Extension of functions satisfying Lipschitz conditions

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Introduction

Let F be a compact set in Euclidean n-space R^n. Suppose that the function $\varphi(P)$ is defined on F and that it satisfies a Lipschitz condition.[1] It is known that φ can be extended to R^n in such a way that the new function f satisfies the same Lipschitz condition. An explicit construction was given in [1], where even more general situations were treated. It is easy to give upper and lower bounds for the solutions. We shall do this, and also discuss questions of uniqueness. It turns out that all solutions agree on a set E which has a very simple structure (Theorem 2).

In analogy with the treatment in [2] and [3], we shall consider a subclass of solutions (called absolute minimals) which have the additional property of being solutions of a corresponding problem on each subregion of $R^n - F$. The partial differential equation $\sum_{i,j=1}^{n} \phi_{x_i} \phi_{x_j} \phi_{x_i x_j} = 0$ is derived in a purely formal manner and it turns out that a smooth function satisfies this equation if and only if it is an absolute minimal (Theorem 8). We shall also give an existence proof for absolute minimals.

In a later paper, the two-dimensional case, and in particular the differential equation $\phi_x^2 \phi_{xx} + 2 \phi_x \phi_y \phi_{xy} + \phi_y^2 \phi_{yy} = 0$, will be studied more closely.

Notations and conventions

$(x_1, x_2, ..., x_n)$ are Cartesian coordinates in R^n.
As a rule, the points will be called A, B, C, ...
$(x_1, x_2, ..., x_n)$ will sometimes be written as x.
$\overline{PQ}$ = the Euclidean distance = $(\sum_{i=1}^{n} (x_i(P) - x_i(Q))^2)^{1/2}$.
$\overrightarrow{Q_1 Q_2}$ is the vector with components $x_i(Q_2) - x_i(Q_1)$.
If the function g is defined on the set E, then

$$\mu(g, E) = \sup_{\substack{P_1, P_2 \in E \\ P_1 \neq P_2}} \frac{|g(P_1) - g(P_2)|}{\overline{P_1 P_2}}.$$

The function φ to be extended is given on the compact set F. We write $\lambda = \mu(\varphi, F)$ and assume $0 < \lambda < \infty$. ∂G = the boundary of G; $\overline{G} = G \cup \partial G$; $CG = R^n - G$. Only real quantities will be considered.

A *solution* of the extension problem is an extension f of φ to some specified set $F_1 \supset F$, such that $\mu(f, F_1) = \lambda$. First, we take $F_1 = R^n$.

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[1] I.e. a Hölder condition with the exponent = 1.

1. The upper and lower functions

Consider now the extension problem which was proposed above. We have $\lambda = \mu(\varphi, F) > 0$. Assume that $f(P)$ is a solution of the problem. This means that

$$\varphi(Q) - \lambda \, \overline{PQ} \leqslant f(P) \leqslant \varphi(Q) + \lambda \, \overline{PQ}$$

for all $Q \in F$ and $P \in R^n$, and consequently

$$\sup_{Q \in F} (\varphi(Q) - \lambda \overline{PQ}) \leqslant f(P) \leqslant \inf_{Q \in F} (\varphi(Q) + \lambda \overline{PQ}).$$

It is easy to verify that the functions

$$u(P) = \sup_{Q \in F} (\varphi(Q) - \lambda \overline{PQ}) \tag{1}$$

and

$$v(P) = \inf_{Q \in F} (\varphi(Q) + \lambda \overline{PQ}) \tag{2}$$

agree with φ on F and satisfy Lipschitz conditions with the constant λ in R^n. Hence they are solutions of the extension problem. This method of extension was proposed in [1] and [4] (footnote on p. 63).

We have thus

Theorem 1. *The functions u and v, defined by (1) and (2), are solutions of the extension problem and an arbitrary solution f satisfies the inequalities $u \leqslant f \leqslant v$ (in R^n).*

We call u and v the *lower* and *upper* functions, respectively. The following assertions follow easily from the fact that u and v are the extreme solutions of the extension problem.

