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Distribution of a Sum of Weighted Noncentral Chi-Square Variables

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Abstract

We derive Laguerre expansions for the density and distribution functions of a sum of positive weighted noncentral chi-square variables. The procedure that we use is based on the inversion of Laplace transforms. The formulas so obtained depend on certain parameters, which adequately chosen will give some expansions already known in the literature and some new ones. We also derive precise bounds for the truncation error.

Key Words: Laguerre expansion, chi-square distribution, truncation error, inverse Laplace transform.

AMS subject classification: 62E15, 62E17.

1 Introduction

Positive quadratic forms in normal variables arise naturally in many problems of estimation and testing related to normal distributions and Gaussian processes. Also in non-normal cases, these quadratic forms appear as limits of certain statistics used in the inference. Under suitable transformations, positive quadratic forms in normal variates can be expressed as linear combinations of independent non-central chi-square variables. Numerous

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applications of these distributions can be found, for instance, in Jensen and Solomon (1972) and Mathai and Provost (1992).

The problem of obtaining the distribution of a quadratic form in normal variables has been addressed by many authors. Hence, several representations for the cumulative distribution function and the density can be found out in the literature. These include, among others, power series expansions, see Shah and Khatri (1961); χ^2 series, see Ruben (1962) and Laguerre series, see Shah (1963) and Kotz et al. (1967). Some of these representations arise in an attempt to generalize the already known formulas for the distribution of linear combinations of central chi-square variables. For instance, Kotz et al. (1967) generalized the expansion that they obtained for the central case, using in essence the same method of Gurland (1955), giving single series expansions in the power series and Laguerre series cases. These formulas are more useful for computational purposes than the double series given by Shah and Khatri (1961) and by Shah (1963), respectively.

One of the most successful approaches for obtaining the distribution and density functions of linear combinations of non-central chi-square variables is the representation in terms of Laguerre series as in Kotz et al. (1967), see also Davis (1977) and Mathai and Provost (1992). In fact, Laguerre series expansions play a very important role in the subject of approximation of distributions, see Tan and Tiku (1999). These expansions have been also used to solve numerous problems in information theory: Borget and Faure (1973); regression analysis: Vomişescu (1999) and Gurmu et al. (1999) and experimental design: Genizi and Soller (1979), Tan (1982) and Tiku (1964).

In this work we derive Laguerre expansions for the density and distribution functions of a non-central χ^2 variable and of positive linear combinations of non-central chi-square variables. The procedure that we use is based on the inversion of their Laplace transforms in terms of Laguerre polynomials.

The paper is structured as follows. In the following section we propose Laguerre expansions for the density and distribution functions of a non-central chi-squared variable. Moreover, we derive bounds on the truncation error in the given expansions. In Section 3 we provide an analogous study for positive linear combinations of non-central chi-squared variables and compare our results with those given in the literature. Some comments and conclusions are included in Section 4. In Section 5, we give an Appendix in which we describe a procedure to invert Laplace transforms and also the proofs of some necessary results in order to study the truncation error of the proposed expansions.

2 Expansions for a non-central chi-square variable

The non-central chi-square distribution was obtained by Fisher (1928) as a limiting case of the distribution of the multiple correlation coefficient. After him, this distribution has been derived in several different ways. The first direct derivation was given by Tang (1938). A geometric derivation was obtained by Patnaik (1949) who emphasized the relevance of this distribution to approximate the power of the χ^2 test and also suggested approximations to the non-central χ^2 distribution itself. Patnaik represented the non-central chi-square distribution as a mixture of central χ^2 variables with weights equal to the probabilities of a Poisson distribution. Tiku (1965) obtained an expression for the density function in terms of the generalized Laguerre polynomials, which we will obtain as a particular case of our procedure. Gideon and Gurland (1977) provided another Laguerre expansion with coefficients which are rather complicated. Other representations and approximations for this distribution can be found in Johnson et al. (1995).

In the next subsection, we propose a Laguerre expansion for the density and distribution functions of a non-central χ^2 variable, respectively. Basically, the method consists in the inversion of the Laplace transform of the density (or distribution function). The procedure to obtain the inverse Laplace transform is described in the Appendix.

2.1 Computation of the distribution of $\chi^2_n(\delta)$ variable

Let $f(y)$ be the density function of $\chi^2_n(\delta)$ variable, where δ is the non-centrality parameter. Its Laplace transform is:

$$\mathcal{L}\left(f(y)\right)(\lambda) = \exp\left\{-\frac{\delta\lambda}{1+2\lambda}\right\}(1+2\lambda)^{-n/2} = G(\lambda).$$

Consider the function:

$$H(\lambda) = \exp\left\{-\frac{\delta(\lambda-1)}{2\lambda}\right\}\lambda^{-n/2} = G\left(\frac{\lambda-1}{2}\right).$$

