gauge group $\mathcal{G}_0$ acts on $\mathcal{A}_P$ and $\mathcal{C}_P$ smoothly. Unfortunately, our primary interest $F(P)$ is not smooth in general. But nevertheless, it is a complex analytic variety.

Definition 2.1: *An infinite-dimensional space $\mathcal{V}$ is said to be an affine variety if there are complex Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, a smooth holomorphic map $\Phi : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\mathcal{V} = \Phi^{-1}(0)$. $\mathcal{V}$ is said to be irreducible if there is a dense open subset $\mathcal{V}^0 \subset \mathcal{V}$ such that $\mathcal{V}^0$ is connected and smooth.*

Lemma 2.2: *$F(P)$ is an infinite dimensional affine variety.*

Proof: $\mathcal{A}_P$ is a complex Banach space and $F(P)$ is a subset of $\mathcal{A}_P$. Fix a $D \in \mathcal{A}_P$, then $F(P) \subset \mathcal{A}_P$ is the set of connections $D + \psi + \omega$, where $(\psi, \omega) \in \Omega^{1,0}(X, adP) \times \Omega^{0,1}(X, adP)$, such that

$$\widetilde{\Theta}((\psi, \omega)) = \Theta(D) + D(\omega + \psi) + [\omega, \psi] = 0.$$

The map $\widetilde{\Theta} : \mathcal{A}_P \to \Omega^{1,1}(X, adP)$ is smooth and holomorphic. By definition, $F(P)$ is an affine subvariety of $\mathcal{A}_P$. It is easy to see that the complex structure so defined is independent of the choice of D. $\square$

Lemma 2.3: *Let $\mathrm{Hom}(\pi, G) \subseteq G \times \cdots \times G$ be the complex subvariety defined as the preimage $\gamma^{-1}(e)$ of the holomorphic map $\gamma : (G)^{\times 2g} \to G$, $\gamma : (x_1, \cdots, x_g, y_1, \cdots, y_g) \mapsto \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1}$. Then*

$$\mathcal{H} : F(P) \to \mathrm{Hom}(\pi, G)$$

is holomorphic.

Proof: Let $\mathbf{v}$ be any holomorphic tangent vector of $F(P)$ at P. Since $\mathcal{A}_P$ is smooth, there is a holomorphic family of connections D_z, $\bar{\partial}_z D_z = 0$ such that $\partial_z D_z|_{z=0} = \mathbf{v}$. On the other hand, because $\mathbf{v}$ belongs to the tangent space of

$F(P)$, $\Theta(D_z)$ vanishes up to first order. That is, $\Theta(D_z) = O(|z|^2)$. Let $\gamma(t)$ be any fixed smooth arc in X and let $h_z(t)$ be smooth sections of P over $\gamma(t)$ parameterized by z with $h_z(0)$ fixed such that $\frac{d}{dt}h_z(t)$ is parallel via D_z. The lemma will be proved if we can show that for any t, $\overline{\partial}_z h_z(t)_{|z=0} = 0$.

Let $\mu : \tilde{X} \to X$ be the universal covering and let $\tilde{P}$ be a trivialization of $\mu^* P$ so that $\mu^* D$ is the trivial connection. Let $\tilde{\omega}_z$ be the connection form of $\mu^* D_z$, let $\tilde{\gamma}(t)$ be a lifting of $\gamma(t)$ and $\tilde{h}_z(t)$ be lifting of $h_z(t)$ with fixed $h_z(0)$. Then since $\frac{d}{dt}h_z(t)$ is parallel, $\tilde{\omega}_z(\frac{d}{dt}\tilde{h}_z(t)) = 0$. On the other hand, $\overline{\partial}_z \omega_z = 0$, so $\tilde{\omega}_0(\frac{d}{dt}\overline{\partial}_z \tilde{h}_z(t)_{|z=0}) = 0$. Assume $\tilde{h}_z(t) = (\tilde{\gamma}(t), f(z,t)) \in \tilde{X} \times G$, then $\frac{d}{dt}\overline{\partial}_z f(z,t)_{|z=0} = 0$. Note that since $f(z,0) = const.$, $\overline{\partial}_z f(z,0)_{|z=0} = 0$. So $\overline{\partial}_z f(z,t)_{|z=0} = 0$ for any t. Therefore $\overline{\partial}_z h_z(t)_{|z=0} = 0$. The lemma has been established. □

The fibration $\mathcal{H} : F(P) \to \mathrm{Hom}(\pi, G)_P$ is very powerful in studying both the local and global geometry of $\mathrm{Hom}(\pi, G)_P$. However, we find the map $\eta : F(P) \to \mathcal{C}_P$, η is induced from the projection $\mathcal{A}_P \to \mathcal{C}_P$, is also helpful in deriving the topological information of $F(P)$.

Lemma 2.4: *With the notation as before, then the map $\eta : F(P) \to \mathcal{C}_P$ is holomorphic. Moreover, for any complex structure $\overline{\partial}_\omega \in \eta(F(P))$, $\eta^{-1}(\overline{\partial}_\omega)$ is an affine space isomorphic to the space of $\overline{\partial}_\omega$ closed forms $\Omega^{1,0}(X, adP)_{\overline{\partial}_\omega} \subset \Omega^{1,0}(X, adP)$. In particular, it is irreducible.*

Proof: Since $F(P)$ is a subvariety of $\mathcal{A}_P$ and $\eta : \mathcal{A}_P \to \mathcal{C}_P$ is holomorphic, the restriction of η to $F(P)$, $\eta : F(P) \to \mathcal{C}_P$ is still holomorphic. To prove the second statement, we assume D is a flat connection with $\eta(D) = \overline{\partial}_\omega$. Let $D_1 = D + \psi$, $\psi \in \Omega^{1,0}(X, adP)$. D_1 is flat if and only if

$$0 = \Theta(D_1) = \Theta(D) + D(\psi) + \frac{1}{2}[\psi, \psi] = \overline{\partial}_\omega(\psi).$$

That is, $\psi \in \Omega^{1,0}(X, adP)_{\overline{\partial}_\omega}$. □

Since $F(P) \subset \mathcal{A}_P$ is a complex variety, it makes sense to talk about subvariety of $F(P)$. Let V be any finite dimensional complex analytic variety. A map $\phi : V \to F(P)$ is said to be holomorphic if $\phi : V \to \mathcal{A}_P$ is holomorphic. We call the image $\phi(V)$ a subvariety of $F(P)$. It is not difficult to see that if $\phi : V \to \mathcal{C}_P$ is a holomorphic map, then there is a holomorphic structure on $P \times V$ such that the induced holomorphic structure on $P \times \{v\}$ is exactly the holomorphic structure given by $\phi(v)$. In this sense, a holomorphic map $\phi : V \to \mathcal{C}_P$ is equivalent to a holomorphic family of holomorphic structures on $P \times V$ parameterized by V.

