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Objective Bayesian Methods for One-Sided Testing

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Abstract

The one-sided testing problem can be naturally formulated as the comparison between two nonnested models. In an objective Bayesian setting, that is, when subjective prior information is not available, no general method exists either for deriving proper prior distributions on parameters or for computing Bayes factor and model posterior probabilities. The encompassing approach solves this difficulty by converting the problem into a nested model comparison for which standard methods can be applied to derive proper priors.
We argue that the usual way of encompassing does not have a Bayesian justification, and propose a variant of this method that provides an objective Bayesian solution. The solution proposed here is further extended to the case where nuisance parameters are present and where the hypotheses to be tested are separated by an interval. Some illustrative examples are given for regular and non-regular sampling distributions.

Key Words: Bayes factor, fractional prior, intrinsic prior, nonnested models, one-sided testing.

AMS subject classification: 62F03, 62F15.

1 Introduction

Consider the sampling model $f(x|\theta)$, where $\theta \in \Theta$ is an unknown parameter that, for simplicity, we assume to be one-dimensional. The one-sided testing problem consists of testing the null hypothesis $H_1 : \theta \leq \theta_0$ *versus* the alternative $H_2 : \theta \geq \theta_0$, where θ_0 is a specified value.

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In a Bayesian setup the problem is formulated as a model selection problem in which we have to compare the model

$$M_1 : f(x|\theta_1),\ \pi_1(\theta_1) = k_1\ \pi(\theta_1) 1_{(-\infty,\theta_0)}(\theta_1),$$

to

$$M_2 : f(x|\theta_2), \pi_2(\theta_2) = k_2\ \pi(\theta_2) 1_{(\theta_0,\infty)}(\theta_2),$$

where $\pi(\theta)$ is the prior for parameter θ in $f(x|\theta)$, and k_1, k_2 are normalizing constants.

Suppose that the data $\mathbf{x} = (x_1, ..., x_n)$ are independently drawn from either model M_1 or M_2. Given a prior P on the set $\{M_1, M_2\}$, say $P(M_1|\xi) = \xi$, $0 < \xi < 1$, Bayes theorem provides the posterior probability

$$P(M_1|\mathbf{x},\xi) = \frac{m(\mathbf{x}|M_1)\ \xi}{m(\mathbf{x}|M_1)\ \xi + m(\mathbf{x}|M_2)(1-\xi)},$$

where $m(x|M_i) = \int_{\Theta_i} f(x|\theta_i)\pi_i(\theta_i)d\theta_i$ is the marginal density of x conditional on model M_i.

In choosing between the two models it is easily seen that the optimal decision under 0-1 loss function is to choose M_1 if the inequality $P(M_1|\mathbf{x},\xi)/P(M_2|\mathbf{x},\xi) \geq 1$ holds. This ratio can also be written as

$$\frac{P(M_1|\mathbf{x},\xi)}{P(M_2|\mathbf{x},\xi)} = B_{12}(\mathbf{x})\frac{\xi}{1-\xi},$$

where $B_{12}(\mathbf{x}) = m(\mathbf{x}|M_1)/m(\mathbf{x}|M_2)$ is the Bayes factor of model M_1 *versus* M_2, and contains all the information the data provides on the posterior odds. Other interesting loss functions for model choice can be found in San Martini and Spezzaferri (1984) and Bernardo and Smith (1994).

When subjective priors are not available, objective priors, also called automatic or default priors, are often used instead. That is, ξ is set to $1/2$ and $\pi(\theta)$ is usually taken as the Jeffreys or the reference prior (Jeffreys, 1961; Berger and Bernardo, 1992). This prior is typically improper and hence it follows that $\pi_i^N(\theta_i) = c_i h(\theta_i)$, where $h(\theta_i)$ is a nonintegrable function and, consequently, c_i is an arbitrary positive constant. In this situation, the posterior probability of M_1

$$P(M_1|\mathbf{x}) = \frac{1}{1 + B_{21}^N(\mathbf{x})}, \tag{1}$$

is ill-defined since $B^N_{21}(\mathbf{x}) = m^N_2(\mathbf{x})/m^N_1(\mathbf{x})$ depends on the arbitrary constant c_2/c_1.

To overcome this difficulty, solutions based on empirical measures such as intrinsic Bayes factor (Berger and Pericchi, 1996b) and fractional Bayes factor (O'Hagan, 1995), have been proposed in the literature. Although these measures correspond to actual Bayes factors for nested models (Berger and Pericchi, 1996b; Moreno, 1997; Moreno et al., 1998) the same cannot be said for the nonnested models of the one-sided testing problem.

This prompts the need for converting the nonnested one-sided testing problem into a nested one. In the nested formulation of the problem the encompassing approach suggested by Cox (1961) plays a central role. In fact, (Berger and Mortera, 1999, p. 545) argue, "*Arithmetic Intrinsic Bayes factors are often not suitable for nonnested situations, especially when testing one-sided hypotheses as here (see Dmochowski, 1996). An attractive alternative, given by Berger and Pericchi (1996a,b), is to embed the competing models in a large encompassing model, say H_0, so that all of the H_i are nested within H_0*"

However, this does not appear to be fully satisfactory. In Section 2 we argue that this form of encompassing yields a procedure that does not correspond to an actual Bayes factor for the original one-sided testing problem. To avoid this difficulty, an alternative form of encompassing is proposed in Section 3. It is shown that the resulting model selection procedure provides an objective Bayesian solution. It is also shown that this latter procedure can be generalized to the one-sided testing problem in the presence of nuisance parameters, and to the case of testing the null $H_1 : \theta \leq \theta^1_0$ versus $H_2 : \theta \geq \theta^2_0$, where θ^1_0, θ^2_0 are specified values such that $\theta^1_0 < \theta^2_0$. We give examples with regular and nonregular sampling distributions. Concluding remarks are made in the last section.