Using standard properties of the inverse Laplace transform we have:

$$\begin{aligned} f(y) &= \mathcal{L}^{-1}\left(G(\lambda)\right)(y) = \mathcal{L}^{-1}\left(H(1+2\lambda)\right)(y) \\ &= \frac{e^{-y/2}}{2}\mathcal{L}^{-1}(H(\lambda))(y/2). \end{aligned} \tag{2.1}$$

Now, we invert the function H using the expansion (5.3) given in the Appendix. For that, let $g(\mu) = (p/\mu)^p H(p/\mu)$. In particular, for $p = n/2$, we obtain

$$g(\mu) = \exp\left\{-\frac{\delta\,(n/2-\mu)}{n}\right\},$$

with derivatives

$$g^{(k)}(\mu) = \left(\frac{\delta}{n}\right)^k \exp\left\{-\frac{\delta\,(n/2-\mu)}{n}\right\},\ k \geq 0. \tag{2.2}$$

So,

$$\begin{aligned} \mathcal{L}^{-1}\left(H(\lambda)\right)(y) = \frac{y^{n/2-1}}{\Gamma(n/2)} &\exp\left\{-\frac{\delta\,(n/2-\mu_0)}{n}\right\} \\ &\cdot\sum_{k\geq 0}\frac{\left(\frac{-\mu_0\delta}{n}\right)^k}{(n/2)_k}\mathrm{L}_k^{(n/2-1)}\left(\frac{ny}{2\mu_0}\right), \end{aligned}$$

with $L_k^{(\alpha)}$ the k-th generalized Laguerre polynomial (see Appendix). Then, from (2.1) we have

$$\begin{aligned} f(y) = \frac{e^{-y/2}}{2^{n/2}}\frac{y^{n/2-1}}{\Gamma(n/2)} &\exp\left\{-\frac{\delta\,(n/2-\mu_0)}{n}\right\} \\ \cdot\sum_{k\geq 0}&\frac{\left(\frac{-\mu_0\delta}{n}\right)^k}{(n/2)_k}\mathrm{L}_k^{(n/2-1)}\left(\frac{ny}{4\mu_0}\right),\ \forall \mu_0 > 0. \end{aligned} \tag{2.3}$$

If we consider $\mu_0 = n/2$ in (2.3), we obtain the expansion given by Tiku (1965) using another procedure:

$$f(y) = \frac{e^{-y/2}}{2^{n/2}}\frac{y^{n/2-1}}{\Gamma(n/2)}\sum_{k\geq 0}\frac{\left(\frac{-\delta}{2}\right)^k}{(n/2)_k}\mathrm{L}_k^{(n/2-1)}\left(\frac{y}{2}\right). \tag{2.4}$$

Let $F(y)$ be the distribution function of $\chi_n^2(\delta)$. Its Laplace transform is:

$$\mathcal{L}\left(F(y)\right)(\lambda)=\frac{1}{\lambda}\exp\left\{-\frac{\delta\lambda}{1+2\lambda}\right\}(1+2\lambda)^{-n/2}=G\left(\lambda\right).$$

Using similar arguments as before, we obtain:

$$F(y)=\frac{e^{-y/2}}{2^{n/2+1}}\frac{y^{n/2}}{\Gamma\left(n/2+1\right)}\sum_{k\geq 0}\frac{k!c_k}{(n/2+1)_k}\mathrm{L}_k^{(n/2)}\left(\frac{(n+2)\,y}{4\mu_0}\right), \tag{2.5}$$

with

$$g(\mu)=\frac{2p}{p-\mu}\exp\left\{-\frac{\delta\left(p-\mu\right)}{2p}\right\},\ p=\frac{n}{2}+1, \tag{2.6}$$

and $c_k=(-\mu_0)^k g^{(k)}(\mu_0)/k!$. These coefficients satisfy the recurrent relations:

$$c_k=\frac{1}{k}\sum_{j=0}^{k-1}c_j d_{k-j},\ k\geq 1,\quad c_0=\frac{2p}{p-\mu_0}\exp\left(\frac{-\delta\left(p-\mu_0\right)}{2p}\right), \tag{2.7a}$$

$$d_j=\left(\frac{-\mu_0}{p-\mu_0}\right)^j,\ j\geq 2,\quad d_1=-\mu_0\left(\frac{1}{p-\mu_0}+\frac{\delta}{2p}\right). \tag{2.7b}$$

Ashour and Abdel-Samad (1990) derived a different expression as a double series; Gideon and Gurland (1977) proposed another Laguerre expansion with coefficients which are difficult to calculate. Tiku (1965) obtained another expression in terms of Laguerre polynomials by direct integration of the density function. With our method, we provide a Laguerre expansion without the explicit knowledge of the density function or the distribution function. Moreover, the expansion given in (2.3) depends on a parameter, μ_0, that can be chosen arbitrarily. Some adequate choices may give computationally efficient formulas for the calculation of the distribution function. Even, uniform convergence can be achieved by an adequate choice, as we will see in Subsection 2.2.

In Table 1 we compare the expansion obtained in (2.5) for $\mu_0=p/4$ with that given by Tiku (1965), and the exact values given in Patnaik (1949), being j the number of terms considered in the expansions.

