BULLETIN Of the International Association of ENGINEERING GEOLOGY de l'Association Internationale de GÉOLOGIE DE L'INGÉNIEUR N° 39 PARIS 1989

HYDRAULIC ANISOTROPY OF HOMOGENEOUS SOILS AND ROCKS : INFLUENCE OF THE DENSIFICATION PROCESS

ANISOTROPIE HYDRAULIQUE DES SOLS ET ROCHES HOMOGÈNES : INFLUENCE DE LA DENSIFICATION

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Abstract

Many values of the ratio k_{max}/k_{min} are available for clays and rocks which can be cut for tests in different directions. In comparison, few reliable results are available for non-cohesive materials. The hydraulic anisotropy ratios of homogeneous clays, rocks and granular soils appear to be very similar. In particular, k_{max}/k_{min} seems to be lower than 4, which confirms that this ratio has an upper limit related to the shape of particles, their arrangement, or the directional tortuosity within the pore space. In the bedding plane of sedimentary rocks, the ratio $k_{h,max}/k_{h,min}$ is usually lower than 1.5, thus these rocks are nearly isotropic in their bedding plane. In granular soils, the ratio k_h/k_v, contrary to common opinion, is not always higher than 1. Experimental values for sands and gravels are in the 0.75 to 4.1 range. The influence of densification on hydraulic anisotropy is found to be similar for a sand and a clay, and probably for any soil having settled in still water and influenced subsequently only by gravity. The hydraulic anisotropy of sandstone is found to be in continuity with that of sand, and it increases with densification.

Key words : soil, rock, permeability, anisotropy, densification.

Résumé

De nombreuses valeurs du rapport k_{max}/k_{min} sont disponibles pour les argiles et les roches, taillables pour des essais dans différentes directions. Comparativement, il y a peu de résultats fiables pour les matériaux sans cohésion. Les coefficients d'anisotropie hydraulique des argiles, roches et sols grenus homogènes paraissent très semblables. En particulier, k_{max}/k_{min} semble inférieur à 4, ce qui confirme que ce rapport a une limite supérieure dictée par la forme des particules, leur arrangement, ou la tortuosité directionnelle dans l'espace des vides. Dans le plan de stratification des roches sédimentaires, le rapport d'anisotropie $k_{h,max}/k_{h,min}$ est généralement inférieur à 1.5, donc ces roches sont à peu près isotropes dans ce plan. Dans les sols grenus, le rapport k_h/k_v, contrairement à une opinion courante, ne dépasse pas toujours 1. Les valeurs expérimentales pour des sables et graviers vont de 0.75 à 4.1. On trouve que la densification influence le rapport d'anisotropie de la même façon pour un sable, une argile, et probablement pour tout sol déposé en eau immobile, et densifié ensuite par la gravité seulement. L'anisotropie hydraulique du grès s'avère être en continuité avec celle du sable, et elle augmente avec la densification.

Mots clés : sol, roche, perméabilité, anisotropie, densification.

Introduction

Physical and mechanical properties of soils and sedimentary rocks are generally anisotropic. Hydraulic conductivity is no exception. This property is usually described by a second-rank, symmetric, positive definite tensor (Ferrandon 1948; Irmay 1951; Liakopoulos 1965a, 1965b).

The hydraulic anisotropy of soils and sedimentary rocks has a great effect on fluid flow and contaminant transport. Consequently, a knowledge of the in-situ values of this tensor, k, is important for many subsurface fluid-associated problems such as, for example, underflow beneath dams and dykes, internal erosion in soil masses or settlement rates of consolidating clays. Knowledge of this tensor is also useful in designing cut-off structures or pressure-relief systems, and it is very important in optimizing the design of oil or water well fields. The design of drainage systems either for roads, airfields (Cedergren 1974) or agricultural needs also requires a knowledge of hydraulic anisotropy.

This paper deals exclusively with the hydraulic anisotropy of saturated soils and sedimentary rocks. For unsaturated conditions, the $k(S_r)$ functions (S_r = degree of saturation) are known to be hysteresic (Nielsen et al. 1986), which makes hydraulic anisotropy more complicated (Mualem 1984). In particular, the anisotropy ratio of unsaturated soil may reach values much higher than those at saturation.

In the field, hydraulic anisotropy is mainly due to stratification in sediments and to fracturing in rocks. The anisotropy induced by continuous or discontinuous isotropic layers has been studied extensively (Terzaghi 1943; Maasland 1957; Marcus and Evenson

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1961; Evans 1962; Kenney 1963; Dupuy and Lefebvre Du Prey 1968; Basak and Anandakrishnan 1970). The geostatistics approach to the permeability of heterogeneous media was developed by Matheron (1966, 1967, 1968) and used in numerical simulations (Desbarats 1987).

On a small scale, anisotropy may occur from : (1) stratification due to the formation process of the soil (Griffiths 1950), or a directional rock fracturation (Snow 1968) and, (2) orientation of non spherical soil particles during deposition (Hughes 1951). In sedimentary soils or rocks, the anisotropy axes are the direction perpendicular to bedding and, within the bedding plane, the direction of deposition and its perpendicular.

The general opinion is that hydraulic conductivity is higher in the bedding plane than in the perpendicular direction, except for shallow depth soils which contain root and worm holes.

In the case of homogeneous materials, anisotropy is due to particle orientations only and can be related to a directional tortuosity (Wyllie and Rose 1950). Theoretical approaches are available for clays (Olsen 1962) and for granular soils. For the latter, theoretical geometric considerations (Witt and Brauns 1981) yield a maximum value of 2.5 for the ratio k_{max}/k_{min} : if correct, this would mean that higher values, as frequently quoted (see for example Bowles 1970), would be due to microstratifications.

In this paper, the following notations are used for the different hydraulic conductivities :

k_h in the horizontal plane,
k_v in the vertical direction,
$k_=$ in the bedding plane,
$k_\perp$ in the direction perpendicular to bedding.

Many laboratory results have been published for the ratio k_h/k_v of cohesive soils, and the ratio $k_=/k_\perp$ of sedimentary rocks which can be cut and tested in different directions. However, there are few experimental values of k_v and k_h for granular soils, including those calculated from aquifer test results. The reason is that few horizontal permeameters have been designed for granular soils.