2 Encompassing approach in one-sided testing

A variety of techniques have been introduced to remove the dependence of the Bayes factor on the constant c_2/c_1. Here we will briefly mention those which allow not only the calculation of empirical Bayes factors, but also the construction of suitable priors for computing actual Bayes factors.

2.1 Intrinsic Bayes factor

Berger and Pericchi (1996b) proposed replacing $B_{21}^N(\mathbf{x})$ in (1) with $B_{21}^{AI}(\mathbf{x})$ which is justified as follows. The sample $\mathbf{x}$ is split into two parts $(x(\ell), x(n-\ell))$. The part $x(\ell)$, called, the *training sample*, is designed to convert the improper prior into a proper posterior. That is,

$$\pi_i^N(\theta_i|x(\ell)) = \frac{f(x(\ell)|\theta_i)\pi_i^N(\theta_i)}{m_i^N(x(\ell))}, \ i = 1, 2, \tag{2}$$

where $x(\ell)$ is such that $0 < m_i^N(x(\ell)) < \infty$. With the remainder of the data, $x(n-\ell)$, the Bayes factor is computed using (2) as the prior. This gives

$$B_{21}^P(\mathbf{x}) = \frac{\int f(x(n-\ell)|\theta_2)\pi_2^N(\theta_2|x(\ell))d\theta_2}{\int f(x(n-\ell)|\theta_1)\pi_1^N(\theta_1|x(\ell))d\theta_1} = B_{21}^N(\mathbf{x})\ B_{12}^N(x(\ell)), \tag{3}$$

which is called a partial Bayes factor (PBF) by O'Hagan (1995). Note that the PBF corrects $B_{21}^N(\mathbf{x})$ with the term $B_{12}^N(x(\ell))$, and that the arbitrary constants c_1 and c_2 cancel out in (2).

It should be noted that for a given sample $\mathbf{x}$ we can consider different training samples $x(\ell)$, and hence there exists a multiplicity of PBFs, one for each training sample. To avoid dependence on a particular training sample, Berger and Pericchi (1996b) first suggested considering all possible subsamples $x(\ell)$ for which there is no proper subsample satisfying the inequalities $0 < m_i^N(x(\ell)) < \infty$ for any c_i. They termed this subsample a *minimal training sample*. Second, they take the arithmetic mean of the PBFs for all minimal training samples. This produces the so-called arithmetic intrinsic Bayes factor (AIBF), defined as

$$B_{21}^{AI}(\mathbf{x}) = B_{21}^N(\mathbf{x})\frac{1}{L}\sum_{\ell=1}^{L} B_{12}^N(x(\ell)),$$

where L is the number of minimal training samples contained in the sample. Other ways of "averaging" PBFs are possible (see Berger and Pericchi, 1996b,c, 1998).

2.2 Fractional Bayes factor

An alternative approach to avoiding the arbitrariness of choosing the training sample for which the PBF is computed was developed by O'Hagan

(1995). He replaces the correction term $B^N_{21}(x(\ell))$ in (3) with

$$F_{21}(\mathbf{x}) = \frac{\int \{f(\mathbf{x}|\theta_2)\}^{\ell/n} \pi_2^N(\theta_2) d\theta_2}{\int \{f(\mathbf{x}|\theta_1)\}^{\ell/n} \pi_1^N(\theta_1) d\theta_1},$$

where $\ell < n$. In this way, he defines the fractional Bayes factor (FBF) as

$$B^F_{12}(\mathbf{x}) = B^N_{12}(\mathbf{x})\ F_{21}(\mathbf{x}).$$

Other fractions apart from ℓ/n can be considered. In fact, O'Hagan (1995) argues that a larger fraction would reduce sensitivity to the prior and he also proposes using the fractions $\log n/n$ or $\sqrt{n}/n$. However, compelling arguments exist in favor of the fraction ℓ/n (Berger and Mortera, 1995; Moreno, 1997).

The above "Bayes factors" have been extensively studied, see, for instance, O'Hagan (1995, 1997); Berger and Pericchi (1996b,c, 1997, 1998); Sansó et al. (1996); De Santis and Spezzaferri (1999).

2.3 Intrinsic and fractional priors

Note that the intrinsic and fractional Bayes factors are not *actual* Bayes factors. Further, stability of the AIBF is a matter of concern. Conceivably, for a given sample $\mathbf{x}$, the number of minimal training samples might be small and minor changes in the data could cause this number to vary substantially. Moreover, the equality $B^{AI}_{21}(\mathbf{x}) = 1/B^{AI}_{12}(\mathbf{x})$ is not necessarily satisfied, so that the coherent equality $P(M_1|\mathbf{x}) = 1 - P(M_2|\mathbf{x})$ does not hold.

To be coherent, it is important to know whether $B^{AI}_{21}(\mathbf{x})$ corresponds to an actual Bayes factor for sensible priors. If so, consistency of the $B^{AI}_{21}(\mathbf{x})$ is assured. With the so-called *intrinsic* priors, the above question has been answered asymptotically by Berger and Pericchi (1996b). There are priors $\pi_1^I(\theta_1)$ and $\pi_2^I(\theta_2)$ for which the corresponding Bayes factor

$$B_{21}(\mathbf{x}) = \frac{\int_{\Theta_2} f(\mathbf{x}|\theta_2) \pi_2^I(\theta_2) d\theta_2}{\int_{\Theta_1} f(\mathbf{x}|\theta_1) \pi_1^I(\theta_1) d\theta_1},$$

and $B^{AI}_{21}(\mathbf{x})$ are asymptotically equivalent under the two models M_1 and M_2. Note that if we use intrinsic priors for computing the Bayes factor, instead of the improper priors we started from, coherency is assured.