We can observe in these examples that the expansion that we propose converge faster than the expansion given by Tiku.

Table 1: Distribution function of $\chi^2_n(\delta)$

n	δ	y	j	$\mu_0 = p/4$ (2.5)	Tiku (1965)	$F(y)$ (Exact)
4	10	10	10	0.3148368	0.3141404	0.3148
7	16	10.257	10	0.04999622	-0.45<0	0.05
24	24	36	15	0.15671754	0.1276852	0.1567

2.2 Bound for the truncation error

As we are interested in the implementation of these formulas in a computer, we study the errors produced when the infinite series are truncated. To get bounds on the truncation error in the expansions above, we need to bound the Laguerre polynomials and also the coefficients c_k given in (2.7a):

Lemma 2.1. *A classical global uniform (w.r.t. n, y and α) estimate given by Szegő (1975) is*

$$\begin{aligned} \left|\mathrm{L}_k^{(\alpha)}(y)\right| &\leq \frac{(\alpha+1)_k}{k!}\exp\left(\frac{y}{2}\right), \alpha \geq 0, \\ \left|\mathrm{L}_k^{(\alpha)}(y)\right| &\leq \left(2-\frac{(\alpha+1)_k}{k!}\right)\exp\left(\frac{y}{2}\right), -1<\alpha<0. \end{aligned} \tag{2.8}$$

To bound the coefficients $c_{k,}$ the following lemma will be useful:

Lemma 2.2. *Consider c_k as given in (2.7a) and p=n/2+1, then*

$$\begin{aligned} |c_k| \leq \frac{2p}{|p-\mu_0|}\exp\left(-\delta\left(\frac{p-(1+1/\xi)\,\mu_0}{2p}\right)\right) \\ \cdot\left(\frac{k}{\xi(1+k)}\right)^{-k}\left(\frac{1}{1+k}\right)^{-1}, \end{aligned} \tag{2.9}$$

with $\xi = \left|\dfrac{\mu_0}{p-\mu_0}\right|$.

Proof. See Appendix. □

Consider the truncation error for the density function as:

$$\mathcal{E}_N\left(f, y, \mu_0, \delta\right) = \Bigg| \frac{e^{-y/2}}{2^{n/2}} \frac{y^{n/2-1}}{\Gamma\left(n/2\right)} \exp\left\{-\frac{\delta\left(n/2-\mu_0\right)}{n}\right\} \cdot \sum_{k=N+1}^{\infty} \frac{\left(\frac{-\mu_0\delta}{n}\right)^k}{(n/2)_k} \mathrm{L}_k^{(n/2-1)}\left(\frac{ny}{4\mu_0}\right) \Bigg|,$$

for $\mu_0 > 0$.

From Lemma 2.1 and $\nu \geq 2$ we get

$$\mathcal{E}_N\left(f, y, \mu_0, \delta\right) \leq \frac{e^{-y/2}}{2^{n/2}} \frac{y^{n/2-1}}{\Gamma\left(n/2\right)} \exp\left\{-\frac{\delta\left(n/2-\mu_0\right)}{n}\right\} \exp\left(\frac{ny}{8\mu_0}\right) \cdot \sum_{k=N+1}^{\infty} \frac{\left(\frac{\mu_0\delta}{n}\right)^k}{k!}.$$

The above series is absolutely convergent for $\mu_0 > 0$, and as a consequence we have that the expansion (2.3) converges uniformly in any finite interval of $(0, \infty)$. However it is possible to get uniform convergence when $n > 2$, for all $y > 0$, by choosing $\mu_0 > n/4$.

In order to get bounds on the truncation error in the expansion given for the distribution function (2.5) we use again Lemmas 2.1 and 2.2, to obtain

$$\mathcal{E}_N\left(F, y, n, \delta, \mu_0\right) \leq \frac{e^{-y/2}y^{n/2}}{2^{n/2+1}} \frac{2p}{|p-\mu_0|} \exp\left(-\delta\left(\frac{p-(1+1/\xi)\,\mu_0}{2p}\right)\right) \cdot \exp\left(\frac{(n+2)y}{8\mu_0}\right) \sum_{k=N+1}^{\infty} b_k, \tag{2.10}$$

with

$$b_k = \xi^k \left(\frac{1+k}{k}\right)^k (1+k).$$

The bound (2.10) is well defined for $\mu_0 < p/2$, since the series $\sum_k b_k$ is absolutely convergent if $0 < \xi < 1$.

3 The distribution of a linear combination of independent non-central chi-square variables

In this section we propose Laguerre expansions for the density and distribution functions of $Q_n = \sum_{i=1}^n \alpha_i X_i$, with $\alpha_i > 0$ and $X_i \sim \chi^2_{\nu_i}(\delta_i)$, independent random variables. We proceed in a similar way as we did in the previous section for a non-central chi-square variable.