The hydraulic anisotropy of clays (Bazak 1972; Al-Tabbaa and Wood 1987) is known to increase during consolidation. However, no information is available on how densification influences the hydraulic anisotropy of sands and sedimentary rocks. The purpose of this paper is to provide new information on this influence, and to compare recent results for a sand (Chapuis et al. 1989b) with those for clay and sandstone.

Common methods for measuring hydraulic anisotropy-laboratory

A variety of field and laboratory techniques have been used to determine the anisotropic hydraulic conductivity of soils and rocks. Laboratory tests may be performed with either rigid-wall or flexible-wall permeameters (Olson and Daniel 1981; Zimmie et al. 1981). In the case of horizontal permeability, only rigid-wall permeameters can be used for granular soils, whereas flexible-wall permeameters can be used for cut samples of rocks or fine-grained soils (Daniel et al. 1984).

For rocks, laboratory testing may be done by :

– cutting specimens in different directions (Fig. 1) and testing them in a permeameter or a triaxial cell to determine a k-value along the specimen axis;

– measuring inward or outward radial flow between a hole under pressure (Bernaix 1967; Ballivy et al. 1976; American Petroleum Institute 1956, ASTM D 2434-68, ASTM D 4525-85) and the periphery of the cylindric rock sample (Fig. 2);

– injecting a fluid through the central hole of a cylindrical sample inserted in an impervious jacket with a longitudinal slot (Fig. 3);

For cohesive soils, laboratory testing may be done by :

– cutting high quality (virtually undisturbed) specimens in different directions and testing them in a triaxial cell or in an oedometer to determine a k-value along the specimen axis (Fig. 1);

– performing special oedometer tests (Shields and Rowe 1965) in which the drainage is radial towards a draining central core (Fig. 4);

– injecting air from a pressure tank into the top, then into the side, of a cubic clod through a glass tube sealed to the clod (De Boodt and Kirkham 1953; Maasland and Kirkham 1955);

In the case of non-cohesive soils, laboratory tests are performed on samples remolded during sampling and then recompacted in permeameters. These samples do not represent, therefore, in situ conditions since infor-

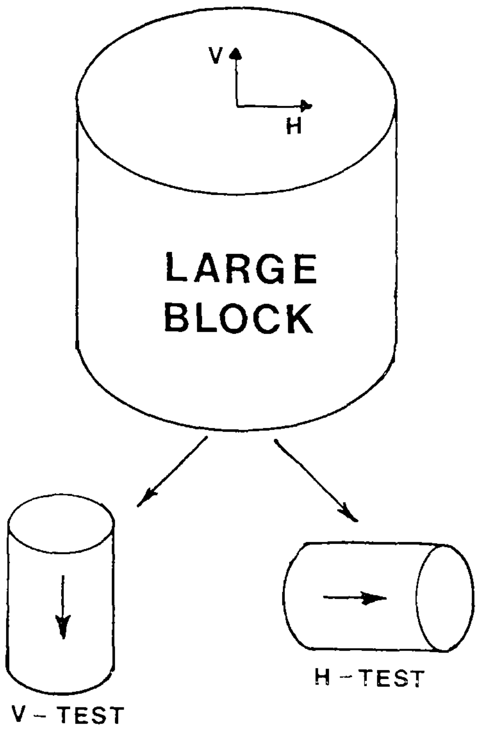

Fig. 1 : Laboratory technique No. 1 to determine directional hydraulic conductivity : cut samples.

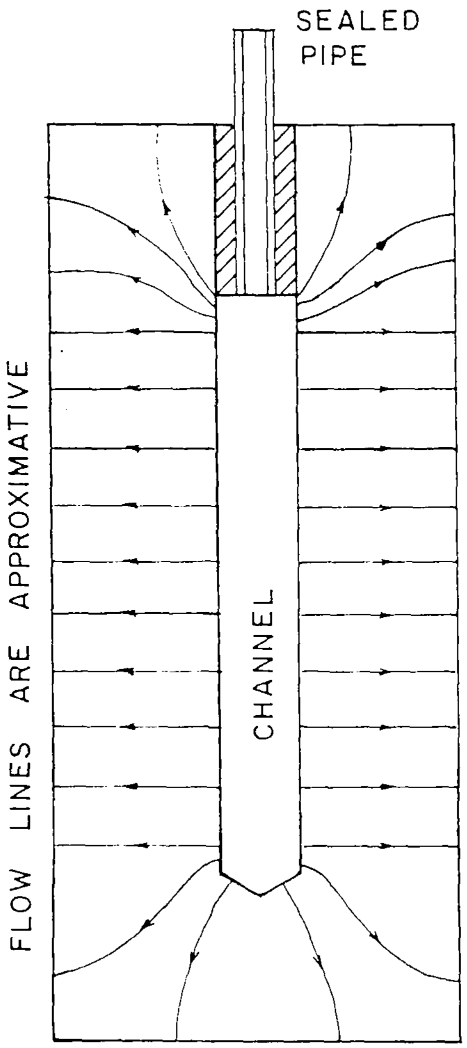

Fig. 2 : Laboratory technique No. 2 to determine directional hydraulic conductivity : triaxial test with central hole.

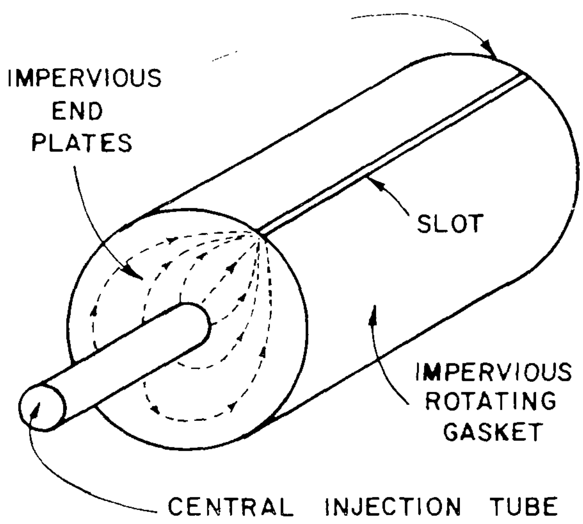

Fig. 3 : Laboratory techniques No. 3 to determine directional hydraulic conductivity : central hole and slotted jacket.