By equating the limit of $B_{21}^{AI}(\mathbf{x})$ and $B_{21}(\mathbf{x})$ as $n \to \infty$ under the two models, we have

$$B_{21}(\mathbf{x}) = B_{21}^{AI}(\mathbf{x})(1 + o_P(1)),$$

Berger and Pericchi (1996b) showed that intrinsic priors satisfy the functional equations

$$\begin{aligned} E^{M_1}_{x(\ell)|\theta_1} B^N_{12}(x(\ell)) &= \frac{\pi_2^I(\psi_2(\theta_1))}{\pi_2^N(\psi_2(\theta_1))} \frac{\pi_1^N(\theta_1)}{\pi_1^I(\theta_1)}, \\ E^{M_2}_{x(\ell)|\theta_2} B^N_{12}(x(\ell)) &= \frac{\pi_2^I(\theta_2)}{\pi_2^N(\theta_2)} \frac{\pi_1^N(\psi_1(\theta_2))}{\pi_1^I(\psi_1(\theta_2))}. \end{aligned} \tag{4}$$

The expectations in these equations are taken with respect to $f(x(\ell)|\theta_1)$ and $f(x(\ell)|\theta_2)$, respectively; $\psi_2(\theta_1)$ denotes the limit of the maximum likelihood estimator $\hat{\theta}_2(\mathbf{x})$ under model M_1 at point θ_1, and $\psi_1(\theta_2)$ the limit of $\hat{\theta}_1(\mathbf{x})$ under model M_2 at point θ_2 (see also Cox, 1961; Huber, 1967; Dmochowski, 1996).

For nested models, the equations in (4) collapse into a single equation. Although the solution (π_1^I, π_2^I) to this single equation is not unique, and the resulting class is not robust (Moreno, 1997), a sensible selection is the pair

$$\pi_1^I(\theta_1) = \pi_1^N(\theta_1),\ \pi_2^I(\theta_2) = \pi_2^N(\theta_2) E^{M_2}_{x(\ell)|\theta_2} B^N_{12}(x(\ell)). \tag{5}$$

Some reasons for choosing this pair are: (i) They are a unique limit of proper priors (Moreno et al., 1998), (ii) the associated Bayes factor

$$B_{21}(\mathbf{x}) = \frac{\int_{\Theta_2} f(\mathbf{x}|\theta_2)\pi_1^N(\theta_1) E^{M_2}_{x(\ell)|\theta_2} B^N_{12}(x(\ell)) d\theta_2}{\int_{\Theta_1} f(\mathbf{x}|\theta_1)\pi_1^N(\theta_1) d\theta_1},$$

does not depend on the arbitrary constants c_1, c_2, (iii) this Bayes factor has been proved to behave well for some important nested testing problems, see Berger and Pericchi (1996b, 1998); Casella and Moreno (2002a,b, 2005); Moreno et al. (1998, 1999, 2000, 2005); Moreno and Liseo (2003), among others.

A similar construction to the intrinsic priors, but using the fractional methodology, is given by Moreno (1997). Fractional priors are

$$\pi_1^F(\theta_1) = \pi_1^N(\theta_1),\ \pi_2^F(\theta_2) = \pi_2^N(\theta_2)\tilde{F}_{12}(\theta_2), \tag{6}$$

where

$$\tilde{F}_{12}(\theta_2) = \lim_{n\to\infty} \, [P_{\theta_2}] \, F_{12}(\mathbf{x}),$$

provided that $\int_{\Theta_2} \pi_2^F(\theta_2)d\theta_2 = 1$. The latter condition is not always satisfied, so that fractional priors do not necessarily exist.

In a nonnested situation, however, equations in (4) do not collapse into a single equation and their solution depends on the behaviour of functions $h_1(\theta_1) = \psi_1\psi_2(\theta_1)$ and $h_2(\theta_2) = \psi_2\psi_1(\theta_2)$. In general, the solution does not necessarily exist and if it does it is not necessarily unique (Dmochowski, 1996; Moreno, 1997; Cano et al., 2004).

This difficulty can be overcome by reducing the nonnested problem to a nested one for which the solution certainly does exist. To do this Berger and Pericchi (1996b) adopted the encompassing approach. However, this technique is not fully satisfactory when applied to the one-sided testing problem as demostrated in the following section.

2.4 The encompassing approach in one-sided testing

The encompassing approach proposes embedding M_1 and M_2 in the larger model $M_3 = M_1 \cup M_2 : \{f(x|\theta),\ \pi_3^N(\theta) = c_3 h(\theta)\}$. Thus, M_1 and M_2 are nested in M_3 so that intrinsic (or fractional) priors can be constructed to compute $B_{31}(\mathbf{x})$ and $B_{23}(\mathbf{x})$. Finally, $B_{21}(\mathbf{x})$ is defined as $B_{21}(\mathbf{x}) = B_{23}(\mathbf{x})B_{31}(\mathbf{x})$.

However, there is no basis for accepting the latter equality. This is due to the fact that when computing $B_{23}(\mathbf{x})$ the intrinsic prior for the parameter θ in M_3 comes from the comparison between M_2 and M_3. This prior is not necessarily the same as the one obtained when computing $B_{31}(\mathbf{x})$ which comes from the comparison between M_1 and M_3. Hence, in the product $B_{23}(\mathbf{x})B_{31}(\mathbf{x})$ the denominator of $B_{23}(\mathbf{x})$ does not cancel out with the numerator of $B_{31}(\mathbf{x})$ and consequently $B_{21}(\mathbf{x})$ does not correspond to an actual Bayes factor.