Let $f(y)$ be the density function of Q_n. Its Laplace transform is:

$$\mathcal{L}\left(f(y)\right)(\lambda) = \exp\left\{-\sum_{i=1}^n \frac{\delta_i \alpha_i \lambda}{1+2\alpha_i\lambda}\right\}\prod_{i=1}^n (1+2\alpha_i\lambda)^{-\nu_i/2} = G(\lambda).$$

Then,

$$f(y) = \frac{e^{-\frac{y}{2\beta}}}{2\beta}\mathcal{L}^{-1}\left(H(\lambda)\right)\left(\frac{y}{2\beta}\right), \tag{3.1}$$

with $H(\lambda) = G\left((\lambda-1)/(2\beta)\right)$ and $\beta > 0$.

Therefore, it is enough to invert $H(\lambda)$. Using (5.3) given in Appendix and (3.1), for $p = \nu/2$, we obtain

$$f(y) = \frac{e^{-\frac{y}{2\beta}}}{(2\beta)^{\nu/2}}\frac{y^{\nu/2-1}}{\Gamma(\nu/2)}\sum_{k\geq 0}\frac{k! c_k}{(\nu/2)_k}\mathrm{L}_k^{(\nu/2-1)}\left(\frac{\nu y}{4\beta\mu_0}\right), \forall \mu_0 > 0, \tag{3.2}$$

with

$$g(\mu) = \left(\frac{\nu}{2}\beta\right)^{\nu/2}\exp\left\{-\frac{1}{2}\sum_{i=1}^n \frac{\delta_i\alpha_i(p-\mu)}{\beta\mu+\alpha_i(p-\mu)}\right\}\prod_{i=1}^n \left(\beta\mu+\alpha_i(p-\mu)\right)^{-\nu_i/2}, \tag{3.3}$$

where $\nu = \sum_{i=1}^n \nu_i$ and $c_k = (-\mu_0)^k g^{(k)}(\mu_0)/k!$. These coefficients satisfy the recurrent relations:

$$\begin{aligned} c_k &= \frac{1}{k}\sum_{j=0}^{k-1} c_j d_{k-j},\ k \geq 1, \\ c_0 &= \left(\frac{\nu}{2\mu_0}\right)^{\nu/2}\exp\left\{-\frac{1}{2}\sum_{i=1}^n \frac{\delta_i\alpha_i(p-\mu_0)}{\beta\mu_0+\alpha_i(p-\mu_0)}\right\} \\ &\quad \cdot\prod_{i=1}^n \left(1+\frac{\alpha_i}{\beta}(p/\mu_0-1)\right)^{-\nu_i/2}, \end{aligned} \tag{3.4a}$$

$$\begin{aligned} d_j = & -\frac{j\beta p}{2\mu_0}\sum_{i=1}^{n}\delta_i\alpha_i\left(\beta-\alpha_i\right)^{j-1}\left(\frac{\mu_0}{\beta\mu_0+\alpha_i(p-\mu_0)}\right)^{j+1} \\ & +\sum_{i=1}^{n}\frac{\nu_i}{2}\left(\frac{1-\alpha_i/\beta}{1+\left(\alpha_i/\beta\right)\left(p/\mu_0-1\right)}\right)^{j}, j\geq 1. \end{aligned} \tag{3.4b}$$

Obviously if we consider $\delta_i = 0, i = 1\dots n$, in the expression (3.2), we obtain the corresponding expansion for the central case.

If we consider in (3.2), $\mu_0 = \nu/2 = p$ we obtain the expansion given by Kotz et al. (1967). However we can consider other choices of the parameter in order to improve the speed of convergence of this series.

Similarly, since the Laplace transform of the distribution function, $F(y)$, of Q_n is given by

$$\mathcal{L}\left(F(y)\right)(\lambda) = \frac{1}{\lambda}\exp\left\{-\sum_{i=1}^{n}\frac{\delta_i\alpha_i\lambda}{1+2\alpha_i\lambda}\right\}\prod_{i=1}^{n}(1+2\alpha_i\lambda)^{-\nu_i/2},$$

we obtain the following expansion for the distribution function:

$$F(y) = \frac{e^{-\frac{y}{2\beta}}}{(2\beta)^{\nu/2+1}}\frac{y^{\nu/2}}{\Gamma\left(\nu/2+1\right)}\sum_{k\geq 0}\frac{k!m_k}{(\nu/2+1)_k}\mathrm{L}_k^{(\nu/2)}\left(\frac{(\nu+2)y}{4\beta\mu_0}\right), \tag{3.5}$$

for $\mu_0 > 0$ and $p = \nu/2 + 1$. The coefficients m_k satisfy the following recurrent relation:

$$\begin{aligned} m_k = & \frac{1}{k}\sum_{j=0}^{k-1}m_j d_{k-j},\ k\geq 1, \\ m_0 = & 2\left(\frac{\nu}{2}+1\right)^{\nu/2+1}\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{\delta_i\alpha_i(p-\mu_0)}{\beta\mu_0+\alpha_i(p-\mu_0)}\right\} \\ & \cdot\frac{\beta^{\nu/2+1}}{p-\mu_0}\prod_{i=1}^{n}\left(\beta\mu_0+\alpha_i(p-\mu_0)\right)^{-\nu_i/2}, \end{aligned} \tag{3.6a}$$