Corollary. *If G is a bounded region such that G and F have no points in common, then*

$$u(P) = \sup_{R \in \partial G} (u(R) - \lambda \overline{PR}) \text{ for all } P \in \bar{G},$$

$$v(P) = \inf_{R \in \partial G} (v(R) + \lambda \overline{PR}) \text{ for all } P \in \bar{G}.$$

2. Questions of uniqueness

It is clear from Theorem 1 that all solutions of the extension problem agree at a point $P \notin F$ if and only if $u(P) = v(P)$. In this case there exist Q_1 and $Q_2 \in F$ such that

$$\varphi(Q_1) + \lambda \, \overline{PQ_1} = \varphi(Q_2) - \lambda \, \overline{PQ_2}, \text{ i.e.}$$

$$\lambda(\overline{PQ_1} + \overline{PQ_2}) = \varphi(Q_2) - \varphi(Q_1) \leqslant \lambda \, \overline{Q_1 Q_2} \leqslant \lambda(\overline{PQ_1} + \overline{PQ_2}).$$

It follows that

$$\overline{PQ_1} + \overline{PQ_2} = \overline{Q_1 Q_2}$$

and

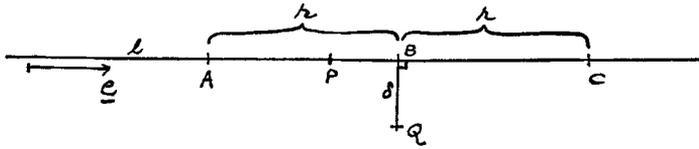$$\varphi(Q_2) - \varphi(Q_1) = \lambda \, \overline{Q_1 Q_2}.$$

Fig. 1

It is seen from the first relation that P, Q_1 and Q_2 are situated on a straight line, with P between Q_1 and Q_2.

Conversely, if Q_1 and Q_2 are points in F such that $\left|\varphi(Q_1)-\varphi(Q_2)\right| = \lambda\,\overline{Q_1Q_2}$, then it is clear that all solutions agree on the straight segment $\overline{Q_1Q_2}$ (which we call a *critical segment*). It is also clear that the solution on Q_1Q_2 is a linear function of arc length.

We introduce the notation $E = \{P\,|\,P \notin F,\; u(P) = v(P)\}$, i.e. E is the set of uniqueness. Consider a point $P \in E$ and let l be a critical segment through P. Let f be an arbitrary solution of the extension problem. We assert that f is differentiable at P and that grad $f(P) = \lambda\,\mathbf{e}$, where $\mathbf{e}$ is a unit vector along l in the direction of increasing f.

In order to verify this we consider a point Q in some neighbourhood of P, and its projection B on l. See Fig. 1. We have

$$f(C) - \lambda\sqrt{r^2 + \delta^2} \leqslant f(Q) \leqslant f(A) + \lambda\sqrt{r^2 + \delta^2},$$

$$f(C) - \lambda r = f(B) = f(A) + \lambda r.$$

Hence $\left|f(Q) - f(B)\right| \leqslant \lambda(\sqrt{r^2 + \delta^2} - r)$. If Q is restricted to a suitable neighbourhood of P, then r can be chosen independently of Q and we get

$$\left|f(Q) - f(B)\right| \leqslant \lambda r\left(\sqrt{1 + \left(\frac{\delta}{r}\right)^2} - 1\right) = O(\delta^2).$$

The rest of the proof follows easily, since f is linear on l.

It follows from this that the vector $\mathbf{e}$ is uniquely determined by $P \in E$, and if there are several critical segments through P, then they all constitute parts of one and the same line through P. Finally, since $\mathbf{e} = \mathbf{e}(P)$ is unique, it is easily proved (indirectly, by a selection argument) that $\mathbf{e}(P)$ is a continuous function on E.

The above considerations are summarized in the following theorem:

Theorem 2. *The set of uniqueness E is determined by φ and F in the following way: Let $\mathcal{L}$ be the collection of straight segments between points $Q_1, Q_2 \in F$ for which* $\left|\varphi(Q_1)-\varphi(Q_2)\right| = \lambda\,\overline{Q_1Q_2}$. *Let L be the point set covered by $\mathcal{L}$.*
   Then

$$E = L \cap CF.$$

There exists a continuous vector function $\mathbf{e}(P)$, defined on E, with the following property: if $P \in E$ and Q_1, Q_2 are the endpoints of any critical segment passing through P, then

$$\mathbf{e}(P) = \operatorname{sign}\,(\varphi(Q_2) - \varphi(Q_1))\,\frac{\overrightarrow{Q_1\,Q_2}}{\overline{Q_1\,Q_2}}.$$

Finally, every solution f of the extension problem is differentiable on E and satisfies the relation

$$\operatorname{grad} f(P) = \lambda \cdot \mathbf{e}\,(P).$$

It can occur that E is empty, as can be seen from the example in the end of the next section.