Example 2.1. *Consider the problem of testing $H_1 : 0 \le \theta \le 1$ versus $H_2 : 1 \le \theta$, where θ is the parameter of the exponential density. In other words, we want to compare the models*

$$M_1 : f(x|\theta_1) = \frac{1}{\theta_1}\exp\{-x/\theta_1\},\ \pi_1^N(\theta_1) = \frac{c_1}{\theta_1}1_{(0,1)}(\theta_1),$$

and

$$M_2 : f(x|\theta_2) = \frac{1}{\theta_2}\exp\{-x/\theta_2\},\ \pi_2^N(\theta_2) = \frac{c_2}{\theta_2}1_{(1,\infty)}(\theta_2),$$

where $\pi^N(\theta) = c/\theta$ is the Jefreys prior for θ. It is easily seen that no fractional or intrinsic priors exist for this problem. This example proves that formulating the one sided testing problem as a nonnested model comparison does not necessarily yield a solution.

The encompassing approach requires transforming the problem into a nested one although the above mentioned difficulty arises. Indeed, the encompassing model is

$$M_3 : f(x|\theta),\ \pi_3^N(\theta) = \frac{c_3}{\theta} 1_{(0,\infty)}(\theta).$$

The minimal training sample is a single replication of X. From (5) it follows that the intrinsic priors for comparing M_1 and M_3 are

$$\pi_1^I(\theta_1) = \frac{c_1}{\theta_1} 1_{(0,1)}(\theta_1),\ \pi_{13}^I(\theta) = \frac{c_1}{\theta(1+\theta)} 1_{(0,\infty)}(\theta).$$

Similarly, the intrinsic priors for comparing M_2 with M_3 are

$$\pi_2^I(\theta_2) = \frac{c_2}{\theta_2} 1_{(1,\infty)}(\theta_2),\ \pi_{23}^I(\theta) = \frac{c_2}{(1+\theta)} 1_{(0,\infty)}(\theta).$$

Since $\pi_{13}^I(\theta) \neq \pi_{23}^I(\theta)$, the product $B_{31}(\mathbf{x})B_{23}(\mathbf{x})$ is not a Bayes factor.

3 An alternative solution

Here we propose a solution for one sided testing that exploits the fact that the two hypotheses being compared can be joined by means of a point.

3.1 One-side testing

The solution we propose considers the auxiliary model defined by the singular point θ_0,

$$M_0 : f(x|\theta), \pi(\theta) = 1_{\{\theta_0\}}(\theta),$$

which is nested in M_3. Thus, according to (5) the intrinsic priors for comparing M_0 and M_3 are

$$\pi_0^I(\theta) = 1_{\{\theta_0\}}(\theta),\ \pi^I(\theta|\theta_0) = \pi_3^N(\theta) E_{x(\ell)|\theta}^{M_3} B_{03}^N(x(\ell)), \tag{7}$$

where $B_{03}^N(x(\ell)) = f(x(\ell)|\theta_0) / \int_{-\infty}^{\infty} f(x(\ell)|\theta)\pi_3^N(\theta)d\theta$.

Note that $\pi^I(\theta|\theta_0)$ is a proper prior. Therefore, we can consider its restriction to the set $\{\theta \leq \theta_0\}$ and $\{\theta \geq \theta_0\}$. So we have models with proper priors, say

$$M_1 : f(x|\theta_1),\ \pi_1^I(\theta_1|\theta_0) = \frac{1}{k_1}\pi^I(\theta_1|\theta_0)1_{(-\infty,\theta_0)}(\theta_1),$$

$$M_2 : f(x|\theta_2),\ \pi_2^I(\theta_2|\theta_0) = \frac{1}{k_2}\pi^I(\theta_2|\theta_0)1_{(\theta_0,\infty)}(\theta_2),$$

where $k_1 = \int_{-\infty}^{\theta_0} \pi^I(\theta_1|\theta_0)d\theta_1$, and $k_2 = \int_{\theta_0}^{\infty} \pi^I(\theta_2|\theta_0)d\theta_2$.

For a sample $\mathbf{x}$ we can now compute the Bayes factor

$$B_{12}(\mathbf{x}) = \frac{\int f(\mathbf{x}|\theta_1)\pi_1^I(\theta_1|\theta_0)d\theta_1}{\int f(\mathbf{x}|\theta_2)\pi_2^I(\theta_2|\theta_0)d\theta_2}.$$

A similar construction can be done with the fractional priors given in (6).

Example 3.1 (Example 2.1 (continued)). *The auxiliary model in this example is*

$$M_0 : f(x|\theta) = \frac{1}{\theta}\exp\{-\frac{x}{\theta}\},\ \pi_0(\theta) = 1_{\{1\}}(\theta).$$

Thus, using (7), the intrinsic priors for M_1 and M_2 are

$$\pi_1^I(\theta_1|1) = \frac{2}{(1+\theta_1)^2}1_{(0,1)}(\theta_1),\ \pi_2^I(\theta_2|1) = \frac{2}{(1+\theta_2)^2}1_{(1,\infty)}(\theta_2).$$

By construction, both intrinsic priors are probability densities.

Proceeding in a similar fashion using (6) gives the fractional priors

$$\pi_1^F(\theta_1|1) = \frac{1}{1.72}\exp\{1-\theta_1\}1_{(0,1)}(\theta_1),\ \pi_2^F(\theta_2|1) = \exp\{1-\theta_2\}1_{(1,\infty)}(\theta_2).$$

While the intrinsic priors do not exhibit discontinuity, the fractional priors present a discontinuity at point $\theta_1 = \theta_2 = 1$.

