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On the boundary behavior of solutions to a class of elliptic partial differential equations

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1. The object of this paper is to investigate the behavior at the boundary of solutions to the uniformly elliptic, semi-linear equation

$$a^{ij}(X)\,u_{ij}(X) = F(X, u, u_i, u_{ij}), \tag{1.1}$$

where a^{ij} are continuous or Hölder continuous and F satisfies

$$|F(X, u, u_i, u_{ij})| \leqslant \frac{\beta(\delta(X))}{\delta^2(X)} + \frac{\alpha(\delta(X))}{\delta^2(X)}|u| + \frac{\alpha(\delta(X))}{\delta(X)}|u_i| + \alpha(\delta(X))|u_{ij}|.$$

Here $\delta(X)$ denotes the distance from X to the boundary, and $\beta(t)$ and $\alpha(t)$ are functions which in most of the cases considered tend to zero with a prescribed speed, as $t \searrow 0$.

In particular our results are valid for the linear equation

$$a^{ij}u_{ij} + b^i u_i + cu = f$$

if b^i, c, and f satisfy corresponding inequalities.

An important feature of this class of equations is that, in a certain sense, it is invariant under mappings between Liapunov regions, and this makes it possible to get results e.g. about harmonic functions in Liapunov regions which have been obtained earlier by different methods. For these results see Keldyš and Lavrent'ev [13], and Widman [27]. It may be noted that all the results of [27] are contained in this paper.

Section 2 and 3 contain basic assumptions and definitions, and some lemmata of various types, respectively.

Section 4 contains theorems assuring the finiteness of weighted integrals of derivatives of solutions, given some information about the integrability of the solution itself. These theorems are formulated for quite general regions. Specializing to the case of a half space, some other estimates of derivatives and integrals of derivatives are given. Finally we prove two theorems on solutions in cones, at least one of which is previously known for the case of harmonic functions. As a corollary we get a generalization of a theorem by Wallin.

In *Section 5* we give the generalization to solutions of (1.1) of the theorem that a positive harmonic function in the unit disc belongs to the Hardy class H^1.

Section 6 contains results on the boundary behavior of Green potentials, one of which is needed in the sequel.

Section 7 contains theorems on the existence and type of assumption of boundary values of solutions of (1.1). Apart from the case of harmonic functions results in this direction have earlier been obtained by Łojasiewicz [16] and implicitely by Serrin [19].

Section 8 finally gives a necessary and sufficient condition for the existence of boundary values to a solution of (1.1). In the case of harmonic functions, this theorem can be found in [22].

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2. We place ourselves in R^n, the points of which are denoted by $X, Y, ..., X = (x_1, ..., x_n)$ etc., $|X-Y|^2 = \sum_{i=1}^n (x_i - y_i)^2$. Points of R^{n-1} will often be denoted by X', Y', and often X' will be the orthogonal projection of X on R^{n-1}. In general our methods will be applicable in R^n for $n \geqslant 2$, but since there are often special methods from the theory of generalized analytic functions available in the plane, see e.g. Vekua [24], and since some minor complications arise from the logaritmic singularity of the fundamental solution of the Laplacian in this case, we will concentrate on R^n with $n \geqslant 3$. Integrals over n-dimensional regions will be denoted by $\int\int(\cdot)dX$, over n-1-dimensional surfaces by $\int(\cdot)dS$, dS being the surface element. $\int(\cdot)dX_{(i)}$ can be interpreted as $\int(\cdot)\cos\gamma_i dS$, where $\cos\gamma_i$ is the scalar product of the ith unit vector and the normalized outer normal of the surface.

By a Liapunov surface we mean a closed, bounded n-1-dimensional surface S satisfying the following conditions:

1° At every point of S there exists a uniquely defined tangent (hyper-)plane, and thus also a normal.

2° There exist two constants $C>0$ and γ, $0<\gamma\leqslant 1$, such that if θ is the angle between two normals, and r is the distance between their foot points, then the inequality $\theta < C \cdot r^\gamma$ holds.

3° There is a constant $\varrho > 0$ such that if Σ_ϱ is a sphere with radius ϱ and center $X_0 \in S$, a line parallel to the normal at X_0 meets S at most once inside Σ_ϱ.

For the properties of Liapunov surfaces in R^3, see Gunther [10]. It is easy to see that the simple facts about Liapunov surfaces in R^n that we need can be derived in the same way as in [10]. A Liapunov region is a region the boundary of which is a Liapunov surface.

The boundary of any set D will be denoted by ∂D, and $\bar{D}$ is the closed hull of D. $\delta(X)$ is the distance from X to ∂D. R^n_+ will as usual be the set $\{X \mid x_n > 0\}$. $C^\infty(\Omega)$, $C^m(\Omega)$ denote the space of infinitely, and m times, continuously differentiable functions in Ω, respectively, and $C^\gamma(\bar{\Omega})$ will be the space of Hölder continuous (with exponent γ) functions in $\bar{\Omega}$.

The assumptions on the equation will be

(i) a^{ij} and F are measurable functions of their arguments.

(ii) a^{ij} are defined in $\bar{\Omega}$ and there exists a constant $\lambda > 0$, the ellipticity constant, such that

$$\lambda |\xi|^2 \leqslant a^{ij}(X)\xi_i\xi_j \leqslant \frac{1}{\lambda}|\xi|^2$$

for all $X \in \Omega$ and all vectors $\xi = (\xi_1, \xi_2, ..., \xi_n) \neq 0$.

(iii) $a^{ij}=a^{ji}$. We also assume $\det(a^{ij})=1$, which is no further restriction.

(iv) $$|a^{ij}(X)-a^{ij}(Y)|\leqslant K\cdot\alpha(|X-Y|),\ X\in\partial\Omega,\ Y\in\overline{\Omega}.$$

(iv)′ $$|a^{ij}(X)-a^{ij}(Y)|\leqslant K\cdot\alpha(|X-Y|),\ X,\ Y\in\overline{\Omega}.$$

(v) $$|F(X,u,u_i,u_{ij})|\leqslant|f(X)|+\delta^{-1}(X)\cdot\alpha(\delta(X))|u_i|+\delta^{-2}(X)\cdot\alpha(\delta(X))|u| + \alpha(\delta(X))|u_{ij}|.$$

(vi) $$|f(X)|\leqslant K\cdot\delta^{-2}(X)\cdot\beta(\delta(X)).$$

We shall work with three types of equations:

(A): where we assume (i)–(vi) with $\alpha(t)$ satisfying $\lim_{t\to+0}\alpha(t)=0$ and $\beta(t)$ bounded;

(B): (i)–(vi) with $\alpha(t)=t^\alpha$, $\alpha>0$, $\beta(t)$ nondecreasing and satisfying $\int_0^1(\beta(t)/t)\,dt<\infty$;

(C): where we assume the same as in (B) and in addition (iv)′ and that F is independent of u_{ij}.

When we say that u is a solution of (1.1 A) we mean that u is a solution of the equation (1.1), about which we assume the conditions A, etc.

With a solution of (1.1) in a general region Ω, we mean a function u belonging to $C^2(\Omega)$, and satisfying (1.1) almost everywhere. When we work with solutions in R^n_+, or parts thereof, we can allow a weaker concept of solution; u is a solution of (1.1) in R^n_+ if u has distributional derivatives of order $\leqslant 2$ which are locally bounded functions, and satisfies (1.1) almost everywhere. In some of the theorems it is even sufficient to assume the second derivatives to be in L^p locally, for some $p>n$. Although we shall not stress this point, we note that it is well known, see e.g. [8], that in both these cases the first derivatives of u are continuous functions which are locally absolutely continuous on all straight lines parallel to one of the coordinate axis except those issuing from a set of n-1-dimensional Lebesgue measure zero on the orthogonal hyperplane.

In the case of (1.1 B and C) we will often have occasion to rewrite the equation in regular regions Ω. To that effect we use Lemma 3.9 to extend the functions $a^{ij}(X)$ on $\partial\Omega$, into Ω in such a way that the new functions $\bar a^{ij}$ satisfy

$$\bar a^{ij}\in C^\infty(\Omega),\quad \bar a^{ij}\in C^\alpha(\overline{\Omega}),\quad \bar a^{ij}=a^{ij}\text{ on }\partial\Omega,$$
$$|\operatorname{grad}\bar a^{ij}(X)|\leqslant K\cdot\delta^{\alpha-1}(X).$$

The Hölder constant and K will not depend on Ω, which is seen from Lemma 3.9. A solution $u(X)$ of (1.1) will then also be a solution of the equation

$$\bar a^{ij}(X)u_{ij}(X)=F(X,u,u_i,u_{ij})+[\bar a^{ij}(X)-a^{ij}(X)]u_{ij}.$$

H^p, $1\leqslant p\leqslant\infty$, will be the class of solutions u of (1.1) in R^n_+, satisfying

$$\sup_{0<x_n<1}\int_{|X'|\leqslant\varrho}|u(x_1,x_2,\dots,x_{n-1},x_n)|^p dX'<\infty,\quad p<\infty$$

and $$\operatorname*{ess\,sup}_{|X|\leqslant\varrho}|u(X)|<\infty,\quad p=\infty$$

respectively, for every $\varrho=0$.

When $X' \in R^{n-1}$ and $h > 0$ we shall denote by $V_h(X')$ the truncated cone

$$\{Y \mid y_n^2 \geqslant h^2 \cdot \textstyle\sum_{i=1}^{n-1} (y_i - x_i)^2,\ 0 < y_n < 1\},$$

and when $X_0 \in \partial\Omega$ $V_h(X_0)$ will be a cone congruent to $V_h(O)$, with axis along the inner normal to $\partial\Omega$ at X_0, the normal always assumed to exist in case we use this notation, and with the convention that we truncate the cone more, if necessary, in order that $V_h(X_0)$ lie inside Ω. If $F \subset R^{n-1}$ we define

$$W_h(F) = \bigcup_{X' \in F} V_h(X').$$

With some obvious exceptions, subindices denote differentiation.

The summation convention is used freely. We shall also use the convention that when the summation convention does not apply, u_i and u_{ij} are vectors in R^n and R^{n^2}, i.e. u_i is the gradient vector, and $|u_i|$ and $|u_{ij}|$ are the respective Euclidean norms.

K will denote a generic constant which constantly changes its value. If doubtful, we shall try to indicate the important variables on which K does or does not depend.

3. Lemmata

Lemma 3.1. *Let D be any bounded, open region in R^n, and let $\{S\}$ be the set of spheres $S = S(X, \delta(X)/4)$ with center X and radius $\delta(X)/4$, $\delta(X)$ being the distance from X to ∂D. Then there exists a denumerable sequence of spheres $\{S_\nu\}_1^\infty$, $S_\nu = S(X_\nu, \delta(X_\nu)/4)$ with the property that $\bigcup S_\nu = D$ and such that every point of D is inside at most $K(n)$ of the spheres $\{S'_\nu\}_1^\infty$, $S'_\nu = S(X_\nu, 3\delta(X_\nu)/4)$. $K(n)$ depends only on n.*

Remark. From the proof follows a crude upper bound of $K(n)$: $K(n) \leqslant (343/3)^n$.

Proof. It is sufficient to consider connected regions D. We use the following lemma of Aronszajn and Smith [1], p. 162: It is possible to find a sequence S_ν such that $\bigcup S_\nu = \Omega$ and such that the spheres $S''_\nu = S(X_\nu, \delta(X_\nu)/16)$ are pairwise disjoint.

Now let X_0 be any point of D. A sphere S'_ν containing X_0 has radius $\leqslant 4\delta(X_0)$. This implies that the corresponding S''_ν lies in a sphere with center X_0 and radius $49\delta(X_0)/12$, i.e. all the S''_ν of this type cover a region of volume $\leqslant \omega_n \cdot (49/12)^n \cdot \delta^n(X_0)$. On the other hand, S'_ν may not have radius $< 3\delta(X_0)/7$ if it is to contain X_0, i.e. the corresponding S''_ν has radius $\geqslant \delta(X_0)/28$. Since the S''_ν are pairwise disjoint we get

$$K \cdot \omega_n \left(\frac{\delta(X_0)}{28}\right)^n \leqslant \omega_n \left(\frac{\delta(X_0) \cdot 49}{12}\right)^n$$

if K is the number of spheres S'_ν containing X_0. Hence $K \leqslant (343/3)^n$.

The following two lemmata are essentially contained in Stein [22].

Lemma 3.2. *Assume f is measurable, locally bounded in R^n_+ and such that*

$$\iint_{W_h(E)} |f|\, dX < \infty$$

for some measurable set $E \subset R^{n-1}$. Then

$$\iint_{V_k(X')} x_n^{1-n}|f|\,dX<\infty$$

for all $k>0$ *and almost all* $X'\in E$.

Lemma 3.3. *Assume* f *is measurable, locally bounded in* R^n_+ *and such that*

$$\iint_{V_h(X')} |f|\,dX<\infty$$

for X' *belonging to some bounded measurable set* $E\subset R^{n-1}$, *and where* h *may vary with* X'. *Then to every* $\varepsilon>0$ *and every* $k>0$ *there is a closed set* $F\subset E$, $\operatorname{mes}(E-F)<\varepsilon$ *such that*

$$\iint_{W_k(F)} x_n^{n-1}|f|\,dX<\infty.$$

The proof of Lemma 3.3 uses the following Egorov like theorem by Calderón, which we shall use several times. See Lemma 1 in [22].

Lemma 3.4. *Let* $f(X)$ *be locally bounded and measurable in* R^n_+. *Suppose we are given a bounded, measurable set* $E\subset R^{n-1}$ *with the following property. Whenever* $X_0'\in E$, $f(X)$ *is bounded as* X *ranges in some cone* $V_h(X_0')$. (*The bound and* h *may depend on* X_0'.) *For any* $\varepsilon>0$ *there exists a closed subset* F, $F\subset E$ *such that*

(1) $\operatorname{mes}(E-F)<\varepsilon$,

(2) *if* k *is fixed,* $f(X)$ *is uniformly bounded in* $W_k(F)$.

It is also clear that if we assume that $f\to 0$ as $x_n\to 0$ in $V_h(X_0')$ for every $X_0'\in E$, with the same method of proof we can find an F such that $\operatorname{mes}(E-F)<\varepsilon$ and f tends to zero uniformly in $W_k(F)$ when $x_n\to 0$.

Lemma 3.5. *Let* $L=a^{ij}\partial^2/\partial x_i\partial x_j$ *be a differential operator with constant coefficients satisfying* $|\xi|^2\lambda\leqslant a^{ij}\xi_i\xi_j\leqslant 1/\lambda|\xi|^2$ *for all* $(\xi_1,\xi_2,\dots,\xi_n)\neq 0$, *and let* $\det(a^{ij})=1$. *If* $G(X,Y)$ *is the Green function of* L *in* R^n_+, *then* G *satisfies the following inequalities:*

$$G(X,Y)\leqslant K\cdot|X-Y|^{2-n}, \tag{i}$$

$$G(X,Y)\leqslant K\cdot\frac{x_n\cdot y_n}{|X-Y|^n}, \tag{ii}$$

$$|G_{x_i}(X,Y)|\leqslant K\cdot|X-Y|^{1-n}, \tag{iii}$$

$$|G_{x_i}(X,Y)|\leqslant K\cdot\frac{y_n}{|X-Y|^n}, \tag{iv}$$

$$|G_{x_ix_j}(X,Y)|\leqslant K\cdot\frac{y_n}{|X-Y|^{n+1}}, \tag{v}$$

where K *depends on* n *and* λ *only.*

Proof. Let A be the matrix (a^{ij}) and define B by $B\cdot B=A$. By the coordinate transformation $X'=XB^{-1}$, L is transformed into the Laplace operator Δ, i.e.

$G'(X', Y') = G(X'B, Y'B)$ is the Green function of Δ in some region, the boundary of which is a hyperplane.