So far, we have considered the problem to extend φ from an arbitrary compact set F to the whole space R^n. However, one can consider each component G of $R^n - F$ separately, and it is then sufficient to know φ on ∂G. *In the rest of this paper, we shall assume that φ is given on the boundary ∂D of a bounded region D and discuss the extension (interpolation) of φ into D.* The results derived so far hold in this case with obvious modifications. (We have

$$\lambda = \sup_{\substack{P,\,Q\,\in\,\partial D \\ P\,\neq\,Q}} \frac{|\varphi(P) - \varphi(Q)|}{\overline{PQ}},$$

$$u(P) = \sup_{Q\,\in\,\partial D} (\varphi(Q)) - \lambda\overline{PQ}), \quad v(P) = \inf_{Q\,\in\,\partial D} (\varphi(Q) + \lambda\overline{PQ})$$

and the set of uniqueness $E = L \cap D$.)

The following theorem is a consequence of Theorem 2:

Theorem 3. *Suppose that the solution is unique in D (i.e. $E = D$ and $u = v$ in D). Then $u \in C^1(D)$ and $|\operatorname{grad} u| = \lambda$ in D.*

This theorem should be compared with the corollary of Theorem 9 in [2].

Lemma 1. *Let Ω be an arbitrary region in R^n. Assume that $\phi \in C^1(\Omega)$ and that $|\operatorname{grad}\phi| = M$ in Ω, where M is a constant > 0.*

Then each point of Ω belongs to one and only one trajectory of the vector field $\operatorname{grad}\phi$. Every such trajectory is a straight line, and it can be continued up to the boundary $\partial\Omega$.

(The trajectories coincide with the characteristics of the differential equation $(\operatorname{grad}\phi)^2 = M^2$. By a trajectory we mean a smooth curve, such that its tangent is always parallel to $\operatorname{grad}\phi$.)

Proof. The differential equation $(\operatorname{grad}\phi)^2 = M^2$ is treated by means of characteristic theory in [5], p. 40, and [6], p. 88. However, the following simple argument leads directly to the desired result: Consider a point $P \in \Omega$. According to Peano's existence theorem, there is a trajectory γ through P. Let Q, R be two points on γ, such that the segment $QR \subset \Omega$, and let $\phi(Q) < \phi(R)$. Then

$$\int_{QR} \operatorname{grad}\phi \cdot ds = \int_{\gamma_{QR}} \operatorname{grad}\phi \cdot ds = M \cdot L(\gamma_{QR}),$$

where $L(\gamma_{QR}) = $ the length of the arc γ_{QR}. But

$$\int_{QR} \operatorname{grad}\phi \cdot ds \leqslant M \cdot \overline{QR}.$$

Hence $L(\gamma_{QR}) \leqslant \overline{QR}$, i.e. γ_{QR} is a straight line, and the rest of the proof is obvious. Observe that existence of second-order derivatives is not needed in the proof.

Theorem 4. (I) *If one of the upper and lower functions belongs to $C^1(D)$, then they are identical, $u \equiv v$, and $E = D$.*

(II) *In each case below, f is assumed to be a function in $C^1(D) \cap C(\bar{D})$ with the proper boundary values:*

A. *If $|\text{grad } f|$ is constant in D, then f is a solution of the extension problem (but not necessarily the only one).*

B. *If $|\text{grad } f| = \lambda$ in D, then $E = D$ and $u \equiv v \equiv f$.*

C. *If $|\text{grad } f|$ is constant in D, and D is a convex region, then $E = D$ and $u \equiv v \equiv f$.*

Proof. (I) Suppose that $v \in C^1(D)$. It is evident that $|\text{grad } v| = \lambda$ in D, and it follows from Lemma 1 that the solution is unique in D. (Clearly, the trajectories of grad v are critical segments.)

