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# A PRIMAL-DUAL ALGORITHM FOR THE FERMAT-WEBER PROBLEM INVOLVING MIXED GAUGES

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We give a new algorithm for solving the Fermat-Weber location problem involving mixed gauges. This algorithm, which is derived from the partial inverse method developed by J.E. Spingarn, simultaneously generates two sequences globally converging to a primal and a dual solution respectively. In addition, the updating formulae are very simple; a stopping rule can be defined though the method is not dual feasible and the entire set of optimal locations can be obtained from the dual solution by making use of optimality conditions.

When polyhedral gauges are used, we show that the algorithm terminates in a finite number of steps, provided that the set of optimal locations has nonempty interior and a counterexample to finite termination is given in a case where this property is violated.

Finally, numerical results are reported and we discuss possible extensions of these results.

*Key words*: Location problems, Fermat-Weber problem, partial inverse method, proximal point algorithm, polyhedral gauges, finite convergence.

## 1. Introduction

The famous location problem, known as the Fermat-Weber problem, is to find a point such that the sum of weighted distances from $m$ given points is minimized.

In the framework of Location Theory, the distances are usually derived from norms such as $l^p$-norms, $1 \leq p \leq +\infty$, polyhedral norms, or more generally from gauges. For a justification to work with polyhedral norms or gauges, see for instance [11, 30, 33].

In general, the choice of a gauge (or a norm) implies the use of a specific method for solving the problem:

- in the Euclidean case, the first method was given in 1937 by Weiszfeld [31]. The convergence properties (global convergence, rate of convergence ...) of this iterative procedure have been studied at length [17, 18, 19, 21, 22]. Some improvements of Weiszfeld's algorithm and various other gradient or non-gradient methods have also been proposed [2, 3, 4, 5, 23].

- in the $l^1$-norm case, the problem can be solved by linear programming [1, 16], (see also [15] for a selective bibliography). More generally, any problem involving mixed polyhedral norms can be formulated as a linear program [30].

- in addition some alternative methods have been proposed. They include a dual method [24] and a cutting plane method [25].

In this paper, we describe a new algorithm which has the following advantages: different gauges (polyhedral, $l^2, \ldots$) can be mixed; it generates simultaneously two sequences globally converging to a primal and a dual solution respectively; the update rules are very simple; a lower and an upper bound converging to the optimal value of the objective function can be defined to give a rule for stopping the process; the entire set of optimal locations can be obtained from the dual solution by making use of general optimality conditions; contrary to cutting plane methods and to the dual method given in [24] the size of the sub-problems to be solved is fixed with iterations; the method can be viewed as a dual decomposition method which permits the possibility to make parallel computations.

The paper is divided as follows:

- in Section 2, we give the formulation of the problem with some results on duality, and we recall the partial inverse method on which our algorithm is based.
- in Section 3, we describe the algorithm and we give some details about the implementation.
- in Section 4, we study the finite convergence of the algorithm when polyhedral gauges are used.
- in Section 5, some numerical results are reported.
- finally in Section 6, we discuss some possible extensions of our algorithm.

## 2. Problem description and partial inverse method

The Fermat-Weber problem (in short FW) is to

$$\underset{x \in \mathbb{R}^n}{\text{Minimize}} \quad \sum_{a \in \mathscr{A}} \omega_a \gamma_a(x-a)$$

where $\mathscr{A}$ is a finite subset of $m$ elements of $\mathbb{R}^n$, which represents the locations of existing facilities, $x$ is the location of a new facility to be placed, $\omega_a$ is a positive weight and $\gamma_a$ is a norm or more generally the gauge of a convex closed bounded set $B_a$ (the unit ball of $\gamma_a$) containing zero in its interior [26].

In the sequel, $\mathbb{R}^n$ is equipped with its usual Euclidean structure for which the scalar product is denoted by $(\cdot,\cdot)$. Let $\gamma_a^0$ be the polar gauge of $\gamma_a$ defined by $\gamma_a^0(y) = \operatorname{Sup}\{(x, y), x \in B_a\}$. We shall denote by $B_a^0$ the unit ball associated with $\gamma_a^0$.

Let $H = \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($m$ times) whose elements are denoted by $\hat{p} = (p_a)_{a \in \mathscr{A}}$.

It is known [33] that the dual of FW is given by

$$\underset{\hat{p} \in H}{\text{Maximize}} \quad -\sum_a \omega_a(a, p_a)$$

$$\text{subject to} \quad \sum_a \omega_a p_a = 0,$$

$$\gamma_a^0(p_a) \leq 1 \quad \text{for all } a,$$

Note that it is not possible in this problem to substitute $p_a$ by $-p_a$ when non-symmetric gauges are used.

The following theorem gives links between the primal and the dual problems.

**Theorem 2.1.** *We have the equivalence: $x \in \mathbb{R}^n$ is a primal solution and $\hat{p} \in H$ is a dual solution if and only if*

$$\sum_a \omega_a p_a = 0, \tag{2.1}$$

$$p_a \in \partial\gamma_a(x-a) \quad \textit{for all } a. \tag{2.2}$$

Although the Fermat–Weber problem has been studied for a long time, these necessary and sufficient conditions for optimality have only recently been used in Location Theory from a theoretical or a practical point of view. For instance, in [11], a geometrical description of the set of optimal solutions to FW, denoted by $M_\omega(\mathscr{A})$, has been derived from (2.1) and (2.2). This description is of particular interest for problems involving polyhedral gauges and will explicitly be used in Section 4. Therefore we need to recall the following.

Let $N_a(p_a)$ be the normal cone to $B_a^0$ at $p_a \in \mathbb{R}^n$ and let $\mathscr{C}_{\hat{p}} = \bigcap_a (a + N_a(p_a))$, with $\hat{p} \in H$ such that $p_a \in B_a^0$ for each $a$.

**Definition 2.1.** A nonempty convex set $\mathscr{C}$ is said to be an elementary convex set if there exists $\hat{p}$ ($p_a \in B_a^0$ for each $a$) such that $\mathscr{C} = \mathscr{C}_{\hat{p}}$.