The corresponding inequalities for G', i.e. where x_n and y_n are replaced by $\delta(X)$ and $\delta(Y)$ respectively, are either well known or easy to derive, since we know G' explicitly. Here K depends on n only. Now (i)–(v) follow easily, since the dilation of distance is bounded above and below with $1/\lambda$ and λ.

Lemma 3.6. *Let L be the differential operator of Lemma 3.5, and let $G(X, Y)$ be the Green function of L for the sphere $|X| \leq \varrho$. Then*

$$|G(X, Y)| \leq K \cdot |X-Y|^{2-n}, \tag{i}$$

$$|\operatorname{grad}_X G(X, Y)| \leq K \cdot |X-Y|^{1-n}, \tag{ii}$$

for $|X|$, $|Y| \leq \varrho$, and

$$\left|\frac{\partial}{\partial y_i} \operatorname{grad}_X G(X, Y)\right| \leq K \cdot \varrho^{-n}, \tag{iii}$$

$$\left|\frac{\partial^2}{\partial y_i \partial y_j} \operatorname{grad}_X G(X, Y)\right| \leq K \cdot \varrho^{-n-1}, \tag{iv}$$

for $|Y| \leq \varrho/2$, $|X| = \varrho$, where K depends on λ and n only.

Proof. The inequalities are evidently true for $\varrho = 1$ (cf. the proof of Lemma 3.5), and the general case follows with a homothety.

Lemma 3.7. *Let D be an open bounded region, and let X_0 and X_0^* be arbitrary points in D and $\bar{D}$ respectively. Put $l = \delta(X_0)/4$ and let $p > 1$. Then the following inequalities hold for any solution of* (1.1) *in D.*

$$\left.\begin{aligned} &\iint_{|X-X_0|\leq l} \delta^p(X)|u_i|^p dX \leq && \text{(i)} \\ &\iint_{|X-X_0|\leq l} \delta^{2p}(X)|u_{ij}|^p dX \leq && \text{(ii)} \end{aligned}\right\} K \iint_{|X-X_0|\leq 3l} \{|u|^p + \delta^{2p}(X)\,[|F|^p + |h^*|^p]\} dX,$$

$$\iint_{|X-X_0|\leq l} |u|^p dX \leq K \cdot l^{n-np}\left[\iint_{|X-X_0|\leq 3l} |u|\, dX\right]^p + K \iint_{|X-X_0|\leq 3l} \delta^{2p}(X)\{|F|^p + |h^*|^p\} dX, \tag{iii}$$

where $h^* = [a^{ij}(X) - a^{ij}(X_0^*)]\,u_{ij}$ and where K does not depend on u, X_0, or X_0^*.

Proof. We rewrite the equation (1.1):

$$a^{ij}(X_0^*)\,u_{ij} = F + h^*.$$

According to a well-known formula, almost everywhere

$$\omega_n u(Y) = \int_{|X-X_0|\leq\varrho} \frac{\partial G}{\partial \nu_X}(X, Y)\, u(X)\, dS_X + \iint_{|X-X_0|\leq\varrho} G(X, Y)\{F + h^*\} dX = v_1^\varrho + v_2^\varrho, \tag{3.7.4}$$

where $2l \leqslant \varrho \leqslant 3l$, G is the Green function of $L = a^{ij}(X_0^*)\partial^2/\partial x_i \partial x_j$ in $|X| \leqslant \varrho$ and $\partial/\partial v$ denotes the corresponding co-normal derivative. We observe that the formula is valid because u and u_i are absolutely continuous on almost every line parallel to one of the coordinate axis, and thus partial integration is allowed. Now using Lemma 3.6,

$$\left|\frac{\partial v_1^\varrho}{\partial y_k}(Y)\right|^p \leqslant \left[K \cdot \varrho^{-n} \int_{|X-X_0|=\varrho} |u|\, dS_X\right]^p \leqslant K \cdot l^{1-p-n} \int_{|X-X_0|=\varrho} |u|^p dS$$

from which follows

$$\iint_{|Y-X_0|\leqslant l} \left|\frac{\partial v_1^\varrho}{\partial y_k}(Y)\right|^p \delta^p(Y)\, dY \leqslant K \cdot l \cdot \int_{|X-X_0|=\varrho} |u|^p dS.$$

On the other hand, also by Lemma 3.5

$$\left|\frac{\partial v_2^\varrho}{\partial y_k}(Y)\right|^p = \left|\iint_{|X-X_0|\leqslant\varrho} G_{y_k}(X, Y)\,[F + h^*]\, dX\right|^p$$

$$\leqslant K \cdot \left[\iint_{|X-X_0|\leqslant 3l} |X-Y|^{\gamma-n} dX\right]^{p-1} \left[\iint_{|X-X_0|\leqslant 3l} |Y-X|^{p-\gamma p+\gamma-n} |F+h^*|^p dX\right]$$

or $$\iint_{|Y-X_0|\leqslant l} \left|\frac{\partial v_2^\varrho}{\partial y_k}(Y)\right|^p \delta^p(Y)\, dY \leqslant K \cdot l^{2p} \iint_{|X-X_0|\leqslant 3l} |F+h^*|^p dX.$$

Adding, integrating with respect to ϱ from $2l$ to $3l$ and dividing by l, we get

$$\iint_{|Y-X_0|\leqslant l} \delta^p(Y)\,|u_i(Y)|^p dY$$

$$\leqslant K \cdot \iint_{|X-X_0|\leqslant 3l} |u|^p dX + K \cdot l^{2p} \iint_{|X-X_0|\leqslant 3l} \{|F|^p + |h^*|^p\}\, dX$$

which is equivalent to (i).

To prove (ii) we use the same method as above to get

$$\iint_{|Y-X_0|\leqslant l} \delta^{2p}(Y) \left|\frac{\partial^2 v_1^\varrho}{\partial y_k \partial y_m}(Y)\right|^p dY \leqslant K \cdot l \cdot \int_{|X-X_0|=\varrho} |u|^p dS.$$

Now $$\frac{\partial^2 G}{\partial y_k \partial y_m}(X, Y) = K(X, Y) + L(X, Y),$$

where K is a Calderón–Zygmund type kernel [6] and $L(X, Y)$ satisfies $|L(X, Y) \leqslant K \cdot l^{-n}$ for $|Y - X_0| \leqslant l$. We use the well-known Calderón–Zygmund theory from [6] to conclude that

$$\frac{\partial^2 v_2^\varrho}{\partial y_k \partial y_m}(Y) = \iint_{|X-X_0|\leqslant\varrho} \frac{\partial^2 G}{\partial y_k \partial y_m}(X, Y)\,[F + h^*]\, dX$$

exists almost everywhere and that

$$\iint_{|Y-X_0|\leqslant l}\left|\frac{\partial^2 v_2^\varrho}{\partial y_k \partial y_m}(Y)\right|^p dY$$

$$\leqslant \iint_{|X-X_0|\leqslant 3l}|F+h^*|^p dX + K\cdot l^{-n(p-1)+n(p-1)}\cdot\iint_{|X-Y_0|\leqslant 3l}|F+h^*|^p dX.$$

After multiplication by l^{2p}, adding, integrating with respect to ϱ, and dividing by l, (ii) follows.

Finally, still by Lemma 3.6 (ii) and (iv)

$$|u(Y)|\leqslant K\cdot\varrho^{1-n}\int_{|X-X_0|=\varrho}|u|dS + K\iint_{|X-X_0|\leqslant 3l}|X-Y|^{2-n}|F+h^*|dX$$

and after integration with respect to ϱ between $2l$ and $3l$

$$|u(Y)|\leqslant K\cdot l^{-n}\iint_{|X-X_0|\leqslant 3l}|u|dX + K\iint_{|X-X_0|\leqslant 3l}|X-Y|^{2-n}|F+h^*|dX.$$

By Hölder's and Minkowski's inequalities

$$\iint_{|X-X_0|\leqslant l}|u(Y)|^p dY$$

$$\leqslant K\cdot l^{n-np}\left[\iint_{|X-X_0|\leqslant 3l}|u|dX\right]^p + K\cdot\iint_{|X-X_0|\leqslant 3l}\delta^{2p}(X)|F+h^*|^p dX$$

which proves (iii).

Lemma 3.8. *Assume $f\in L^1(R^n)$. Then for every i, $i=1, \dots, n-1$, and every $\gamma>0$*

$$\iint_{R^n}|x_i-t|^{\gamma-1}|f(X)|dX$$

is finite for almost every $t\in R^1$.

Proof. Obvious by Fubini's theorem.

We need the following modification of the Whitney extension theorem.[1] We temporarily change the notation and put $\alpha=(\alpha_1, \alpha_2, \dots, \alpha_n)$ α_i non-negative integers, $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_n$,

$$D^{(\alpha)}f=f^{(\alpha)}=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_n^{\alpha_n}}.$$

Lemma 3.9. *Let $\mathcal{A}$ be a compact set in R^n and let $f\in C^l$ in R^n. Assume also*

$$|f^{(\alpha)}(X)-f^{(\alpha)}(Y)|\leqslant\omega(|X-Y|),\quad |\alpha|=l,$$

where $\omega(t)$ is a non-decreasing function, $\lim_{t\to+0}\omega(t)=0$, satisfying $\omega(2t)\leqslant 2\cdot\omega(t.)$ Then there exists a function $\Phi(X)$ with the following properties

[1] At this point the author has profited from a discussion with Dr. L. I. Hedberg.

1° $\Phi(X)\in C^\infty$, $X\notin \mathcal{A}$,

2° $\Phi(X)\in C^l(R^n)$,

3° $\Phi^{(\alpha)}(X)=f^{(\alpha)}(X)$, $X\in\mathcal{A}$, $|\alpha|\leqslant l$,

4° $|\Phi^{(\alpha)}(X)|\leqslant K\cdot\delta^{-1}(X)\cdot\omega(\delta(X))$, $|\alpha|=l+1$, $|X|\leqslant\varrho$,

5° $|\Phi^{(\alpha)}(X)-\Phi^{(\alpha)}(Y)|\leqslant K\cdot\omega(|X-Y|)$, $|\alpha|=l$, $|X|, |Y|\leqslant\varrho$,

where $\delta(X)$ denotes the distance to $\mathcal{A}$ and K depends on n, ϱ, l and ω only.

Proof. We follow the presentation of Whitney's extension theorem in Hörmander [12]. By Lemma 3 of that paper there is a sequence of functions $\varphi_j\in C_0^\infty$ with support in the complement of $\mathcal{A}$ with the following properties.

$\varphi_j(X)\geqslant 0,\quad \Sigma_j\varphi_j=1,\quad X\notin\mathcal{A}.$ (i)

A compact set in $C\mathcal{A}$ intersects only a finite number of the support of φ_j. (ii)

$\Sigma_j|\varphi_j^{(\alpha)}(X)|\leqslant C_\alpha(\delta^{-|\alpha|}(X)+1)$, where C_α is independent of $\mathcal{A}$. (iii)

There is a constant C independent of j and $\mathcal{A}$ such that the diameter of the support of φ_j is $\leqslant C$ times the distance to $\mathcal{A}$. (iv)

If X^* is a point of $\mathcal{A}$ satisfying $\delta(X)=|X-X^*|$ and X^j is any fixed point in the support of φ_j, we define Φ by

$$\Phi(X)=\sum_j\varphi_j(X)f_l(X,X^j)=f_l(X,X^*)+\sum_j\varphi_j(X)\{f_l(X,X^j)-f_l(X,X^*)\},\quad X\notin\mathcal{A},$$

$$\Phi(X)=f(X),\quad x\in\mathcal{A},$$

where $f_l(X, Y)$ is the Taylor expansion of order l at Y;

$$f(X)=f_l(X, Y)+R_l(X, Y).$$

Our assumptions about f imply

$$|R_l^{(\alpha)}(X, Y)|\leqslant\frac{|X-Y|^{l-|\alpha|}}{(l-|\alpha|)!}\,\omega(|X-Y|),\quad |\alpha|\leqslant l.$$

Taking $|\alpha|=l+1$ we have

$$\Phi^{(\alpha)}(X)=\sum_{\substack{\beta+\eta=\alpha\\ |\beta|>0}}\sum_j\varphi_j^{(\beta)}\{f_l^{(\eta)}(X,X^j)-f_l^{(\eta)}(X,X^*)\}$$
$$=\sum_{\beta,\eta}\sum_j\varphi_j^{(\beta)}\{R_l^{(\eta)}(X,X^j)-R_l^{(\eta)}(X,X^*)\}$$

which implies, using that $|X-X^j|\leqslant(C+1)\delta(X)$ if X is in the support of φ_j,

$$|\Phi^{(\alpha)}(X)|\leqslant\sum_{\beta,\eta}\sum_j|\varphi_j^{(\beta)}|\cdot K\cdot\delta^{l-|\eta|}(X)\cdot\omega(\delta(X))$$
$$\leqslant K\sum_{\beta,\eta}\delta^{l-|\eta|}\cdot\delta^{-|\beta|}\cdot\omega(\delta)\leqslant K\cdot\delta^{-1}(X)\cdot\omega(\delta(X))$$

and 4° is proved.

To prove 5°, first assume that $|X-Y|\leqslant\frac{1}{2}\delta(X)$. Then using 4°, for $|\alpha|=l$,

$$|\Phi^{(\alpha)}(X)-\Phi^{(\alpha)}(Y)|\leqslant|X-Y|\cdot\sup|\Phi^{(\beta)}(Z)|\leqslant K\cdot\omega(|X-Y|),$$

where the supremum is taken over those Z and β for which $|Z-X|\leqslant\frac{1}{2}\delta(X)$ and $|\beta|=|\alpha|+1=l+1$ respectively. On the other hand, if $\delta(X)<2|X-Y|$ we have $\delta(Y)\leqslant 4|X-Y|$ and

$$|\Phi^{(\alpha)}(X)-f^{(\alpha)}(X^*)|\leqslant|f_l^{(\alpha)}(X,X^*)-f^{(\alpha)}(X^*)|$$
$$+\sum_{\beta,\eta}\sum_j|\varphi_j^{(\beta)}|\,|R_l^{(\eta)}(X,X^j)-R_l^{(\eta)}(X,X^*)|\leqslant K\cdot\omega(\delta(X))\leqslant K\cdot\omega(|X-Y|).$$

Similarly,

$$|\Phi^{(\alpha)}(Y)-f^{(\alpha)}(Y^*)|\leqslant K\cdot\omega(\delta(Y))\leqslant K\cdot\omega(|X-Y|).$$

Since by assumption

$$|f^{(\alpha)}(X^*)-f^{(\alpha)}(Y^*)|\leqslant\omega(|X^*-Y^*|)\leqslant K\cdot\omega(|X-Y|),$$

the lemma follows with the triangle inequality.

Remark. If $\mathcal{A}$ is the boundary of a convex set Ω, say, and f is defined and has the properties required in theorem in $\bar{\Omega}$ only, we can extend f from $\mathcal{A}$ to $\bar{\Omega}$ by using only those φ_j which have support in Ω.

4. In this section we shall be concerned with the connections between the solution and its derivatives.

Theorem 4.1. *If $u(X)$ is a solution of (1.1A) in an open bounded region D which has the property that*

$$\iint_D \delta^{\gamma-1}(X)\,dX<\infty\quad \textit{for all}\quad \gamma>0,$$

then, if $p>1$, $\gamma>0$, the finiteness of the first of the following integrals implies the finiteness of the two others.

$$\iint_D \delta^{\gamma-1}(X)\,|u(X)|^p dX, \tag{4.1.1}$$

$$\iint_D \delta^{p-1+\gamma}(X)\,|u_i(X)|^p dX, \tag{4.1.2}$$

$$\iint_D \delta^{2p-1+\gamma}(X)\,|u_{ij}(X)|^p dX. \tag{4.1.3}$$

Remark 1. One important type of permissible regions are those whose boundary admits a local representation satisfying a Lipschitz condition, i.e. to every point $X_0\in\partial D$ there is a sphere Σ such that the part of ∂D which is inside Σ may be represented as $\xi_n=\varphi(\xi_1,\ldots,\xi_{n-1})$, where the coordinate system $(\xi_1,\ldots,\xi_n)$ has X_0 as origin and φ satisfies a Lipschitz condition of order one.