Now we study the following question. Suppose P is a holomorphic principal G-bundle and that D is a compatible flat connection. Let t be the complex parameter and let $\omega_t \in \Omega^{0,1}(X, adP)$ be a smooth family of forms with $\omega_0 = 0$. $D + \omega_t$ induces a smooth deformation of complex structure on P. The question is under what condition can we find a family $\psi_t \in \Omega^{1,0}(X, adP)$, $\psi_0 = 0$, such that $D + \omega_t + \psi_t$ is a family of flat connections.

It is obvious that in order to have $D + \omega_t + \psi_t$ flat, ψ_t must satisfy the equation

$$\Theta(D) + D(\omega_t) + [\omega_t, \psi_t] + \overline{\partial}_D(\psi_t) = 0. \tag{2.2}$$

We solve this equation by using the method developed by Kuranishi and Taubes. In the following, we fix a D and denote $\overline{\partial} = \overline{\partial}_D$. Let $H^i(X, adP \otimes T^*_X)$ be the space of $\overline{\partial}$ harmonic forms in $\Omega^{1,i}(X, adP)$ (with respect to the Hermitian metric introduced in §1). We have the following orthogonal decomposition $\Omega^{1,i}(X, adP) = \Omega^{1,i}(X, adP)_0 \oplus H^i(X, adP \otimes T^*_X)$. Let $\Pi : \Omega^{1,1}(X, adP) \to H^1(X, adP \otimes T^*_X)$ be the orthogonal projection. Π is complex linear.

Lemma 2.5: *Let D be any flat connection, then there is an open neighborhood U of $0 \in \Omega^{0,1}(X, adP)$ and a smooth $f : U \to \Omega^{1,0}(X, adP)_0$ such that for any $\omega \in U$, $f(\omega)$ is the solution of the equation*

$$(I - \Pi)\big(\Theta(D) + D\omega + [\omega, f(\omega)] + \overline{\partial} f(\omega)\big) = 0. \tag{2.3}$$

Moreover, f is unique and holomorphic.

Proof. Let $Q : \Omega^{0,1}(X, adP) \times \Omega^{1,0}(X, adP)_0 \to \Omega^{1,1}(X, adP)_0$ be defined by

$$Q(\omega, \psi) = (I - \Pi)(\Theta(D) + D\omega + [\omega, \psi] + \overline{\partial}\psi). \tag{2.4}$$

Since D is flat, $Q(0,0) = 0$. When ω is small enough, the first order variation of Q along the second variable ψ,

$$\delta_\psi Q(\omega, \psi)(\dot{\psi}) = (I - \pi)([\omega, \dot{\psi}] + \overline{\partial}\, \dot{\psi})$$

is an isomorphism between $\Omega^{1,0}(X, adP)_0$ and $\Omega^{1,1}(X, adP)_0$. Applying the implicit function theorem, for some neighborhood U of $0 \in \Omega^{0,1}(X, adP)$, there is a unique function $f : U \to \Omega^{1,0}(X, adP)_0$, $f(0) = 0$, such that (2.3) holds.

To show that f is holomorphic, let $\widetilde{\partial}$ be the $\overline{\partial}$-operator of $\Omega^{0,1}(X, adP)$. Then

$$\begin{aligned} 0 &= \widetilde{\partial}\big((I - \Pi)(\Theta(D) + D\omega + [\omega, f(\omega)] + \overline{\partial} f(\omega))\big) \\ &= (I - \Pi)\big([\omega, \widetilde{\partial} f(\omega)] + \overline{\partial}(\widetilde{\partial} f(\omega))\big). \end{aligned} \tag{2.5}$$

Therefore, $\widetilde{\partial} f$ must be zero in a neighborhood of 0. □

An easy consequence is the following corollary which is our main tool in constructing deformation of flat connections.

Proposition 2.6: *Let $Z \subset \Omega^{0,1}(X, adP)$ be any complex subvariety, $0 \in Z$ and $Z \subset U$ where U is the open neighborhood of 0 introduced in lemma 2.5. Then the subset*

$$Z_0 = \{\omega \in Z | \ \Theta(D + \omega + f(\omega)) = 0\}$$

is a complex subvariety whose complex dimension is no less than $\dim Z - h^1(X, adP \otimes T_X^*)$. In particular, $V = \{(f(\omega), \omega) \mid \omega \in Z_0\} \subset F(D)$ is a complex subvariety of dimension no less than $\dim Z - h^1(X, adP \otimes T_X^*)$.

Proof: Since D is flat, Z_0 is non-empty. Further

$$\Theta(D + \omega + f(\omega)) = \Pi(\Theta(D) + D\omega + [\omega, f(\omega)] + \overline{\partial} f(\omega)) \tag{2.6}$$

is a holomorphic map from $U \subset \Omega^{0,1}(X, adP)$ to $H^1(X, adP \otimes T_X^*)$. By dimension comparison, $\dim Z_0 \geq \dim Z - \dim H^1(X, adP \otimes T_X^*)$. □

Since G is semisimple, the Killing form $B(\cdot, \cdot)$ provides a non-degenerate bilinear map $adP \times adP \to \mathbb{C}$. This is a holomorphic correspondence. Therefore adP is isomorphic to its own dual. By Serre duality, $H^1(X, adP \otimes T_X^*) = H^0(X, adP)^\vee$ and the induced pairing

$$(\cdot, \cdot) : H^0(X, adP) \times H^1(X, adP \otimes T_X^*) \longrightarrow \mathbb{C} \tag{2.7}$$

is nondegenerate. Therefore we have proved the following corollary.

Corollary 2.7: *Suppose $h^0(X, adP) = 0$, then there is an open neighborhood $0 \in U \subset \Omega^{0,1}(X, adP)$ such that for any $\omega \in U$, $D + \omega + f(\omega)$ is a flat connection.* □

3. Standard filtration of $s \in H^0(X, adP)$

The goal of the following two sections is to show that for any flat connection D, there is a (smooth) deformation $D + \omega_t + \psi_t$ of flat connections such that for generic t, $H^0(X, adP_{\overline{\partial}_t}) = \{0\}$. Let us first examine the effect of the existence of sections $s \in H^0(X, adP)$ on the structure of P.