For the sample $\mathbf{x}$, the Bayes factor $B_{21}(\mathbf{x})$ for intrinsic priors is

$$B_{21}(\mathbf{x}) = \frac{\int_1^{\infty} \theta_2^{-n}(1+\theta_2)^{-2}\exp\{-n\ \bar{x}/\theta_2\}\ d\theta_2}{\int_0^1 \theta_1^{-n}(1+\theta_1)^{-2}\exp\{-n\ \bar{x}/\theta_1\}\ d\theta_1},$$

and for fractional priors is

$$\tilde{B}_{21}(\mathbf{x}) = 1.72 \, \frac{\int_1^\infty \theta_2^{-n} \exp\{-n\,\bar{x}/\theta_2 - \theta_2\}\, d\theta_2}{\int_0^1 \theta_1^{-n} \exp\{-n\,\bar{x}/\theta_1 - \theta_1\}\, d\theta_1}.$$

The above Bayes factors are computed and reported in the second column of Table 1 for different values of the statistic $(n,\,\bar{x})$. *The posterior probability of* M_1 *when using intrinsic priors is denoted as* $P = P(M_1|\mathbf{x})$ *and when using fractional priors is denoted as* $\tilde{P} = \tilde{P}(M_1|\mathbf{x})$.

Table 1: Bayes factors and posterior probabilities

$(n,\,\bar{x})$	$B_{21}, \tilde{B}_{21}$	$P, \tilde{P}$
$(5, 0.1)$	$\approx 0, \approx 0$	$1.00, 1.00$
$(5, 0.6)$	$0.24, 0.41$	$0.81, 0.71$
$(5, 1.1)$	$1.71, 2.70$	$0.37, 0.27$
$(5, 1.83)$	$16.3, 21.9$	$0.06, 0.04$

Table 1 shows that both the intrinsic and the fractional analyses convey essentially the same message. For small values of $\bar{x}$, *that intuitively favor* M_1, *the posterior probability of* M_1 *is high, more so as the sample size increases. As the sample mean increases the posterior probability decreases. For a large sample size and* $\bar{x} = 1$, *it can be seen that the posterior probability of* M_1 *is* 0.5, *which is sensible. For* $(n,\,\bar{x}) = (5, 1.83)$ *the p-value is equal to* 0.05 *and the posterior probability of* M_1 *for the intrinsic prior is* 0.06 *and* 0.04 *for the fractional prior, so that the three measures of evidence agree.*

Our procedure is applicable to one-sided testing problems associated with nonregular sampling distributions as the following example shows.

Example 3.2. *Suppose that* X *is a random variable with uniform density* $f(x|\theta) = \theta^{-1} 1_{(0,\theta)}(x)$, $\theta \in \mathbb{R}^+$. *The reference prior for this model is* $\pi^N(\theta) = c/\theta$. *Suppose that we are interested in testing* $H_1 : \theta \leq \theta_0$ *versus* $H_2 : \theta \geq \theta_0$. *The two default models being compared are*

$$M_1 : f(x|\theta_1) = \theta_1^{-1} 1_{(0,\theta_1)}(x), \;\; \pi_1^N(\theta_1) = \frac{c_1}{\theta_1} 1_{(0,\theta_0)}(\theta_1),$$

and

$$M_2 : f(x|\theta_2) = \theta_2^{-1} 1_{(0,\theta_2)}(x),\ \pi_2^N(\theta_2) = \frac{c_2}{\theta_2} 1_{(\theta_0,\infty)}(\theta_2).$$

The auxiliary model is

$$M_0 : f(x|\theta) = \frac{1}{\theta} 1_{(0,\theta)}(x),\ \pi_0(\theta) = 1_{\{\theta_0\}}(\theta).$$

Using (7), the intrinsic priors for M_1 and M_2 are

$$\pi_1^I(\theta_1|\theta_0) = \frac{1}{\theta_0} 1_{(0,\theta_0)}(\theta_1),\ \pi_2^I(\theta_2|\theta_0) = \frac{\theta_0}{\theta_2^2} 1_{(\theta_0,\infty)}(\theta_2).$$

Fractional priors do not exist in this problem.

For a sample $\mathbf{x}$*, consider the sufficient statistic* $t_n = \max\{x_1, ..., x_n\}$*. The Bayes factor* $B_{21}(n, t_n) = \infty$ *if* $t_n > \theta_0$*. For* $t_n \leq \theta_0$ *the Bayes factor is*

$$B_{21}(n, t_n) = \begin{cases} \frac{1}{2}\left(\log \frac{\theta_0}{t_n}\right)^{-1} & \text{if } n = 1, \\ \frac{n-1}{n+1}\left(\left(\frac{\theta_0}{t_n}\right)^{n-1} - 1\right)^{-1} & \text{if } n \geq 2. \end{cases}$$

For $\theta_0 = 3$*,* $n = 5$ *and different values of* t_5 *the corresponding Bayes factor and the posterior probability of* M_1 *are given in the second and third row of Table 2, respectively.*

Table 2: Values of t_5, $B_{21}(5, t_5)$ and $P(M_1|t_5)$

t_5	1.00	1.70	2.97	> 3
$B_{21}(5, t_5)$	8×10^{-3}	7×10^{-2}	16.2	∞
$P(M_1\|t_5)$	0.99	0.93	0.06	0

Table 2 shows that for increasing values of t_n *the posterior probability of* M_1 *decreases, as expected. Note that* $P(M_1|t_5) \leq 1/2$ *for the region* $t_5 \geq 2.64$*, so these points favor model* M_2*. In this example the p-value for* $t_5 = 2.97$ *is* 0.05 *and the posterior probability of* M_1 *for that point is* 0.06*. The two measures of evidence agree.*

3.2 One-sided testing in the presence of a nuisance parameter

The procedure described above generalizes to the case of one-sided testing problems in presence of a nuisance parameter, assuming that the parameter of interest and the nuisance parameter are *a priori* independent. Consider the sampling model $f(x|\theta,\psi)$ and suppose we are interested in testing H_1 : $\theta \leq \theta_0$ *versus* $H_2 : \theta \geq \theta_0$, where ψ is a nuisance parameter.