Remark 2. If D satisfies the condition of Remark 1, then the finiteness of any one of the integrals (4.1.1–3) implies the finiteness of the two others. This is a consequence of the representation of u (and u_i) as an indefinite integral and the following inequality of Hardy;

$$\int_0^1 x^s \left| \int_x^1 f(t)\,dt \right|^p dx \leqslant \left(\frac{p}{s+1}\right)^p \int_0^1 x^{s+p} |f(x)|^p dx,$$

which is valid for $p \geqslant 1$ and $s > -1$, see [11], Theorem 330.

Proof. Consider the region D_t defined by

$$D_t = \{X \mid \delta(X) > t\},$$

where $\delta(X)$ is the boundary distance function of D while δ_t will be that of D_t.

We shall first prove that there exists a sequence $\{t_i\}_1^\infty$, $t_i \to 0$, such that

$$\iint_{D_{t_i}} \delta_{t_i}^{\gamma-1} |u|^p dX \leqslant K < \infty.$$

Suppose there is no such sequence. Then the function

$$g(t) = \iint_{D_t} \delta_t^{\gamma-1} |u|^p dX$$

tends to infinity as $t \to 0$. It is then easy to see that there is a function $\varepsilon_1(t) \to 0$ such that

$$\int_0 \frac{\varepsilon_1(t)}{t}\,dt < \infty$$

while

$$\int_0 \frac{\varepsilon_1(t)\,g(t)}{t}\,dt = \infty.$$

In fact, if $a_\nu = \inf g(t)$ where the infimum is taken over $(2^{-\nu-1}, 2^{-\nu})$ we can always find a convergent positive series $\Sigma\, b_\nu$ with the property that $\Sigma\, a_\nu b_\nu = \infty$, since $a_\nu \to \infty$. Then define $\varepsilon_1(t) = b_\nu$ for $2^{-\nu-1} \leqslant t < 2^{-\nu}$.

Now we get

$$\infty = \int_0 \frac{\varepsilon_1(t)\,g(t)}{t}\,dt \leqslant \iint_D |u|^p dX \int_0^{\delta(X)} \frac{\varepsilon_1(t)}{t}\,\delta_t^{\gamma-1} dt$$

$$\leqslant \iint_D |u|^p \left[\int_{-\infty}^{\infty} \frac{\varepsilon_1(t)}{t} |\delta(X) - t|^{\gamma-1} dt\right] dX \leqslant \iint_D |u|^p \delta^{\gamma-1}(X)\,dX < \infty,$$

an obvious contradiction.

Choose a covering $\{S_\nu\}_1^\infty$ of D_t in the sense of Lemma 3.1. Assuming the centers of the spheres in the covering to be $\{X_\nu\}_1^\infty$, define X_ν^* as one of the points satisfying $X_\nu^* \in \partial D$, $|X_\nu - X_\nu^*| = \delta(X_\nu)$. Then apply Lemma 3.7 (i) for each ν with $X_0 = X_\nu$ and $X_0^* = X_\nu^*$. Since $K_1 l_\nu \leqslant \delta_t(X) \leqslant K_2 l_\nu$ for $|X - X_\nu| \leqslant 3 l_\nu$, $l_\nu = \delta_t(X_\nu)/4$, we get

$$\iint_{|X-X_\nu|\leqslant l_\nu} \delta^{p-1+\gamma}(X)|u_i|^p dX \leqslant K \iint_{|X-X_\nu|\leqslant 3l_\nu} \delta_t^{\gamma-1}|u|^p dX$$

$$+K\cdot \iint_{|X-X_\nu|\leqslant 3l_\nu} \delta_t^{2p-1+\gamma}(X)\{|F|^p+\alpha^p(\delta(X))|u_{ij}|^p\}\,dX.$$

Now sum over ν on both sides:

$$\iint_{D_t} \delta_t^{\gamma-1}|u_i|^p dX \leqslant K\iint_{D_t}\delta_t^{\gamma-1}|u|^p dX + K\iint_{D_t}\delta_t^{2p-1+\gamma}\{|F|^p+\alpha^p(\delta)|u_{ij}|^p\}dX. \tag{4.1.4}$$

We have $|F^p|\leqslant K|u|^p\delta^{-2p}(X)\cdot\alpha^p(\delta)+K|u_i|^p\delta^{-p}\cdot\alpha^p(\delta)+K\alpha^p(\delta)|u_{ij}|^p+|f|^p$, and since K does not depend on t, and $\alpha(\delta)$ tends to zero with δ, we can find some t' which is independent of t, and is such that $K\cdot\alpha^p(\delta(X))<\frac{1}{2}$ if $\delta(X)<t'$. Then if $t<t'/2$

$$K\cdot\iint_{D_t}\delta_t^{2p-1+\gamma}\delta^{-p}\alpha^p(\delta)|u_i|^p dX \leqslant \frac{1}{2}\iint_{D_t\cap\{\delta\leqslant t'\}}\delta_t^{p-1+\gamma}|u_i|^p dX$$

$$+K\iint_{D_t\cap\{\delta>t'\}}\delta_{t'/2}^{p-1+\gamma}\delta^{-p}\alpha^p|u_i|^p dX \leqslant \frac{1}{2}\iint_{D_t}\delta_t^{p-1+\gamma}|u_i|^p dX + K(t') = I_1+I_2.$$

If we combine this inequality with (4.1.4) and move I_1 to the left hand side in the resulting inequality, we get

$$\iint_{D_t}\delta_t^{\gamma-1+p}|u_i|^p dX \leqslant K(t')+K\iint_{D_t}\delta_t^{\gamma-1}|u|^p+\delta_t^{2p-1+\gamma}\{\alpha^p(\delta)|u_{ij}|^p+|f|^p\}\,dX, \tag{4.1.5}$$

where $K(t')$ depends on u_i and t', but not on t, and K does not depend on u, t, or t'. Now use Lemma 3.7 (ii) in the same way as above to get

$$\iint_{D_t}\delta_t^{2p-1+\gamma}|u_{ij}|^p dX \leqslant K\iint_{D_t}\delta_t^{\gamma-1}|u|^p dX + K\iint_{D_t}\delta_t^{2p-1+\gamma}\{|F|^p+\alpha^p(\delta)|u_{ij}|^p\}\,dX$$

in which we use (4.1.5):

$$\iint_{D_t}\delta_t^{2p-1+\gamma}|u_{ij}|^p dX \leqslant K(t')+K\iint_{D_t}\delta_t^{\gamma-1}|u|^p dX + K\iint_{D_t}\delta_t^{2p-1+\gamma}\{|f|^p+\alpha^p|u_{ij}|^p\}\,dX.$$

We choose t'' such that $K\cdot\alpha^p(\delta(X))<\frac{1}{2}$ when $\delta(X)<t''$ and get

$$\iint_{D_t}\delta_t^{2p-1+\gamma}|u_{ij}|^p dX \leqslant K(t'')+K\iint_{D_t}\delta_t^{\gamma-1}|u|^p dX + K\iint_{D_t}\delta_t^{2p-1+\gamma}|f|^p dX. \tag{4.1.6}$$

If we put $t=t_\nu$ and let $\nu\to\infty$, (4.1.6) implies that (4.1.3) is finite, with Fatou's lemma, since it is easy to see that

$$\iint_{D_t}\delta_t^{2p-1+\gamma}|f|^p dX \leqslant K<\infty.$$

The finiteness of (4.1.2) follows from (4.1.5) and (4.1.6). The theorem is proved.

Corollary 4.2. *If* $u \in H^p$, $p > 1$, *then*

$$\iint_\Omega x_n^{p+1-\gamma} |u_i|^p dX < \infty, \tag{4.2.1}$$

$$\iint_\Omega x_n^{2p-1+\gamma} |u_{ij}|^p dX < \infty, \tag{4.2.2}$$

for every bounded subdomain Ω *of* R_+^n *and every* $\gamma > 0$.

Proof. Put $D = \{X \mid \varrho > x_n > 0,\ |x_i| < \varrho,\ i = 1, \ldots, n-1\}$. By Lemma 3.8, for almost every $\varrho \in R_+^1$,

$$\sum_{i=1}^{n-1} \iint_{R_+^n} |x_i \pm \varrho|^{\gamma-1} |u|^p dX < \infty$$

and (4.2.1) and (4.2.2) follow from the theorem with

$$\Omega = \{X \mid \varrho/2 > x_n > 0,\ |x_i| < \varrho/2,\ i = 1, \ldots, n-1\}.$$

It is clear that the region used here fulfils the hypothesis of the theorem.

Theorem 4.3. *Let* D *be a region satisfying the assumptions of Theorem 4.1. Assume that* u *is a solution of* (1.1 A) *which for some* $\gamma > 0$ *satisfies*

$$\iint_D \delta^{\gamma-1}(X) |u| dX < \infty. \tag{4.3.1}$$

Then if $\gamma_1 \geq n\gamma/(1-\gamma)$ *and* $p \leq 1 + \gamma_1/n$ *we have*

$$\iint_D \delta^{\gamma_1 - 1} |u|^p dX < \infty, \tag{4.3.2}$$

$$\iint_D \delta^{p-1+\gamma_1} |u_i|^p dX < \infty, \tag{4.3.3}$$

$$\iint_D \delta^{2p-1+\gamma_1} |u_{ij}|^p dX < \infty. \tag{4.3.4}$$

Proof. We cover D_t using Lemma 3.1, the centers of the spheres being $\{X_\nu\}_1^\infty$ as before. By Lemma 3.7 (iii) where we put $X_0 = X_\nu$ and $X_0^* = X_\nu^*$, we get, since δ_t is bounded above and below by l_ν times a positive constant,

$$\iint_{|X-X_\nu| \leq l_\nu} \delta_t^{\gamma_1 - 1} |u|^p dX \leq K \cdot l_\nu^{p'} \left[\iint_{|X-X_\nu| \leq 3l_\nu} \delta_t^{\gamma-1} |u| dX \right]^p$$
$$+ K \cdot \iint_{|X-X_\nu| \leq 3l_\nu} \delta_t^{2p-1+\gamma} \{|F|^p + |h^*|^p\} dX.$$

Here $p' = \gamma_1 - 1 - p\gamma + p + n - np \geq 0$ by the assumptions. Now sum over ν and use the elementary inequality $\Sigma |a_\nu|^p \leq (\Sigma |a_\nu|)^p$ to get

$$\iint_{D_t} \delta_t^{\gamma_1 - 1} |u|^p dX \leq K \left[\iint_{D_t} \delta_t^{\gamma - 1} |u| \, dX \right]^p + K \iint_{D_t} \delta_t^{2p-1+\gamma} \{ |F|^p + \alpha^p |u_{ij}|^p \} \, dX. \tag{4.3.5}$$

If we combine (4.3.5) with the inequalities (4.1.5) and (4.1.6) from the proof of Theorem 4.1 we get

$$\iint_{D_t} \delta_t^{\gamma_1 - 1} |u|^p dX \leq K(t'') + K \left[\iint_{D_t} \delta_t^{\gamma - 1} |u| \, dX \right]^p$$
$$+ K \cdot \iint_{D_t} \delta_t^{2p-1+\gamma_1} \{ |f|^p + \alpha^p \cdot \delta^{-2p} |u|^p \} \, dX.$$

There is a t''' such that $K \cdot \alpha^p(\delta) < \frac{1}{2}$ when $\delta < t'''$ which with the moving of a suitable part of the right hand side gives rise to

$$\iint_{D_t} \delta_t^{\gamma_1 - 1} |u|^p dX \leq K(t''') + K \left[\iint_{D_t} \delta_t^{\gamma - 1} |u| \, dX \right]^p + K \cdot \iint_{D_t} \delta^{\gamma_1 - 1}(X) \, dX. \tag{4.3.6}$$

There is also a sequence $t_\nu \searrow 0$ such that

$$\iint_{D_{t_\nu}} \delta_{t_\nu}^{\gamma - 1} |u| \, dX \leq K < \infty$$

which proves (4.3.2). The rest of the theorem follows from Theorem 4.1.

Corollary 4.4. *Let $u \in H^1$. Then to every $\gamma > 0$ there is a $p > 1$ such that for every bounded subdomain Ω of R^n_+*

$$\iint_{\Omega} x_n^{\gamma - 1} |u|^p dX < \infty, \tag{4.4.1}$$

$$\iint_{\Omega} x_n^{p-1+\gamma} |u_i|^p dX < \infty, \tag{4.4.2}$$

$$\iint_{\Omega} x_n^{2p-1+\gamma} |u_{ij}|^p dX < \infty. \tag{4.4.3}$$

The following estimate is well known for a more restrictive class of elliptic equations, see [18].

Theorem 4.5. *Let D be a region satisfying the assumptions of Theorem 4.1. Assume that u is a bounded solution of* (1.1B) *in D. Then*

$$|\operatorname{grad} u(X)| \leq K \cdot \delta^{-1}(X). \tag{4.5.1}$$

Proof. Since $H^p \supset H^\infty$ we have by Theorem 4.1

$$\iint_D \{ \delta^{p-1+\gamma} |u_i| + \delta^{2p-1+\gamma} |u_{ij}|^p \} \, dX < \infty. \tag{4.5.2}$$

Making obvious estimates in the representation formula (3.7.4) of Lemma 3.7 we get

$$\left|\frac{\partial u}{\partial x_k}(Y)\right| \leqslant \frac{K}{\delta(Y)} + K\cdot\iint_{|X-Y|\leqslant\frac{1}{2}\delta(Y)} |X-Y|^{1-n}\{\delta^{\alpha-1}(X)|u_i| + \delta^{\alpha}(X)|u_{ij}|\}\,dX.$$

Choose $p > n/\alpha \geqslant n$, use Hölder's inequality and apply (4.5.2)

$$\left|\frac{\partial u}{\partial x_k}(Y)\right| \leqslant \frac{K}{\partial(Y)} + K\cdot\delta^{\alpha-2+(1-\gamma)/p}(Y)\left[\iint_{|X-Y|\leqslant\frac{1}{2}\delta(Y)}|X-Y|^{(1-n)q}dX\right]^{1/q}$$

$$\times\left[\iint_D \delta^{2p-1+\gamma}\{|u_{ij}|^p + \delta^{-p}|u_i|^p\}\,dX\right]^{1/p} \leqslant \frac{K}{\delta(Y)} + K\cdot\delta^{\alpha-2+(1-\gamma)/p-(n/p)+1}(Y) \leqslant K\cdot\delta^{-1}(Y)$$

if γ is small enough.

Remark. Whether $|u_{ij}| = O(\delta^{-2})$ is also true remains an open question. We will not need this result in the sequel, however.