Let P be any holomorphic principal G-bundle. Assume $H^0(X, adP) \neq \{0\}$. Let $s \in H^0(X, adP)$ be a non-trivial section. Then $ad(s) : adP \to adP$ is holomorphic. The characteristic polynomial $\det(\lambda \cdot id - ad(s))$ of $ad(s)$ is a polynomial of λ whose coefficients are holomorphic functions of X. So they must be constant functions. The Jordan decompositions of $\rho(s)$ at points $x \in X$ provide a decomposition of the vector bundle adP. The proof of the following lemma can be found in [Gu].

Lemma 3.1: *There are sub-bundles $E_0, \cdots, E_l$ of adP, distinct complex numbers $\lambda_0, \cdots, \lambda_l$ and nilpotent endomorphism $N_j : E_j \to E_j$ such that*
a). $\oplus_{j=0}^{l} E_j = adP$,
b). $ad(s)(E_j) \subseteq E_j$,
c). $ad(s)|_{E_j} = \lambda_j \cdot id + N_j$. □

Since zero is always an eigenvalue of $ad(s)$, we agree $\lambda_0 = 0$. We call $s \in H^0(X, adP)$ a nilpotent element if $ad(s)$ is nilpotent. The nilpotent endomorphism $N_0 : E_0 \to E_0$ further defines a filtration of E_0 as follows: Let $\mathcal{O}(\mathcal{F}_i)$ be the subsheaf of $\mathcal{O}(E_0)$ defined by

$$\mathcal{O}(\mathcal{F}_i) = \{h \in \mathcal{O}(E_0) \mid N_0^i(h) = 0\}.$$

Since $\dim X = 1$, $\mathcal{O}(\mathcal{F}_i)$ is the sheaf of a subbundle of E_0 which we denote by F_i. We call filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = E_0 \tag{3.1}$$

the canonical filtration of (E_0, s) and call decomposition

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = E_0, E_1, \cdots, E_l \tag{3.2}$$

the canonical s-decomposition of adP. We denote $r(s) = r$ and $l(s) = l$. Let $n = \dim g$. We define the length of $s \in H^0(X, adP)$ by

$$\text{length}(s) = n^{n+2}(n - l(s)) + \sum_{i=1}^{r(s)} n^{n-i} \text{rank}\, F_i. \tag{3.3}$$

We have the following observation.

Lemma 3.2: *Let P_i be holomorphic G-bundles and $s_i \in H^0(X, adP_i)$, $i = 1, 2$. Then* $\text{length}(s_1) \geq \text{length}(s_2)$ *if the first nonzero integer of*

$$-(l(s_1) - l(s_2)), \text{rank}\, F_1(s_1) - \text{rank}\, F_1(s_2), \text{rank}\, F_2(s_1) - \text{rank}\, F_2(s_2), \cdots$$

is positive.

Proof: The lemma follows directly from the fact that $l(s)$ and $\operatorname{rank} F_i$ are no more than n. □

Lemma 3.3: *The length function is upper-semi-continuous in both zariski topology and classical topology. That is, if $(P, \overline{\partial}_t)$ is any holomorphic (resp. smooth) family of holomorphic structures and $s_t \in H^0(X, adP_t)$ is any holomorphic (resp. smooth) family of sections parameterized by complex variety V, then for any k, $\{t \in V \mid \operatorname{length}(s_t) \geq k\}$ is a closed subset of V in zariski (resp. classical) topology.*

Proof: It is obvious that the number of distinct eigenvalues of $ad(s_t)$ is a lower-semi-continuous function and $\operatorname{rank} F_i = \dim \operatorname{Ker}(ad(s_t))^i$ is an upper-semi-continuous function in both topologies. Therefore, by Lemma 3.2, $\operatorname{length}(s_t)$ is an upper-semi-continuous function in both topologies. □

We now state in what sense a flat connection $D \in F(P)$ is generic in its irreducible component. Let $\mathcal{M} \subseteq F(P)$ be any irreducible component and since $F(P) \to \operatorname{Hom}(\pi, G)_P$ is a fiber bundle, there is a corresponding irreducible component $M \subset \operatorname{Hom}(\pi, G)_P$. Let $\tau \in M$ be a generic point such that M is smooth at τ (without loss of generality, we can assume M is reduced). Let $U \subset M$ be an open neighborhood of τ such that $h^0(X, adP_\tau) = h^0(X, adP_{\tau'})$ for $\tau' \in$ U. We claim that there is an analytic subvariety $V \subseteq F(P)$ such that $U \subseteq \mathcal{H}(V)$. Indeed, let $U_0 \subset F(P)$ be a (finite dimensional) submanifold surjects onto U via $\mathcal{H} : F(P) \to \operatorname{Hom}(\pi, G)_P$ and let $W_0 = \eta(U_0) \subset \mathcal{C}_P$. Shrinking U (and U_0) if necessary, we can find a smooth complex subvariety $W \subset \mathcal{C}_P$ such that the image of $W_0 \subset \mathcal{C}_P \to \mathcal{C}_P/\mathcal{G}_0$ is contained in the image $W \subset \mathcal{C}_P \to \mathcal{C}_P/\mathcal{G}_0$ [AB, §14]. Let $V = \widetilde{\Theta}^{-1}(\{0\} \times W)$, where $\widetilde{\Theta} : \mathcal{A}_P \to \Omega^{1,1}(X, adP) \times \mathcal{C}_P$ is defined by $\widetilde{\Theta} : (\psi, \omega) \mapsto (\Theta(D + \psi + \omega), \omega)$. A standard argument shows that $\widetilde{\Theta}$ is Fredholm and holomorphic. Therefore, V is a finite-dimensional subvariety of $\mathcal{A}_P$. It is clear that $V \subset F(P)$ and $U \subset \mathcal{H}(V)$.