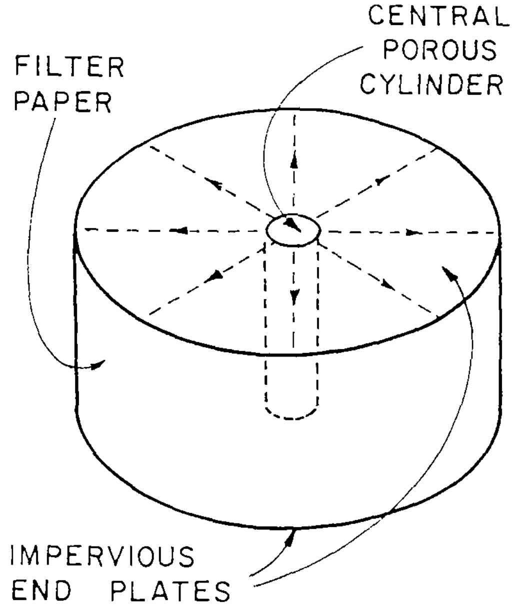

Fig. 4 : Laboratory technique No. 4 to determine directional hydraulic conductivity : radial drainage during an œdometer test.

mation on possible horizontal stratification is lost during sampling. Vertical permeameters are routine equipment but only a limited number of horizontal permeameters (Figs. 5 to 8) have been built to date for research (Wit 1966; Rice et al. 1970; Moore 1979; Paré et al. 1982). The quality of the results given by these permeameters is limited by at least one of these factors : (1) the porous plates are smaller than the sample, thus theoretical correction factors must be used, (2) there is an important lateral leakage along the upper rigid wall, thus the permeameter must be "calibrated" with known soils, and (3) there is no real control on the saturation degree (Chapuis et al. 1989a) which has a great influence on the k-values.

Common methods for measuring hydraulic anisotropy - field

Field determination techniques for hydraulic anisotropy include :

— comparison of piezometer test results for cylindrical injection zones having different lengths (Hvorslev 1951; Reeve and Kirkham 1951),

— interpretation of pumping test results (Arnold et al. 1962; Mansur and Dietrich 1965; Schneebeli 1966; Hantush 1966a, 1966b; Hantush and Thomas 1966; Dagan 1967; Weeks 1969; Hsieh and Neuman 1985), and

— matching an experimental flow net with numerical results (Kenney and Chan 1973; Yokota 1963).

The bedded character of alluviums imparts a strong anisotropy to the deposits if the average properties of large volumes are considered. Anisotropy is usually more marked (higher values) in the field, because soils are more heterogeneous than the small homogeneous samples tested in the laboratory.

Consequently, laboratory determination of hydraulic conductivity may be of little use in studying large-scale real problems in natural deposits. They are, however, representative of field conditions in the case of compacted soils for dams and dykes, or structural layers for highways and airfields.

Interpretation methods and problems

When directional permeabilities are known, it is theoretically easy to determine the principal directions of the permeability tensor. The most common method is to plot the results in terms of $k^{1/2}$ or $k^{-1/2}$ on polar coordinate paper : in both cases an ellipse is obtained with its axes in the principal directions of the k tensor (Vreedenburgh 1936; Scheidegger 1974). Another method is to use the directional values of k for drawing a Mohr's circle (Yang 1948; Schneebeli 1966).

In practice, the interpretation of laboratory results is not easy, because the velocity vector and the hydraulic gradient are generally not parallel everywhere within the sample, except when testing in a principal direction. Directional permeability is conventionally defined as permeability in the direction of flow lines. However, it is possible to define the latter as permeability in the direction of the gradient vector, as done by Scheidegger (1974). In fact, the directional k-value of a sample is an apparent value depending on the L/W ratio (length/width) of the sample (Scheidegger 1957; Marcus 1962; Parsons 1964). The true value of directional permeability is obtained for very small values of L/W, whereas Scheidegger's k-value corresponds to high values of L/W.

As a result, apparent directional k-values must be corrected according to the sample L/W ratio if true directional k-values are needed. The correction applies for any direction, except the three principal directions. Consequently, published ratios $r_{kb} = k_{b,max}/k_{b,min}$ for rocks in the bedding plane can always be used without correction. However, many published anisotropy ratios, $r_k = k_{=}/k_{\perp}$ for rocks, or $r_k = k_h/k_v$ for cohesive or granular soils, are pieces of information only because k_h or $k_{=}$ is one apparent value within the bedding plane. Specifically, there is no reason to have the same k_h value in the direction of deposition and in the perpendicular direction, which are deemed to be principal directions of tensor k.

Available results

Published anisotropy ratios are given in Tables 1 to 4. These are for :

- cohesive soils (Table 1),
- non-cohesive soils (Table 2),
- rocks (Table 3), and
- rocks in the bedding plane (Table 4).

Many data are available for the ratio k_h/k_v of cohesive soils and rocks which can be cut and tested in different directions. However, as indicated previously, laboratory experiments are difficult to conduct with non-cohesive soils for which few reliable results are available as compared to clays and rocks.

Similarity of anisotropy ratios

In the case of clays (Table 1), k_h is usually higher than k_v. The anisotropy ratio is in the 0.7 to 4 range in most experimental results. There are a few exceptions, all associated with inhomogeneous clays, like certain varved clays which have an anisotropy ratio higher than 4. Shallow-depth clays may have a very low anisotropy ratio (k_v much higher than k_h) : this anomaly can be related to the very high values of k_v (10^{-3} to 10^{-5} cm/s) for such clays sampled at shallow depths (0 to 1.0 m), which probably contained root and worm holes.

The hydraulic anisotropy of homogeneous rocks (Tables 3 and 4) seems very similar to that of homogeneous clays. Several hundred results are available. The anisotropy ratio defined as k_{max}/k_{min} is usually lower than 4, with a few exceptions up to 42. In the bedding plane of sedimentary rocks, the anisotropy ratio $k_{b,max}/k_{b,min}$ is usually lower than 1.5, which means that these rocks are nearly isotropic in that plane.

Few results are available for granular materials (Table 2). Their ratio k_h/k_v, contrary to common belief, is not always higher than 1. Experimental values for sands and gravels are in the 0.75 to 4.1 range.