$$\begin{aligned} d_j = & -\frac{j\beta p}{2\mu_0}\sum_{i=1}^{n}\delta_i\alpha_i\left(\beta-\alpha_i\right)^{j-1}\left(\frac{\mu_0}{\beta\mu_0+\alpha_i(p-\mu_0)}\right)^{j+1} \\ & +\left(\frac{-\mu_0}{p-\mu_0}\right)^{j}+\sum_{i=1}^{n}\frac{\nu_i}{2}\left(\frac{\mu_0\left(\beta-\alpha_i\right)}{\beta\mu_0+\alpha_i(p-\mu_0)}\right)^{j}, j\geq 1. \end{aligned} \tag{3.6b}$$

In this case, we offer an alternative expression for the distribution function without knowing the density function. Most of the authors in the literature obtain an expression for the distribution function by direct integration of the density function, such as Kotz et al. (1967). If $\delta_i = 0, \forall i$, in (3.5), we obtain an expression for the distribution function of a linear combination of independent central chi-square variables.

3.1 Bounds for the truncation error

In a similar way to the case of one variable, our objective is to implement these formulas in a computer so we study the errors produced when the infinite series given in (3.2) and (3.5) are truncated.

Firstly we need to bound the coefficients:

Lemma 3.1. *Consider* c_k *as given in* (3.4a), *then*

$$|c_k| \leq \left(\frac{\nu}{2\mu_0}\right)^{\nu/2} \prod_{i=1}^{n} \left|1 + \frac{\alpha_i}{\beta}\left(\frac{p}{\mu_0} - 1\right)\right|^{-\nu_i/2} \exp\left\{\frac{\mu_0 \sum_{i=1}^{n} \delta_i}{2p\zeta}\right\} \tag{3.7}$$

$$\cdot \exp\left\{-\frac{1}{4}\sum_{i=1}^{n} \frac{\delta_i \left(\alpha_i/\beta\right)\left(p/\mu_0 - 1\right)}{1 + \left(\alpha_i/\beta\right)\left(p/\mu_0 - 1\right)}\right\} \left(\frac{2k+\nu}{2k}\right)^k \left(\frac{2k+\nu}{\nu}\right)^{\nu/2} \zeta^k,$$

with $\zeta = \max_i \left|\frac{1-(\alpha_i/\beta)}{1+(\alpha_i/\beta)(p/\mu_0-1)}\right|$ *and* $p = \nu/2$.

Proof. See Appendix. □

Remark 3.1. *If* $\mu_0 < \nu/4$, *then* $0 < \zeta < 1$, *for all* $\beta > 0$ *and if* $\mu_0 \geq \nu/4$, *then* $0 < \zeta < 1$, *for* $\beta > (2 - \nu/(2\mu_0))\, \alpha_{(n)}/2$ *and* $\alpha_{(n)} = \max_i \alpha_i$.

Lemma 3.2. *Let* m_k *as given in* (3.6a), *then*

$$|m_k| \leq \frac{2\beta p}{|p - \mu_0|} \exp\left\{-\frac{1}{4}\sum_{i=1}^{n} \frac{\delta_i \left(\alpha_i/\beta\right)\left(p/\mu_0 - 1\right)}{1 + \left(\alpha_i/\beta\right)\left(p/\mu_0 - 1\right)}\right\} \exp\left\{\frac{\mu_0 \delta}{2p\varepsilon}\right\} \tag{3.8}$$

$$\cdot \left(\frac{p}{\mu_0}\right)^{\nu/2} \prod_{i=1}^{n} \left|1 + \frac{\alpha_i}{\beta}\left(\frac{p}{\mu_0} - 1\right)\right|^{-\nu_i/2} \left(\frac{2k+\nu+2}{2k}\right)^k \left(\frac{2k+\nu+2}{\nu+2}\right)^{\nu/2+1} \varepsilon^k$$

with $p = \nu/2 + 1$, $\delta = \sum_{i=1}^{n} \delta_i$, $\varepsilon = \max(a, \zeta)$,

$$a = \left|\frac{-\mu_0}{p - \mu_0}\right| \;\; and \;\; \zeta = \max_i \left|\frac{1 - (\alpha_i/\beta)}{1 + (\alpha_i/\beta)\,(p/\mu_0 - 1)}\right| .$$

Note that $0 < \varepsilon < 1$ *if* $\mu_0 < p/2$.

Proof. See Appendix. □

From the bounds of the Laguerre polynomials given in Lemma 2.1 and of the coefficients given in Lemmas 3.1 and 3.2 we will study the truncation error in the proposed expansions.