By further shrinking V (and U) if necessary, we can assume V is smooth, connected and $U = \mathcal{H}(V)$. Let P_V be the holomorphic principal G-bundle over $X \times V$ such that for any $D \in V$, $P|_{X \times \{D\}} = P_D$. Let $H_V = p_{V*}(adP_V)$ be the direct image sheaf over V, where p_V is the projection $X \times V \to V$. Since $h^0(X, adP_v)$ is constant for $v \in V$, by base change theorem, $p_{V*}(adP_V)$ is locally free. Let $\mathbf{P}(H_V)$ be the projective bundle of H_V over V. Since every point of $\mathbf{P}(H_V)$ corresponds to a multiple of global section of adP, the length function defined in (3.3) provides a stratification of $\mathbf{P}(H_V)$ as follows:

$$S_k(V) = \{s \in \mathbf{P}(H_V) \mid \text{length}(s) \geq k\}. \tag{3.4}$$

If we agree that $S_k(V)$ have reduced scheme structures, by lemma (3.3), $S_k(V)$ are closed (in Zariski topology) subset of $\mathbf{P}(H_V)$.

Definition 3.4: *$D \in V$ is said to be generic if for any $s \in H^0(X, adP_D)$ and any (smooth) deformation $D_t \in V$ of D, there is a smooth deformation $s_t \in H^0(X, adP_{D_t})$ of s such that length(s_t)=length(s) for t small enough. In general, $D \in F(P)$ is said to be generic if same conclusion holds when V is replaced by $F(P)$.*

We show that the set of generic points of V is a dense subset of V. (Then the set of generic points of $F(P)$ is also dense in $F(P)$.) Let $p_k : S_k(V) \to V$ be induced from the projection. Since $S_k(V) \subset \mathbf{P}(H_V)$ is closed, p_k is proper. Let $q_k : \widetilde{S}_k(V) \to S_k(V)$ be the desingularization and let $\tilde{p}_k = p_k \circ q_k : \widetilde{S}_k(V) \to V$. Define

$$\widetilde{S}_k(V)^{deg} = \{v \in \widetilde{S}_k(V) \mid \tilde{p}_{k*} : T_v\widetilde{S}_k(V) \to T_{\tilde{p}_k(v)}V \text{ is not surjective}\}.$$

$\widetilde{S}_k(V)^{deg}$ is a closed subvariety of $\widetilde{S}_k(V)$ and moreover, $\tilde{p}_k(\widetilde{S}_k(V)^{deg})$ is a proper subvariety of V. Let $V^k = V \setminus \tilde{p}_k(\widetilde{S}_k(V)^{deg})$. V^k is a dense open subset of V.

Lemma 3.5: *Let $D \in V^k$ and $s \in H^0(X, adP_D)$ with length(s)=k. Assume D_t is a smooth deformation of D. Then there is a family $s_t \in H^0(X, adP_{D_t})$, $s_0 = s$, such that length(s_t) $=k$ for t small enough.*

Proof. Since $D \in V^k$, there is $\tilde{s} \in \widetilde{S}_k(V)$, $p_k(\tilde{s}) = s$ such that $p_{k*} : T_{\tilde{s}}\widetilde{S}_k(V) \to T_DV$ is surjective. Since both $\widetilde{S}_k(V)$ and V are smooth, for any deformation D_t of D, there is a family $\tilde{s}_t \in \widetilde{S}_k(V)$ such that $\tilde{p}_k(\tilde{s}_t) = D_t$. Put $s_t = q_k(\tilde{s}_t)$, then $s_t \in H^0(X, adP_{D_t})$ is the family with the desired property. □

Corollary 3.6: *The set of generic points of V is a dense open subset of V.*

Proof. Since V^k is a dense open subset of V and $\Lambda = \{k \mid \tilde{S}_k(V) \neq \emptyset\}$ is a finite set. $V_0 = \cap_{k \in \Lambda} V_k$ is a dense open subset of V. It is clear that any point in V_0 is a generic point of V. □

Our intention is to show that if D is generic in $F(P)$, then $H^0(X, adP_D) = \{0\}$. Assume D is generic and $s \in H^0(X, adP_D) \neq \{0\}$, $s \neq 0$. Let (3.2) be the canonical s-decomposition of adP_D. There is a subsheaf $\mathcal{E}nd(adP_D)_s \subset$

$\mathcal{E}nd(adP_D)$,

$$\mathcal{E}nd(adP_D)_s = \left\{\rho \in \mathcal{E}nd(adP_D) \mid \begin{array}{c}\rho(E_i) \subseteq E_i,\ 0 \le j \le l \text{ and} \\ \rho(F_j) \subseteq F_j, 0 \le j \le r\end{array}\right\}.$$

It is well-known that the infinitesimal deformation of holomorphic structures (up to gauge equivalence) on P is $H^1(X, \mathcal{E}nd(adP_D))$. In the following, we say $v \in H^1(X, \mathcal{E}nd(adP_D))$ is a direction that preserves the canonical s-decomposition (3.2), where $s \in H^0(X, adP_D)$, if there is a smooth deformation $\overline{\partial}_t$, $P_{\overline{\partial}_0} = P_D$, $\frac{d}{dt}\overline{\partial}_{t|t=0} = v$, and a family $s_t \in H^0(X, adP_{\overline{\partial}_t})$, $s_0 = s$, such that for t small enough,

$$\text{length}(s_t) = \text{length}(s_0). \tag{3.5}$$

Lemma 3.7: *Let $s \in H^0(X, adP)$ and let $v \in H^1(X, \mathcal{E}nd(adP))$ be any vector that preserves the canonical s-decomposition, then $v \in H^1(X, \mathcal{E}nd(adP)_s)$. In particular, if $D \in V$ is a generic point, then the set*

$$\text{Im}\{T_D V \to H^1(X, \mathcal{E}nd(adP_D))\}$$

is contained in $H^1(X, \mathcal{E}nd(adP_D)_s)$ for any $s \in H^0(X, adP_D)$.