Theorem 4.6. *Let u be a solution of* (1.1B) *belonging to H^p, $p \geqslant 1$. Then*

$$\int_{|Y'|\leqslant\varrho} |u_i(y_1,\ldots,y_{n-1},y_n)|^p dY' \leqslant K(\varrho)\cdot y_n^{-p}.$$

Proof. By formula (3.7.4)

$$\left|\frac{\partial u}{\partial x_k}(Y)\right| \leqslant K\cdot y_n^{1-n}\int_{|X-Y|=y_n/2}|u|\,dS + K\iint_{|X-Y|\leqslant y_n/2}|X-Y|^{1-n}\{|F| + x_n^{\alpha}|u_{ij}|\}\,dX.$$

If $p > 1$ choose $\gamma > 0$ and use Hölder's inequality:

$$\left|\frac{\partial u}{\partial x_k}(Y)\right|^p \leqslant K\cdot y_n^{1-n-p}\int_{|X-Y|=y_n/2}|u|^p dS + K\cdot y_n^{p-1-\gamma}\iint_{|X-Y|\leqslant y_n/2}|X-Y|^{1-n+\gamma}$$

$$\times\{|F|^p + x_n^{\alpha p}|u_{ij}|^p\}\,dX$$

or

$$\int_{|Y'|\leqslant\varrho} y_n^p\left|\frac{\partial u}{\partial x_k}(Y)\right|^p dY' \leqslant K + \iint x_n^{2p-1}\{|F|^p + x_n^{\alpha p}|u_{ij}|^p\} \leqslant K < \infty$$

by Corollary 4.2 if the double integral is taken e.g. over $|X| \leqslant 2\varrho$, $x_n > 0$.

If $p = 1$,

$$\int_{|Y'|\leqslant\varrho} y_n\left|\frac{\partial u}{\partial x_k}(Y)\right|dY' \leqslant K\cdot y_n^{1-n}\int_{|Y'|\leqslant\varrho}dY'\int_{|X-Y|=y_n/2}|u|\,dS_X$$

$$+ K\cdot y_n\int_{|Y'|\leqslant\varrho}dY'\iint_{|X-Y|\leqslant y_n/2}|X-Y|^{1-n}|f|\,dX$$

$$+ K\cdot y_n\int_{|Y'|\leqslant\varrho}dY'\iint_{|X-Y|\leqslant y_n/2}|X-Y|^{1-n}\{x_n^{\alpha-2}|u| + x_n^{\alpha-1}|u_i| + x_n^{\alpha}|u_{ij}|\}\,dX$$

$$\leqslant K + \iint |\log|x_n - y_n|| \{x_n^{\alpha-1}|u| + x_n^{\alpha}|u_i| + x_n^{1+\alpha}|u_{ij}|\}\, dX$$

$$\leqslant K + K\left[y_n^{\alpha-1}\iint_{(y_n/2)<x_n<(3y_n/2)} |\log|x_n-y_n||^{q'}dX\right]^{1/q'}\left[\iint\{x_n^{\alpha-1}|u|^{p'} + x_n^{p'-1+\alpha}|u_i|^{p'} + x_n^{2p'-1+\alpha}|u_{ij}|^{p'}\}\,dX\right]^{1/p'}$$

$$\leqslant K + Ky_n^{\alpha/2q'}\iint_{|X|\leqslant 2\varrho}\{x_n^{\alpha-1}|u|^{p'} + x_n^{p'-1+\alpha}|u_i|^{p'} + x_n^{2p'-1+\alpha}|u_{ij}|^{p'}\}\,dX \leqslant K < \infty$$

by Corollary 4.4 if p' is small enough.

Theorem 4.7. *If u is a solution of* (1.1B), *then $u \in H^2$ if and only if*

$$\iint_{\Omega} x_n|u_i|^2 dX < \infty$$

for every bounded $\Omega \subset R_+^n$.

Proof. Assume $u \in H^2$. It suffices to take $\Omega = Q_\varrho$,

$$Q_\varrho = \{X\,|\,0 < x_n < \varrho,\ |x_i| < \varrho,\ i = 1, \ldots, n-1\}.$$

We will need a special type of test functions $\psi(X) = \psi(\varepsilon, X)$ with the following properties:

1° $\psi \in C_0^\infty(R_+^n)$, $\operatorname{supp}(\psi) \subset Q_{2\varrho}$.

2° When $X \in Q_\varrho$, ψ depends on x_n (and ε) only.

3° $\psi(X) = 0$ for $x_n < \varepsilon/2$.

4° $0 \leqslant \psi(X) \leqslant x_n$ everywhere, $\psi(X) = x_n$ for $X \in Q_\varrho$, $\varepsilon < x_n < \varrho$.

5° $|\psi_i| \leqslant K$, $|\psi_{ij}| \leqslant K\cdot x_n^{-1}$ where K does not depend on ε.

6° $\int_0^{2\varrho}\{\max_{|X'|\leqslant 4\varrho^2}|\psi_{ij}(x_1, x_2, \ldots, x_{n-1}, t)|\}\,dt \leqslant K$ where K does not depend on ε.

Such a function clearly exists, e.g. $\psi(X) = \eta^1(x_n)\cdot\eta^2(\sqrt{x_1^2 + \ldots + x_{n-1}^2})$ where $\eta^2(t) = 1$ for $|t| \leqslant \varrho$ and $=0$, $|t| \geqslant 2\varrho$ and does not depend on ε, while η^1 satisfies 3° and 4° above, and $d^2\eta^1(t)/dt^2$ changes sign, say, at most four times.

Now regularize the equation (1.1) in R_+^n, multiply with $\psi(X)\cdot u(X)$ and integrate partially:

$$\iint_{Q_{2\varrho}} \psi u[\bar{a}^{ij}u_{ij} - F + (a^{ij} - \bar{a}^{ij})\,u_{ij}]\,dX = 0,$$

$$\iint_{Q_{2\varrho}} \psi\bar{a}^{ij}u_i u_j dX = \iint_{Q_{2\varrho}} -\psi_j\bar{a}^{ij}uu_i - \psi\bar{a}_j^{ij}uu_i + \psi u[-F + (a^{ij} - \bar{a}^{ij})\,u_{ij}]\,dX$$

$$= \frac{1}{2}\iint_{Q_{2\varrho}}\{\psi_{ji}u^2\bar{a}^{ij} + \psi_j u^2\bar{a}_i^{ij}\}\,dX + \frac{1}{2}\iint_{Q_{2\varrho}}\{\psi_i u^2\bar{a}_j^{ij} + \psi u^2\bar{a}_{ji}^{ij}\}\,dX$$

$$+ \iint_{Q_{2\varrho}} \psi u[-F + (a^{ij} - \bar{a}^{ij})\,u_{ij}]\,dX.$$

If τ is small enough, $\bar{a}^{ij}(X)\xi_i\xi_j \geqslant \lambda/2|\xi|^2$ for $x_n < \tau$, and hence on the left hand side

$$\iint_{Q_{2\varrho}} \psi u_i u_j \bar{a}^{ij} dX \geqslant \frac{\lambda}{2} \iint_{Q_{2\varrho}\cap\{x_n<\tau\}} \psi |u_i|^2 dX - K(\tau)$$

On the right hand side we have three integrals to estimate. We get using the properties of ψ and Corollary 4.2,

$$\left|\iint_{Q_{2\varrho}} \{\psi_{ij} u^2 \bar{a}^{ij} + \psi_j u^2 \bar{a}_i^{ij}\} dX\right|$$

$$\leqslant K \cdot \int_0^{2\varrho} \max_{|X'|\leqslant 2\varrho} |\psi_{ij}(x_1, \ldots, x_{n-1}, t)| dt \int_{|X'|\leqslant 2\varrho} |u|^2 dX' + K \iint_{Q_{2\varrho}} x_n^{\alpha-1} |u|^2 dX \leqslant K$$

independently of ε.

$$\left|\iint_{Q_{2\varrho}} \{\psi_i \bar{a}_j^{ij} u^2 + \psi u^2 \bar{a}_{ji}^{ij}\} dX\right| \leqslant K \iint_{Q_{2\varrho}} x_n^{\alpha-1} |u|^2 dX,$$

$$\left|\iint_{Q_{2\varrho}} \psi u [F + (\bar{a}^{ij} - a^{ij}) u_{ij}] dX\right| \leqslant K \iint_{Q_{2\varrho}} x_n^{\alpha-1} |u|^2 dX$$

$$+ K \left[\iint_{Q_{2\varrho}} \frac{\beta(x_n)}{x_n} |u|^2 dX\right]^{\frac{1}{2}} \left[\iint_{Q_{2\varrho}} \frac{\beta(x_n)}{x_n} dX\right]^{\frac{1}{2}}$$

$$+ K \left[\iint_{Q_{2\varrho}} x_n^{\alpha-1} |u|^2 dX\right]^{\frac{1}{2}} \left[\iint_{Q_{2\varrho}} x_n^{\alpha+1} |u_i|^2 + x_n^{3+\alpha} |u_{ij}|^2 dX\right]^{\frac{1}{2}}.$$

Thus we get

$$\iint_{Q_{2\varrho}\cap\{x_n<\tau\}} \psi |u_i|^2 dX \leqslant K + K(\tau)$$

or

$$\iint_{Q_{\varrho}\cap\{x_n>\varepsilon\}} x_n |u_i|^2 dX \leqslant K + K(\tau).$$

Since K and $K(\tau)$ do not depend on ε, the necessity part of the theorem is proved.

To prove the sufficiency, assume

$$\iint_{\Omega} x_n |u_i|^2 dX < \infty$$

for every bounded subdomain Ω of R_+^n. An easy application of Schwarz's inequality then shows that

$$\iint_{\Omega} x_n^{\gamma-1} |u|^2 dX < \infty$$

for every $\gamma > 0$ and

$$\iint_{\Omega} \frac{\beta(x_n)}{x_n} |u|^2 dX < \infty.$$

From Theorem 4.1 we conclude that the following integral is bounded when Ω is bounded

$$\iint_{\Omega} x_n^{3+\alpha}|u_{ij}|^2 dX.$$

Let D be a convex bounded region $\subset R_+^n$ the boundary of which is sufficiently regular and contains the set $\{X|x_n=0, |X'|\leqslant\varrho\}$ for an arbitrarily chosen $\varrho>0$. Such a region is easily constructed. If we denote by $\delta(X)$ the boundary distance function of D, while δ_τ is that of D_τ, we can construct a positive function $\Delta(X)$, coinciding with $\delta(X)$ for $\delta(X)$ sufficiently small, and belonging to $C^2(\bar{D})$. We observe that $\Delta(X)-\tau\leqslant\delta_\tau(X)$ if δ is small enough.

Integrating partially, we find

$$\iint_{D_\tau} \bar{a}^{ij}2uu_i\Delta_j(X)\,dX = -\iint_{D_\tau}\{\bar{a}_j^{ij}2uu_i(\Delta-\tau)+\bar{a}^{ij}2u_iu_j(\Delta-\tau)+\bar{a}^{ij}2uu_{ij}(\Delta-\tau)\}\,dX.$$

On the other hand

$$\iint_{D_\tau}\bar{a}^{ij}2uu_i\Delta_j(X)\,dX=\int_{\partial D_\tau}\bar{a}^{ij}u^2\Delta_j dX_{(i)}-\iint_{D_\tau}\{\bar{a}_i^{ij}u^2\Delta_j+\bar{a}^{ij}u^2\Delta_{ji}\}\,dX.$$

Thus using the fact that u is a solution and that

$$\bar{a}^{ij}\Delta_j dX_{(i)}\geqslant\lambda/2\frac{\partial\Delta}{\partial n}dS=\frac{\lambda}{2}dS$$

on that part of ∂D_τ which is below $x_n=\varepsilon$ for some $\varepsilon>0$ we get

$$\begin{aligned}\int_{\partial D_\tau\cap\{x_n<\varepsilon\}}u^2dS &\leqslant K\int_{\partial D_\tau\cap\{x_n>\varepsilon\}}u^2dS+K\iint_{D_\tau}\{x_n^{\alpha-1}|u|^2+x_n^{\alpha}|u||u_i|+x_nu_i^2\\&\quad+x_n|u|[|F|+x_n^{\alpha}|u_{ij}|]\}dX\leqslant K\cdot\max_{\{x_n>\varepsilon\}}|u|^2+K\cdot\iint_D\{x_n^{\alpha-1}|u|^2+x_n|u_i|^2\}\,dX\\&\quad+K\left[\iint_D x_n^{2\alpha-1}|u|^2dX\right]^{\frac{1}{2}}\left[\iint_D x_n|u_i|^2dX\right]^{\frac{1}{2}}+K\iint_D\frac{\varepsilon(x_n)}{x_n}|u|\,dX\\&\quad+K\left[\iint_D x_n^{\alpha-1}|u|^2dX\right]^{\frac{1}{2}}\left[\iint_D x_n^{3+\alpha}|u_{ij}|^2dX\right]^{\frac{1}{2}}.\end{aligned}$$

Since the right hand side is finite and independent of τ, we have

$$\int_{|X'|\leqslant\varrho}u^2(x_1,\ldots,x_{n-1},\tau)\,dX'\leqslant\int_{\partial D_\tau\cap\{x_n<\varepsilon\}}u^2dS\leqslant K<\infty$$

and the theorem is proved.

Theorem 4.8. *Consider the equation* (1.1 A) *in* R_+^n. *If u is a solution, possibly defined only in* $V_h(X')$,[1] *satisfying*

[1] In this case we assume that u, u_i, and u_{ij} are bounded in $\{x_n>\frac{1}{2}\}$, say.

$$\iint_{V_h(X')} x_n^{\gamma-n}|u|\,dX<\infty$$

for some $\gamma>0$, *then for all* $p\geqslant 1$ *and all* $h'>h$

$$\iint_{V_{h'}(X')} x_n^{p\gamma-n}|u|^p dX<\infty,$$

$$\iint_{V_{h'}(X')} x_n^{p+p\gamma-n}|u_i|^p dX<\infty,$$

$$\iint_{V_{h'}(X')} x_n^{2p-n+p\gamma}|u_{ij}|^p dX<\infty.$$

Proof. We introduce the sets $V^\tau=V_h^\tau(X')=V_h(X')\cap\{X\,|\,x_n>\tau>0\}$. Let $\delta_\tau(X)$ denote the distance from X to ∂V^τ. Using Lemma 3.1, we cover V^τ with spheres having centers X_ν, then apply Lemma 3.7 (i) and (ii) with $X_0^*=X'$. After multiplication by $x_n^{p\gamma-np}\cdot\delta_\tau^{np-n}$ we find

$$\left.\begin{aligned}&\iint_{|X-X_\nu|\leqslant l_\nu}\delta_\tau^{p+np-n}\cdot x_n^{p\gamma-np}|u_i|^p dX\leqslant\\ &\iint_{|X-X_\nu|\leqslant l_\nu}\delta_\tau^{2p+np-n}\cdot x_n^{p\gamma-np}|u_{ij}|^p dX\leqslant\end{aligned}\right\} K\iint_{|X-X_\nu|\leqslant 3l_\nu}\delta_\tau^{np-n}x_n^{p\gamma-np}|u|^p dX$$

$$+K\iint_{|X-X_\nu|\leqslant 3l_\nu}\delta_\tau^{2p+np-n}x_n^{p\gamma-np}\{|F|^p+\alpha^p(x_n)|u_{ij}|^p\}\,dX.$$

After summing over ν we note that the constants do not depend on τ and in a manner by now familiar we get the inequalities

$$\iint_{V^\tau}\delta^{2p+np-n}x_n^{p\gamma-pn}|u_{ij}|^p dX$$

$$\leqslant K(\tau')+K\iint_{V^\tau}\{\delta_\tau^{np-n}x_n^{p\gamma-np}|u|^p+\delta_\tau^{p+np-n}\cdot x_n^{p\gamma-np}\alpha^p(x_n)|u_i|^p\}\,dX,$$

$$\iint_{V^\tau}\delta_\tau^{p+np-n}x_n^{p\gamma-np}|u_i|^p dX$$

$$\leqslant K(\tau'')+K\cdot\iint_{V^\tau}\{\delta_\tau^{np-n}x_n^{p\gamma-np}|u|^p+\delta_\tau^{2p+np-n}\cdot x_n^{p\gamma-np}\alpha^p(x_n)|u_{ij}|^p\}\,dX$$

which combined give rise to

$$\left.\begin{aligned}&\iint_{V^\tau}\delta_\tau^{p+np-n}x_n^{p\gamma-np}|u_i|^p dX\leqslant\\ &\iint_{V^\tau}\delta_\tau^{2p+np-n}x_n^{p\gamma-np}|u_{ij}|^p dX\leqslant\end{aligned}\right\} K(\tau''')+K\iint_{V^\tau}\delta_\tau^{np-n}\cdot x_n^{p\gamma-np}|u|^p dX. \qquad \begin{aligned}&(4.8.2)\\ &(4.8.3)\end{aligned}$$