(II A) The relation

$$\mu(f, \bar{D}) = \max \left(\mu(f, \partial D), \sup_{P \in \bar{D}} |\text{grad } f(P)| \right) \tag{*}$$

is easy to verify, and the assertion follows from this and Lemma 1.

The reasoning is similar in the other cases. This theorem will be illustrated in Example 3 below.

The assertion II B, in combination with Theorem 3, leads to

Theorem 5. *Let f be an admissible function. Then f is the only solution of the extension problem if and only if $f \in C^1(D)$ and $|\text{grad } f| = \lambda$ in D.*

Examples. (1) Let D be an arbitrary bounded region in R^n and let the boundary function be given by $\varphi = A_0 + \sum_{\nu=1}^n A_\nu x_\nu$ (A_ν = constants). It is clear that $\lambda = (\sum_{\nu=1}^n A_\nu^2)^{\frac{1}{2}}$, and it follows from II B in Theorem 4 that *there is a unique solution, namely* $f(x_1, ..., x_n) = A_0 + \sum_{\nu=1}^n A_\nu x_\nu$. The critical segments are precisely those which have the direction $(A_1, A_2, ..., A_n)$.

(2) Choose $\varphi(Q) = \overline{P_0 Q}$, where P_0 is a fixed point *not in D*. Clearly, $\lambda = 1$, and the critical segments are those which are parts of "half-rays" emanating from P_0. The solution is unique, namely $f(P) = \overline{PP_0}$.

(3) Consider the case $n = 2$ and write $x_1 = x$, $x_2 = y$. Let Ω be the region obtained from the xy-plane by deleting the semi-axis $y = 0$, $x \geqslant 0$. Define the function $f(x, y)$ as

$$f(x, y) = \begin{cases} \sqrt{x^2 + y^2}, \text{ for } y \geqslant 0 \\ \sqrt{(x-1)^2 + y^2} - 1, \text{ for } y < 0. \end{cases}$$

It is easy to verify that $f \in C^1(\Omega)$, and $|\text{grad } f| = 1$. Thus Lemma 1 is applicable. See Fig. 2, where the straight lines are the trajectories of grad f.

We shall now consider the extension problem on various domains D. In each case we assume that $\bar{D} \subset \Omega$ and put $\varphi = f$ on ∂D.

1. If D_1 is a region in the half-plane $y > 0$, then it follows (as in the previous example) that $\lambda = 1$ and that f represents the unique solution. (Case II B in Theorem 4.)

2. Let D_2 be a convex region. Then the segment between any two points in D_2 lies in Ω and it is clear that $\lambda = 1$. Here, too, f represents the unique solution, and the critical segments are the trajectories of grad f. (Case II C.)
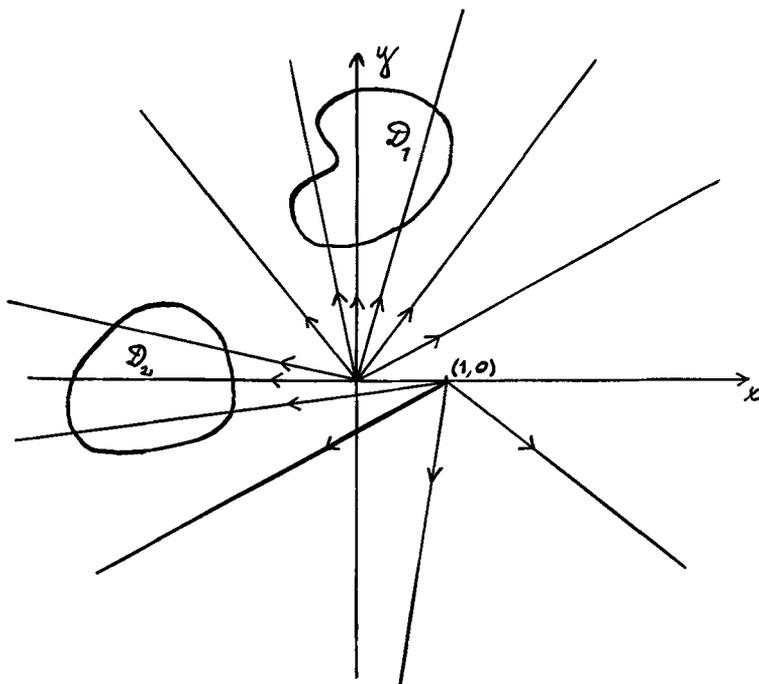
Fig. 2

3. The function f is discontinuous across the positive x-axis and one can find a region D_3 (as shown in Fig. 3) for which $\lambda = \mu(\varphi, \partial D_3) > 1$. It follows from Theorems 4 and 5 that f is a solution of the extension problem but not the only one. (Case II A.) Clearly, E is empty in this case.