Proof: By definition, there is a family of holomorphic structures $\overline{\partial}_t$, $\frac{d}{dt}\overline{\partial}_{t|t=0} = v$, and a family of sections $s_t \in H^0(X, adP_{\overline{\partial}_t})$ such that (3.5) holds for small t. Let

$$0 = F_0(t) \subset F_1(t) \subset \cdots \subset F_r(t) = E_0(t), E_1(t), \cdots, E_r(t)$$

be the canonical filtration of s_t. Since length(s_t) =length(s), by lemma 3.3, $\dim F_i(t)$ and $\dim E_i(t)$ are constants for t small enough. Then $F_i(t)$ and $E_i(t)$ are smooth families of holomorphic vector bundles over X. Therefore, D_t is a deformation of complex structures that preserves the filtration (3.2). By [AB, §2], the image of v in $H^1(X, \mathcal{E}nd(adP_D))$ is contained in $H^1(X, \mathcal{E}nd(adP_D)_s)$. □

4. Proof of the theorem 2

We adapt the notation developed in the previous sections. Let $s \in H^0(X, adP)$, $D \in V$ and $P = P_D$, be a generic point. Let $E_0 \oplus E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}$ be the spectral decomposition of adP. For any $v_\lambda \in E_\lambda$ and $v_\mu \in E_\mu$, a consequence of Jacobi-identity shows that $[v_\lambda, v_\mu] \in E_{\lambda+\mu}$. Therefore, any $v \in \oplus_{\lambda \neq 0} H^0(X, E_\lambda)$ is nilpotent and,

Lemma 4.1: *The pairing*

$$(\cdot,\cdot): H^1(X, E_\lambda) \otimes H^0(X, E_\mu \otimes T^*_X) \to \mathbb{C}$$

is non-trivial only if $\lambda + \mu = 0$. *In such cases, the pairings are non-degenerate.*

Proof: The first part is obvious. The second part is the consequence of the fact that $(\cdot,\cdot): H^1(x, \oplus_\lambda E_\lambda) \otimes H^0(X, \oplus_\lambda E_\lambda \otimes T^*_X) \to \mathbb{C}$ is a non-degenerate pairing. □.

Let $\Lambda_0 \subset H^1(X, adP)$ be the largest linear subspace consisting of directions that preserve the canonical decomposition (3.2) of all $s \in H^0(X, adP)$. We have the following proposition which provides a bound of the codimension of Λ_0.

Proposition 4.2: *Let* $D \in V$ *be a generic point. Suppose* $H^0(X, adP) \neq \{0\}$, *then*

$$\operatorname{codim}(\Lambda_0, H^1(X, adP_D)) \geq h^0(X, adP_D) + 1.$$

Before going into the detail of the proof, let us state several technical lemmas which we need. Let

$$H^0_{nil} = \{s \in H^0(X, adP) \mid ad(s) \text{ is a nilpotent endomorphism}\}. \tag{4.1}$$

It is clear that $H^0_{nil} \subseteq H^0(X, adP)$ is an algebraic subvariety. Let W be the linear space spanned by H^0_{nil} and $W^\perp$ be a linear compliment of W in $H^0(X, adP)$.

Lemma 4.3: *Let* $l(s)$ *be the number of distinct nonzero eigenvalues of* $ad(s)$ *and* $l(D) = \max\{l(s) \mid s \in H^0(X, adP)\}$. *Then*

$$l(D) \geq \dim W^\perp.$$

Proof: Let $\rho(s, \lambda) = \lambda^n + a_1(s)\lambda^{n-1} + \cdots + a_n(s)$ be the characteristic polynomial of $ad(s)$. $a_i(s)$ are holomorphic. If we restrict the polynomial $\rho(s, \lambda)$ to $W^\perp$, we can find a branched covering $\varphi: Z \to \widetilde{W}^\perp$ and holomorphic functions f_i on Z such that

$$\rho(\varphi(\tilde{s}), \lambda) = \lambda^{n_0} \prod_{i=1}^{l(D)} (\lambda - f_i(\tilde{s}))^{n_i}. \tag{4.2}$$

Let $(f) = (f_1, f_2, \cdots, f_{l(D)}): Z \to \mathbb{C}^{l(D)}$ be the holomorphic map. Clearly, the set $(f)^{-1}(0)$ in Z corresponds to nilpotent elements in $H^0(X, adP)$. Thus $(f)^{-1}(0)$ is discrete. Therefore, $l(D) \geq \dim Z = \dim W^\perp$. □

Lemma 4.4: *Let E be a vector bundle on X and let $\Lambda \subset H^0(X,E)$ be any linear subspace. Then the dimension of the image $\Lambda \otimes H^0(X,T_X^*) \to H^0(X, E\otimes T_X^*)$ is at least $\dim \Lambda + (g-1)$.*

Proof: Without loss of generality, we can assume E is a line bundle. Let $x \in X$ and let $s_1, \cdots, s_k$ be a basis of $H^0(X,E)$ such that s_i has vanishing order α_i at x with $\alpha_1 < \cdots < \alpha_k$. Let $t_1, \cdots, t_g \in H^0(X,T_X^*)$ be a basis of $H^0(X,T_X^*)$ of the same natural. Then $s_1t_1, \cdots, s_kt_1, s_kt_2, \cdots, s_kt_g$ are linearly independent. Thus

$$\dim \mathrm{Im}\{\Lambda \otimes H^0(X,T_X^*) \to H^0(X,E\otimes T_X^*)\} \geq \dim \Lambda + (g-1). \qquad \square$$

Lemma 4.5: *Assume $s \in H^0(X, adP_D)$ with $ad(s)$ nilpotent. Then the following pairing induced by integrating the trace over X*

$$tr_X : H^1(X, \mathcal{E}nd(adP)_s) \otimes (ad(s) \otimes H^0(T_X^*)) \to \mathbb{C}$$

is trivial.

Proof: Let $0 = F_0 \subset F_1 \subset \cdots \subset F_r = adP_D$ be the canonical s-decomposition of adP. For any $\nu \in \mathcal{E}nd(adP)_s \otimes \Omega_X^{1,0}$, $\nu(F_i) \subset F_i \otimes \Omega_X^{1,0}$. On the other hand, $ad(s)(F_i) \subset F_{i-1}$. Thus

$$\nu \circ ad(s)(F_i) \subset F_{i-1} \otimes \Omega_X^{1,0}.$$

Therefore $tr_X(\nu \circ ad(s) \otimes h) = 0$ for any $h \in H^0(X,T_X^*)$. $\square$

Proof of proposition 4.2: By definition, any $v \in \Lambda_0$ preserves the canonical filtration (3.2) of all $s \in H^0(X, adP)$. In particular, if s is nilpotent, by lemma 3.7 and lemma 4.5, $(v, s\otimes h) = 0$ for any $h \in H^0(X,T_X^*)$. Since W is spanned by nilpotent elements,

$$(\cdot,\cdot) : \Lambda_0 \otimes (W \otimes H^0(X,T_X^*)) \to \mathbb{C} \tag{4.3}$$

is trivial. Let $s \in H^0(X, adP)$ be a generic point, $l(s) = l(D)$. Let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = E_0, \cdots, E_l$$

be the canonical s-decomposition. Since Λ_0 preserve the decomposition, $\Lambda_0 \subset H^1(X,E_0)$. By lemma 4.1, and lemma 4.4, if we let $W_0 = W \cap H^0(X,E_0)$, then

$$\begin{aligned}\mathrm{codim}(\Lambda_0, H^1(X,E_0)) &\geq \dim \mathrm{Im}\{W_0 \otimes H^0(X,T_X^*) \to H^0(X, E_0 \otimes T_X^*)\} \\ &\geq \dim W_0 + (g-1).\end{aligned}$$