From Theorem 2.1, it may be seen [11] that $M_\omega(\mathscr{A})$ is a bounded elementary convex set, since condition (2.2) is equivalent to the conditions:

$$\gamma_a^0(p_a) \leqslant 1, \tag{2.3}$$

$$x \in a + N_a(p_a) \quad \text{for all } a. \tag{2.4}$$

Thus the entire set $M_\omega(\mathscr{A})$ of optimal solutions to FW can be obtained from a dual solution, which is of interest for the decision maker.

Hence we have [11]:

**Corollary 2.1.** (1) *$M_\omega(\mathscr{A})$ is a bounded elementary convex set $\mathscr{C}_{\hat{p}}$ associated with some $\hat{p}$ such that $\sum_a \omega_a p_a = 0$.*

(2) *Let $\mathscr{C}_{\hat{p}}$ be an elementary convex set associated with some $\hat{p}$ such that $\sum_a \omega_a p_a = 0$. Then $M_\omega(\mathscr{A}) = \mathscr{C}_{\hat{p}}$ and $\mathscr{C}_{\hat{p}}$ is bounded.*

**Remark 2.1.** In general an elementary convex set $\mathscr{C}$ is not necessarily associated with a single $\hat{p} \in H$, with $p_a \in B_a^0$, except if the interior of $\mathscr{C}$ is nonempty. However, if $\mathscr{C}$ is not reduced to the singleton $\{a\}$, and if for $\hat{p}$ and $\hat{q}$ we have $\mathscr{C} = \mathscr{C}_{\hat{p}} = \mathscr{C}_{\hat{q}}$, then, for each $a$, $p_a$ and $q_a$ belong to the same exposed face of $B_a^0$.

Now making use of optimality conditions, we shall construct a primal–dual algorithm to solve FW. The problem of finding a zero of a sum of maximal monotone multifunctions, i.e. a problem such as (2.1), (2.2) has been underlined by J.E. Spingarn [28] as a particular application of the method of partial inverses. Let us recall the framework.

Let the space $H$ be equipped with the scalar product defined by $(\hat{x}, \hat{p}) = \sum_a \omega_a(x_a, p_a)$. The corresponding norm of $H$ will be denoted by $\|\cdot\|$. Putting

$$A = \{\hat{x} \in H, \hat{x} = (x, x, \ldots, x) \text{ with } x \in \mathbb{R}^n\},$$

$$B = \left\{\hat{p} \in H, \sum_a \omega_a p_a = 0\right\},$$

we obtain two complementary subspaces of $H$ ($A = B^{\perp}$) and conditions (2.1), (2.2) are equivalent to:

$$\hat{x} \in A, \quad \hat{p} \in B \quad \text{and} \quad \hat{p} \in T(\hat{x})$$

where $T$ is the maximal monotone multifunction defined on $H$ by $T = \prod_a T_a$ with $T_a(\hat{y}) = \partial\gamma_a(y_a - a)$.

Now, from Spingarn's work [28], for $\hat{x} \in A$ and $\hat{p} \in B$ we have

$$\hat{p} \in T(\hat{x}) \quad \text{if and only if} \quad 0 \in T_A(\hat{x} + \hat{p})$$

where $T_A$ is the (maximal monotone) multifunction defined by its graph $\Gamma$:

$$\Gamma = \{(\hat{x}_A + \hat{p}_B; \hat{p}_A + \hat{x}_B), \hat{p} \in T(\hat{x})\}$$

$\hat{z}_A$ and $\hat{z}_B$ denoting the orthogonal projection of $\hat{z}$ onto $A$ and $B$ respectively. The proximal point algorithm generating the sequence

$$\hat{z}_{k+1} = (I + T_A)^{-1}\hat{z}_k \tag{2.5}$$

converges to a zero $\hat{z}$ of $T_A$ which gives a solution to (2.1), (2.2) by calculating

$$\hat{x} = \hat{z}_A \quad \text{and} \quad \hat{p} = \hat{z}_B.$$

This provides a theoretical primal-dual method for solving FW but we have to know, as noted in [28], if procedure (2.5) can be executed in a computationally feasible manner. This problem will be studied in the next section.

## 3. The primal–dual algorithm

This section is devoted to the description of an algorithm to solve FW. In addition, we discuss some details of the implementation in relation to Location Theory and give a rational termination rule. We suppose without loss of generality that $\sum_a \omega_a = m$.

### 3.1. Algorithm description

Starting with $\hat{x}^0 = (x^0, \ldots, x^0)$ and $\hat{p}^0$, ($\hat{z}^0 = \hat{x}^0 + \hat{p}^0$) such that $\sum_a \omega_a p_a^0 = 0$, the partial inverse method computes at step $k$: $\hat{x}^{k+1} \in A$ and $\hat{p}^{k+1} \in B$, ($\hat{z}^{k+1} = \hat{x}^{k+1} + \hat{p}^{k+1}$) such that

$$\hat{x}^{k+1} + \hat{p}^{k+1} = (I + T_A)^{-1}(\hat{x}^k + \hat{p}^k) \tag{3.1}$$

which can also be rewritten as

$$\hat{x}^k - \hat{x}^{k+1} + \hat{p}^k - \hat{p}^{k+1} \in T_A(\hat{x}^{k+1} + \hat{p}^{k+1})$$

or equivalently from the definition of $T_A$

$$\hat{x}^k - \hat{x}^{k+1} + \hat{p}^{k+1} \in T(\hat{p}^k - \hat{p}^{k+1} + \hat{x}^{k+1}).$$

Putting

$$\hat{p}'^k = \hat{x}^k - \hat{x}^{k+1} + \hat{p}^{k+1}, \tag{3.2}$$

$$\hat{x}'^k = \hat{p}^k - \hat{p}^{k+1} + \hat{x}^{k+1}, \tag{3.3}$$

we obtain

$$\hat{p}'^k \in T(\hat{x}'^k) \tag{3.4}$$

or equivalently

$$\hat{p}'^k \in T(\hat{x}^k + \hat{p}^k - \hat{p}'^k)$$

since

$$\hat{x}^k + \hat{p}^k = \hat{x}'^k + \hat{p}'^k. \tag{3.5}$$

In other words, $\hat{x}'^k$ is the image of $\hat{x}^k + \hat{p}^k$ under the proximal mapping for $T$. This means that

$$p_a'^k \in \partial\gamma_a(x^k - a + p_a^k - p_a'^k) \quad \text{for all } a$$

or in other words

$$(x^k - a + p_a^k) - p_a'^k \in N_a(p_a'^k) \quad \text{with } p_a'^k \in B_a^0 \text{ for all } a.$$