Using the same covering of V^τ as above, we now apply Lemma 3.7 (iii)

$$\iint_{|X-X_\nu|\leqslant l_\nu} \delta_\tau^{np-n} x_n^{p\gamma-np}|u|^p dX \leqslant K\left[\iint_{|X-X_\nu|\leqslant 3l_\nu} x_n^{\gamma-n}|u|dX\right]^p$$
$$+K\iint_{|X-X_\nu|\leqslant 3l_\nu} \delta_\tau^{2p+np-n} x_n^{p\gamma-np}\{|F|^p+\alpha^p(x_n)|u_{ij}|^p\}\,dX.$$

Sum over ν and use the inequality $\Sigma|a_n|^p \leqslant (\Sigma|a_n|)^p$:

$$\iint_{V^\tau} \delta_\tau^{np-n} x_n^{p\gamma-np}|u|^p dX \leqslant K(\tau^{(iv)})+K\left[\iint_{V^\tau} x_n^{\gamma-n}|u|dX\right]^p$$
$$+K\iint_{V^\tau}\{\delta_\tau^{np-n}x_n^{p\gamma-np}\alpha^p(x_n)|u|^p+\delta_\tau^{p+np-n}x_n^{p\gamma-np}\alpha^p(x_n)|u_i|^p$$
$$+\delta_\tau^{2p+np-n}x_n^{p\gamma-np}\alpha^p(x_n)|u_{ij}|^p\}\,dX.$$

Applying (4.8.2) and (4.8.3) while noticing that $K\alpha^p(x_n)\to 0$ independently of τ we get

$$\iint_{V_h^\tau} \delta_\tau^{np-n} x_n^{p\gamma-np}|u|^p dX \leqslant K(\tau^{(v)})+K\left[\iint_{V_h} x_n^{\gamma-n}|u|dX\right]^p.$$

Since the right hand side is independent of τ, Fatou's lemma gives

$$\iint_{V_h} \delta^{np-n} x_n^{p\gamma-np}|u|^p dX<\infty.$$

Now (4.8.2) and (4.8.3) give the corresponding results for u_i and u_{ij}, and the theorem follows with the observation that for every $h'>h$, $\delta(X)\geqslant K\cdot x_n$ for $X\in V_{h'}$.

Remark. In the case of Laplace's equation, Theorem 4.8 follows much easier if we use the Poisson representation. In fact, Fubini's theorem implies that there is an h'', $h<h''<h'$ such that

$$\int_{\partial V_{h''}} x_n^{\gamma+1-n}|u|dS<\infty.$$

Since $V_{h''}$ is convex, $|\partial G/\partial n(X,Y)|\leqslant K\cdot|X-Y|^{1-n}$ and with Hölder's inequality

$$|u(Y)|^p \leqslant \left[\int_{\partial V_{h''}} |X-Y|^{1-n}x_n^{\gamma}|u(X)|dS_X\right]^{p/q}\cdot\left[\int_{\partial V_{h''}} |X-Y|^{1-n}x_n^{-\gamma(p-1)}|u|dS\right].$$

As $Y\in V_{h'}$, $|X-Y|\geqslant x_n\cdot K$, and hence the first integral is finite. Multiplication with $y_n^{p\gamma-n}$ and integration over $V_{h'}$ gives

$$\iint_{V_{h'}} y_n^{p\gamma-n}|u(Y)|^p dY\leqslant K\cdot\int_{\partial V_{h''}} x_n^{-\gamma(p-1)}|u(X)|dX\iint_{V_{h'}} y_n^{p\gamma-n}|X-Y|^{1-n}dY.$$

The proof follows with the observation that

$$\iint_{V_{h'}} y_n^{p\gamma-n}|X-Y|^{1-n}dY \leqslant K\cdot x_n^{\gamma p+1-n}.$$

The following theorem for harmonic functions can e.g. be found in [22].

Theorem 4.9. *Consider the equation* (1.1B) *in* R_+^n. *If* u *is a solution, possibly defined in* $V_h(X')$ *only, satisfying*

$$\iint_{V_h(X')} x_n^{2-n}|u_i|^2 dX < \infty.$$

Then

$$x_n|u_i(X)| \to 0$$

as $x_n \to 0$ *in* $V_{h'}(X')$ *for every* $h' > h$.

Proof. Using Schwarz's inequality, we see that the assumption implies

$$\iint_{V_h(X')} x_n^{\gamma-n}|u|^2 dX < \infty$$

for every $\gamma > 0$. By Theorem 4.8, we can conclude the finiteness of the following integrals for every $\gamma > 0$ and every $h'' > h$.

$$\iint_{V_{h''}(X')} x_n^{\gamma-n}|u|^p dX, \tag{4.9.1}$$

$$\iint_{V_{h''}(X')} x_n^{p+\gamma-n}|u_i|^p dX, \tag{4.9.2}$$

$$\iint_{V_{h''}(X')} x_n^{2p+\gamma-n}|u_{ij}|^p dX. \tag{4.9.3}$$

Choose an arbitrary $h' > h$ and let $h' > h'' > h$. Then there is a constant k such that a sphere around $Y \in V_{h'}$ with radius $k\cdot y_n$ lies inside $V_{h''}$. In this sphere we can write u as the sum of two functions

$$u(Z) = \frac{1}{\omega_n}\int_{|X-Y|=k\cdot y_n} \frac{\partial G}{\partial \nu}(X,Z)\,u(X)\,dS_X$$
$$+\frac{1}{\omega_n}\iint_{|X-Y|\leqslant k\cdot y_n} G(X,Z)\,[F+\{a^{ij}(X)-a^{ij}(X')\}\,u_{ij}]\,dX = u^{(1)}+u^{(2)},$$

where G is the Green function of the operator $a^{ij}(X')\partial^2/\partial x_i\partial x_j$ in $|X-Y| \leqslant k\cdot y_n$. Since $u^{(1)}$ is the solution of an operator with constant coefficients we find

$$\left|\frac{\partial u^{(1)}}{\partial x_l}(Y)\right|^2 \leqslant \left[\sum_{i=1}^{n}\frac{K}{y_n^n}\iint_{|X-Y|\leqslant k\cdot y_n}\left|\frac{\partial u^{(1)}}{\partial x_i}\right|dX\right]^2 \leqslant K\cdot y_n^{-n}\iint_{|X-Y|\leqslant k\cdot y_n}|u_i^{(1)}|^2 dX.$$

By the Dirichlet principle

$$\iint_{|X-Y|\leqslant k\cdot y_n} |u_i^{(1)}|^2 dX \leqslant K\cdot \iint_{|X-Y|\leqslant k\cdot y_n} |u_i|^2 dX.$$

On the other hand

$$\left|\frac{\partial u^{(2)}}{\partial x_l}(Y)\right| \leqslant K \iint_{|X-Y|\leqslant y_n\cdot k} |X-Y|^{1-n}\{|F| + x_n^{\alpha}|u_{ij}|\}\,dX$$

$$\leqslant K\left[\iint_{|X-Y|\leqslant k\cdot y_n} |X-Y|^{(1-n)q}dX\right]^{1/q}\left[\iint_{|X-Y|\leqslant k\cdot y_n}\{|F|^p + x_n^{\alpha p}|u_{ij}|^p\}\,dX\right]^{1/p}$$

$$\leqslant K\cdot y_n^{-1}\left[\iint_{|X-Y|\leqslant k\cdot y_n} x_n^{2p-n}\{|F|^p + x_n^{\alpha p}|u_{ij}|^p\}\,dX\right]^{1/p}$$

if $p > n$. Combining these inequalities we get

$$y_n|u_i(Y)| \leqslant K\cdot\left[\iint_{V_{h''}\cap\{x_n\leqslant 2y_n\}} x_n^{2-n}|u_i|^2 dX\right]^{\frac{1}{2}}$$

$$+K\left[\iint_{V_{h''}\cap\{x_n\leqslant 2y_n\}} x_n^{2p-n}\{|f|^p + x_n^{p\alpha}|u_{ij}|^p + x_n^{(\alpha-1)p}|u_i|^p + x_n^{(\alpha-2)p}|u|^p\}\,dX\right]^{1/p}.$$

By (4.9.1–3) above we find that the right hand side tends to zero as $y_n \to 0$.

Our next theorem is a corollary of Theorem 4.9 and the following theorem by Wallin [25].

If u is a continuous Beppo–Levi function in R_n^+ such that for some γ, $0\leqslant\gamma<n$

$$\iint_{\Omega} x_n^{\gamma}|u_i|^2 dX < \infty$$

for every bounded subdomain Ω of R_+^n, then $\lim_{x_n\to 0} u(X)$ exists and is finite for all $X'\in R^{n-1}$ except when X' belongs to a certain Borel set E of $m-2+\gamma$-capacity zero.

A Beppo–Levi function is a function which is absolutely continuous on almost every line parallel to some coordinate axis. A Borel set E is said to be of $m-2+\gamma$-capacity zero if for every non-trivial non-negative mass function μ with support in E the potential

$$u^{\mu}(X) = \int_{R^n} |X-Y|^{2-m-\gamma}d\mu(Y)$$

is unbounded.

Theorem 4.10. *If u is a solution of* (1.1B) *in R_+^n with the property that*

$$\iint_{\Omega} x_n^{\gamma}|u_i|^2 dX < \infty$$

for every bounded subdomain Ω of R_+^n, then u has a nontangential finite limit at every $X'\in R^{n-1}$ except in a set of $m-2+\gamma$-capacity zero.

Proof. Since, by our definition of solution, u is automatically a continuous Beppo–Levi function, Wallin's theorem shows that perpendicular limits exist except in a set of the right size. Moreover, it can be proved that

$$\iint_{V_h(X')} x_n^{2-n} |u_i|^2 dX < \infty$$

for every $h>0$ and $X' \in R^{n-1}$ except for those X' belonging to a set of $n-2+\gamma$-dimensional Hausdorff measure zero. See Wallin [25], Lemma 5. Combining these facts with Theorem 4.8, the present theorem is proved.

5. It is well known that a positive harmonic function in the unit disc belongs to H^1. The traditional way of proving a theorem of this sort is to use the Poisson representation of the harmonic function and show that the normal derivative of Green's function is bounded away from zero. See e.g. [27]. However, it is possible to do without the Poisson representation.

Theorem 5.1. *Suppose u is a non-negative solution of* (1.1C). *Then* $u \in H^1$.

Proof. Choose an arbitrary $\varrho > 0$. We shall prove that

$$\int_{|X'|\leqslant \varrho} |u(x_1, \dots, x_{n-1}, x_n)|\, dX' \leqslant K < \infty \quad \text{as} \quad x_n \to 0.$$

Let D be the convex bounded region $\subset R_+^n$ whose boundary contains

$$\{X \mid x_n = 0, \ |X'| < \varrho\}$$

that was constructed in the proof of Theorem 4.7. D_t, δ, δ_t, and $\Delta(X)$ will have the same meaning as there. Define

$$m(t) = \max_{\tau \geqslant t} \left[\int_{\delta(X)=\tau} |u(X)|\, dS_X \right].$$

We shall prove the inequality

$$\iint_{\delta \geqslant t} \delta_t^{p-1+\alpha} |u_i|^p dX + \iint_{\delta \geqslant t} \delta_t^{2p-1+\alpha} |u_{ij}|^p dX \leqslant K(t_0) + \varepsilon(t_0)\,[m(t)]^p \tag{5.1.1}$$

for some $p>1$, $t<t_0$, where $\varepsilon(t_0)$ and $K(t_0)$ do not depend on t, and $\varepsilon(t_0)$ does not depend on u, while $\lim_{t_0 \to 0} \varepsilon(t_0) = 0$.

To do that choose $p < 1 + \alpha/n$ and $\gamma < 1 - 1/p$, and combine the inequalities (4.1.5) and (4.1.6) from the proof of Theorem 4.1, with $\gamma = \alpha$, and (4.3.6) of Theorem 4.3, with $\gamma_1 = \alpha$. We get

$$\iint_{D_t} \delta_t^{p-1+\alpha} |u_i|^p dX + \iint_{D_t} \delta_t^{2p-1+\alpha} |u_{ij}|^p dX \leqslant K(t_0) + K \left[\iint_{D_t} \delta_t^{\gamma-1} |u|\, dX \right]^p.$$

But $$\iint_{D_t} \delta_t^{\gamma-1} |u|\, dX \leqslant K(t_0) + K m(t) \cdot \int_t^{t_0} (s-t)^{\gamma-1} ds \leqslant K(t_0) + K \cdot m(t) \cdot t_0^\gamma,$$

which proves inequality (5.1.1).

Now we regularize the equation in the region D_t the boundary of which is sufficiently regular if t is small enough, due to our assumptions. Thus we can find functions $\bar{a}^{ij}(t, X) \in C^\infty(D_t)$, $|\partial/\partial x_k \bar{a}^{ij}| \leqslant K \cdot \delta_t^{\alpha-1}$, $\bar{a}^{ij} \in C^\alpha(\bar{D}_t)$, $\bar{a}^{ij}(t, X) = a^{ij}(X)$ for $\delta(X) = t$, where K and the Hölder constants of $\bar{a}^{ij}$ are independent of t. It is also clear that $2\lambda^{-1}\xi_i^2 \geqslant \bar{a}^{ij}\xi_i\xi_j \geqslant 2^{-1}\lambda\xi_i^2$ when $t < \delta(X) < \tau$ for τ small enough. We get

$$\left| \iint_{D_t} \bar{a}^{ij} u \Delta_{ij}(X)\, dX \right| \leqslant K \iint_{D_t} |u|\, dX \leqslant K(t_0) + K \cdot t \cdot [m(t)]. \tag{5.1.2}$$

On the other hand, using partial integration

$$\iint_{D_t} \bar{a}^{ij} u \Delta_{ij}\, dX = \int_{\partial D_t} \bar{a}^{ij} u \Delta_i\, dX_{(j)} - \iint_{D_t} \{\bar{a}_j^{ij} u \Delta_i + \bar{a}^{ij} u_j \Delta_i\}\, dX.$$

In the surface integral

$$\bar{a}^{ij} \Delta_i\, dX_{(j)} \geqslant \frac{\lambda}{2} \frac{\partial \Delta}{\partial n}\, dS = \frac{\lambda}{2}\, dS$$

for $\delta(X) < \tau$, from which we conclude, using the positiveness of u

$$\int_{\partial D_t} \bar{a}^{ij} u \Delta_i\, dX_{(j)} \geqslant K \int_{\partial D_t} |u|\, dS. \tag{5.1.3}$$

The first part of the double integral admits the estimate

$$K \cdot \iint_{D_t} \delta_t^{\alpha-1} |u|\, dX \leqslant K(t_0) + K \cdot \varepsilon(t_0) \cdot m(t). \tag{5.1.4}$$

In the second part, we integrate partially and use the fact that u is a solution:

$$\begin{aligned}\left| \iint_{D_t} \bar{a}^{ij} u_j \Delta_i\, dX \right| &= \left| \int_{\partial D_t} \bar{a}^{ij} u_j (\Delta - t)\, dX_{(j)} - \iint_{D_t} \{\bar{a}_i^{ij} u_j (\Delta - t) + \bar{a}^{ij} u_{ij} (\Delta - t)\}\, dX \right| \\ &\leqslant \iint_{D_t} \{\delta_t^\alpha |u_i| + \delta_t [|F| + (\bar{a}^{ij} - a^{ij}) u_{ij}]\}\, dX \\ &\leqslant K + K \iint_{D_t} \delta_t^{\alpha-1} |u|\, dX + K \left[\iint_{D_t} \{\delta_t^{p-1+\alpha} |u_i|^p + \delta_t^{2p-1+\alpha} |u_{ij}|^p\}\, dX \right]^{1/p}.\end{aligned}$$

If we use the inequality (5.1.1) we get

$$\left| \iint_{D_t} \bar{a}^{ij} u_j \Delta_i\, dX \right| \leqslant K(t_0) + \varepsilon(t_0)\, m(t). \tag{5.1.5}$$

Combining (5.1.2)–(5.1.5) we get

$$\int_{\delta(X)=t} |u|\, dS \leqslant K(t_0) + \varepsilon(t_0)\, [m(t)].$$

As we may assume that $m(t) = \int_{\delta=t} |u|\, dS$ for some sequence $\{t_\nu\}_1^\infty$, $t_\nu \searrow 0$,

$$m(t_\nu) \leqslant K(t_0) + \varepsilon(t_0)\, m(t_\nu), \quad \nu = 1, 2, \dots .$$

If we choose t_0 so small that $\varepsilon(t_0) < \frac{1}{2}$, $m(t_\nu) \leqslant K(t_0)$, and since $m(t)$ is a non-increasing function, it follows that

$$\int_{\delta(X)=t} |u|\,dS \leqslant K < \infty \quad \text{as} \quad t \to 0$$

which implies the theorem.