With the usual notation we have:

$$\begin{aligned} \mathcal{E}_N\left(f, y, \mu_0, \beta\right) \leq\ & \frac{e^{-\frac{y}{2\beta}}}{(2\beta)^{\nu/2}} \frac{y^{\nu/2-1}}{\Gamma(\nu/2)} \prod_{i=1}^{n} \left|1 + \frac{\alpha_i}{\beta}\left(\frac{p}{\mu_0} - 1\right)\right|^{-\nu_i/2} \\ & \cdot \left(\frac{\nu}{2\mu_0}\right)^{\nu/2} \exp\left\{\frac{\mu_0 \delta}{2p\zeta}\right\} \exp\left\{-\frac{1}{4}\sum_{i=1}^{n} \frac{\delta_i\,(\alpha_i/\beta)\,(p/\mu_0 - 1)}{1 + (\alpha_i/\beta)\,(p/\mu_0 - 1)}\right\} \\ & \cdot \exp\left(\frac{\nu y}{8\beta\mu_0}\right) \sum_{k=N+1}^{\infty} \zeta^k \left(\frac{2k+\nu}{2k}\right)^k \left(\frac{2k+\nu}{\nu}\right)^{\nu/2} . \end{aligned} \tag{3.9}$$

This bound is well defined since the above series is absolutely convergent if $0 < \zeta < 1$, see Remark 3.1. As a consequence the expansion given for the density function converges uniformly in any finite interval, for all μ and β chosen in an adequate way.

In the particular case of $\mu_0 = \nu/2 = p$ in (3.9), we have:

$$\begin{aligned} \mathcal{E}_N(f, y, \beta) \leq\ & \frac{e^{-\frac{y}{4\beta}}}{(2\beta)^{\nu/2}} \frac{y^{\nu/2-1}}{\Gamma(\nu/2)} \exp\left\{\frac{\delta}{2\zeta}\right\} \\ & \cdot \sum_{k=N+1}^{\infty} \zeta^k \left(\frac{2k+\nu}{2k}\right)^k \left(\frac{2k+\nu}{\nu}\right)^{\nu/2} . \end{aligned} \tag{3.10}$$

Kotz et al. (1967) proposed another different bound given by

$$\mathcal{E}_N(f,y,\beta) \le \frac{e^{-\frac{y}{4\beta}}}{(2\beta)^{\nu/2}} \frac{y^{\nu/2-1}}{\Gamma(\nu/2)} \exp\left\{\frac{\delta}{2\zeta}\right\} (1-\rho\zeta)^{-\nu/2} \cdot (1-R)^{-\nu/2} \sum_{k=N+1}^{\infty} (\rho R)^{-k}, \tag{3.11}$$

with $\rho = \zeta^{-2/3}$ and $R = \zeta^{1/3}$.

In Table 2, we show how the bound that we propose is better than the one given by Kotz et al. (1967) for $Q_2 = 0.7\chi_1^2(6) + 0.3\chi_1^2(2)$ (see Imhof, 1961) with $\beta = (0.3+0.7)/2$ and $N = 20$.

Table 2: Bounds for the truncation error of the density of Q_2

	$y=1$	$y=6$	$y=15$
(3.11) Kotz et al.	1200.579848	98.54959519	1.094787111
(3.10)	0.005900164346	0.0004843149820	$0.5380253460 \cdot 10^{-5}$
(3.9) ($\mu_0 = p/3$)	$1.349683601 \cdot 10^{-10}$	$0.1644251231 \cdot 10^{-9}$	$0.1480107789 \cdot 10^{-7}$

Similarly we obtain the following bound for the truncation error of the distribution function. Using Lemmas 2.1 and 3.2:

$$\mathcal{E}_N(F,y,\mu_0,\beta) \le \frac{e^{-\frac{y}{2\beta}} y^{\nu/2}}{\Gamma(\nu/2+1)} \prod_{i=1}^{n} \left(1 + \frac{\alpha_i}{\beta}\left(\frac{p}{\mu_0} - 1\right)\right)^{-\nu_i/2} \cdot \frac{p}{|p-\mu_0|} \exp\left\{\frac{\mu_0\delta}{2p\varepsilon}\right\} \exp\left\{-\frac{1}{4}\sum_{i=1}^{n} \frac{\delta_i(\alpha_i/\beta)(p/\mu_0 - 1)}{1+(\alpha_i/\beta)(p/\mu_0-1)}\right\} \cdot \left(\frac{p}{2\beta\mu_0}\right)^{\nu/2} \exp\left(\frac{(\nu+2)y}{8\beta\mu_0}\right) \sum_{k=N+1}^{\infty} b_k, \tag{3.12}$$

where

$$b_k = \varepsilon^k \left(\frac{2k+\nu+2}{2k}\right)^k \left(\frac{2k+\nu+2}{\nu+2}\right)^{\nu/2+1}. \tag{3.13}$$

The above series is absolutely convergent for $\mu_0 < p/2$ ($p = \nu/2+1$) since $0 < \varepsilon < 1$, then in a similar way to the density function we have the uniform convergence of the expansion given for the distribution function.