Therefore $p_a'^k$ is uniquely determined by

$$p_a'^k = \mathrm{Proj}_{B_a^0}(x^k - a + p_a^k) \tag{3.6}$$

where $\mathrm{Proj}_{B_a^0}(x)$ is the orthogonal projection of $x$ onto $B_a^0$. By (3.5) $\hat{x}'^k$ is found as $\hat{x}'^k = \hat{x}^k + \hat{p}^k - \hat{p}'^k$ and according to (3.2), (3.3) $\hat{x}^{k+1}$ and $\hat{p}^{k+1}$ are given by

$$\hat{x}^{k+1} = (\hat{x}'^k)_A, \tag{3.7}$$

$$\hat{p}^{k+1} = (\hat{p}'^k)_B. \tag{3.8}$$

To summarize, the algorithm for solving FW is the following:
Starting point: choose

$$x^0 \in \mathbb{R}^n,$$

$$\hat{p}^0 \in H \quad \text{such that} \quad \sum_a \omega_a p_a^0 = 0.$$

Step $k$: compute

$$p_a'^k = \mathrm{Proj}_{B_a^0}(x^k - a + p_a^k) \quad \text{for all } a,$$

$$x_a'^k = x^k + p_a^k - p_a'^k \quad \text{for all } a,$$

$$x^{k+1} = \frac{1}{m}\sum_b \omega_b x_b'^k,$$

$$p_a^{k+1} = p_a'^k - \frac{1}{m}\sum_b \omega_b p_b'^k \quad \text{for all } a.$$

**Remark 3.1.** In the general partial inverse method [28], as in the application developed in [29], $\hat{x}'^k$ is computed and then $\hat{p}'^k$ is determined by $\hat{p}'^k = \hat{x}^k + \hat{p}^k - \hat{x}'^k$. In our method however, we compute first $\hat{p}'^k$ and then $\hat{x}'^k$ by (3.5).

### *3.2. Comments about the implementation*

(a) Actually, $x^{k+1}$ can be determined without computing $\hat{x}'^k$. Since $\hat{p}^k \in B$ we have

$$x^{k+1} = x^k - \frac{1}{m}\sum_b \omega_b p_b'^k. \tag{3.9}$$

(b) The algorithm needs to compute the projection of points onto the unit balls $B_a^0$. This problem does not present any difficulty when $\gamma_a^0$ is the Euclidean norm, but if $\gamma_a^0$ is an $l^p$-norm, $1 < p < +\infty$, the projection cannot be explicitly obtained.

With polyhedral gauges, in dimension $n > 2$, the projection can be done by well known methods for solving linearly constrainted least squares problems. However, location problems most frequently occur in the plane. In this case, if we denote by $\mathrm{Ext}(C)$ the finite set of extreme points of a polyhedral convex $C$, the projection of a point $x \notin B_a^0$ onto $B_a^0$ is given by

$$\mathrm{Proj}_{B_a^0}(x) = \begin{cases} e^0 & \text{if } x - e^0 \in N_a(e^0) \text{ for some } e^0 \in \mathrm{Ext}(B_a^0), \\ x - (\mu_{\bar{e}}/\|\bar{e}\|_2)\bar{e} & \text{otherwise,} \end{cases}$$

where $\bar{e}$ is uniquely determined by

$$\mu_{\bar{e}} = \mathrm{Max}\{\mu_e,\ \mu_e = ((x, e) - 1)/\|e\|_2,\ \mu_e > 0,\ e \in \mathrm{Ext}(B_a)\}$$

$\|\cdot\|_2$ denoting the Euclidean norm.

(c) In practice, we also need a rule to determine when to stop the iterations according to a specified accuracy.

Recently, some lower and upper bounds have been developed [7, 8, 14, 20, 32] when the Fermat-Weber problem involving $l^p$-norms, $1 < p < +\infty$, is solved by an iterative procedure. Unfortunately these lower bounds do not converge to the optimal value of the objective function when non-differentiable norms are used. An advantage of our algorithm is that a rational stopping rule can be formulated by making use of converging lower and upper bounds.

As we can suppose without restriction that $\hat{p}^k \neq 0$, putting

$$\alpha_k = \operatorname*{Max}_a \{\gamma_a^0(p_a^k)\},$$

$$\hat{q}^k = (1/\alpha_k)\hat{p}^k,$$

we obtain a dual feasible sequence $\{\hat{q}^k\}$ converging to the dual solution $\hat{p}^\infty$, which is the limit of the sequence $\{\hat{p}^k\}$ (see Section 3.3). Then a lower bound $m_k$ and an upper bound $M_k$ can be defined by

$$m_k = -\sum_a \omega_a(a, q_a^k),$$

$$M_k = \sum_a \omega_a \gamma_a(x^k - a)$$

and these bounds converge to the optimal value of the objective function of FW.

### 3.3. Convergence

The general convergence of the proximal point algorithm associated with an arbitrary maximal monotone multifunction $S$ has been carefully studied in [27]. Consequently the proximal point algorithm applied to $T_A$ converges and allows us to state without need of proof the theorem:

**Theorem 3.1.** *The algorithm, described in Section* (3.1), *converges in the following sense*:

(1) $\hat{x}^k \to \hat{x}^\infty \in A$,

(2) $\hat{p}^k \to \hat{p}^\infty \in B$,

(3) $\|(\hat{x}^{k+1} + \hat{p}^{k+1}) - (\hat{x}^\infty + \hat{p}^\infty)\| \leq \|(\hat{x}^k + \hat{p}^k) - (\hat{x}^\infty + \hat{p}^\infty)\|$,

*where* $x^\infty$ *and* $\hat{p}^\infty$ *are respectively a primal and dual solution to* FW.

Actually Rockafellar proved that, if the multifunction $S$ is the subdifferential of a polyhedral convex closed function, the proximal point algorithm converges in a finite number of steps (see [27]); unfortunately, this is not the case for the multifunction $T_A$. However, the finite convergence of our algorithm will be studied in Section 4.