We might ask ourselves whether this theorem is sharp or if it is possible to weaken the conditions on the equation. It seems probable that the coefficients in front of u_i and u in the assumptions on F may be replaced by $\varepsilon(\delta(X))/\delta(X)$ and $\varepsilon(\delta(X))/\delta^2(X)$ respectively, where $\varepsilon(t)$ is a Dini function, i.e. is monotonic and satisfying

$$\int_0 \frac{\varepsilon(t)}{t}\,dt < \infty.$$

However, in general it is not possible to go further with this type of assumption, as is seen from the following theorem. For further discussions, see after Theorem 7.3.

Theorem 5.2. *Let $\varepsilon(t)$ be a non-decreasing function on $(0, \infty)$, satisfying*

$$\int_0^1 \frac{\varepsilon(t)}{t}\,dt = \infty, \quad \varepsilon(2t) \leqslant 2 \cdot \varepsilon(t).$$

Then there exists a positive function $u(x, y)$, $y > 0$ such that $u \to \infty$ as $y \to 0$ for all $x \in (-1, 1)$, and such that

$$|\Delta u| \leqslant K \cdot \frac{\varepsilon(y)}{y} |u_y| + K \frac{\varepsilon(y)}{y^2} |u|. \tag{5.2.1}$$

Proof. Define $\varphi(x) = \int_0^x \varepsilon(|t|)\,dt$, $|x| \leqslant 2$, and use Lemma 3.9 to construct a function $\Phi(x, y)$, $y > 0$ with the following properties:

$$\Phi(x, 0) = \varphi(x),$$

$$\left|\frac{\partial \Phi}{\partial y}(x, y) - 1\right| \leqslant K \cdot \varepsilon(y),$$

$$|D^{(2)}\Phi| \leqslant K \cdot \frac{\varepsilon(y)}{y},$$

$$\left|\frac{\partial \Phi}{\partial x}(x, y) - \varepsilon(x)\right| \leqslant K \cdot \varepsilon(y),$$

$$K \cdot \varphi \leqslant \Phi \leqslant K \cdot y + K \cdot x.$$

If we put
$$u(x, y) = \int_{-1}^1 \frac{\Phi(x+t, y)}{\Phi^2(x+t, y) + (x+t)^2}\,dt,$$

elementary calculations show that u satisfies (5.2.1). Also

$$u(x,y)\geqslant\int_{|x-t|\geqslant y}\frac{\Phi}{\Phi^2+(x+t)^2}\,dt\geqslant K\int_{|t|\geqslant y}\frac{\varphi(t)}{t^2}\,dt-K\to\infty\quad\text{as}\quad y\to 0$$

if $|x|\leqslant 1$, which proves the theorem.

6. Before we go on with solutions to the equation (1.1) we shall investigate the boundary behavior of a special type of solutions to the equation $\Delta u = f$, namely so called Green potentials

$$u(Y)=\iint_{R^n_+}G(X,Y)f(X)\,dX,$$

where G is the Green function of the Laplacian, or, more generally, of any linear homogeneous second order elliptic operator with constant coefficients.

We note first that in order that the defining integral exist as an absolutely convergent integral it is necessary that

$$\iint_{R^n_+}\frac{x_n|f|}{1+|X|^n}\,dX<\infty.$$

As the behavior of u in the neighborhood of a boundary point depends on the values of f in a neighborhood of this point, we shall assume in this section that f has compact support, say in $\{X \,\|\, X|\leqslant 1, x_n\geqslant 0\}$. The necessary condition above is also sufficient to guarantee the existence of perpendicular boundary values of u, a fact which was first proved by Littlewood [15] for $n=2$. The proof in the general case is similar and we state here without proof:

Theorem 6.1. *If*

$$\iint_{R^n_+}x_n|f(X)|\,dX<\infty,$$

then the Green potential u of f satisfies

$$\lim_{y_n\to 0}u(y_1,y_2,\ldots,y_{n-1},y_n)=0$$

for almost every $(y_1,\ldots,y_{n-1})\in R^{n-1}$.

In order to ensure the existence of non-tangential boundary values, we have to assume higher order integrability of f. One such condition was given by Solomencev [20], namely essentially

$$\iint_{R^n_+}x_n^p|f|^p\,dX<\infty$$

for some $p>n/2$. This result can be improved. We also believe our method of proof to be simpler than the ones used earlier.

Theorem 6.2. *If*

$$\iint_{R^n_+}x_n|f(X)|\,dX<\infty \tag{6.2.1}$$

and if there is a $p > n/2$ such that to every X' belonging to some (measurable) set $E \subset R^{n-1}$ there is an $h > 0$ such that

$$\iint_{V_h(X')} x_n^{2p-n} |f|^p dX < \infty, \tag{6.2.2}$$

then the Green potential u of f has non-tangential limit zero at almost all points of E.

Proof. By Lemmata 3.2 and 3.3 there is a set $E' \subset E$ such that $E - E'$ has measure zero and

$$\iint_{V_h(X')} x_n^{2p-n} |f|^p dX < \infty \tag{6.2.3}$$

for all $h > 0$ and all $X' \in E'$. If

$$\varepsilon(\tau) = \iint_{0 < x_n < \tau} x_n |f(X)| \, dX,$$

then $\lim_{\tau \to 0} \varepsilon(\tau) = 0$.

Define the set function $\Phi(e)$, $e \subset R^{n-1}$, by

$$\Phi(e) = \iint x_n |f| \, dX,$$

where the integration is performed over

$$\{X \,|\, 0 < x_n < \tau, \ (x_1, \ldots, x_{n-1}) \in e\}.$$

By a well-known theorem from the theory of integration (6.2.1) implies that Φ has a finite regular derivative almost everywhere, and a simple argument shows that

$$|\Phi'(X')| \leqslant \eta(\tau) = \sqrt{\varepsilon(\tau)}$$

except in a set E'' of measure at most $\eta(\tau)$. Let $X_0' \in E' \cap CE''$. The proof shows that it is no restriction to assume $X_0' = O$. We shall prove that

$$\lim_{\substack{Y \to O \\ Y \in V_h(O)}} \sup |u| \leqslant K \cdot \eta(\tau) \tag{6.2.4}$$

for every $h > 0$. By choosing a suitable sequence τ_ν tending to zero sufficiently fast, it is not difficult to see that the theorem is hereby proved.

To prove (6.2.4) choose δ so small that

$$\frac{\Phi(e)}{\varrho^{n-1}} \leqslant 2\omega_{n-1} \cdot \eta \tag{6.2.5}$$

if $\varrho < \delta$ and $e = \{X \,|\, x_n = 0, \ |X'| \leqslant \varrho\}$.

We have with a fixed $Y \in V_h$

$$|u(Y)| \leqslant \iint_{R_+^n} G(X, Y) |f| \, dX \leqslant I_\tau + I_\delta + I_0' + I_0'' + \sum_{\nu=1}^{N} I_\nu,$$

where the regions of integration are

$$
\begin{aligned}
&I_\tau: && \{X \mid x_n > \tau\},\\
&I_\delta: && \{X \mid |X'| \geqslant \delta\},\\
&I_0': && \left\{X \mid 0 < x_n < \tau,\ |X'| \leqslant 2\varrho,\ |X - Y| \geqslant \frac{y_n}{2}\right\},\\
&I_0'': && \left\{X \mid |X - Y| \leqslant \frac{y_n}{2}\right\},\\
&I_\nu: && \{X \mid 0 < x_n < \tau,\ 2^\nu \varrho \leqslant |X'| \leqslant 2^{\nu+1}\varrho\},
\end{aligned}
$$

$\varrho = |Y|$, and N is chosen so that $\delta \leqslant 2^{N+1}\varrho < 2\delta$.

Now
$$\limsup_{y_n \to 0} I_\tau \leqslant \lim\sup K \cdot \tau^{-n} \cdot y_n \iint_{x_n > \tau} x_n |f|\, dX = 0,$$

$$\limsup_{y_n \to 0} I_\delta \leqslant \lim\sup K \cdot \delta^{-n} \cdot y_n \iint_{|X'| \geqslant \delta} x_n |f|\, dX = 0.$$

Moreover, by (6.2.5)

$$|I_\nu| \leqslant K \cdot \frac{y_n}{(2^\nu \varrho)^n} \iint_{|X'| \leqslant 2^{\nu+1}\varrho} x_n |f|\, dX \leqslant K \cdot \frac{y_n}{(2^\nu \varrho)^n} 2\eta (2^{\nu+1}\varrho)^{n-1} = K \cdot 2^{-\nu} \eta$$

and so
$$\sum_{\nu=1}^{N} I_\nu \leqslant K \cdot \eta \sum_{\nu=1}^{N} 2^{-\nu} = K \cdot \eta,$$

$$|I_0'| \leqslant K \cdot \varrho^{1-n} \iint_{|X'| \leqslant 2\varrho} x_n |f|\, dX \leqslant K \cdot \eta.$$

Finally,

$$
\begin{aligned}
|I_0''|^p &\leqslant \left[K \cdot \iint_{|X-Y| \leqslant y_n/2} |X - Y|^{2-n} |f(X)|\, dX\right]^p\\
&\leqslant K \cdot \iint_{|X-Y| \leqslant y_n/2} x_n^{2p-n} |f|^p dX \cdot \left[y_n^{(n-2p)/(p-1)} \iint_{|X-Y| \leqslant y_n/2} |X - Y|^{(2-n)q} dX\right]^{p-1}\\
&\leqslant K \iint x_n^{2p-n} |f|^p dX,
\end{aligned}
$$

where the last integral is performed over $V_{h/n}(O) \cap \{x_n < 2y_n\}$. This integral tends to zero with y_n by (6.2.3). The theorem is proved.

By Lemma 3.2 we get the following Corollary. While this manuscript was in its final stage of preparation, Arsove and Huber [2] announced a similar result for $n = 2$.

Corollary 6.3. *If*

$$\iint_{R_+^n} x_n |f|\, dX < \infty,$$

$$\iint_{R_+^n} x_n^{2p-1} |f|^p dX < \infty$$

for some $p>n/2$, then u has non-tangential boundary values zero almost everywhere.

By Hölder's inequality we get

Corollary 6.4. *If*

$$\iint_{R^n_+} x_n^{2p-1-\varepsilon}|f|^p dX < \infty$$

for some $p>n/2$ and some $\varepsilon>0$, then u has non-tangential boundary values zero almost everywhere.

It is not possible to allow $p=n/2$ in Theorem 6.2, which is seen from the following example.

Example 6.5. *To every positive, locally bounded weight function $g(t)$, $0<t\leqslant 1$, there exists an f such that*

$$\iint_{R^n_+} x_n|f|\,dX < \infty,$$

$$\iint_{R^n_+} g(x_n)|f|^{n/2}dX < \infty$$

while the Green potential u of f does not have non-tangential boundary values anywhere in $|X'|\leqslant 1$, with the possible exception of a set of measure zero.

Proof. For each ν, construct a grid of points $X_{\nu i}$, lying in the plane $x_n=2^{-\nu}$, the $n-1$ first coordinates of which are integral multiples of $2^{-\nu}$. Inside $|X|\leqslant 1$ there are roughly $2^{-\nu(1-n)}$ such points for each ν. Let $B_{\nu i}$ be the ball having center $X_{\nu i}$ and radius $2^{-\nu-2}$. Define the sequence $\{k_\nu\}_1^\infty$, $k_\nu\geqslant 1$, in such a way that

$$2^{-\nu k_\nu((n/2)-1)+\nu n}\sup g(t)\leqslant 1,$$

where the supremum is taken over those t for which $t\geqslant 2^{-\nu-1}$. Now define

$$\begin{aligned} f(X) &= |X-X_{\nu i}|^{-2}2^{-k_\nu \nu} && \text{if } X\in B_{\nu i} \text{ and } \exp(-2^{k_\nu \nu})\leqslant |X-X_{\nu i}|\leqslant 2^{-\nu-2}\\ &= 0 && \text{elsewhere.}\end{aligned}$$

Then $$\iint x_n|f|\,dX\leqslant \sum_\nu 2^{\nu(n-1)}2^{-\nu}\cdot 2^{-k_\nu \nu}\iint_{B_{\nu i}}|X-X_{\nu i}|^{-2}dX\leqslant \sum_\nu 2^{-k_\nu\nu}<\infty,$$

$$\iint g(x_n)|f|^{n/2}dX=\sum_\nu 2^{-\nu}\cdot 2^{-\nu k_\nu((n/2)-1)+\nu n}\sup g(t)\leqslant \sum_\nu 2^{-\nu}<\infty.$$

But $$G(X,Y)\geqslant K\cdot|X-Y|^{2-n}\ \text{ in } B_{\nu i}\quad \text{and hence}$$

$$u(X_{\nu i})\geqslant K\cdot 2^{-k_\nu \nu}\iint |X-X_{\nu i}|^{-n}dX\geqslant K\cdot 2^{-k_\nu\nu}\int_{\exp(-2^{k_\nu \nu})}^{2^{-\nu-1}}\frac{dr}{r}\geqslant K>0.$$

Now if X' is any point satisfying $|X'|\leqslant 1$ and h is small enough, $V_h(X')$ contains at least one grid point $X_{\nu i}$ for each ν, and our assertion is proved, since by Theorem 6.2

$$\lim_{\substack{X\to X'\\ X\in V_h(X')}}\inf u(X)=0.$$

On the other hand, for a given $p > n/2$, x_n^{2p-1} is the largest weight function we can allow:

Example 6.6. *If for some* $p > n/2$, $\lim_{t \to 0} g(t) \cdot t^{1-2p} = 0$, *then there exists a function* $f \geq 0$ *such that*

$$\iint_{R_+^n} x_n |f| \, dX < \infty,$$

$$\iint g(x_n) |f|^p dX < \infty$$

but whose Green potential u does not have non-tangential limits anywhere in $|X'| \leq 1$ *with the possible exception of a set of measure zero.*

Proof. Choose a suite t_ν, $t_{\nu+1} < 2^{-1} t_\nu$ such that $g(t_\nu) \cdot t_\nu^{1-2p} \leq 2^{-\nu}$. In the plane $x_n = t_\nu$ we construct a grid of points as in 6.5. If $B_{\nu i}$ is the ball with center $X_{\nu i}$ and radius $t_\nu \cdot \nu^{-2}$ we define $f = \nu^4 \cdot t_\nu^{-2}$ in $B_{\nu i}$ and $f = 0$ elsewhere. That f satisfies the hypothesis and assertion of the theorem follows as in 6.5.