Consequently, the hydraulic anisotropy ratios of homogeneous clays, rocks and sands appear to be quite similar. In particular, the ratio k_{max}/k_{min} seems to be lower than 4. In a sense, this finding confirms that such a ratio has some upper limit related to the shape of particles, their arrangement, or to the directional tortuosity within the pore space.

Similar influence of densification for sand and clay

The hydraulic anisotropy ratio of clay and sand is a function of preparation mode and consolidation in the laboratory.

For clays, Bazak (1972) and Al-Tabbaa and Wood (1987) have published k_v- and k_h-results for artificial homogeneous clays. According to their results, clays are hydraulically isotropic in the slurry state and their hydraulic anisotropy is increased by consolidation. The hydraulic anisotropy of a clay also depends on its structure. According to Basak (1972), the same clay may have different hydraulic anisotropy ratios k_h/k_v (e) in dispersed and flocculated conditions.

A comparison can be made between the anisotropy ratio, $r_k = k_h/k_v$, of a sand and that of two clays. The results for the sand (Chapuis et al. 1988b) are plotted in Fig. 9 whereas the results for the clays (Basak 1972; Al-Tabbaa and Wood 1987) are plotted in Fig. 10. There are two similarities in the results.

Firstly, these soils are hydraulically isotropic in their loosest state, which seems to be intuitively correct. For the two clays, isotropy is verified in the slurry condition. For the sand, instability occurred (segregration of fines) during k-tests performed at $e > 0.62$, a value lower than the maximum void ratio, $e_{max} = 0.824$. However, if the results are extrapolated as indicated in Fig. 9, it may be considered that this sand is hydraulically isotropic in its loosest state.

Table 1 : Values of the anisotropy ratio $r_k = k_h/k_v$ for cohesive soils

Type of soil	Method	$r_k = k_h/k_v$	Reference
CLAY-BOSTON	cut samples	1.7 to 3.6	Mitchell (1956)
BLUE	cut samples	0.9 to 4.0	Olsen (1962)
»	»	0.7 to 3.3	Haley & Aldrich (1969)
CHICAGO	»	1.4	Mitchell (1956)
CINCINNATI	»	2.2	Mitchell (1956)
DOW FIELD	»	1.2	Mitchell (1956)
ENKOPING	»	1.1 to 1.3	Jakobson (1955)
FORE RIVER	»	2.2	Mitchell (1956)
GOOSE BAY	»	3.4	Mitchell (1956)
LONDON	»	2	Leroueil (1988, pers. com.)
LOUISIANA	»	0.9	Mitchell (1956)
MEXICO	cut samples	0.60	Mitchell (1956)
PUMP SITE	»	0.3	Mitchell (1956)
SILTY	»	1 to 15	Johnson & Morris (1962)
TEXAS	»	3.9	Mitchell (1956)
VASBY	»	0.8 to 1.3	Jakobson (1955)
CLAY-FISSURED	cut samples	0.7 to 4.6	Garga (1988)
CLAY-MARINE	cut samples	0.7 to 1.4	Lumb and Holt (1968)
»	»	1.05	Subbaraju et al. (1973)
Backebol	»	1.0	Larsson (1981)
many clays	»	0.9 to 1.4	Tavenas et al. (1983)
Atchafalaya	»	2.2 to 2.5	Tavenas et al. (1983)
CLAY-SENSITIVE	cut samples	8 to 12	Wu et al. (1978)
CLAY-VARVED	cut samples	1.5	Bazett and Brodie (1961)
»	cut s. + rad. perm.	1.5 to 3.7	Chan & Kenney (1973)
»	flow nets matching	same range	Kenney & Chan (1973)
»	cut samples	3 to 8	Wu et al. (1978)
»	»	4 to 40	Casagrande & Poulos (1969)
»	»	1.2 to 1.3	Tavenas et al. (1983)
KAOLIN	radial permeameter		Basak (1972)
flocculated	»	1.0	»
dispersed	»	1.0 to 1.6	»
»	radial permeameter	1.0 to 2.8	Al-Tabbaa & Wood (1987)
»	radial permeameter	0.8	Wilkinson & Shipley (1969)
»	air inj. by tubes	1.1 to 2.1	De Boodt & Kirkham (1953)
»	air inj. by tubes	1.6 to 13	Maasland & Kirkham (1955)
KAOLINITE	cut samples	1.3 to 1.7	Olsen (1962)
	oedometer tests	2.3 to 2.8	Morgenstern & Tchalenko (67)
SHALLOW	Fig. 8 (see (1))	0.07 to 0.11	Wit (1966)
DEPTH CLAYS	Fig. 8 (see (2))	1.4	»
PEAT	cut samples	1.2 to 1.7	Tsien (1955)
SILT	cut samples	0.7	MacGary & Lambert (1962)

NOTES : Very often, k_v is measured instead of k_h.
(1) "Clays" having k_v – values between 1.1 and 3.5 × 10^{-3} cm/s.
(2) "Clays" having k_v – values of 5.7 × 10^{-4} and 8.1 × 10^{-5} cm/s.

Table 2 : Values of the anisotropy ratio $r_k = k_h/k_v$ for non-cohesive soils.

Type of soil	Method	$r_k = k_h/k_v$	Reference
GRANULAR	pumping test, two	1.4 to 2.4	Mansur & Dietrich (1965)
	interp. methods	1.4 to 4.1	»
SAND & GRAVEL	horiz. and vertical	0.75 to 1.7	Paré et al. (1982)
TILL	permeameters	4.7	»
SAND	»	0.87 to 1.83	Chapuis et al. (1989b)
SAND	bi-direct. permeameter	1.04 to 1.06	Latini (1967)
CRUS. LIMEST.	»	1.13 to 1.23	»
SAND	2D horiz. permeameter	0.93 to 1.36	Fontugne (1969)
CRUS. LIMEST.	»	0.80 to 1.22	»
MICA	»	1.9	»
ACTIV. CARB.	»	2.2	»
FIBERS	air permeameter	1.80 to 2.12	Sullivan (1941)
	⊥ or = to flow		
SAND	Fig. 8 (see (1))	0.38	Wit (1966)
TIDAL SAND	Fig. 8 (see (2))	1.3	»
FLAT LENSES	⊥ or = to sediment.	2.3	Witt & Brauns (1983)
THEORY	orientation only	0.4 to 2.5	Witt & Brauns (1981)

(1) "Sand" sampled at a depth of 75 to 100 cm, k_h = 1.9 × 10^{-4} cm/s.
(2) "Sand" sampled at a depth of 150 to 285 cm, k_h = 3.9 × 10^{-4} cm/s.