## 4. Finite convergence in the polyhedral case

In this section, we suppose that all the gauges $\gamma_a$ are polyhedral. Our aim is to prove that, if the interior of the set of optimal solutions is nonempty, the algorithm, previously given, converges in a finite number of steps; in other words, $\hat{x}^{k+m} = \hat{x}^\infty$ and also $\hat{p}^{k+m} = \hat{p}^\infty$ for some $k$ and every $m \geq 0$.

For this, we need some lemmas using the following property of polyhedral convex functions.

**Lemma 4.1.** *A polyhedral convex function f has the Diff-Max property meaning that for every $x \in \mathbb{R}^n$, there exists a neighbourhood V of x such that $\partial f(y) \subset \partial f(x)$ for all $y \in V$.*

**Proof.** This property has already been proved and/or used [9, 10, 12]. Here, we give only the proof for gauges.

Let $\gamma$ be a polyhedral gauge. If $x \neq 0$, the subdifferential $\partial\gamma(x)$ is (geometrically) an exposed face of $B^0$ the polar of the unit ball associated with $\gamma$. Since $\partial\gamma(0) = B^0$ only the case $x \neq 0$ needs to be proved.

Let $D(x)$ be the convex hull of the extreme points of $B^0$ which do not belong to $\partial\gamma(x)$. There exists a hyperplane $G$ strictly separating $\partial\gamma(x)$ and $D(x)$. As the multifunction $y \to \partial\gamma(y)$ is upper semicontinuous there exists a neighbourhood $V$ of $x$ such that $G$ separates $\partial\gamma(y)$ and $D(x)$ for $y \in V$. Consequently $\partial\gamma(y)$ is necessarily included in $\partial\gamma(x)$.

**Lemma 4.2.** *The sequences $\{\hat{x}'^k\}$ and $\{\hat{p}'^k\}$ generated by the algorithm converge to $\hat{x}^\infty$ and $\hat{p}^\infty$ respectively.*

*Furthermore, for all k sufficiently large, we have:*

(i) $x_a'^k \in a + N_a(p_a^\infty)$ *for all a,*

(ii) $x^\infty \in a + N_a(p_a'^k)$ *for all a.*

**Proof.** From equation (3.9) we deduce that $\lim_{k\to+\infty} \sum_a \omega_a p_a'^k = 0$, since the sequence $\{x^k\}$ converges to $x^\infty$. Hence equation (3.8) gives

$$\lim_{k\to+\infty} p_a'^k = \lim_{k\to+\infty} p_a^{k+1} = p_a^\infty$$

and equation (3.5) gives

$$\lim_{k\to+\infty} x_a'^k = \lim_{k\to+\infty} x^k = x^\infty.$$

Now the Diff-Max property of the gauges $\gamma_a$ shows that, for $k$ sufficiently large, we have:

$$p_a'^k \in \partial\gamma_a(x_a'^k - a) \subset \partial\gamma_a(x^\infty - a)$$

which means that $p_a'^k \in B_a^0$ and $x^\infty - a \in N_a(p_a'^k)$.

Similarly, the conjugate of $\gamma_a$, denoted $\gamma_a^*$, has the Diff-Max property since it too is polyhedral. Thus for $k$ sufficiently large, we have

$$x_a'^k - a \in \partial\gamma_a^*(p_a'^k) \subset \partial\gamma_a^*(p_a^\infty)$$

or in other words, $x_a'^k - a \in N_a(p_a^\infty)$.

Let us denote by $\mathscr{C}$ the elementary convex set which coincides (see [11]) with the set $M_\omega(\mathscr{A})$ of optimal solutions to FW.

**Lemma 4.3.** *Let $N(\mathscr{C}, x^{\infty})$ be the normal cone to $\mathscr{C}$ at $x^{\infty}$. Then we have $x^k - x^{k+1} \in N(\mathscr{C}, x^{\infty})$ for all $k$ sufficiently large.*

**Proof.** Let $x \in \mathscr{C} = \bigcap_a (a + N_a(p_a^{\infty}))$. Using (3.3) and (3.5), we deduce for all $a$ that:

$$x^k - x^{k+1} = p_a'^k - p_a^{k+1}.$$

Then we have

$$\begin{aligned}(x^k - x^{k+1}, x - x^{\infty}) &= (p_a'^k - p_a^{k+1}, x - x^{\infty})\\ &= (p_a'^k - p_a^{\infty}, x - x^{\infty}) + (p_a^{\infty} - p_a^{k+1}, x - x^{\infty}).\end{aligned}$$

After multiplying each equation by $\omega_a$ and summing on $a$, we deduce that

$$\begin{aligned}m(x^k - x^{k+1}, x - x^{\infty}) &= \sum_a \omega_a (p_a'^k - p_a^{\infty}, x - x^{\infty})\\ &= \sum_a \omega_a (p_a'^k - p_a^{\infty}, x - a) - \sum_a \omega_a (p_a'^k - p_a^{\infty}, x^{\infty} - a).\end{aligned}$$

Using Lemma 4.2, for $k$ sufficiently large, we get $\sum_a \omega_a (p_a'^k - p_a^{\infty}, x^{\infty} - a) = 0$ and as $x \in \mathscr{C}$, we obtain

$$\begin{aligned}m(x^k - x^{k+1}, x - x^{\infty}) &= \sum_a \omega_a (p_a'^k, x - a) - \sum_a \omega_a \gamma_a (x - a)\\ &\leqslant 0 \quad \text{since } p_a'^k \in B_a^0\end{aligned}$$

which completes this proof.

Now, as $N(\mathscr{C}, x^{\infty})$ is a closed convex cone, we easily deduce the following corollary.