However, it is also clear that if $g \in C^1(D_3)$ and $g = \varphi = f$ on ∂D_3, then $\sup_{D_3} |\operatorname{grad} g| > 1$, unless $g \equiv f$. (Compare Section 5.)

4. Consider again the case $n = 2$ and let D be an open circle. Let A and B be two diametrically opposite points on ∂D. Put $\varphi(A) = 1$, $\varphi(B) = 0$ and let φ be a linear function of arc length on each of the two semi-circles AB.

It is then easy to verify that E consists of the straight segment AB, which is the only critical segment.

3. The connection between the extension problem and the partial differential equation

$$\sum_{i,j=1}^{n} \frac{\partial \phi}{\partial x_i} \cdot \frac{\partial \phi}{\partial x_j} \cdot \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0.$$

Consider, for a moment, a convex region D and a function $\phi \in C^1(D) \cap C(\bar{D})$. It is evident that

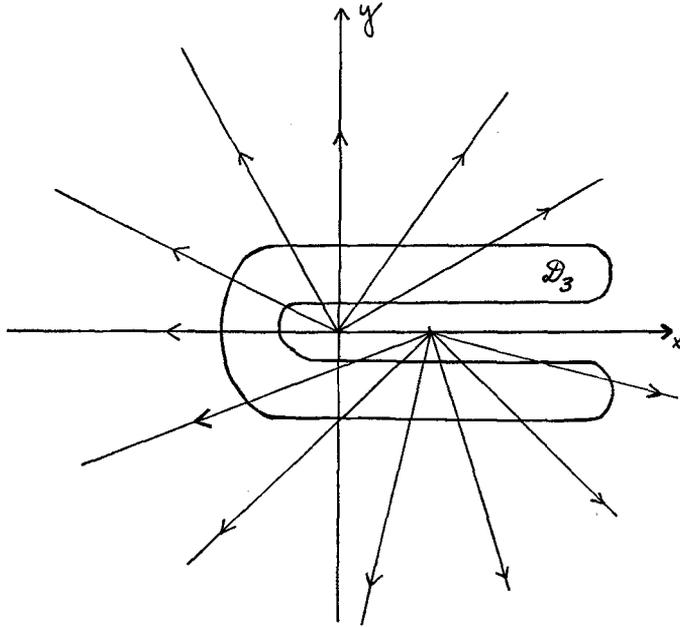$$\mu(\phi, \bar{D}) = \sup_{P \in D} |\operatorname{grad} \phi(P)|.$$

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Fig. 3

Therefore, for such regions and such functions, our minimization problem (for $\mu(\phi, \bar{D})$) is equivalent to a minimization problem for the functional sup $|\text{grad } \phi|$. Because of this, some of the results of this section will have corresponding interpretations for the latter problem. This will be treated more generally in the supplement.

It is natural to consider the functional $H(\phi) = \sup_D |\text{grad } \phi|$ as a "limit" of the sequence of functionals

$$I_N(\phi) = \left(\int_D |\text{grad } \phi|^{2N} dx \right)^{\frac{1}{2N}}; \quad N = 1, 2, 3, \ldots .$$

The Euler equation for the problem $I_N(\phi) = \text{minimum}$ is

$$|\text{grad } \phi|^{2(N-2)} \cdot \left\{ \frac{1}{2(N-1)} \cdot |\text{grad } \phi|^2 \cdot \Delta\phi + \sum_{i, j=1}^{n} \phi_{x_i}\phi_{x_j}\phi_{x_i x_j} \right\} = 0.$$

Remove the first factor and let N tend to infinity: This leads to the new equation $\sum_{i, j=1}^{n} \phi_{x_i} \phi_{x_j} \phi_{x_i x_j} = 0$, which is quasi-linear and parabolic. It is the object of this section to study the connection between the basic extension problem and the above differential equation.