Kotz et al. (1967) proposed the following bound:

$$\mathcal{E}_N \leq 2\left(1-\xi\rho\right)^{-n/2}\left(1-R\right)^{-1-n/2} e^{\lambda/2\xi} e^{y/4\beta} \cdot g\left(n+2, y/\beta\right)\left[\left(\rho R\right)^{-(N+1)} / \left(1-1/\rho R\right)\right], \tag{3.14}$$

with $\xi = \max_i \left|1-\alpha_i/\beta\right|$, $\lambda = \sum_{i=1}^{n} \delta_i$ and $g\left(n+2, y/\beta\right)$ is the central χ^2 density function with $n+2$ degrees of freedom, $\rho = \xi^{-2/3}$ and $R = \xi^{1/3}$.

We can observe that the expression (3.14) is different than (3.12) that we propose. In Table 3 we compare these bounds for $Q_2 = 0.7\chi_1^2\left(6\right) + 0.3\chi_1^2\left(2\right)$ with $\beta = \left(\alpha_1+\alpha_2\right)/2$ and $N = 20$. Again, we observe the improvement produced by our bound.

Table 3: Bounds for the truncation error of the distribution of Q_2

	$y=1$	$y=6$	$y=10$
(3.14)	4561.582748	2246.625094	506.7460696
(3.12) ($\mu_0 = p/4$)	$0.2211225252 \cdot 10^{-5}$	0.001969049548	0.1791774378

4 Comments and conclusions

We propose Laguerre series expansions for the density and distribution functions of non-central χ^2 variables and of positive linear combinations of non-central chi-square variables.

Our expansions depend on some parameters that can be chosen arbitrarily. Parameter p has been chosen in such a way that the derivatives of an auxiliar function (g) are easily calculated. The other parameter, μ_0, is chosen to obtain uniform convergence in the expansion, see Subsections 2.2 and 3.1.

The terms of our expansions are easily calculated using recurrent formulas, so that with no much computational effort we can obtain many terms for these expansions. Also we provide precise bounds for the truncation errors.

An IMSL subroutine is available for calculating the probability integral of a chi-square distribution with both integer as well as fractional degrees of freedom. It is, therefore, easy to evaluate the Patnaik and Pearson approximations. The latter gives remarkably accurate approximations except perhaps for small values of y (Table 4). However, small values of y are not of much interest since non-central chi-square distribution arises in the context of determining the power of a chi-square test. The test uses the tail on the right hand side of a chi-square distribution. Consequently, the value of y is greater than 1 in which case the Pearson approximation is remarkably accurate.

In Table 4 we compare the probabilities of a noncentral chi-squared variable, $\chi_n^2(\delta)$, obtained from the first j terms of our series and compare them with the values based on two and three moments approximations given in Patnaik (1949) and Pearson (1959), respectively. We note that our approximations seem to be better than other approximations to the non-central χ^2 distribution.

Table 4: Approximations to the distribution function of $\chi_n^2(\delta)$

n	δ	y	j	(2.5) $\mu_0 = p/4$	Patnaik's approx.	Pearson's approx.	Exact
2	1	0.17	3	0.050028	0.061760	0.069248	0.05
2	4	0.65	3	0.050440	0.02777	0.0581	0.05
2	4	14.72	8	0.949881	0.948975	0.950862	0.95
4	4	1.77	3	0.050274	0.040042	0.053059	0.05
4	10	10	5	0.314904	0.3178	0.3118	0.3148
4	16	7.88	5	0.05135	0.039995	0.05027	0.05
7	4	3.66	5	0.049848	0.04542	0.050788	0.05
7	16	10.257	5	0.0509	0.0430	0.0503	0.05

An alternative method is the one given by Kotz et al. (1967). They provide a Laguerre expansion and as Mathai and Provost (1992) state "it is computationally the most convenient and effective through the range of interesting value y", for this reason we mainly compare our results on truncation errors with those given by Kotz et al. (1967), see Tables 2 and 3.

5 Appendix

In the next we describe a method to invert Laplace transforms that is based on properties of unbiased estimation in the Gamma distribution.

Consider a random variable X distributed as a gamma with convolution parameter, $p>0$, known, and shape parameter, $\lambda>0$, unknown. As it is known this distribution belongs to the natural exponential family (NEF) with quadratic variance function. We parametrize the family in terms of the mean, $\mu=p/\lambda$, as in Morris (1982).

In this situation, we will say that a function $h(\mu)$ is MVU-estimable if there exists a function T such that, $E_\mu(T^2(X))<\infty, \forall\mu>0$, and $E_\mu(T(X))=h(\mu), \forall\mu>0$. So, $T(X)$ is the minimum variance unbiased estimator (MVUE) of $h(\mu)$.