We consider the auxiliary model $M_0 : f(x|\theta_0,\psi_0)$, where ψ_0 is an arbitrary but fixed coordinate, and compute the intrinsic prior for the parameter (θ,ψ) that results from comparing M_0 and $M_3 = M_1 \cup M_2$. This prior is given by

$$\pi^I(\theta,\psi|\theta_0,\psi_0) = \pi^N(\theta,\psi) E^{M_3}_{x(\ell)|\theta,\psi} \frac{f(x(\ell)|\theta_0,\psi_0)}{\int\int f(x(\ell)|\theta,\psi)\pi^N(\theta,\psi)d\theta d\psi},$$

where $\pi^N(\theta,\psi) = \pi^N(\theta)\pi^N(\psi)$ is the reference prior for the sampling model $f(x|\theta,\psi)$. We denote the restriction of $\pi^I(\theta,\psi|\theta_0,\psi_0)$ to the set $\{\theta \leq \theta_0\}$ and $\{\theta \geq \theta_0\}$ as $\pi_i^I(\theta,\psi|\theta_0,\psi_0)$, $i = 1,2$ respectively. Hence, for the sample $\mathbf{x}$ the Bayes factor is

$$B_{21}(\mathbf{x}) = \frac{\int_{(\psi_0)}\int_{(\psi)}\int_{\theta\geq\theta_0} f(\mathbf{x}|\theta,\psi)\pi_1^I(\theta,\psi|\theta_0,\psi_0)\pi^N(\psi_0)d\theta d\psi d\psi_0}{\int_{(\psi_0)}\int_{(\psi)}\int_{\theta\leq\theta_0} f(\mathbf{x}|\theta,\psi)\pi_2^I(\theta,\psi|\theta_0,\psi_0)\pi^N(\psi_0)d\theta d\psi d\psi_0},$$

where the arbitrary coordinate ψ_0 has been integrated out with respect to the reference prior $\pi^N(\psi_0)$. Note that this Bayes factor is well-defined as long as the arbitrary constant involved in $\pi^N(\psi_0)$ cancels out in the ratio.

Example 3.3. *Let X be a random variable with normal density $f(x|\theta)$ $= N(x|\mu,\sigma^2\}$, where μ and σ are unknown. Suppose that we are interested in testing $H_1 : \mu \leq 0$ versus $H_2 : \mu \geq 0$, so that σ is a nuisance parameter. The two default models being compared are*

$$M_1 : f(x|\theta_1) = N\left(x|\mu_1,\sigma_1^2\right), \pi_1^N(\theta_1) = \frac{c_1}{\sigma_1} 1_{\mathbb{R}^- \times \mathbb{R}^+}(\mu_1,\sigma_1),$$

and

$$M_2 : f(x|\theta_2) = N\left(x|\mu_2,\sigma_2^2\right), \pi_2^N(\theta_2) = \frac{c_2}{\sigma_2} 1_{\mathbb{R}^+ \times \mathbb{R}^+}(\mu_2,\sigma_2).$$

We first consider the auxiliary model $M_0 : N\left(x|0,\sigma_0^2\right)$, where σ_0 is an arbitrary but fixed point and derive intrinsic priors for comparing

$$M_0 : N\left(x|0,\sigma_0^2\right) \quad versus \quad M_3 = M_1 \cup M_2.$$

It can be shown that the intrinsic prior for parameter (μ, σ) *in* M_3 *is*

$$\pi_3^I(\mu, \sigma|0, \sigma_0) = N\left(\mu|0, \frac{\sigma_0^2 + \sigma^2}{2}\right) HC^+(\sigma|0, \sigma_0),$$

where HC^+ *denotes a half Cauchy on* $\mathbb{R}^+$.

The restriction of $\pi_3^I(\mu, \sigma|0, \sigma_0)$ *to* $\mu \le 0$ *and* $\mu \ge 0$*, the subspaces of model* M_1 *and model* M_2*, gives*

$$\pi_1^I(\mu_1, \sigma_1|0, \sigma_0) = 2N\left(\mu_1|0, \frac{\sigma_0^2 + \sigma_1^2}{2}\right) 1_{\mathbb{R}^-}(\mu_1) HC^+(\sigma_1|0, \sigma_0),$$

$$\pi_2^I(\mu_2, \sigma_2|0, \sigma_0)) = 2N\left(\mu_2|0, \frac{\sigma_0^2 + \sigma_2^2}{2}\right) 1_{\mathbb{R}^+}(\mu_2) HC^+(\sigma_2|0, \sigma_0).$$

Some algebra shows that for a sample $\mathbf{x}$*, the Bayes factor for these priors can be written*

$$B_{21}(\mathbf{x}) = \frac{\int_0^\infty d\mu \int_0^{\pi/2} g(\mathbf{x}, \varphi, \mu) d\varphi}{\int_{-\infty}^0 d\mu \int_0^{\pi/2} g(\mathbf{x}, \varphi, \mu) d\varphi},$$

where

$$g(\mathbf{x}, \varphi, \mu) = \sin^{-n}\varphi \left(\mu^2 + \frac{ns^2 + n(\bar{x} - \mu)^2}{2\sin^2\varphi}\right)^{-(n+1)/2}.$$

Note that the Bayes factor depends on the sample through the sufficient statistic $s^2 = \sum(x_i - \bar{x})^2/n, \bar{x} = \sum x_i/n$*, and* n.