We introduce the notation $A(\phi) \equiv \sum_{i, j=1}^{n} \phi_{x_i} \phi_{x_j} \phi_{x_i x_j}$. It is an easy matter to verify that

$$A(\phi) = \tfrac{1}{2} \text{ grad } \{(\text{grad } \phi)^2\} \cdot \text{grad } \phi.$$

The meaning of the differential euqation $A(\phi) = 0$ is therefore simply that $|\text{grad } \phi|$ *is constant along every trajectory of the vector field* grad ϕ. Thus ϕ is a linear function of arc

length on every such trajectory. It is also clear that every function $\phi \in C^2$, for which $|\text{grad }\phi|$ is constant, satisfies $A(\phi) = 0$.

Now let D be a region with smooth boundary and let $\phi \in C^2$ in a neighbourhood of $\bar{D}$. Assume that grad $\{(\text{grad }\phi)^2\} \neq 0$ on $\bar{D}$. Now $|\text{grad }\phi|$ must take its maximum M (in $\bar{D}$) at some point $P_0 \in \partial D$ and at such a point, grad $\{(\text{grad }\phi)^2\}$ is orthogonal to ∂D. If $A(\phi) = 0$, then it follows that grad ϕ is tangential to ∂D at P_0 and then $\mu(\phi, \partial D) \geqslant M$, which means that $\mu(\phi, \bar{D}) = \mu(\phi, \partial D)$. (Compare the relation (*) in the proof of Theorem 4.)

However, if $A(\phi) \neq 0$, then grad ϕ is not tangential to ∂D and we may have

$$\mu(f, \partial D) < \mu(f, \bar{D}).$$

This suggests the next theorem.

Theorem 6. *Assume that $f \in C^2$ in a neighbourhood of the point P_0 and that $A(f) \neq 0$ at P_0. If D is a sphere with center at P_0 and with sufficiently small radius, then $\mu(f, \bar{D}) > \mu(f, \partial D)$.*

Proof. The proof is almost clear from above. Let us complete it by means of a variational technique.

Let D be a sphere with center at P_0 such that $A(f) \neq 0$ on $\bar{D}$. Assume, for example, that $A(f) < 0$ on $\bar{D}$. Put $M = \max_{\bar{D}} |\text{grad } f|$ and $E = \{P \mid P \in \bar{D}, |\text{grad } f(P)| = M\}$. Then $E \subset \partial D$. If $P \in E$, then grad $\{(\text{grad } f)^2\}$, evaluated at P, is parallel to the outer normal $n(P)$ of ∂D. It follows, since $A(f) < 0$, that grad $f \cdot n < 0$ at all points of E.

Therefore, there exists a function $g(x)$, of the form $h(|x - x_{P_0}|)$, such that

(1) $g(x) \in C^1$ in an open set containing $\bar{D}$.
(2) $g(x) = 0$ on ∂D.
(3) grad $g(x) \cdot$ grad $f(x) < C < O$ on E.

Now it follows by a standard argument that $\max_{\bar{D}} |\text{grad }(f + \lambda g)| < \max_{\bar{D}} |\text{grad } f|$ if the parameter λ is positive and small enough. This technique is quite analogous to a method used in approximation theory, compare [7], p. 14, Satz 17.

The reasoning which preceded Theorem 6, also indicated a result in the converse direction. However, the following theorem is more general:

Theorem 7. *Let D be an arbitrary bounded region and assume that $f \in C^2(D) \cap C(\bar{D})$, $f = \varphi$ on ∂D and $A(f) = 0$ in D. Then $\mu(f, \bar{D}) = \mu(\varphi, \partial D)$, i.e. f is a solution of the extension problem.*

Proof. Since $\mu(f, \bar{D}) = \max(\mu(\varphi, \partial D), \sup_D |\text{grad } f|)$, it is sufficient to prove that $\sup_D |\text{grad } f| \leqslant \mu(\varphi, \partial D)$. Choose a point $P_0 \in D$ such that $|\text{grad } f(P_0)| = k_0 > 0$. The functions $\partial f / \partial x_i$ are in C^1, and hence there is a unique trajectory γ through P_0. Now γ cannot terminate inside D, because $|\text{grad } f| = k_0$ on γ. However, f is a linear function of arc length on γ, and f is bounded. Therefore, γ must have finite length S. Consequently, γ must approach ∂D in both directions from P_0, and have two definite limit points Q and R on ∂D. It follows that $\varphi(R) - \varphi(Q) = k_0 \cdot S \geqslant k_0 \cdot \overline{QR}$. Hence $\mu(\varphi, \partial D) \geqslant k_0$ and the rest of the proof is evident.