From the results in Morris (1983), it can be showed that T admits the following expansion:

$$T(y)=\sum_{j=0}^{\infty}\frac{(-\mu)^j g^{(j)}(\mu)}{(p)_j}\mathrm{L}_j^{(p-1)}\left(\frac{py}{\mu}\right),\ \forall\mu>0,\ (a.e.) \tag{5.1}$$

with $g(\mu)=h(p/\mu)$, $g^{(j)}(\mu)=\dfrac{d^j}{d\mu^j}g(\mu)$ and $\mathrm{L}_j^{(\alpha)}(x)=\sum_{m=0}^{j}\binom{j+\alpha}{j-m}\frac{(-x)^m}{m!}$, $\alpha>0$ is the j-th generalized Laguerre polynomial.

On the other hand, from the unbiasedness condition $E_\mu(T(X))=h(\mu)$, for all $\mu>0$, we obtain an alternative expression for the unbiased estimator based on the inverse Laplace transform (denoted by $\mathcal{L}^{-1}$):

$$T(y)=\frac{\Gamma(p)}{y^{p-1}}\mathcal{L}^{-1}\left(\left(\frac{p}{\mu}\right)^{-p}h\left(\frac{p}{\mu}\right)\right)(y),\quad y>0. \tag{5.2}$$

And from the uniqueness (a.s) of the MVU estimators, equating (5.1) and (5.2), we obtain the following expression for the inverse Laplace transform of a function $G(\lambda)$, such that for certain $p>0$, $h(\lambda)=\lambda^pG(\lambda)$ is MVU-estimable function:

$$\mathcal{L}^{-1}\left(G\left(\lambda\right)\right)(y)=\frac{y^{p-1}}{\Gamma\left(p\right)}\sum_{j=0}^{\infty}\frac{\left(-\mu_0\right)^j g^{(j)}\left(\mu_0\right)}{\left(p\right)_j}\mathrm{L}_j^{(p-1)}\left(\frac{py}{\mu_0}\right),\ \text{(a.e.)} \tag{5.3}$$

for any $\mu_0>0$, with $g(\mu)=h\left(p/\mu\right)$.

Note that the choice of μ_0 is irrelevant, so adequate choices of this parameter may yield formulas computationally efficient.

The generalized Laguerre polynomials can be obtained recurrently by the relations:

$$
\begin{aligned}
j\mathrm{L}_j^{(\alpha)}(x) &= (2j+\alpha-1-x)\mathrm{L}_{j-1}^{(\alpha)}(x)-(j+\alpha-1)\mathrm{L}_{j-2}^{(\alpha)}(x), \quad j\geq 1,\\
\mathrm{L}_{-1}^{(\alpha)}(x) &= 0, \qquad \mathrm{L}_0^{(\alpha)}(x)=1.
\end{aligned}
$$

Proof of the Lemmas 2.2, 3.1 and 3.2.

The proofs of these lemmas are similar. So we only give the proof of Lemma 2.2.

Proof. By definition $c_k = (-\mu_0)^k\, g^{(k)}(\mu_0)/k!$, and $g(\cdot)$ given in (2.6) is an analytic function, so for each $\mu_0>0$ we have:

$$
\begin{aligned}
g\left((1-\theta)\mu_0\right) &= \frac{2p}{p-\mu_0}\left(1+\theta\left(\frac{\mu_0}{p-\mu_0}\right)\right)^{-1} \\
&\quad \cdot \exp\left(-\delta\left(\frac{p-(1-\theta)\,\mu_0}{2p}\right)\right) \\
&= \sum_{k\geq 0} c_k\theta^k, |\theta|<\frac{1}{\xi},
\end{aligned}
\tag{5.4}
$$

with $\xi=\left|\dfrac{\mu_0}{p-\mu_0}\right|$ and $(p=n/2+1)$.

Applying Cauchy's inequality to (5.4), we bound the coefficients c_k:

$$
|c_k| \leq \rho^{-k}\max_{|\theta|=\rho}\left|g\left((1-\theta)\mu_0\right)\right|,\ \forall\rho,\ 0<\rho<\frac{1}{\xi}. \tag{5.5}
$$

Considering (5.4),

$$
\begin{aligned}
\max_{|\theta|=\rho}\left|g\left((1-\theta)\mu_0\right)\right| &\leq \frac{2p}{p-\mu_0}\left(1-\rho\xi\right)^{-1} \\
&\quad \cdot \exp\left(-\delta\left(\frac{p-(1+1/\xi)\,\mu_0}{2p}\right)\right).
\end{aligned}
\tag{5.6}
$$

From (5.5) and (5.6):

$$
|c_k| \leq \frac{2p}{|p-\mu_0|}\exp\left(-\delta\left(\frac{p-(1+1/\xi)\,\mu_0}{2p}\right)\right)\rho^{-k}\left(1-\rho\xi\right)^{-1}.
$$

We find a better bound if we consider:

$$\inf_{0<\rho<1/\xi} \rho^{-k} (1-\rho\xi)^{-1} = \left(\frac{k}{\xi(1+k)}\right)^{-k} \left(\frac{1}{1+k}\right)^{-1}, \tag{5.7}$$

so we have (2.9). □

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