Table 3 : Values of the anisotropy ratio $r_k = k_{\parallel}/k_{\perp}$ for rocks.

Type of rock	Method	$r_k = k_{\parallel}/k_{\perp}$	Reference
ARKOSE			
coarse grain	cut samples	1.5	Rima et al. (1962)
medium grain	»	2.0	»
fine grain	»	10.0	»
COAL	pumping tests	1.8	Stone et Snoeberger (1977)
»	»	2.5	Stoner (1981)
CONGLOMERATE	cut samples	1.3	Rima et al. (1962)
METAMORPHIC	aquifer tests	2	Stuart et al. (1954)
	aquifer tests	> 1.0	Stewart (1964)
	flow net. dam	1.0	Yokota (1963)
SANDSTONE			
oil sands	cut samples	ave. 2.0	Fancher et al. (1933)
»	»	0.14 to 42	»
Woodbine	cut samples	5.6 to 6.9	»
Bradford	»	1.1 to 42	»
Venango	»	0.66 to 1.09	»
Wilcox	»	0.96 to 4.7	»
Johnson	»	1.35 to 31	»
School Land	»	0.28 to 2.32	»
Layton	»	1.44	»
Prue	»	0.14	»
Hammer-Hain.	»	3.22	»
Gilcrease	»	0.70 to 1.36	»
Cromwell	»	0.34 to 6.74	»
Wanette	»	1.21 to 12.9	»
Glen Rose	»	2.7	»
various	cut samples	ave. 1.5	Piersol et al. (1940)
»	»	12 % > 3.0	»
»	»	6 % < 1.0	»
various	cut samples	43 % ≈ 1.0	Greenkorn et al. (1964)
»	»	16 % < 1.0	»
»	»	41 % > 1.0	»
cretaceous	cut samples	0.03	Young et al. (1964)
six diff.	»	all > 1.0	Rühl & Schmid (1957)
»	»	most < 10	»
Berea	air injec. by tubes	0.62 to 0.68	Gray et al. (1963)
Boise	air injec. by tubes	0.62 to 0.65	Gray et al. (1963)
SAND ? (1)			
depth 767 ft	cut samples	30	Johnson & Morris (1962)
depth 911 ft	»	3.9	»
SAND ? (1)	cut samples	0.8 and 1.0	McGary & Lambert (1962)
SILSTONE	cut samples	0.75	Rima et al. (1962)

(1) Very often, sandstone is called sand in papers related to geology and oil industry.

Table 4 : Values of the anisotropy ratio $r_{kh} = k_{h\,max}/k_{h\,min}$ for rocks in the bedding plane.

Type of rock	Method	Ratio r_{kh}	Reference
SANDSTONE	cut samples	< 1.78	Fettke (1938)
NEW YORK	cut samples and	1.42 & 2.59	Johnson & Hughes (1948)
BRADFORD	method of Fig.3	1.18 to 2.7	»
VENANGO	»	1.22 to 1.42	»
RUSHFORD	»	1.38 to 1.63	»
BRADFORD	»	ave. 1.5	Johnson & Breston (1951)
VENANGO	»	ave. 1.25	»
HOMEWOOD	»	ave. 1.40	»
ALLEGAN	»	ave. 1.5	»
BEREA	»	1.0	»
BARTLESVILLE	»	1.0	Johnson & Breston (1951)
SQUIRREL	»	1.0	»
DEESE	»	1.0	»

Secondly, the hydraulic anisotropy of soils increases when they are statically and vertically densified. This finding is in agreement with recent test results. It explains why clays in a "loose" condition like the sensitive clays of Eastern Canada (Tavenas et al. 1983) or Sweden (Larsson 1981) are nearly isotropic, whereas the overconsolidated homogeneous London clay has a r_k-value close to 2 (Leroueil 1988, personal communication).

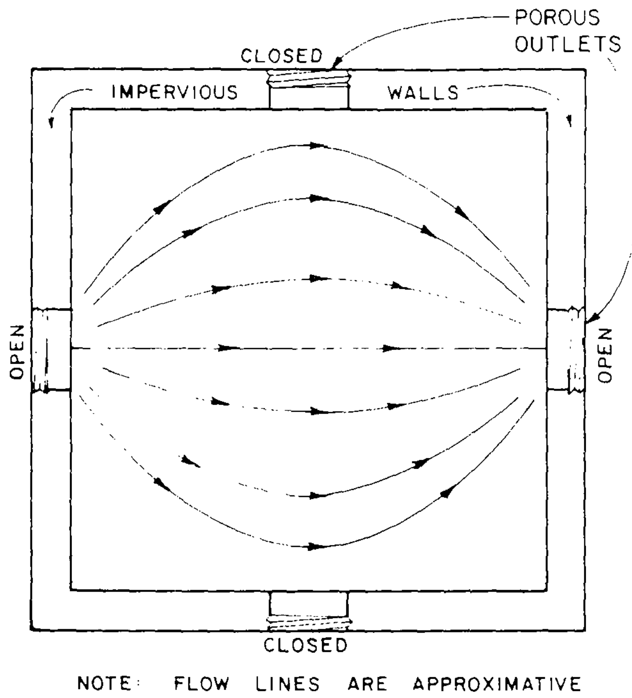

Fig. 5 : Laboratory technique No. 5 to determine horizontal hydraulic conductivity : horizontal permeameter of Moore (1979).

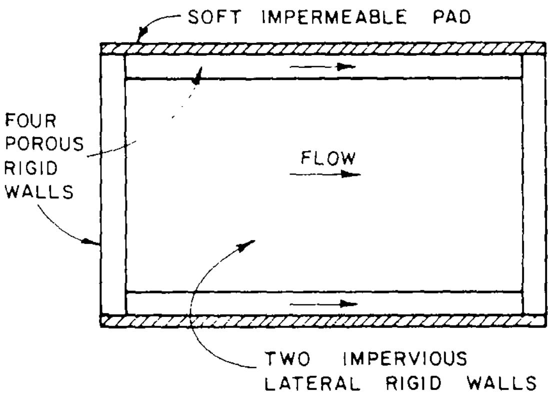

Fig. 6 : Laboratory technique No. 6 to determine horizontal hydraulic conductivity : horizontal permeameter of Latini (1967).