Remark. Suppose that $g \in C^1(D) \cap C(\bar{D})$ and $g = \varphi$ on ∂D. Then it follows from the proof that $\sup_D |\text{grad } g| \geqslant \sup_D |\text{grad } f|$.

The concept of absolutely minimizing functions (absolute minimals, a.s. minimals), was introduced in [2] for a class of minimization problems for functions of one variable. It can be carried over to the present case in the following way: Let Ω be an arbitrary region in R^n. The function f is said to be an *absolute minimal in Ω* if $\mu(f, \bar{D}) = \mu(f, \partial D) < \infty$ for every bounded region D such that $\bar{D} \subset \Omega$. We have then the following theorem:

Theorem 8. *If $f \in C^2(\Omega)$, then f is an absolute minimal in Ω if and only if $A(f) = 0$ in Ω.*

Proof. One half of the theorem follows from Theorem 6 and the other half follows from Theorem 7 and the fact that f satisfies a Lipschitz condition on every compact subset of Ω.

Theorem 9. *Let D be a bounded region, and let $f \in C^2(D) \cap C(\bar{D})$. Assume that $A(f) = 0$ in D and*

$$f(x_1, x_2, \ldots, x_n) \leqslant A_0 + \sum_{\nu=1}^{n} A_\nu x_\nu \quad on \ \partial D.$$

Then the same inequality holds in D.

Proof. If the proposition were not true, then there would be a subregion $D_1 \subset D$, such that $f(x) = L(x) \equiv A_0 + \sum A_\nu x_\nu$ on ∂D_1, and $f(x) > L(x)$ in D_1. Consider then the extension problem on D_1, with $\varphi(x) = f(x) = L(x)$. It follows from Theorem 7 that f is a solution and it follows from Example 1, Section 2, that $L(x)$ is the only solution. This gives a contradiction and the proof is complete. Naturally, the inequality sign may be reversed in the theorem.

It is clear that the theorem remains valid for any function in $C(\bar{D})$ which is absolutely minimizing in D.

Example. Let Ω be the region obtained from the xy-plane by deleting the negative x-axis and let ϕ be an arbitrary continuous determination of arctg y/x in Ω. The trajectories of grad ϕ are then circles with center at the origin, and $|\operatorname{grad} \phi|$ is constant on each such circle. Therefore, $A(\phi) = 0$. Let D be a bounded subdomain of Ω such that $\bar{D} \subset \Omega$ and consider the extension problem on D with the boundary function ϕ. One solution is given by ϕ itself, according to Theorem 7. Since the trajectories of grad ϕ are circles, it follows that the set of uniqueness E is empty.

4. The existence of absolutely minimizing functions

The object of this section is to carry out an existence proof for a.s. minimals. The first part of the proof is analogous to the corresponding part of the proof of Theorem 2.1 in [3] and it will therefore be brief.

Consider then a *fixed* extension problem, on the region D with boundary values φ. Let f be a solution of this problem and consider the extension problem on a subregion $D' \subset D$, with boundary values f. Let u' and v' be the lower and upper functions on D'.

Definition. f is said to have the property A in D, if $f \geqslant u'$ on D', for every choice of $D' \subset D$. Further, f has the property B on D, if $f \leqslant v'$ on D', for every $D' \subset D$.

Let G be the class of those solutions of the extension problem on D which have the property A on D. G is not empty, since the upper function belongs to G.

We introduce the function

$$h(x_1, \ldots, x_n) = \inf_{g \in G} g(x_1, \ldots, x_n)$$

and assert that it is an a.s. minimal. Evidently, h is a minimizing function (i.e. a solution of the extension problem). Further, h has the properties A and B in D. Since the verification of this is analogous to the corresponding arguments in [3], it will be omitted.