So

$$\operatorname{codim}(\Lambda_0, H^1(X, adP)) \geq \dim W_0 + (g-1) + \sum_{\lambda\neq 0} h^1(X, E_\lambda). \tag{4.4}$$

On the other hand, $E_\lambda = E^\vee_{-\lambda}$. By Riemann-Roch theorem,

$$\sum_{\lambda\neq 0} h^1(X, E_\lambda) = \sum_{\lambda\neq 0}(-\deg E_\lambda + (g-1)\cdot \operatorname{rank}(E_\lambda) + h^0(X, E_\lambda)).$$

Therefore,

$$\begin{aligned} &\operatorname{codim}(\Lambda_0, H^1(X, adP) \\ &\qquad \geq \dim W_0 + (g-1) + \sum_{\lambda\neq 0}((g-1)\cdot \operatorname{rank} E_\lambda + h^0(X, E_\lambda)) \\ &\qquad \geq \dim W_0 + (g-1) + l(s) + \sum_{\lambda\neq 0} h^0(X, E_\lambda) \\ &\qquad \geq \dim W_0 + (g-1) + \dim W^\perp + \sum_{\lambda\neq 0} h^0(X, E_\lambda) \\ &\qquad = h^0(X, adP) + (g-1). \end{aligned}$$

The third inequality follows from lemma 4.3 and the last equality holds since $H^0(X, E_0) = W_0 \oplus W^\perp$. □

Now we are ready to prove the first part of theorem 2.

Proposition 4.6: *Assume $F(P)$ is non-empty, then $F(P)$ is irreducible.*

Proof: We first show that for any generic point D in $F(P)$, $H^0(X, adP) = \{0\}$. Suppose $H^0(X, adP_D) \neq \{0\}$. By proposition 2.6, there is a germ of subvariety $V' \subset F(P)$, $D \in V'$ such that $\dim V' \geq h^1(X, adP_D) - h^0(X, adP_D)$ and

$$\dim \operatorname{Im}\{T_D V' \to H^1(X, adP_D)\} \geq h^1(X, adP_D) - h^0(X, adP_D). \tag{4.5}$$

Now let $D' \in V'$ be a generic point in V' so that V' is smooth at D and so that (4.5) still holds. Since D is generic and since $h^0(X, adP_D)$ is an upper-semicontinuous function when D varies, $h^0(X, adP_D) = h^0(X, adP_{D'})$. On the other hand, we have $\operatorname{Im}\{T_{D'}V' \to H^1(X, adP_{D'})\} \subset \Lambda_0$, $\Lambda_0 \subseteq H^1(X, adP_{D'})$, and then thanks to proposition 4.2, if $h^0(X, adP_{D'}) \neq 0$, then

$$\dim \operatorname{Im}\{T_{D'}V \to H^1(X, adP_{D'})\} \leq h^1(X, adP_{D'}) - h^0(X, adP_{D'}) - 1. \tag{4.6}$$

This contradicts to (4.5). Therefore, $h^0(X, adP_D) = h^0(X, adP_{D'}) = 0$.

Let $F(P)_1 = \{D \in F(P) \mid h^0(X, adP_D) \neq 0\}$ and $F(P)_0 = F(P) \setminus F(P)_1$. We just showed that $F(P)_0$ is open and dense in $F(P)$. Further, $F(P)_0$ is a smooth Banach manifold because $\widetilde{\Theta} : \mathcal{A}_P \to \Omega^{1,1}(X, adP) \times \mathcal{C}_P$ is regular at D when $h^0(X, adP_D) = 0$. $F(P)$ will be irreducible if we can show that $F(P)_0$ is connected. By lemma 2.4, $\eta : F(P)_0 \to \mathcal{C}_P$ is a fiber bundle over its image. Applying corollary 2.7, $\eta(F(P)_0)$ is dense in $\mathcal{C}_P$. Indeed, it is dense in the Zariski topology. Therefore, $\eta(F(P)_0)$ is connected and so $F(P)_0$ is connected. □.

5 The Topology of $\mathrm{Hom}(\pi, G)_P$

The goal of this section is to complete the proof of theorem 0.2. First we state a generalization of Weil's theorem which says that a holomorphic vector bundle is flat if it is indecomposable.

Proposition 5.1 (Weil): *Let P be a holomorphic principal G-bundle. Assume $H^0(X, adP)$ is spanned by nilpotent elements, then P admits holomorphic connections (compatible flat connections).*

Proof: Let D be the unitary connection of P and $\Theta(D)$ be its curvature. P admits a holomorphic connection if there is a $\psi \in \Omega^{1,0}(X, adP)$ such that $\Theta(D + \psi) = 0$. Clearly, the curvature of the connection adD on adP induced by D is $\widetilde{\Theta} = ad(\Theta) \in \Omega^{1,1}(X, \mathcal{E}nd(adP))$. By Weil's theorem [Gu]

$$\int_X tr_X(\widetilde{\Theta} \circ \rho) = 0 \tag{5.1}$$

for any nilpotent endomorphism $\rho \in H^0(X, \mathcal{E}nd(adP))$. Since $H^0(X, adP)$ is spanned by nilpotent elements, for any $s \in H^0(X, adP)$,

$$\int_X tr_X(ad(\Theta) \circ ad(s)) = 0.$$

Thus Θ is $\bar{\partial}$-exact by Serre duality. In particular, there is $\psi \in \Omega^{1,0}(X, adP)$ such that $\Theta(D + \psi) = 0$. □

By lemma 2.4, the map $\eta : F(P) \to \mathcal{C}_P$ is a fiber bundle near $\bar{\partial}$ if $h^1(X, adP_{\bar{\partial}})$ is locally constant. By Riemann-Roch, $h^1(X, adP_{\bar{\partial}}) = (g-1) \cdot \mathrm{rank}(g)$ if $h^0(X, adP_{\bar{\partial}}) = 0$. Let Z be the set of exceptional points, that is

$$Z = \{\bar{\partial} \in \mathcal{C}_P \mid h^0(X, adP_{\bar{\partial}}) \neq 0 \}. \tag{5.2}$$

We have the estimate,

Lemma 5.2: *$Z \subset \mathcal{C}_P$ is a closed subvariety of finite codimension. If Z is a proper closed subset, then* $\operatorname{codim}(Z, \mathcal{C}_P) \geq 2$.