Correlating r_k to density for vertically and statically compacted soils

The influence of densification on the anisotropy ratio $r_k = k_h/k_v$ of dispersed clay and sand can be compared by introducing a density index having a similar meaning for these soils. This density index, I_e, will have the same zero value for the loosest state (e_0 for clay as a slurry, and $e_O = e_{max}$ for sand) and a value of 1 for a reference dense state. Only one compaction method is applicable to both sands and clays, the (standard or modified) Proctor test, which corresponds to a fixed amount of compaction energy. Thus, the density index, I_e, takes its value of 1 for the void ratio $e_1 = e_{opt}$ at the optimum density ρ_{opt} of the modified Proctor test.

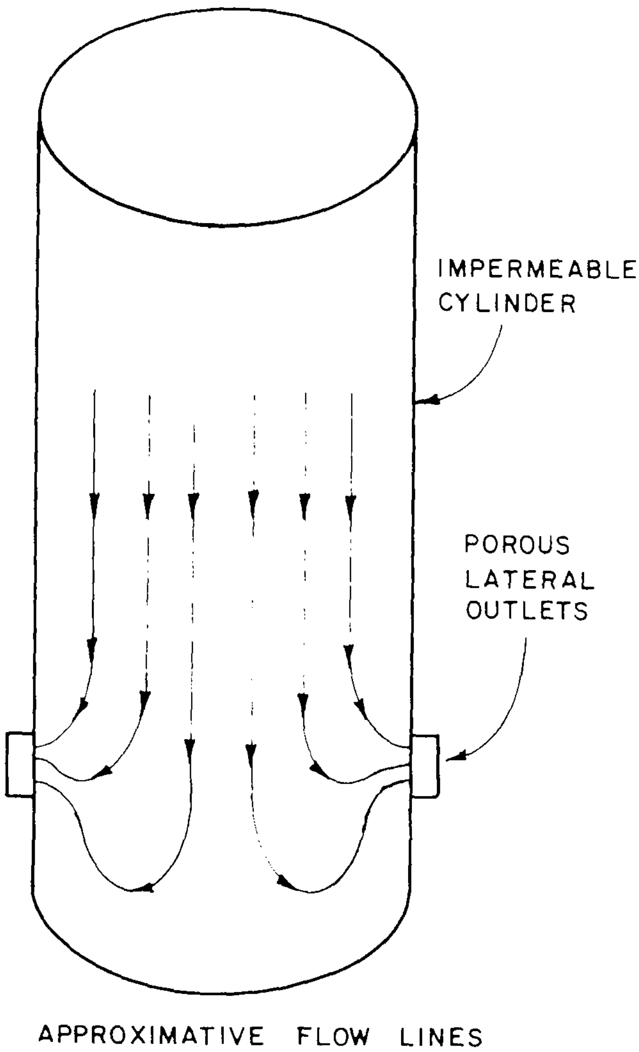

Fig. 7 : Laboratory technique No. 7 to determine horizontal hydraulic conductivity : horizontal permeameter of Fontugne (1969).

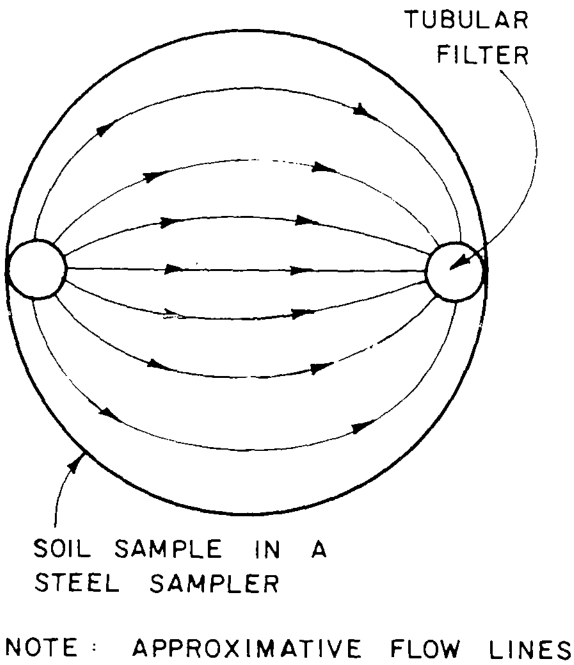

Fig. 8 : Laboratory technique No. 8 to determine horizontal hydraulic conductivity : horizontal permeameter of Wit (1966).

For any void ratio, e, the density index, I_e, is defined as :

$$I_e = (e_0 - e)/(e_0 - e_1)$$

For the sand in Fig. 9, the void ratios corresponding to $I_e = 0$ and 1 are $e_0 = 0.824$ and $e_1 = 0.361$ respectively. For the clays in Fig. 10, the values of ρ_{opt} are unknown. However, the Atterberg's limits for the kaolin (Al-Tabbaa and Wood 1987) are available ($w_P = 38\,\%$ and $w_L = 69\,\%$). Based on experience, it may be assumed that the ρ_{opt} for this clay corresponds approximately to a water content $w = w_P$ and a saturation degree $S_r = 80\,\%$. Then, assuming that this kaolin has a $G_s = 2.64$, the e-values corresponding to $I_e = 0$ and 1 are $e_0 = 2.50$ and $e_1 = 1.25$ respectively.

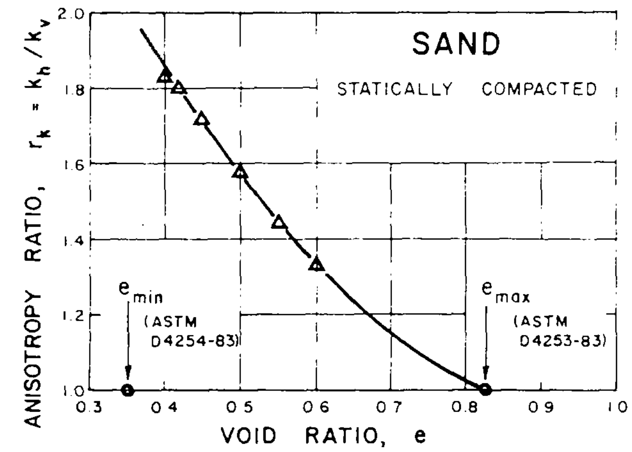

Fig. 9 : Anisotropy ratio, $r_k = k_h/k_v$, versus the void ratio, e, of a sand statically compacted (Chapuis et al. 1989b).