To complete the proof, consider a subregion $D_1 \subset D$ and put $m = \mu(h, \partial D_1)$, $M = \mu(h, D_1)$. Assume that $M > m$. Then there must be two points A, $B \in D_1$, such that the chord $AB \subset D_1$ and such that $h(A) - h(B) > m \cdot \overline{AB}$.[1] If S denotes arc length on the chord AB, measured from B, there must be a point P where dh/ds exists and $dh/ds > m$. Choose a number M_1 such that $m < M_1 < (dh/ds)_P$.

Now at least one of these two statements is true:

$$\begin{cases} h(Q) < h(P) + M_1 \, \overline{PQ} & \text{for every } Q \in \partial D_1, \\ h(Q) > h(P) - M_1 \, \overline{PQ} & \text{for every } Q \in \partial D_1. \end{cases}$$

In the opposite case there would be $Q_1, Q_2 \in \partial D_1$ such that

$$h(Q_1) \geqslant h(P) + M_1 \, \overline{PQ_1}$$

and

$$h(Q_2) \leqslant h(P) - M_1 \, \overline{PQ_2},$$

which gives $h(Q_1) - h(Q_2) \geqslant M_1(\overline{PQ_1} + \overline{PQ_2}) > m \, \overline{Q_1 Q_2}$. This contradicts the definition of m.

Suppose then, for example, that $h(Q) < h(P) + M_1 \, \overline{PQ}$ for every $Q \in \partial D_1$. Put $\psi(Q) = h(P) + M_1 \, \overline{PQ}$. Then $\psi(P) = h(P)$ and $\psi > h$ on ∂D_1. Since $(dh/ds)_P > M_1$, there is a subregion $D_2 \subset D_1$ such that $\psi < h$ in D_2 and $\psi = h$ on ∂D_2. The function ψ is the only solution of the extension problem on D_2 with the boundary values $\psi = h$, as can be seen from Example 2 in Section 2. But then the relation $\psi < h$ contradicts the fact that h has the property B. Hence the assumption $M > m$ leads to a contradiction and h is therefore absolutely minimizing on D.

Theorem 10. *For a given extension problem (D, φ), there is a solution which is absolutely minimizing in D.*

5. Supplement

It should be mentioned that some of the preceding ideas can be carried over in a natural way to the following problem: minimize the functional $H(\phi) = \sup_{P \in D} |\operatorname{grad} \phi(P)|$ over those functions which take on given values φ at ∂D and which are sufficiently wellbehaved in D. It is convenient to consider only those functions ϕ which satisfy a Lipschitz condition in some neighbourhood of every point $P \in D$ (the constant in this condition may depend on ϕ and P). Such functions are differentiable a.e. in D, and $H(\phi)$ is well-defined (see [8]).

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[1] Compare the relation (*) in the proof of Theorem 4.

Now the following distance is defined for points in D: $d(P, Q) = \inf_{\gamma \in \Gamma} L(\gamma)$, where Γ is the class of polygonal arcs which connect P and Q *and which lie in D.* $L(\gamma) =$ the length of γ. ($d(P, Q) =$ the distance, measured within D.) After excluding some pathological types of regions, $d(P, Q)$ can be properly defined for $P, Q \in \bar{D}$.

Now, the method of the upper and lower functions can be carried over if the Euclidean distance $\overline{PQ}$ is replaced by $d(P, Q)$ in the formulas. It turns out that there is an admissible ϕ with $H(\phi) < \infty$ if and only if φ satisfies on ∂D a Lipschitz condition *with respect to the distance within D* (i.e. $d(P, Q)$), and in that case, the (best) constant in this Lipschitz condition is equal to $\min_\phi H(\phi)$.

It is also easy to obtain this theorem: An admissible function f represents a unique solution of the minimization problem if and only if $f \in C^1(D)$ and $|\operatorname{grad} f|$ is constant in D. (Compare the corollary of Theorem 9 in [2].) One half of the theorem follows from Lemma 1, and the other half follows from Theorem 3, when applied to convex subregions of D. (For convex regions, the functionals $\mu(\phi)$ and $H(\phi)$ are identical).

Compare Example 3 in Section 2, the proof of Theorem 6 and the remark to Theorem 7.

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