Similarly, it can be shown that the fractional priors are

$$\pi_1^F(\mu_1, \sigma_1|0, \sigma_0) = 2N\left(\mu_1|0, \frac{\sigma_0^2}{2}\right) 1_{\mathbb{R}^-}(\mu_1) HN^+\left(\sigma_1|0, \frac{\sigma_0^2}{2}\right),$$

$$\pi_2^F(\mu_2, \sigma_2|0, \sigma_0)) = 2N\left(\mu_2|0, \frac{\sigma_0^2}{2}\right) 1_{\mathbb{R}^+}(\mu_2) HN^+\left(\sigma_2|0, \frac{\sigma_0^2}{2}\right),$$

where HN^+ *denotes a half normal distribution on* $\mathbb{R}^+$. *The corresponding Bayes factor is*

$$\tilde{B}_{21}(\mathbf{x}) = \frac{\int_0^\infty d\mu \int_0^{\pi/2} h(\mathbf{x}, \varphi, \mu) d\varphi}{\int_{-\infty}^0 d\mu \int_0^{\pi/2} h(\mathbf{x}, \varphi, \mu) d\varphi},$$

where

$$h(\mathbf{x},\varphi,\mu) = \frac{\exp\{-\tan^2\varphi\}}{\sin^n\varphi}\left(\frac{\mu^2}{\cos^2\varphi} + \frac{ns^2 + n(\bar{x}-\mu)^2}{2\sin^2\varphi}\right)^{-(n+2)/2}.$$

$B_{21}(n,\bar{x},s^2)$, $\tilde{B}_{21}(n,\bar{x},s^2)$ and the posterior probabilities of M_1 are reported for some values of $(n,\bar{x},s^2)$ in Table 3.

Table 3: Bayes factors and posterior probabilities

$(n,\bar{x},s^2)$	$B_{21},\tilde{B}_{21}$	$P,\tilde{P}$
$(5, 0.0, 1)$	$1.00, 1.00$	$0.50, 0.50$
$(5, 0.2, 1)$	$1.83, 1.86$	$0.35, 0.35$
$(5, 0.5, 1)$	$4.36, 4.60$	$0.19, 0.18$
$(5, 1.0, 1)$	$15.07, 17.60$	$0.06, 0.05$

In this example Bayes factors for intrinsic and fractional priors behave almost identically. For a given sample size and sample variance, positive values of the sample mean favor model M_2, more so as the sample size and the sample mean increase. For $(n,\bar{x},s^2) = (5, 1.07, 1)$ the p-value is equal to 0.05. For this symmetric one-sided testing problem the latter measure of evidence is in agreement with the posterior probability of M_1.

3.3 Separated one-sided testing

The procedure in Section 3.1 applies to the case of testing the null $H_1 : \theta \le \theta_0^1$ versus $H_2 : \theta \ge \theta_0^2$, where $\theta_0^1 < \theta_0^2$ are specified values such that $\int_{\Theta_0} \pi^N(\theta)d\theta < \infty$ with $\Theta_0 = (\theta_0^1, \theta_0^2)$. The auxiliary model in this case is

$$M_0 : f(x|\theta_0), \pi_0(\theta_0) = \frac{\pi^N(\theta_0)}{\int_{\theta_0^1}^{\theta_0^2} \pi^N(\theta)d\theta} 1_{(\theta_0^1,\theta_0^2)}(\theta_0).$$

The intrinsic priors for comparing M_0 *versus* $M : f(x|\theta), \pi^N(\theta)$ is, according to (5), given by

$$\pi_0^I(\theta_0) = \pi_0(\theta_0),\ \pi^I(\theta|\Theta_0) = \pi^N(\theta) E^M_{x(\ell)|\theta} B^N(x(\ell)),$$

where $B^N(x(\ell)) = \int_{\Theta_0} f(x|\theta_0)\pi_0(\theta_0)d\theta_0 / \int_{-\infty}^{\infty} f(x(\ell)|\theta)\pi_3^N(\theta)d\theta$.

Note that $\pi^I(\theta|\Theta_0)$ is a proper prior. Therefore, we can consider its restriction to the set $\{\theta \leq \theta_0^1\}$ and $\{\theta \geq \theta_0^2\}$. So we have models with proper priors, say

$$M_1 : f(x|\theta_1),\ \pi_1^I(\theta_1|\Theta_0) = \frac{1}{k_1}\pi^I(\theta_1|\Theta_0)1_{(-\infty,\theta_0^1)}(\theta_1),$$

$$M_2 : f(x|\theta_2),\ \pi_2^I(\theta_2|\Theta_0) = \frac{1}{k_2}\pi^I(\theta_2|\Theta_0)1_{(\theta_0^2,\infty)}(\theta_2),$$

where $k_1 = \int_{-\infty}^{\theta_0^1} \pi^I(\theta_1|\Theta_0)d\theta_1$, and $k_2 = \int_{\theta_0^2}^{\infty} \pi^I(\theta_2|\Theta_0)d\theta_2$. Now, the Bayes factor and model posterior probabilities for comparing M_1 versus M_2 can be computed.

4 Concluding remarks

A fully default Bayesian analysis of the one-sided testing problem involves formalizing the prior distribution on model parameters and the prior on the models themselves. We have shown that in an objective Bayesian setting these priors can be formalized. The method employs a form of encompassing that enables us to perform a Bayesian analysis. Furthermore, the method is quite simple to apply, yields a reasonable answer for regular and nonregular sampling models, and the presence of nuisance parameters does not produce any particular theoretical or computational difficulties.

For most of the examples considered, and for small sample sizes a sample point that rejects the null hypothesis with a p-value of 0.05 gives a posterior probability of the null close to 0.05. Therefore it suggests that the two measures of evidence agree. This conclusion seems to be in disagreement with the numerical results obtained by Berger and Mortera (1999), but we remark that the priors they use differ from our.

Whether or not our default analysis and a robust analysis matches, in the spirit of the paper by De Santis (2002), is a point that deserves exploration but goes beyond the scope of this paper. Also, we point out that extensions to multiple hypotheses testing are straightforward and hence they have not been detailed.