**Corollary 4.1.** *We have $x^k - x^{\infty} \in N(\mathscr{C}, x^{\infty})$ for all $k$ sufficiently large.*

**Lemma 4.4.** *We have $(\hat{x}'^k - \hat{x}^{\infty}, \hat{p}'^k - \hat{p}^{\infty}) = 0$ for all $k$ sufficiently large.*

**Proof.** Lemma 4.2 states that, for large values of $k$,

$$x_a'^k - a \in N_a(p_a^{\infty})$$

and

$$x^{\infty} - a \in N_a(p_a'^k)$$

which give, using (3.4),

$$(x_a'^k - a, p_a^{\infty}) = \gamma_a(x_a'^k - a) = (x_a'^k - a, p_a'^k)$$

and

$$(x^{\infty} - a, p_a'^k) = \gamma_a(x^{\infty} - a) = (x^{\infty} - a, p_a^{\infty}).$$

Subtracting these equations, we obtain

$$(x_a'^k - x^{\infty}, p_a'^k - p_a^{\infty}) = 0$$

which means that

$$(\hat{x}'^k - \hat{x}^{\infty}, \hat{p}'^k - \hat{p}^{\infty}) = 0.$$

**Remark 4.1.** It may be seen, using (3.2) and (3.3) that the equation $(\hat{x}'^k - \hat{x}^\infty, \hat{p}'^k - \hat{p}^\infty) = 0$ can also be written as $(\hat{z}^k - \hat{z}^{k+1}, \hat{z}^{k+1} - \hat{z}^\infty) = 0$.

**Lemma 4.5.** *Let us assume that* $\text{int}(\mathscr{C}) \neq \emptyset$. *There exists* $k_0$ *such that if* $x^k = x^\infty$ *for some* $k \geq k_0$ *then the algorithm converges in a finite number of steps.*

**Proof.** Let $k_0$ be chosen sufficiently large with respect to Lemma 4.2, Lemma 4.3 and Corollary 4.1.

Assuming that $x^k = x^\infty$ for some $k \geq k_0$, we obtain

$$x^\infty - x^{k+1} \in N(\mathscr{C}, x^\infty)$$

and

$$x^{k+1} - x^\infty \in N(\mathscr{C}, x^\infty).$$

As $\text{int}(\mathscr{C}) \neq \emptyset$, the normal cone $N(\mathscr{C}, x^\infty)$ only contains the subspace $\{0\}$ (see [13, 29]), thus $x^{k+1} = x^k$ and by induction we find that $x^{k+m} = x^k$ for every $m \geq 0$. Then equation (3.2) gives for $m \geq 0$

$$\sum_a \omega_a p_a'^{k+m} = 0$$

which means that $\hat{p}'^{k+m}$ is dual feasible.

From Lemma 4.2 we find that

$$x^\infty \in \bigcap_a (a + N_a(p_a'^{k+m})).$$

Consequently, the optimality conditions are satisfied by $x^\infty$ and $\hat{p}'^{k+m}$; it follows from the unicity of the dual solution (Remark 2.1) that $\hat{p}'^{k+m} = \hat{p}^\infty$. Finally, equation (3.8) implies that $\hat{p}^{k+m} = \hat{p}'^{k+m} = \hat{p}^\infty$ which terminates the proof.

**Theorem 4.1.** *If* $\text{int}(\mathscr{C}) \neq \emptyset$, *the algorithm converges in a finite number of steps.*

**Proof.** First of all, let us assume that $x^\infty \in \text{int}(\mathscr{C})$. In this case, $N(\mathscr{C}, x^\infty) = \{0\}$. Thus for large values of $k$, we have $x^k = x^\infty$ and the result is obtained by Lemma 4.5.

Now suppose that $x^\infty$ belongs to the boundary $\text{Bd}(\mathscr{C})$ of $\mathscr{C}$ and let us assume that $x^k \neq x^\infty$ for all $k$ sufficiently large. J.E. Spingarn proved in [29] that the proximal point algorithm with respect to a maximal monotone multifunction cannot generate a sequence $\{\hat{z}^k\}$ satisfying the following properties:

(1) $\hat{z}^k \to \hat{z}^\infty$,

(2) $(\hat{z}^k - \hat{z}^{k+1}, \hat{z}^{k+1} - \hat{z}^\infty) = 0$,

(3) $\{l(\hat{z}^k)\}$ is strictly increasing, where $l: H \to \mathbb{R}$ is a linear function.

The sequence $\{\hat{z}^k\}$ already possesses properties (1) and (2) (see Lemma 4.4 and Remark 4.1), thus in order to prove that our assumption fails, we shall construct a linear function which satisfies property (3).

As $\operatorname{int}(\mathscr{C}) \neq \emptyset$ (and $x^\infty \in \operatorname{Bd}(\mathscr{C})$), there exists, [13, 29], $\eta \in N(\mathscr{C}, x^\infty)$ such that $(\eta, x) > 0$ for all $x \in N(\mathscr{C}, x^\infty)$, $x \neq 0$.

The elementary convex set $\mathscr{C}$ is the intersection of the polyhedral cones $\mathscr{C}_a = a + N_a(p_a^\infty)$, hence we have [26]

$$N(\mathscr{C}, x^\infty) = \sum_a N(\mathscr{C}_a, x^\infty)$$

where $N(\mathscr{C}_a, x^\infty)$ denotes the normal cone to $\mathscr{C}_a$ at $x^\infty$. Then, $\eta$ can be written as

$$\eta = \sum_a \omega_a \eta_a \quad \text{with } \eta_a \in N(\mathscr{C}_a, x^\infty)$$

and a linear function $l: H \to \mathbb{R}$ can be defined by

$$l(\hat{z}) = (\hat{\eta}, \hat{z}_B) \quad \text{with } \hat{\eta} = (\eta_a)_{a \in \mathscr{A}}.$$

From relation (3.3) we obtain

$$\begin{aligned} l(\hat{z}^{k+1}) - l(\hat{z}^k) &= (\hat{\eta}, \hat{p}^{k+1} - \hat{p}^k) \\ &= -\sum_a \omega_a(\eta_a, x_a'^k - x^{k+1}) \\ &= -\sum_a \omega_a(\eta_a, x_a'^k - x^\infty) + (\eta, x^{k+1} - x^\infty). \end{aligned}$$

But for $k$ sufficiently large $(\eta_a, x_a'^k - x^\infty) \leq 0$ since $x_a'^k \in \mathscr{C}_a$ (Lemma 4.2) and $(\eta, x^{k+1} - x^\infty) > 0$ since $x^{k+1} - x^\infty \in N(\mathscr{C}, x^\infty)$ by Corollary 4.1. Hence the sequence $\{l(\hat{z}^k)\}$ is strictly increasing and the assumption $x^k \neq x^\infty$ for all $k$ sufficiently large is contradicted. Thus, the finite convergence is again obtained by Lemma 4.5.