Proof: The first part follows from dim $H^1(X, adP) \leq \infty$. To show the second part, we estimate the dimension of the normal bundle of Z in $\mathcal{C}_P$. Let $\overline{\partial}$ be a generic point of Z, $s \in H^0(X, adP_{\overline{\partial}})$. Let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = E_0, E_1, \cdots, E_l$$

be the canonical s-decomposition. If $l \geq 1$, since $[v_\lambda, v_\mu] \in E_{\lambda+\mu}$, where $v_\lambda \in E_\lambda$, $\oplus_{\lambda \neq 0} H^1(X, E_\lambda)$ is contained in the normal bundle to Z [AB, p566]. By Riemann-Roch,

$$h^1(X, E_\lambda \oplus E_{-\lambda}) = h^0(X, E_\lambda \oplus E_{-\lambda}) + 2 \cdot \operatorname{rank}(E_\lambda)(g-1) \ \geq 2.$$

If $l = 0$, by lemma 4.5, the tangent directions of Z is orthogonal to $s \otimes H^0(X, T_X^*)$. By lemma 4.4, the dimension of the normal bundle is at least $1 + (g-1) \geq 2$. □

We now prove the second part of theorem 0.2.

Proposition 5.3: *When $F(P)$ is non-empty, $F(P)$ is simply connected.*

Proof: We first claim that $\pi_1(F(P)_0) \to \pi_1(F(P))$ is surjective. Let $\phi: S^1 \to F(P)$ be any homotopy class. Since $F(P)$ is an affine variety, we can assume that when ϕ is in generic position, $\phi(S^1) \cap F(P)_1$ is a discrete point set. Moreover, since $\eta(F(P)_0)$ is dense in $\mathcal{C}_P$ and, adding lemma 2.4, $F(P)$ is locally irreducible, we can further perturb ϕ so that $\phi(S^1) \cap F(P)_1 = \emptyset$. Finally, since $F(P)_0 \to \mathcal{C}_P \setminus Z$ is a fibration with affine fiber, $\pi_1(F(P)_0) = \pi_1(\mathcal{C}_P \setminus Z) = \{0\}$. Here, the second equality holds because $\mathcal{C}_P$ is affine and $\operatorname{codim}(Z, \mathcal{C}_P) \geq 2$. Therefore, $\pi_1(F(P)) = \{0\}$. □

Since $F(P) \to \operatorname{Hom}(\pi, G)_P$ is a fiber bundle with fiber $\mathcal{G}_0$. The induced sequence

$$\begin{aligned} &\to \pi_1(F(P)) \to \pi_1(\operatorname{Hom}(\pi, G)_P) \to \pi_0(\mathcal{G}_0) \\ &\qquad \to \pi_0(F(P)) \to \pi_0(\operatorname{Hom}(\pi, G)_P) \to 0 \end{aligned} \tag{5.3}$$

is exact. Combined with theorem 0.2, we see $\pi_0(\operatorname{Hom}(\pi, G)_P) = \{0\}$ and $\pi_1(\operatorname{Hom}(\pi, G)_P) = \pi_0(\mathcal{G}_0)$.

Proposition 5.4: *If $\pi_1(G) = \{0\}$, then $\pi_1(\operatorname{Hom}(\pi, G)_P) = \{0\}$.*

Proof: We only need to show that $\pi_0(\mathcal{G}_0) = \{0\}$. Since when $\pi_1(G) = \{0\}$, the only smooth principal G-bundle is $P = X \times G$. Then $autP = X \times G$. A standard application of obstruction theory shows that $\mathcal{G}_0$ is connected. So the proposition is established. □

6. Existence of flat structures

So far, we have proved that when $F(P) \neq \emptyset$, then $\pi_0(F(P)) = \{0\}$. It is also known that the number of topological G-bundles (and K-bundles) on X is exactly $\pi_1(G) = \pi_1(K)$. In this section, we will show that any topological K-bundle comes from a representation $\rho : \pi \to K$. Combined with lemma 1.3, theorem 0.3 then follows.

We first describe the obstruction map

$$o : \mathrm{Hom}(\pi, K) \to \pi_1(K).$$

Following [Ra, §5], a K-bundle P can be constructed as follows: Let D be a small disk around $p_0 \in X$. Since $P_{|D}$ and $P_{|X \setminus p_0}$ are trivial bundles, P is determined by the transition function $\varphi : D \setminus \{p_0\} \to G$. On the other hand, $D \setminus \{p_0\}$ is homotopy equivalent ot S^1. Therefore, the bundle P is uniquely determined by $[\varphi] \in \pi_1(K)$.

Now let $\rho : \pi \to K$ be any representation and let P_ρ be the associated flat bundle. Let

$$\{x_1, \cdots, x_g, y_1, \cdots, y_g \mid \Pi_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} = 1\}$$

be the canonical presentation of π and let A_i, B_i be simple contours of X so that $[A_i] = x_i$, $[B_i] = y_i$ and $X \setminus \Sigma$, $\Sigma = \cup_{i=1}^g (A_i \cup B_i)$, is homeomorphic to the disk. We assume A_i and B_i are initiated from same point $p_0 \in X$. By definition, $P_\rho = \widetilde{X} \times K/\pi$, where π acts on K via $\rho : \pi \to K$. Fix a $\tilde{p}_0 \in \widetilde{X}$ over p_0 and let $\widetilde{A}_i$ and $\widetilde{B}_i$ be lifting of A_i and B_i respectively with initial point $\tilde{p}_0$. Clearly, any trivialization of P_ρ along Σ is equivalent to a continuous map $h : \cup_{i=1}^g (\widetilde{A}_i \cup \widetilde{B}_i) \to K$ such that if denote by u_i and v_i the end point of $\widetilde{A}_i$ and $\widetilde{B}_i$ other that $\tilde{p}_0$ respectively, then $h(u_i) = \rho(x_i)$ and $h(v_i) = \rho(y_i)$. We fix such a trivialization (denoted by $h_{|\Sigma} : \Sigma \times K \to P_\Sigma$). We let $f_i : [0,1] \to K$ and $g_i : [0,1] \to K$ be induced by $h_{|\widetilde{A}_i}$ and $h_{|\widetilde{B}_i}$, based on a choice of parameterization of $\widetilde{A}_i$ and $\widetilde{B}_i$, respectively, where we agree that $f_i(0) = g_i(0) = e$. We claim that the obstruction class $o(\rho)$ is represented by the loop

$$\Pi_{i=1}^g f_i(\cdot) g_i(\cdot) f_i^{-1}(\cdot) g_i^{-1}(\cdot) : [0,1] \to K. \tag{6.1}$$