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References

BERGER, J. O. and BERNARDO, J. M. (1992). On the development of the reference prior method. In J. M. Bernardo, J. O. Berger, A. P. Dawid, and A. F. M. Smith, eds., *Bayesian Statistics*, vol. 4, pp. 35–60. Oxford University Press, Oxford.

BERGER, J. O. and MORTERA, J. (1995). Discussion of O'Hagan. *Journal of the Royal Statistical Society. Series B*, 57:99–138.

BERGER, J. O. and MORTERA, J. (1999). Default Bayes factors for nonnested hypothesis testing. *Journal of the American Statistical Association*, 94:542–554.

BERGER, J. O. and PERICCHI, L. R. (1996a). The intrinsic Bayes factor for linear models. In J. M. Bernardo, J. O. Berger, A. P. Dawid, and A. F. M. Smith, eds., *Bayesian Statistic*, vol. 5, pp. 23–42. Oxford University Press, Oxford.

BERGER, J. O. and PERICCHI, L. R. (1996b). The intrinsic Bayes factor for model selection and prediction. *Journal of the American Statistical Association*, 91:109–122.

BERGER, J. O. and PERICCHI, L. R. (1996c). On the justification of default and intrinsic Bayes factor. In J. C. Lee, W. Johnson, and A. Zellner, eds., *Modeling and Prediction*, pp. 276–293. Springer-Verlag, New York.

BERGER, J. O. and PERICCHI, L. R. (1997). Accurate and stable Bayesian model selection: the median intrinsic Bayes factor. *Sankyà, B*, 60:1–18.

BERGER, J. O. and PERICCHI, L. R. (1998). On criticism and comparison of default Bayes factors for model selection and hypothesis testing. In W. Racugno, ed., *Proceedings of the Workshop on Model Selection*, pp. 1–50. Pitagora, Bologna.

BERNARDO, J. M. and SMITH, A. F. M. (1994). *Bayesian Theory*. Wiley, New York.

CANO, J. A., KESSLER, M., and MORENO, E. (2004). On intrinsic priors for nonnested models. *Test*, 13(2):445–463.

CASELLA, G. and MORENO, E. (2002a). Objective Bayesian of contingency tables. Technical report, Universidad de Granada. Submitted.

CASELLA, G. and MORENO, E. (2002b). Objective Bayesian variable selection. Technical report, Universidad de Granada. Submitted.

CASELLA, G. and MORENO, E. (2005). Intrinsic meta analysis of contingency tables. *Statistics in Medicine*, 24:583–604.

COX, D. R. (1961). Test of separate families of hypothesis. In J. Neyman, ed., *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, pp. 105–123. University of California Press, Berkeley.

DE SANTIS, F. (2002). Interactive use of default and robust Bayes testing methods in the presence of vague prior information. *Journal of the Royal Statistial Society. Series D*, 51:451–465.

DE SANTIS, F. and SPEZZAFERRI, F. (1999). Methods for default and robust Bayesian model comparison: the fractional Bayes factor approach. *International Statistical Review*, 67:267–286.

DMOCHOWSKI, J. (1996). Intrinsic Bayes factor via Kullback-Leibler geometry. In J. M. Bernardo, J. O. Berger, A. P. Dawid, and A. F. M. Smith, eds., *Bayesian Statistics*, vol. 5. Oxford University Press, London.

HUBER, P. (1967). The behaviour of maximum likelihood estimators under nonstandard conditions. In L. LeCam and N. J., eds., *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1, pp. 221–233.

JEFFREYS, H. (1961). *Theory of Probability*. Clarendon Press, Oxford.

MORENO, E. (1997). Bayes factor for intrinsic and fractional priors in nested models: Bayesian robustness. In D. Yadolah, ed., *L1-Statistical Procedures and Related Topics*, vol. 31 of *Lecture Notes-Monograph Series*, pp. 257–270. Institute of Mathematical Statistics, Hayward, CA.

Moreno, E., Bertolino, F., and Racugno, W. (1998). An intrinsic limiting procedure for model selection and hypotheses testing. *Journal of the American Statistical Association*, 93:1451–1460.

Moreno, E., Bertolino, F., and Racugno, W. (1999). Default Bayesian analysis of the Behrens-Fisher problem. *Journal of Statistical Planning and Inference*, 81:323–333.

Moreno, E., Bertolino, F., and Racugno, W. (2000). Bayesian model selection approach to analysis of variance under heteroscedasticity. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 46:1–15.

Moreno, E. and Liseo, B. (2003). A default Bayesian test for the number of components of a mixture. *Journal of Statistical Planning and Inference*, 111:129–142.

Moreno, E., Torres, F., and Casella, G. (2005). Testing the equality of regression coefficients in heteroscedastic normal regression models. *Journal of Statistical Planning and Inference*, 131:117–134.

O'Hagan, A. (1995). Fractional Bayes factor for model comparison (with discussion). *Journal of the Royal Statistical Society. Series B*, 57:99–138.

O'Hagan, A. (1997). Properties of intrinsic and fractional Bayes factors. *Test*, 6:101–118.

San Martini, A. and Spezzaferri, F. (1984). A predictive model selection criterion. *Journal of the Royal Statistical Society. Series B*, 46:296–303.

Sansó, B., Pericchi, L. R., and Moreno, E. (1996). On the robustness of the intrinsic Bayes factor for nested models (with discussion). In J. O. Berger, B. Betró, F. Ruggeri, E. Moreno, L. Pericchi, and L. Wasserman, eds., *Bayesian Robustness*, vol. 29 of *Lecture Notes-Monograph Series*, pp. 157–176. Institute of Mathematical Satistics, Hayward, CA.