**A counterexample**

In $\mathbb{R}^2$, we consider $\mathscr{A} = \{a_1, a_2, a_3\}$ with $a_1 = (0; 3)$, $a_2 = (-2; 0)$, $a_3 = (2; 0)$.

We take $\omega_a = 1$ and $\gamma_a = \gamma$ for each $a \in \mathscr{A}$, where $\gamma$ is the $l^1$-norm. The set of primal solutions $M_\omega(\mathscr{A})$ is the singleton $\{\bar{x}\}$ with $\bar{x} = (0; 0)$ and the set of dual solutions is made up of the vectors

$$p_1 = (0; -1), \qquad p_2 = (1; \lambda), \qquad p_3 = (-1; 1 - \lambda)$$

with any choice of $\lambda \in [0, 1]$.

Let $x^0 = (0; 1/3)$, $p_1^0 = (0; -1)$, $p_2^0 = (1; \frac{1}{2})$, $p_3^0 = (-1; \frac{1}{2})$ be the starting point. It is easy to prove by induction that the algorithm generates the sequences $x^k$, $p_1^k$, $p_2^k$, $p_3^k$ given by

$$\begin{aligned} x^k &= (0; \alpha_k), \\ p_1^k &= (0; -1 - 2\beta_k), \\ p_2^k &= (1; \beta_k + \tfrac{1}{2}), \\ p_3^k &= (-1; \beta_k + \tfrac{1}{2}) \end{aligned}$$

where

$$\alpha_k = \frac{1}{3}\left(\frac{\sqrt{3}}{3}\right)^k \cos k\theta,$$

$$\beta_k = \frac{\sqrt{2}}{6}\left(\frac{\sqrt{3}}{3}\right)^k \sin k\theta,$$

and $\theta \in [0, 2\pi]$ is defined by $\cos\theta = \sqrt{3}/3$ and $\sin\theta = \sqrt{6}/3$.

The sequences $\{x^k\}$ and $\{\hat{p}^k\}$ converge respectively to $x^\infty$ and $\hat{p}^\infty$ defined by

$$x^\infty = (0; 0), \qquad p_1^\infty = (0; -1), \qquad p_2^\infty = (1; \tfrac{1}{2}), \qquad p_3^\infty = (-1; \tfrac{1}{2})$$

and the limit is not reached in a finite number of iterations.

We can see that neither the convergence of $\{x^k\}$ nor the convergence of $\{\hat{p}^k\}$ is linear. However, using the relations

$$\alpha_{k+1} = (\alpha_k - 2\beta_k)/3,$$

$$\beta_{k+1} = (\alpha_k + \beta_k)/3,$$

we deduce that

$$\left\| \hat{z}^{k+1} - \hat{z}^\infty \right\| = \frac{\sqrt{3}}{3} \left\| \hat{z}^k - \hat{z}^\infty \right\|.$$

This equality means that the sequence $\{\hat{x}^k + \hat{p}^k\}$ converges at a linear rate.

## 5. Numerical tests

The given algorithm has been implemented in FORTRAN using a MATRA 550-CX at the Dijon University Computing Center. Single precision arithmetic was used throughout. The computer executes approximately 1.25 million machine instructions per second. It uses 4 gigabytes of real main memory with approximately 400 ns access time.

A number of test problems were solved and a selection of five (low dimension) problems has been chosen to emphasize the behavior of tthe algorithm. Results concerning the first four problems are summarized in Tables 1 to 4 respectively. The times indicated in seconds do not include input/output times and the value of $\varepsilon$ used in the stopping rule $|M_k - m_k| < \varepsilon$ was fixed to $10^{-5}$.

It is interesting to note that problems 1 and 3 have been solved very efficiently. The algorithm requires few iterations due to the fact that for these problems the convergence is finite according to Theorem 4.1.

Furthermore it is worth noting that in all cases (see Tables 1 to 4) the algorithm generates an optimal location which lies in the relative interior of $M_\omega(\mathscr{A})$ even when the starting point is an optimal location belonging to the boundary of $M_\omega(\mathscr{A})$.

**Problem 1.** We consider $\mathscr{A}=\{a_1, a_2, \ldots, a_6\}$ with $a_1=(1;0)$, $a_2=(3;0)$, $a_3=(\frac{3}{2};\frac{5}{2})$, $a_4=(2;3)$, $a_5=(0;1)$, $a_6=(3;2)$ and $\omega_1=\omega_2=\cdots=\omega_6=1$. We choose a single polyhedral norm $\gamma$, the unit ball of which is a regular octagon inscribed in a circle centered at the origin and with radius one (with respect to the Euclidean norm).

The set of optimal locations is the polygon generated by the points whose coordinates are $(\frac{3}{2};\frac{5}{2})$, $(\frac{3}{2};2)$, $(2;2)$, $(2;1)$ and which is uniquely determined by the dual variables

$$p_1=(\sqrt{2}-1;1),\qquad p_2=(1-\sqrt{2};1),\qquad p_3=(-1+\sqrt{2};-1),$$
$$p_4=(1-\sqrt{2};-1),\qquad p_5=(1;\sqrt{2}-1),\qquad p_6=(-1;1-\sqrt{2}).$$

**Problem 2.** We consider the same data as in problem 1 where the weight $\omega_2=1$ is replaced by $\omega_2=1+\sqrt{2}$. The set $M_\omega(\mathscr{A})$ is the singleton $\{(2;1)\}$ which is (not uniquely) determined by the dual variables

$$p_1=(\sqrt{2}/2;\sqrt{2}/2),\qquad p_2=(1-\sqrt{2};1),\qquad p_3=(-1+\sqrt{2};-1),$$
$$p_4=(1-\sqrt{2};-1),\qquad p_5=(1;1-\sqrt{2}),\qquad p_6=(-\sqrt{2}/2;-\sqrt{2}/2).$$