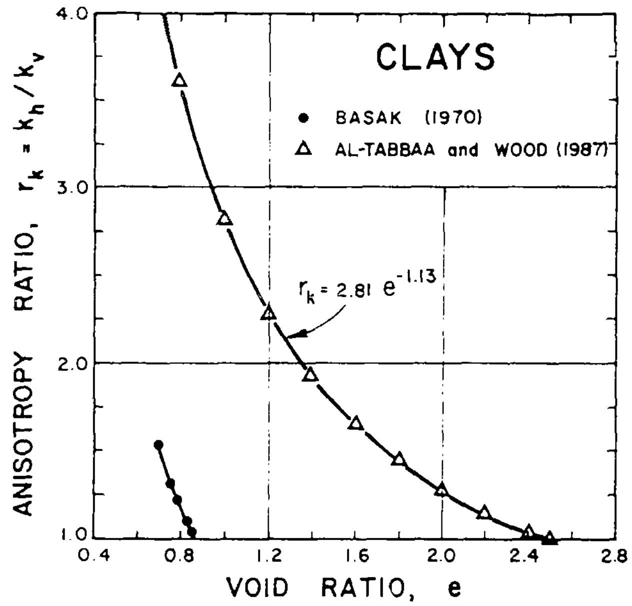

Fig. 10 : Anisotropy ratio, $r_k = k_h/k_v$, versus the void ratio, e, of a dispersed clay (Basak 1972) and a kaolin (Al-Tabbaa and Wood (1987).

The experimental results of r_k versus I_e are given in Fig. 11. The dispersed clay (Al-Tabbaa and Wood 1987) was prepared as a slurry and vertically statically consolidated. The sand was loosely placed and vertically statically densified. The two hydraulic anisotropy ratios take similar values for the same density index.

This finding on how densification influences hydraulic anisotropy may be applicable to natural sediments having settled in still water and influenced subsequently only by gravity.

It will be remembered, however, that the same clay, either flocculated or dispersed, has different hydraulic anisotropies at the same void ratio (Basak 1972). Similarly, according to the results of Chapuis et al. (1989b), the same granular material, either statically or dynamically compacted, has different hydraulic anisotropies at the same void ratio.

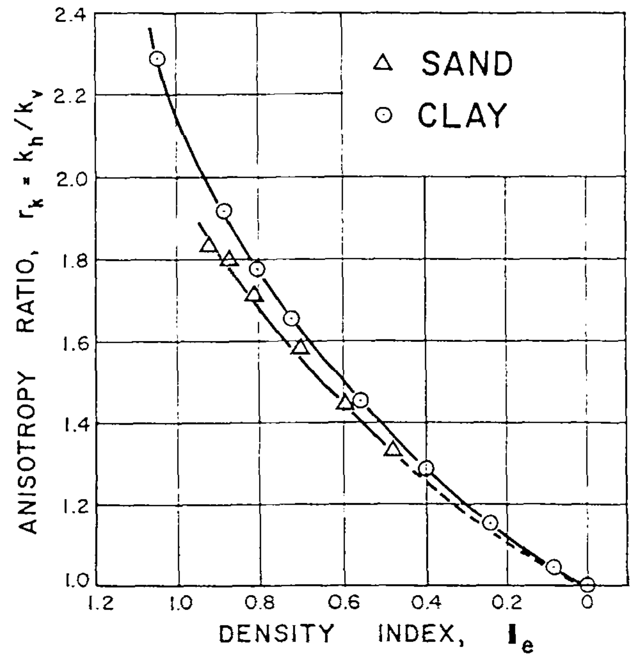

Fig. 11 : Anisotropy ratio, r_k, versus the density index, I_e, for vertically and statically compacted sand and clay.

Correlation between sand and sandstone

Sand and clay have similar hydraulic anisotropies. Then, whether there is some continuity from sand to sandstone in terms of hydraulic anisotropy when the void ratio or the porosity decreases may be verified.

The transformation of sand into sandstone is due to cementing agents (calcite, silica and other minerals) at the points of contact between particles, and eventually deformation of the grains where temperatures and pressures are both high. Excluding experiments which are difficult to interpret, many directional permeability values have been measured on cut samples. Muskat (1937) and Rühl and Schmid (1957) reported respectively 49 and 200 pairs of results which can be used for a statistical analysis of the ratio k_{max}/k_{min} versus a broad range of e or n.

Geostatistical results are available for the law of composition of permeabilities. The principal problem is how to derive the macroscopic K from the statistical parameters of the local k. Matheron (1966, 1967, 1968) has given a general demonstration that the average macroscopic permeability is always formed according to an intermediate method between harmonic and arithmetic ponderations. The law of geometric weighting, frequently used in the petroleum industry is one of the intermediate methods. However, it was mathematically proven that the law of geometric weighting is never valid for heterogeneous porous media because it is equivalent to a principle of equidistribution of energy which generally is not valid. As shown by several examples (Matheron 1968), the weighting law depends on each distribution law and the number of dimensions in the problem. In particular, in the case of a lognormal distribution (often assumed for sediments) it was shown (Matheron 1967) that the average macroscopic permeability is situated half or two thirds of the way between the harmonic and the arithmetic means, for 2-D or 3-D problems respectively.

In studying the transition between sand and sandstone, it was decided that the usual geometric weighting method and the intermediate 3-D weighting method would be applied to the anisotropy ratio. The results are shown in Figs 12 and 13 respectively. Although they are somewhat scattered at low e-values, there is a clear transition between sand and sandstone, the latter becoming progressively more anisotropic (on the average) when its void ratio decreases.