Indeed, over the interior of $X \setminus \Sigma$, there is an obvious trivialization given by $e \in K$ (denoted by $h'_{|X \setminus \Sigma} : (X \setminus \Sigma) \times G \to P_{|X \setminus \Sigma}$). Then if we extend the trivialization $h_{|\Sigma}$ to a tubular neighborhood $T(\Sigma)$ of Σ, say $h_{|T(\Sigma)}$, and let $\alpha: S^1 \to T(\Sigma) \cap (X \setminus \Sigma)$ be the generator of its π_1, then $o(\rho)$ is represented by

$$(h'_{|X\setminus\Sigma}(\alpha(\cdot), e)) \cdot (h_{|T(\Sigma)}(\alpha(\cdot), e))^{-1} : S^1 \to K. \qquad (6.2)$$

One checks directly that (6.1) is homotopic equivalent to the class given by (6.2).

It remains to show that any element of $\pi_1(K)$ can be represented by class of type (6.1). But this follows from the surjectivity of the multiplication map $\widetilde{K} \times \widetilde{K} \to \widetilde{K}$, where $\widetilde{K}$ is the universal covering of K and $(a, b) \mapsto aba^{-1}b^{-1}$, which is true because K is semisimple, compact and for any finite covering $K' \to K$, the same map $K' \times K' \to K'$ is surjective. Thus we have proved

Proposition 6.1: *Let P be any K-bundle, where K is connected, compact and semisimple. Then P is topologically equivalent to P_ρ for some $\rho \in \mathrm{Hom}(\pi, K)$.*

7. Compact group cases

In this section, we assume K is a compact, connected semisimple Lie group. We will combine the argument of [Ra] and [AB] to prove the following

Proposition 7.1: *Let K be a compact, connected semisimple Lie group and let P be any principal K-bundle. Then $\mathrm{Hom}(\pi, K)_P$ is irreducible.*

We first recall that a set $\Gamma \subseteq K$ is called irreducible if we have

$$\{H \in k \mid AD(s)(H) = H, \forall s \in \Gamma\} = \{0\},$$

where k is the Lie algebra of K. A representation $\rho: \pi \to K$ is called irreducible if $\rho(\pi)$ is irreducible. Let P be any principal K-bundle and let $\mathrm{Hom}(\pi, K)_P^{irre} \subseteq \mathrm{Hom}(\pi, K)_P$ be the set of all irreducible homomorphisms. Following the argument of §5, we see that $\mathrm{Hom}(\pi, K)_P^{irre}$ is dense in $\mathrm{Hom}(\pi, K)_P$ when $g > 1$. So to prove that $\mathrm{Hom}(\pi, K)_P$ is irreducible, it suffices to show that $\mathrm{Hom}(\pi, K)_P^{irre}$ is irreducible.

Now let G be the complexification of K. Let $P_G = P \times_K G$ be the associated G-bundle. For any complex structure $\bar{\partial} \in \mathcal{C}_{P_G}$ of P_G, Ramanathan introduced the concept of stable principal bundles. For the precise definition of stability, we refer to [Ra]. We quote the following two properties that we need:

Proposition 7.2: *A holomorphic principle G-bundle is stable if and only if it is isomorphic to P_ρ for some irreducible $\rho \in \mathrm{Hom}(\pi, K)$.*

Proposition 7.3: *The condition of being stable is a Zariski open condition. In particular, the set of all stable holomorphic structures on P (denoted by $\mathcal{C}_P^s$) is a zariski open subset of $\mathcal{C}_P$ and hence is irreducible when it is non-empty.*

By proposition 7.2, any stable holomorphic structure on the G-bundle P associates to a canonical homomorphism $\rho \in \mathrm{Hom}(\pi, K)_P^{irre}/K$ and therefore, this map defines a fibration

$$\mathcal{C}_P^s \to \mathrm{Hom}(\pi, K)_P^{irre}/K.$$

Since $\mathcal{C}_P^s$ is irreducible, $\mathrm{Hom}(\pi, K)_P^{irre}/K$ is irreducible. Therefore, $\mathrm{Hom}(\pi, K)_P^{irre}$ and hence $\mathrm{Hom}(\pi, K)_P$ are irreducible. This proves theorem 0.6.

Bibliography

[AB] Atiyah, M. F. and Bott, R., The Yang-Mills equations over a compact Rieman surface, *Phil. Trans. Roy. Soc. London*, **A 308** (1982), 523-615

[Go] Goldman, W. M., Topological components of spaces of representations, *Invent. Math.* **93** (1988) 557-607

[GM1] Goldman, W. M. and Millson, J. J., Deformations of flat bundles over Kahler manifolds, in *Geometry and Topology, Manifolds, Varieties and Knots*, C. McCrory and T. Shifrin (eds.), *Lecture Notes in Pure and Applied Mathematics*, **105**, Mercer Dekker, New York-Basel (1987), 129-145

[GM2] Goldman, W. M. and Millson, J. J., The deformation theory of representations of fundamental groups of compact Kahler manifolds, *Publ. Math. I.H.E.S.*

[Gu] Gunning, R. C., Lectures on vector bundles over Riemann surfaces, Princeton university press, 1967

[Ko] Koszul, J. L., Lectures on fiber bundles and differential geometry, Tata Institute of fundamental research, Bombay 1960

[Ra] Ramanathan, A., Moduli for principal bundles, in *Algebraic Geometry. Proceeding*, Copenhagen 1978, *Lecture Notes in Mathematics*, **732**. Berlin, Heidelberg, New York: Springer 1979

Mathematical Science Research Institute
1000 Centennial drive
Berkeley, CA 94720

(Received March 16, 1992;
in revised form September 22, 1992)