**Problem 3.** We consider $\mathscr{A}=\{a_1, a_2, \ldots, a_6\}$ with $a_1=(0;2)$, $a_2=(0;1)$, $a_3=(2;3)$, $a_4=(2;0)$, $a_5=(3;2)$, $a_6=(3;1)$ and $\omega_1=\omega_2=\omega_3=\omega_4=1$, $\omega_5=\omega_6=\frac{1}{2}$. We choose three polyhedral norms; the octagonal norm (as in problem 1) associated with $a_1$

Table 1

| | | | | | | |
|---|---|---|---|---|---|---|
| Starting point | 0.<br>0. | 3.<br>0. | 0.<br>3. | 5.<br>5. | −10.<br>0. | 2.*<br>1. |
| Number of iterations | 9 | 8 | 8 | 10 | 18 | 5 |
| CPU (sec.) | 0.08 | 0.08 | 0.08 | 0.08 | 0.14 | 0.04 |
| Optimal location | 1.80935<br>1.44036 | 1.83789<br>1.21137 | 1.72420<br>1.91253 | 1.79514<br>1.77469 | 1.73088<br>1.58802 | 1.89552<br>1.26965 |
| Optimal value | 9.94974 | — | — | — | — | — |

* Starting point in $M_\omega(\mathscr{A})$.

Table 2

| | | | | | | |
|---|---|---|---|---|---|---|
| Starting point | 0.<br>0. | 3.<br>0. | 0.<br>3. | 5.<br>5. | −10.<br>0. | 2.*<br>1. |
| Number of iterations | 63 | 53 | 59 | 63 | 67 | 53 |
| CPU (sec.) | 0.62 | 0.54 | 0.56 | 0.60 | 0.66 | 0.54 |
| Optimal location | 1.99999<br>0.99999 | 2.00000<br>1.00000 | 1.99999<br>0.99999 | 1.99999<br>0.99999 | 2.00000<br>1.00000 | 2.00000<br>1.00000 |
| Optimal value | 11.94974 | — | — | — | — | — |

* Starting point in $M_\omega(\mathscr{A})$

Table 3

| | | | | | |
|---|---|---|---|---|---|
| Starting point | 0.<br>0. | 3.<br>0. | 2.<br>0. | −20.<br>2. | 2.*<br>1.5 |
| Number of iterations | 8 | 6 | 6 | 37 | 3 |
| CPU (sec.) | 0.08 | 0.06 | 0.08 | 0.30 | 0.04 |
| Optimal location | 0.75651<br>1.48756 | 1.72839<br>1.25395 | 1.79404<br>1.24463 | 0.86679<br>1.55735 | 1.77777<br>1.50000 |
| Optimal value | 8.91421 | — | — | — | — |

* Starting point in $M_\omega(\mathscr{A})$.

and $a_2$, the norm defined by $\gamma(x_1; x_2) = \frac{1}{2}|x_1| + |x_2|$ associated with $a_3$ and $a_4$, the $l^1$-norm associated with $a_5$ and $a_6$. The set $M_\omega(\mathscr{A})$ is the polygon generated by the points whose coordinates are $(1; 1)$, $(\frac{1}{2}; \frac{3}{2})$, $(1; 2)$, $(2; 2)$, $(2; 1)$ and which is uniquely determined by the dual variables

$$p_1 = (1; 1-\sqrt{2}), \qquad p_2 = (1; \sqrt{2}-1), \qquad p_3 = (-\tfrac{1}{2}; -1),$$

$$p_4 = (-\tfrac{1}{2}; 1), \qquad p_5 = (-1; -1), \qquad p_6 = (-1; +1).$$

**Problem 4.** We consider the same facilities and the same norms as in problem 3 with the weights $\omega_1 = \omega_2 = \cdots = \omega_6 = 1$. The set $M_\omega(\mathscr{A})$ is the segment of line defined by the points $(2; 1)$, $(2; 2)$ and which is uniquely determined by the dual variables

$$p_1 = (1; 1-\sqrt{2}), \qquad p_2 = (1; \sqrt{2}-1), \qquad p_3 = (0; -1),$$

$$p_4 = (0; +1), \qquad p_5 = (-1; -1), \qquad p_6 = (-1; +1).$$

**Problem 5.** We consider $\mathscr{A} = \{a_1, a_2, \ldots, a_5\}$ with $a_1 = (0; 0)$, $a_2 = (1; 0)$, $a_3 = (1; 1)$, $a_4 = (0; 1)$, $a_5 = (100; 100)$ and $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$, $\omega_5 = 4$. Distances are measured by the Euclidean norm. The set $M_\omega(\mathscr{A})$ is the singleton $\{a_5\}$. This example is known

Table 4

| | | | | | |
|---|---|---|---|---|---|
| Starting point | 0.<br>0. | 3.<br>0. | 2.<br>0. | −20.<br>2. | 2.*<br>1. |
| Number of iterations | 63 | 58 | 50 | 86 | 4 |
| CPU (sec) | 0.54 | 0.50 | 0.44 | 0.70 | 0.04 |
| Optimal location | 2.00000<br>1.41690 | 1.99999<br>1.44695 | 2.00000<br>1.42585 | 2.00000<br>1.60079 | 2.00000<br>1.34325 |
| Optimal value | 10.41421 | — | — | — | — |

* Starting point in $M_\omega(\mathscr{A})$

to be difficult to solve because the objective function decreases very slowly along the diagonal of $\mathbb{R}^2$. Our algorithm starting from (90.; 90.) requires 1001 iterations and 1.16 seconds of CPU time to obtain the location (100.00; 99.99) whereas Weiszfeld's algorithm gives (97.2; 97.2) after 200 000 iterations which required 342 seconds including evaluation of lower and upper bounds of the optimal value as defined in Wendell and Peterson [32].

A referee has informed us that the Calamai-Conn algorithm [4], starting from the same initial point, requires only 3 iterations and considerably less time (14.84 seconds of CPU time on an IBM-PC executing approximately 0.0064 million instructions per second) to solve this problem. These results are not surprising since the Calamai-Conn algorithm belongs to the class of second order methods which are known to be very efficient near an optimal solution.