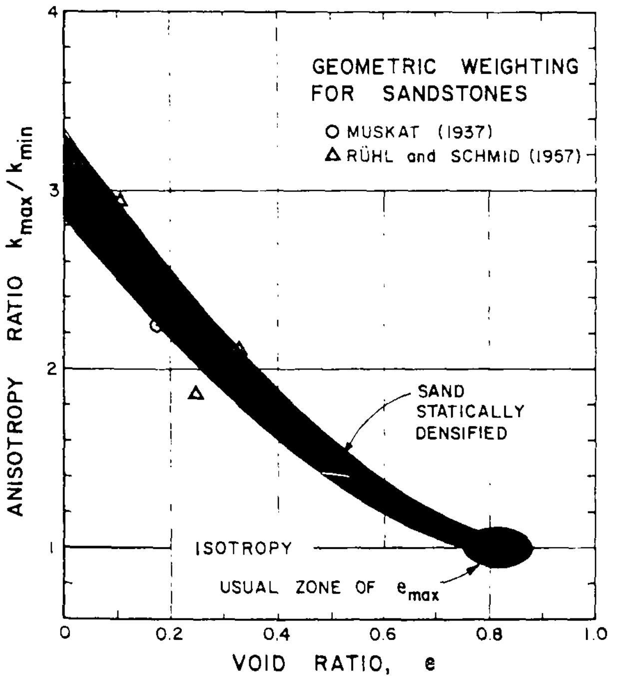

Fig. 12 : Anisotropy ratio k_{max}/k_{min} of sandstone obtained by the geometric weighting method, as compared to that of sand.

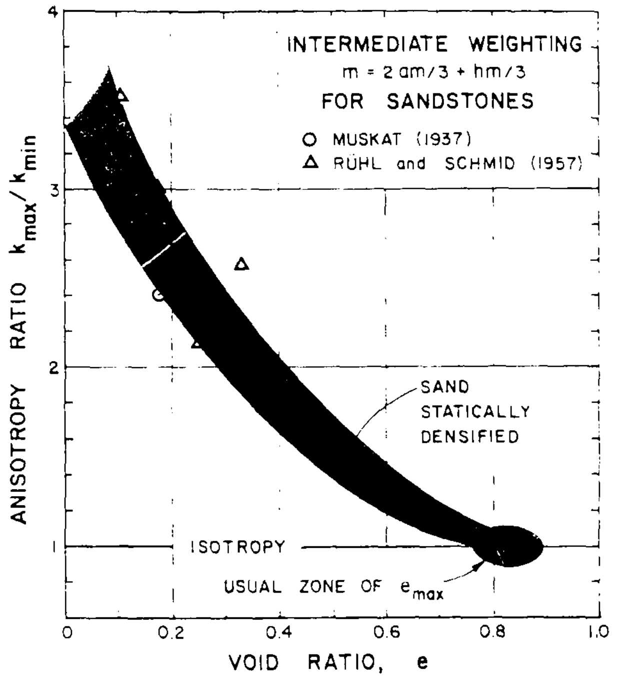

Fig. 13 : Anisotropy ratio k_{max}/k_{min} of sandstone by the intermediate 3-D weighting method ($3K = 2E(k) + [E(k^{-1})]^{-1}$), as compared to that of sand.

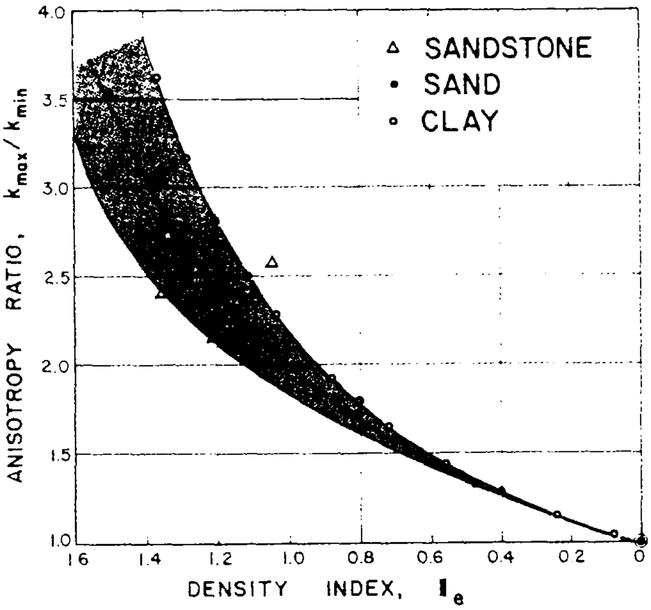

Fig. 14 : Anisotropy ratio (k_h/k_v or k_{max}/k_{min}) of clay, sand and sandstone versus the density index, I_e.

General correlation of anisotropy ratio to density index

In order to compare the influence of densification on sand, clay and sandstone, values of e_0 and e_i must be assumed for sandstones for the times when they were sands. For simplification, it may be assumed that all the different sandstones used in Figs 12 and 13 have the same e_0 and e_i as the sand. These values make it possible to calculate the density index, I_e, for the sandstones.

All experimental values for sand, clay and sandstone are reported in Fig. 14. Even if I_e-values are approximate, there is a general feature well illustrated by Fig. 14. As previously indicated, all homogeneous soils are hydraulically isotropic in their loosest state (at their highest void ratio). Then, the hydraulic anisotropy of homogeneous soils and sandstone increases in a similar manner when these materials are densified.

Conclusion

Many data are available for the ratio k_h/k_v of cohesive soils and sedimentary rocks which can be cut and tested in different directions. However, there are few experimental values of k_v and k_h available for non-cohesive materials, including those calculated from aquifer test results.

The hydraulic anisotropy of homogeneous clays, sedimentary rocks and sands are comparable. Specifically, their ratio k_{max}/k_{min} seems to be lower than 4. This finding confirms that such a ratio has an upper limit related to the shape and arrangement of the particles, or the directional tortuosity within the pore space.

The evolution of the anisotropy ratio, k_h/k_v, versus a density index, I_e, is nearly identical for a sand and a clay, when they are vertically and statically compacted. Similarly, the anisotropy ratio of sandstone is found to be in continuity with that of sand.

These results seem to be applicable to any natural soils having settled in still water and influenced subsequently only by gravity. In short, homogeneous soils appear to be hydraulically isotropic at their highest void ratio and their anisotropy ratio increases when they are densified.

It is shown that, as with the geological history, the placement method influences hydraulic anisotropy. Both must be taken into account in drainage problems related to structural layers of roads and seepage control in earth dams.

Acknowledgements

This paper is the result of a research program aimed at improving the reliability of permeability tests. The studies have been sponsored by NSERC (Canada, grant U-0502), and this financial assistance is gratefully acknowledged.

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