Theorem 6.2 is, however, not the ultimate in this connection.

Theorem 6.7. *Suppose* $M(t)$ *and* $N(t)$ *are complementary in the sense of Young. If f satisfies*

$$\iint_{R_+^n} x_n |f| \, dX < \infty,$$

$$\iint_{R_+^n} x_n^{n-1} M(f(X)) \, dX < \infty,$$

and if

$$\iint_{|X|<1} N(|X|^{2-n}) \, dX < \infty,$$

then the Green potential u of f has non-tangential limit zero at almost all boundary points.

Remark. As examples of $M(t)$ which satisfy these requirements, we mention

$$M(t) \sim t^{n/2} (\log t)^{(n/2)-1} (\log\log t)^{(n/2)-1} \dots (\log\log \dots \log t)^{(n/2)-1+\varepsilon}, \quad \varepsilon > 0,$$

for large values of t. By [14], p. 75, we have

$$N(t) \sim \frac{n-2}{n} t^{(n-2)/n} [(\log t)(\log\log t) \dots (\log\log \dots \log t)^{[(2\varepsilon)/(n-2)]+1}]^{-1}.$$

We also remark that Theorem 6.7 does not contain Corollary 6.3.

Proof. The proof proceeds with Lemma 3.2 as in Theorem 6.2, except that Young's inequality is used instead of Hölder's in the estimate of I_0''.

From the theory of Orlicz spaces, see [14], we know that if

$$\iint_{|X|<1} N(|X|^{2-n}) \, dX = \infty,$$

we can find f such that

$$\iint_{|X|<k} M(f)\,dX \leqslant 1$$

while

$$\iint_{|X|<k} |X|^{2-n} f\,dX$$

can be made arbitrarily large, for each k. Using this, it is possible to prove

Example 6.8. *If M and N are complementary, and*

$$\iint_{|X|<1} N(|X|^{2-n})\,dX = \infty,$$

then to every positive, locally bounded weight function $g(t)$, $0<t\leqslant 1$ it is possible to find an f satisfying

$$\iint_{R^n_+} x_n|f|\,dX < \infty,$$

$$\iint_{R^n_+} g(x_n)\,M(|f|)\,dX < \infty$$

but whose Green potential u does not have non-tangential boundary values in $|X'|\leqslant 1$, disregarding a set of measure zero.

For the sake of completeness, we state the corresponding results in the case $n=2$.

Theorem 6.9. *If for $n=2$, f satisfies*

$$\iint_{R^2_+} x_2|f(X)|\,\overset{+}{\log}\,|f(X)|\,dX < \infty,$$

then the Green potential u of f has non-tangential limit zero almost everywhere on the boundary.

Remark. This theorem was proved by Tolsted [23]. In [20], Solomencev claims to prove a more general condition, namely

$$\iint_{R^2_+} x_2|f|\,\overset{+}{\log}\,[x_2|f|]\,dX < \infty.$$

However, if f satisfies Solomencev's condition, it also satisfies Tolsted's, since

$$x_2|f|\,\overset{+}{\log}\,|f| \leqslant 2\cdot x_2^{-\frac{1}{2}}\,\overset{+}{\log}\,2x_2^{-\frac{3}{2}} + 4x_2|f|\,\overset{+}{\log}\,x_2|f|.$$

Contrary to the case $n\geqslant 3$, $M(t)=t\overset{+}{\log}t$ is the best possible for $n=2$.

Example 6.10. *If $\lim_{t\to\infty} g(t)[t\overset{+}{\log}t]^{-1}=0$, then to every positive locally bounded weight function $h(t)$, $0<t\leqslant 1$, there is an f such that*

$$\iint_{R^2_+} x_2 |f|\, dX < \infty,$$

$$\iint_{R^2_+} h(x_n)\, g(|f|)\, dX < \infty$$

but whose Green potential u does not have non-tangential limits in $|x_1| \leqslant 1$, with the exception of a set of measure zero.

The proof presents no new difficulties, and we omit it.

We shall investigate what the integrability condition on f means when $1 < p \leqslant n/2$. In order to do so, we need some new notation. Consider the "k-dimensional cone"

$$\{X \mid x_k = x_{k+1} = \ldots = x_{n-1} = 0,\ h(x_1^2 + \ldots + x_{k-1}^2) < x_n^2 < 1\},$$

where $2 \leqslant k \leqslant n-1$ and $h > 0$. For a fixed k we denote by $\mathcal{V}_h(O)$ the image of this "cone" after an orthonormal mapping of R^{n-1} into itself leaving the origin fixed. $\mathcal{V}_h(X')$ will be the usual translation of $\mathcal{V}_h(O)$. A typical case where the situation can be visualized is $n = 3$, $k = 2$. The convention that f has support $\subset \{|X| < 1\}$ is still in force.

Theorem 6.11. *If $\mathcal{V}_h(O)$ is a fixed k-dimensional cone, and f satisfies*

$$\iint_{R^n_+} x_n |f|\, dX < \infty$$

and

$$\iint_{R^n_+} x_n^{2p-1} |f|^p dX < \infty \quad \textit{with} \quad p > \frac{k}{2},$$

then

$$\lim_{\substack{Y \to X' \\ Y \in \mathcal{V}_h(X')}} u(Y) = 0$$

for almost every $X' \in R^{n-1}$, if u is the Green potential of f.

Proof. With a suitable coordinate transformation, we can always assume that $\mathcal{V}_h(O)$ is of the original type considered above. Define

$$\varepsilon(\tau) = \iint_{x_n < \tau} x_n^{2p-1} |f|^p dX.$$

An inspection of the proof of Theorem 6.2 shows that it is sufficient to consider the integral I_0'', i.e. the integral over $\{X \mid |X - Y| \leqslant y_n/2\}$.

Put $\quad X = (X', x_n) = (X'', X''', x_n) = (x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n)$

and define

$$\mathcal{W}^\tau(X'') = \{\bigcup \mathcal{V}_{h/n}(X'', X''')\} \cap \{x_n < \tau\},$$

where the union is taken over all $X''' \in R^{n-k}$. If $\psi(Y, X'')$ is the characteristic function of $\mathcal{W}^1(X'')$, we have

$$\int_{R^{k-1}} \psi(Y, X'')\, dX'' \leqslant K \cdot y_n^{k-1}.$$

Hence

$$\int_{R^{k-1}} dX'' \iint_{\mathfrak{w}^\tau(X'')} y_n^{2p-k} |f(Y)|^p dY = \int_{R^{n-k}} dX'' \iint_{x_n<\tau} \psi(Y, X''') y_n^{2p-k} |f|^p dY$$
$$\leqslant K \cdot \iint_{x_n<\tau} y_n^{2p-1} |f|^p dX = K \cdot \varepsilon(\tau),$$

which implies that

$$\iint_{\mathfrak{w}^\tau(X'')} y_n^{2p-k} |f|^p dY < K \cdot \eta(\tau) = K \cdot \sqrt{\varepsilon(\tau)}$$

except for X'' in a set whose $k-1$-dimensional measure is less than $\eta(\tau)$. If X'' is not in this exceptional set, the set function $\Phi(e)$, $e \subset R^{n-k}$, defined by

$$\Phi(e) = \int_e dX''' \int_{\{\mathfrak{v}_{h/n}(X'', X''')\} \cap \{x_n<\tau\}} y_n^{2p-k} |f|^p dY'' dy_n$$

has a derivative $< K \cdot \sqrt{\eta(\tau)}$ except in a set of at most $n-k$-dimensional measure $\sqrt{\eta(\tau)}$. If $X_0' = (X_0'', X_0''')$ and X_0'' and X_0''' do not belong to the exceptional sets above, we have for y_n small enough

$$|I_0''|^p \leqslant \left[\iint_{|X-Y|\leqslant y_n/2} \frac{|f(X)|}{|X-Y|^{n-2}} dX\right]^p \leqslant K \iint \frac{|f|^p}{|X-Y|^{n-k-\gamma}} dX \left[\iint |X-Y|^s dX\right]^{p-1}$$
$$\leqslant K \cdot y_n^{2p-k-\gamma} \cdot \iint \frac{|f|^p}{|X-Y|^{n-k-\gamma}} dX,$$

where $s = (n-k-\gamma+2p-np)/(p-1) > -n$ if γ is small enough.

$$\iint \frac{|f|^p}{|X-Y|^{n-k-\gamma}} dX \leqslant K \cdot \sum_{\nu=1}^{\infty} (2^\nu y_n^{-1})^{n-k-\gamma} \iint_{D_\nu} x_n^{2p-k} |f(X)|^p dX$$
$$\leqslant K \cdot \sqrt{\eta(\tau)} \cdot y_n^{k-2p+\gamma} \sum_{\nu=1}^{\infty} 2^{-\gamma\nu} \leqslant K \cdot \sqrt{\eta(\tau)},$$

where
$$D_\nu = \{\bigcup \mathfrak{v}_{h/n}(X_0'', X''')\} \cap \{x_n < \tau\},$$

the union being taken over those X''' for which $|X''' - X_0'''| \leqslant 2^{-\nu} y_n$. Hence

$$\limsup_{\substack{Y \to X_0' \\ Y \in \mathfrak{v}_h(X_0')}} |I_0''| \leqslant K \cdot [\eta(\tau)]^{1/2p},$$

and since $\eta(\tau) \to 0$, the theorem follows in the usual way.

Theorem 6.12. *Suppose*

$$\iint x_n |f| dX < \infty$$

and
$$\iint x_n^{2p-1} |f|^p dX < \infty, \quad 1 < p \leqslant \frac{n}{2},$$

then for every $p' < p^{**} = np/(n-2p)$ *we have*

$$\iint_{V_h(X')} y_n^{\gamma-n} |u(X)|^{p'} dY < \infty$$

for almost every X', *all* $\gamma > 0$ *and all* $h > 0$.

Proof. Choose an arbitrary $h > 0$, and let $X_0' \in R^{n-1}$ be a point where the set function

$$\Phi(e) = \int_e dX' \int_0^1 x_n |f| \, dX, \quad e \subset R^{n-1},$$

has a finite derivative and where

$$\iint_{V_{h/n}} x_n^{2p-n} |f|^p dX < \infty.$$

Put
$$u(Y) = \iint_{|X-Y| \leqslant y_n/2} + \iint_{|X-Y| \geqslant y_n/2} G(X, Y) f dX = u_1 + u_2.$$

With Minkowski's inequality

$$\left[\iint_{V_h(X_0')} y_n^{\gamma-n} |u_2(Y)|^{p'} dY\right]^{1/p'} \leqslant \iint_{R_+^n} x_n f(X) \left[\iint |X-Y|^{-np'} \cdot y_n^{\gamma+p'-n} dY\right]^{1/p'} dX$$

$$\leqslant K \cdot \iint_{R_+^n} x_n |f| \, |X' - X_0'|^{\gamma+1-n} dX \leqslant K \sum_{\nu=0}^{\infty} 2^{\nu(n-1-\gamma)} \iint_{|X'-X_0'| \leqslant 2^{-\nu+1}} x_n |f| \, dX$$

$$\leqslant K \cdot \sum_{\nu=0}^{\infty} 2^{-\nu\gamma} < \infty,$$

where the double integral without integration limits is taken over

$$\{y_n \leqslant |X' - X_0'| \cdot K\} \cap V_h(X_0').$$

With Hölder's inequality

$$|u_1(Y)| \leqslant \iint_{|X-Y| \leqslant y_n/2} \frac{|f(X)|}{|X-Y|^{n-2}} dX$$

$$\leqslant \left[\iint \frac{dX}{|X-Y|^{n-\gamma}}\right]^{1/q} \left[\iint \frac{x_n^{2p-1} |f|^p}{|X-Y|^{p''}} dX\right]^{1/p'} \left[\iint x_n^{2p-1} |f|^p dX\right]^{1/p-1/p'}$$

$$\leqslant K \cdot y_n^{p'''} \left[\iint \frac{x_n^{2p-1} |f|^p}{|X-Y|^{p''}} dX\right]^{1/p'},$$

where $p'' = p'[n-2+(\gamma-n)q^{-1}]$ and $p''' = \gamma p q^{-1} + (n-1)(p'-p)(pp')^{-1}$ from which follows

$$\iint_{V_h(X_0')} y_n^{-n} |u(Y)|^{p'} dY \leqslant K \iint_{V_{h/n}(X_0')} x_n^{2p-n} |f|^p dX < \infty.$$

The theorem is proved.

Remark. By putting $f(X)=x_n^{-2}(\log(1/x_n))^{-1}(\log\log(1/x_n))^{-2}$ in $0<x_n<1$, $|X'|<1$, we see that it is not possible to allow $\gamma=0$ in the theorem above. Likewise, by using the grid of Example 6.5 and defining $f=2^{4\nu+(\nu n/p')}$ in $|X-X_{\nu i}|<2^{-2\nu}$ we see that we cannot take $p'>np/(n-2p)$. It is probable, however, that $p'=np/(n-2p)$ would do.

7. In this section we state and prove some theorems on the existence of boundary values of solutions to (1.1), and how these are approached. The first ones are extensions of the Fatou theorem [9].

Theorem 7.1. *Suppose u is a solution of* (1.1B) *in R^n_+, and $u \in H^1$. Then at almost every point of R^{n-1}, u has a finite, non-tangential limit. The limit function is locally summable.*

Proof. It is clearly sufficient to prove the existence of limits in bounded sets, $|X'|\leqslant\varrho/2$ for arbitrarily large ϱ. Put

$$D_\tau=\{X\,|\,\tau<x_n<1+\tau,\ |x_i|<\varrho,\ i=1,\ldots,n-1\},\quad \tau\geqslant 0$$

and $D_0=D$. The part of ∂D_τ satisfying $x_n=\tau$ will be denoted by $\partial' D_\tau$.

By definition

$$\iint_\Omega x_n^{\gamma-1}|u|\,dX<\infty \tag{7.1.1}$$

for all $\gamma>0$ and all bounded sets $\subset R^n_+$. From Corollary 4.4 we conclude that

$$\iint_\Omega \{x_n^{\alpha-1}|u|+x_n^\alpha|u_i|+x_n^{1+\alpha}|u_{ij}|\}\,dX<\infty. \tag{7.1.2}$$

On the other hand, (7.1.1) implies that

$$\iint_{V_h(X')} x_n^{\gamma-n}|u|\,dX<\infty$$

for all $h>0$, all $\gamma>0$ and almost all $X'\in R^{n-1}$. By Theorem 4.8

$$\iint_{V_h(X')}\{x_n^{\alpha-n}|u|^p+x_n^{p-n+\alpha}|u_i|^p+x_n^{2p-n+\alpha}|u_{ij}|^p\}\,dX<\infty \tag{7.1.3}$$

for all $h>0$, $\gamma>0$, and almost all $X'\in R^{n-1}$.