## 6. Concluding remarks

The algorithm described in this paper has been obtained from the partial inverse method of J.E. Spingarn which has attracted attention because of its various applications in convex programming.

This suggests that our algorithm can be generalized for the Fermat-Weber problem with polyhedral constraints and for the minisum multifacility location problem. These problems of great interest in the framework of Location Theory will be studied in separate papers.

Furthermore, the procedure for solving the Fermat-Weber problem and the algorithm of J.E. Spingarn for finding a point in a polyhedron generate (when the interior of the set of optimal solutions is nonempty) finite sequences which enjoy the same properties (see Lemmas 4.2-4.4 and [29]).

In a forthcoming paper, it will be shown (under appropriate assumptions) that the proximal point algorithm applied to the partial inverse of a maximal monotone multifunction $T$ always terminates in a finite number of steps when $T$ and $T^{-1}$ possess the following property:

$$\forall x \quad \exists \varepsilon > 0 \quad \text{such that } T(y) \subset T(x) \quad \text{for } \|y - x\| < \varepsilon$$

which means that each point is a local maximum for $T$ with respect to the inclusion relation.

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## References

[1] R.D. Armstrong, "A primal simplex algorithm to solve a rectilinear distance facility location problem," *Naval Research Logistics Quarterly* 24 (1977) 619-626.

[2] L. Cooper and I.N. Katz, "The Weber problem revisited," *Computers and Mathematics with Applications* 7 (1981) 225-234.

[3] P.H. Calamai and A.R. Conn, "A stable algorithm for solving the multifacility location problem involving Euclidean distances," *SIAM Journal on Scientific and Statistical Computing* 1 (1980) 512-526.

[4] P.H. Calamai and A.R. Conn, "A second order method for solving the continuous multifacility location problem," in: G.A. Watson, ed., *Numerical Analysis: Proceedings of the Ninth Biennial Conference. Dundee, Scotland,* Lecture Notes in Mathematics 912 (Springer-Verlag, Berlin, Heidelberg and New York, 1982), pp. 1-25.

[5] F. Cordellier and J.C. Fiorot, "On the Fermat-Weber problem with convex cost functions," *Mathematical Programming* 14 (1978) 295-311.

[6] A. Dax, "The use of Newton's method for solving Euclidean multifacility location problems," Tech. Report. Hydrological Service (Jerusalem, 1985).

[7] P.D. Dowling and R.F. Love, "Bounding methods for facilities location algorithms," Research and Working Paper Series No. 219, McMaster University (Hamilton, Ontario, 1984).

[8] P.D. Dowling and R.F. Love, "An evaluation of the dual as a lower bound in facilities location problems," Research and Working Paper Series No. 236, McMaster University (Hamilton, Ontario, 1985).

[9] R. Durier, "On efficient points and Fermat-Weber problem," Working Paper, Université de Bourgogne (Dijon, France, 1984).

[10] R. Durier, "Weighting factor results in vector optimization," Working Paper, Université de Bourgogne (Dijon, France, 1984).

[11] R. Durier and C. Michelot, "Geometrical properties of the Fermat-Weber problem," *European Journal of Operational Research* 20 (1985) 332-343.

[12] R. Durier and C. Michelot, "Sets of efficient points in a normed space," *Journal of Mathematical Analysis and Applications* 117 (1986), 506-528.

[13] U. Eckhardt, "Theorems on the dimension of convex sets," *Linear Algebra and its Applications* 12 (1975) 63-76.

[14] D.J. Elzinga and D.W. Hearn, "On stopping rules for facilities location algorithms," *IIE Transactions* 15 (1983) 81-83.

[15] R.L. Francis and J.M. Goldstein, "Location theory: A selective bibliography," *Operations Research* 22 (1974) 400-410.

[16] R.L. Francis and J.A. White, *Facility Layout and Location: An Analytical Approach* (Prentice-Hall, Englewood Cliffs, NJ, 1974).

[17] I.N. Katz, "Local convergence in Fermat's problem," *Mathematical Programming* 6 (1974) 89-104.

[18] H.W. Kuhn, "A note on Fermat's problem," *Mathematical Programming* 4 (1973) 98-107.

[19] H.W. Kuhn and R.E. Kuenne, "An efficient algorithm for the numerical solution of the generalized Weber problem in spatial economics," *Journal of Regional Science* 4 (1962) 21-23.

[20] R.F. Love and W.Y. Yeong, "A stopping rule for facilities location algorithms," *AIIE Transactions* 13 (1981) 357-362.

[21] L.M. Ostresch, "On the convergence of a class of iterative methods for solving the Weber location problem," *Operations Research* 26 (1978) 597-609.

[22] L.M. Ostresh, "Convergence and descent in the Fermat location problem," *Transportation Science* 12 (1978) 153-164.

[23] M.L. Overton, "A quadratically convergent method for minimizing a sum of Euclidean norms," *Mathematical Programming* 27 (1983) 34-63.

[24] A. Planchart and A.P. Hurter, "An efficient algorithm for the solution of Weber problem with mixed norms," *SIAM Journal on Control* 13 (1975) 650–665.

[25] F. Plastria, "Continuous location problems solved by cutting planes," Working Paper, Vrije Universiteit Brussels, (Bruxelles, 1982).

[26] R.T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).

[27] R.T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization* 14 (1976) 877–898.

[28] J.E. Spingarn, "Partial inverse of a monotone operator," *Applied Mathematics and Optimization* 10 (1983) 247–265.

[29] J.E. Spingarn, "A primal-dual projection method for solving systems of linear inequalities," *Linear Algebra and its Applications* 65 (1985) 45–62.

[30] J.E. Ward and R.E. Wendell, "Using block norms for location modeling," *Operations Research* 33 (1985) 1074–1090.

[31] E. Weiszfeld, "Sur le point pour lequel la somme des distances de *n* points donnés est minimum," *The Tôhoku Mathematical Journal* 43 (1937) 355–386.

[32] R.E. Wendell and E.L. Peterson, "A dual approach for obtaining lower bounds to the Weber problem," *Journal of Regional Science* 24 (1984) 219–228.

[33] C. Witzgall, "Optimal location of a central facility: Mathematical models and Concepts," National Bureau of Standards, Report 8388 (Washington, 1964).