The next step will be to find a suitable representation formula for u. To that end we first note that by (7.1.2) and Fubini's theorem

$$\int_{D\cap\{x_j=t\}} x_n|u_i|\,dS<\infty$$

for $j=1,\ldots,n-1$ and almost all $t\in R^1$. We can assume that

$$\int_{\partial D-\partial' D} x_n|u_i|\,dS\leqslant\infty \quad\text{and}\quad \int_{\partial D-\partial' D}|u|\,dS<\infty. \tag{7.1.4}$$

Consider the sequence $d\mu^{(k)}(X') = u(x_1, \ldots, x_{n-1}, \tau^{(k)})\,dX'$ on $|X'| \leqslant \varrho$. By the definition of H^1 we can pick out a subsequence $\{\tau^{(k_\nu)}\}_1^\infty$ such that $d\mu^{(k_\nu)}$ converges weakly to $d\mu$, say. By the Lebesgue decomposition theorem, $d\mu = \bar{u}(X')\,dX' + dm$, where $\bar{u} \in L^1(|X'| \leqslant \varrho)$ and dm is a singular measure. Let X_0' be any fixed point in $|X'| \leqslant \varrho/2$ and let $V_h(X')$, $h > 0$, be an arbitrary cone. If Y is a point in this cone, denote by Y_τ the point $Y + (0, \ldots, \tau)$. G^τ will be the Green function in $\{X \mid x_n > \tau\}$ of the operator $a^{ij}(X_0')(\partial^2/\partial x_i \partial x_j)$. Now apply Green's formula in D_τ. We get

$$u(Y_\tau) = \frac{1}{\omega_n} \int_{\partial D_\tau} \frac{\partial G^\tau}{\partial \nu}(X, Y_\tau)\, u(X)\, dS_X - \frac{1}{\omega_n} \int_{\partial D_\tau - \partial' D_\tau} \frac{\partial u}{\partial \nu}(X)\, G^\tau(X, Y_\tau)\, dS_X + \iint_{D_\tau} G^\tau \bar{a}^{ij}(X_0')\, u_{ij}\, dX$$

and after using the fact that u satisfies the regularized equation

$$\bar{a}^{ij}(X_0')\, u_{ij} = F + [\bar{a}^{ij}(X) - a^{ij}(X)]\, u_{ij} + [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u_{ij},$$

$$\begin{aligned} \omega_n \cdot u(Y_\tau) = &\int_{\partial D_\tau} \frac{\partial G^\tau}{\partial \nu}\, u\, dS_X - \int_{\partial D_\tau - \partial' D_\tau} \frac{\partial u}{\partial \nu}\, G^\tau\, dS_X \\ &+ \iint_{D_\tau} G^\tau(X, Y_\tau)\{F + [\bar{a}^{ij}(X) - a^{ij}(X)]\, u_{ij}\}\, dX \\ &+ \iint_{D_\tau} G^\tau [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u_{ij}\, dX. \end{aligned}$$

Before letting τ tend to zero we integrate partially in the last integral over the region $D_\tau - \{X \mid |X - Y_\tau| \leqslant \sigma\} = D_\tau - B_{\sigma\tau}$, where $y_n/4 \leqslant \sigma \leqslant y_n/2$,

$$\begin{aligned} \iint_{D_\tau - B_{\sigma\tau}} \{\ \}\, dX = &\int_{\partial D_\tau} - \int_{\partial B_{\sigma\tau}} G^\tau [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u_i\, dX_{(j)} - \iint_{D_\tau - B_{\sigma\tau}} G^\tau \bar{a}^{ij}_j u_i\, dX \\ &- \left\{ \int_{\partial D_\tau} - \int_{\partial B_{\sigma\tau}} \right\} G^\tau_j [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u\, dX_{(i)} \\ &+ \iint_{D_\tau - B_{\sigma\tau}} \{G^\tau_{ji} [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u + G^\tau_j \bar{a}^{ij}_i u\}\, dX. \end{aligned}$$

Now put $\tau = \tau^{(k)}$ and let $k \to \infty$. If we use (7.1.2), (7.1.4), the fact that $d\mu^{(k)}$ converges weakly and Lebesgue's principle of dominated convergence, we see that all passages to the limit are allowed. After this we integrate with respect to σ between $y_n/4$ and $y_n/2$, divide by $y_n/4$, and get the following representation formula for u:

$$\omega_n \cdot u(Y) = \int_{|X'| \leqslant \varrho} \frac{\partial G}{\partial \nu}(X, Y)\{\bar{u}(X')\, dX' + dm\} \tag{7.1.5}$$

$$+ \int_{|X'| \leqslant \varrho} \frac{\partial G}{\partial x_j}(X, Y)\, [\bar{a}^{nj}(X_0') - \bar{a}^{nj}(X)]\{\bar{u}(X')\, dX' + dm\} \tag{7.1.6}$$

$$+\int_{\partial D_\tau-\partial' D_\tau} \bar{a}^{ij}(X)\,uG_j dX_{(i)} - G\bar{a}^{ij}(X)\,u_i dX_{(j)} \tag{7.1.7}$$

$$+\frac{4}{y_n}\int_{y_n/4}^{y_n/2} d\sigma\left\{\int_{\partial B_\sigma} G[\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\,u_i dX_{(j)}\right.$$

$$\left.+G_j[\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\,u dX_{(i)}\right\} \tag{7.1.8}$$

$$+\iint_D G\{F + [\bar{a}^{ij}(X) - a^{ij}(X)]\,u_{ij}\}\,dX \tag{7.1.9}$$

$$+\frac{4}{y_n}\int_{y_n/4}^{y_n/2} d\sigma \iint_{B_\sigma} G[\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\,u_{ij} dX \tag{7.1.10}$$

$$+\frac{4}{y_n}\int_{y_n/4}^{y_n/2} d\sigma \iint_{D-B_\sigma} \{G_{ji}[\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\,u$$

$$+G_j\bar{a}_i^{ij}u - G\bar{a}_j^{ij}u_i\}\,dX. \tag{7.1.11}$$

We intend to prove that $u \to \bar{u}(X_0')$ at almost all points of $|X'| \leqslant \varrho/2$. To do so we note that

$$\int_{\partial D} \frac{\partial G}{\partial \nu}(X, Y)\,dS_X = \omega_n$$

and hence if X_0' is a point in the Lebesgue set of $\bar{u}dX' + dm$, i.e. a point where

$$h(t) = t^{1-n}\int_{|X'-X_0'|\leqslant t} \{|\bar{u}(X') - \bar{u}(X_0')|\,dX' + dm\} \to 0, \quad t \to 0, \tag{7.1.12}$$

we have

$$\omega_n|(7.1.5) - \bar{u}(X_0')|$$

$$\leqslant \int_{|X'|\leqslant\varrho} \frac{\partial G}{\partial \nu}(X', Y)\,|\bar{u}(X') - \bar{u}(X_0')|\,dX' + \int_{|X'|\leqslant\varrho} \frac{\partial G}{\partial \nu}dm + \int_{\partial D-\partial' D} \frac{\partial G}{\partial \nu}(X, Y)\,\bar{u}(X_0')\,dS_X$$

$$\leqslant K\cdot\sum_{\nu=1}^{N} \frac{y_n}{(2^\nu l)^n}\int_{|X'-X_0'|\leqslant 2^{\nu+1}} \{|\bar{u}(X') - \bar{u}(X_0')|\,dX' + dm\}$$

$$+K\cdot\delta^{-n}\cdot y_n\int_{|X'-X_0'|\geqslant\delta} \{|\bar{u}(X') - \bar{u}(X_0')|\,dX' + dm\} + K\cdot\varrho^{-n}\cdot y_n\int_{\partial D-\partial' D} |u|\,dS_X$$

$$\leqslant K\cdot\varepsilon(\delta)\sum_{\nu=1}^{N} 2^{-\nu} + K_\delta\cdot y_n,$$

where $l = |Y - X_0'|$ and $\varepsilon(\delta) = \sup_{0<t<\delta} h(t)$.

By (7.1.11) $\varepsilon(\delta)$ tends to zero with δ and hence for every $\varepsilon > 0$

$$\lim_{\substack{Y\to X_0' \\ Y\in V_h(X_0')}} \sup\, \omega_n|(7.1.5) - u(X_0')| < \varepsilon,$$

i.e. (7.1.5) tends to $\omega_n \cdot \bar{u}(X_0')$ as $Y \to X_0'$ non-tangentially almost everywhere. In the same way,

$$|(7.1.6)| \leqslant K \cdot \delta^\alpha + K_\delta \cdot y_n,$$

whence (7.1.6) $\to 0$ almost everywhere. The integral (7.1.7) admits the estimate

$$K \cdot y_n \cdot \varrho^{-n} \int_{\partial D - \partial' D} \{x_n |u_i| + |u|\} dS_X$$

which tends to zero for all X_0', by (7.1.4). (7.1.8) and (7.1.10) can be estimated by

$$K \cdot y_n^{1-n} \iint_{|X-Y| \leqslant y_n/2} x_n^\alpha |u_i| dX$$

$$+ K \cdot y_n^{-n} \iint_{|X-Y| \leqslant y_n/2} x_n^\alpha |u| dX + K \iint |X-Y|^{2-n} x_n^\alpha |u_{ij}| dX$$

$$\leqslant K \iint x_n^{1+\alpha-n} |u_i| dX + K \iint x_n^{\alpha-n} |u| dX + K \left[\iint x_n^{2p-n+\alpha p} |u_{ij}|^p \right]^{1/p}$$

if $p > 1$, all the integrations in the last membrum being performed over $V_{h/n}(X_0') \cap \{x_n < 2y_n\}$ from which we conclude, using (7.1.3), that (7.1.8) and (7.1.10) tend to zero for almost every X_0'. Also by (7.1.3), (7.1.2) and Theorem 6.2 we find that (7.1.9) and the last term in (7.1.11) tend to zero almost everywhere in the prescribed way. To estimate the first term of (7.1.11), put

$$\varepsilon(r) = \iint_{\{x_n < r\} \cap D} x_n^{(\alpha/2)-1} |u| dX.$$

Then the set function

$$\int_e dX' \int_0^r x_n^{(\alpha/2)-1} |u| dx_n, \quad e \subset R^{n-1},$$

has a derivative $\leqslant 1$ except in a set of measure $\leqslant \varepsilon(r)$. If X_0' is outside of this set we have

$$\left| \frac{4}{y_n} \int_{y_n/4}^{y_n/2} d\sigma \iint_{D - B_\sigma} G_{ji} [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)] u dX \right|$$

$$\leqslant \iint_{D - By_n/4} \frac{y_n}{|X-Y|^{n+1}} |X - X_0'|^\alpha |u| dX$$

$$\leqslant K \sum_{\nu=1}^{N} \frac{y_n (2^{\nu+1} \cdot y_n)^\alpha}{(2^\nu l)^{n+1}} \iint_{|X - X_0'| \leqslant 2^{\nu+1} l} |u| dX + K_r \cdot y_n \iint_{|X - X_0'| \geqslant r/2} |u| dX$$

$$\leqslant K \sum_{\nu=1}^{N} \frac{y_n \cdot r^{\alpha/2}}{(2^\nu l)^n} \iint_{|X - X_0'| \leqslant 2^{\nu+1} l} x_n^{(\alpha/2)-1} |u| dX + K_r \cdot y_n \leqslant K \cdot r^{\alpha/2} \sum_{\nu=1}^{N} 2^{-\nu} + K_r \cdot y_n,$$

where $l = |Y - X_0'|$ and $r \cdot 2^{-1} \leqslant 2^{N+1} \cdot l \leqslant r$. Hence with the usual argument we see that this integral tends to zero for almost every X_0'. The second term of (7.1.11) is estimated in the same way. Since $\bar{u}(X') \in L^1(|X'| \leqslant \varrho)$, the last statement of the theorem, and thereby the whole theorem, is proved.

An examination of the proof of Theorem 7.1 shows that nowhere has the fact that u is a solution in R_+^n been used, only that u is a solution in a region Ω, one part of the boundary of which lies in a hyperplane H, or, more carefully, to every X in this part of $\partial\Omega$ there is a neighborhood N such that $\partial\Omega \cap N \subset H$. This and the same remark about Theorem 5.1 imply the following theorem.

Theorem 7.2. *If u is a non-negative solution of* (1.1C) *in a region Ω, some part Γ of the boundary of which lies in a hyperplane in the sense stated above, then at almost every point of Γ, u has a finite, non-tangential limit.*

Theorem 7.3. *If u is a non-negative solution of* (1.1C) *in a Liapunov region Ω, then u has a non-tangential, finite limit at almost every point of the boundary $\partial\Omega$.*

Proof. It is sufficient to prove the almost everywhere existence of limits in a neighborhood of an arbitrary point $X_0 \in \partial\Omega$. By the definition of Liapunov surfaces, there is a sphere Σ_ϱ of radius $\varrho > 0$ and center X_0 such that a line parallel to the normal at X_0 intersects $\partial\Omega$ at most once inside Σ_ϱ. We can also choose ϱ so small that any two normals issuing from points of $\partial\Omega$ inside Σ_ϱ form an angle $< \pi/4$, say. It will be no restriction to assume that $X_0 = O$ and that the positive x_n-axis is along the (inner) normal of $\partial\Omega$ at X_0. Then, inside Σ_ϱ, $\partial\Omega$ is described by $x_n = \varphi(x_1, \dots, x_{n-1})$, where $\varphi \in C^{1+\gamma}(|X'| \leqslant \varrho + \varepsilon)$. Let $\mathcal{A}$ be this part of $\partial\Omega$, and use Lemma 3.9 to extend the function $x_n - \varphi(x_1, \dots, x_{n-1})$ from $\mathcal{A}$ into R^n. We assume that we have multiplied the extension $\Phi(X)$ by a function in C_0^∞ which is identically one in $|X| \leqslant 10\varrho$, say. Since $\partial\Phi/\partial x_n = 1$ on $\mathcal{A}$ we can consider the connected region $D =$ that connected component of the set $\{X \mid |X'| < \tfrac{1}{2}\varrho,\ \partial\Phi/\partial x_n > \tfrac{1}{2},\ \Phi > 0\}$ which has $\mathcal{A}$ as part of its boundary. It is clear that Φ has the following properties in D:

1° $\Phi \in C^{1+\gamma}(\bar{D})$,

2° $K_1[x_n - \varphi(X')] \leqslant \Phi \leqslant K_2[x_n - \varphi(X')], \quad K_i > 0,$

3° $|D^{(2)}\Phi| \leqslant K \cdot \Phi^{\gamma - 1}$,

4° For each X', $|X'| < \tfrac{1}{2}\varrho$, Φ is strictly monotonic considered as a function of x_n.

The mapping $Y = H(X) = (h^1, h^2, \dots, h^n) = (x_1, x_2, \dots, x_{n-1}, \Phi(X))$, $X \in \bar{D}$, is one-one and maps $\bar{D}$ onto a region $\bar{D}'$ which contains the set $\{Y \mid |Y'| < \tfrac{1}{2}\varrho,\ 0 < y_n < \tau\}$, for some $\tau > 0$, in such a way that $\mathcal{A}$ and $\{|Y'| \leqslant \tfrac{1}{2}\varrho\}$ correspond. Consider the function $v(Y) = u(H^{-1}(Y))$. We shall prove that v satisfies the following differential equation, which is of admissible type in D':

$$\hat{a}^{kl}(Y)\, v_{kl} = \hat{F}(Y, v(Y), v_k(Y)) - \hat{b}^k(Y) v_k(Y), \tag{7.3.1}$$

where

$$\hat{a}^{kl}(Y) = a^{ij}(H^{-1}(Y))\, h_j^l(H^{-1}(Y))\, h_i^k(H^{-1}(Y)),$$

$$\hat{F} = F(H^{-1}(Y),\ v(Y),\ v_i(Y)\, h_k^i(H^{-1}(Y))),$$

$$\hat{b}^k(Y) = a^{ij}(H^{-1}(Y))\, h_{ij}^k(H^{-1}(Y)).$$

In fact, since $u = v(H(X))$ we get $u_i = v_k(H(X))\, h_i^k(X)$ and $u_{ij} = v_{kl} h_i^k h_j^l + v_k h_{ij}^k$, and after substitution in (1.1):

$$a^{ij}u_{ij} = a^{ij}u_{ij}v_{kl}h_i^k h_j^l + a^{ij}v_k h_{ij}^k = F(X, u(X), u_i(X)).$$

By reordering and putting $X = H^{-1}(Y)$ we get (7.3.1). To see that (7.3.1) is of admissible type we note that the functional determinant $\|\mathcal{H}\|$ of H is $= \partial\Phi/\partial x_n \geqslant \frac{1}{2}$. Hence

$$\inf_{X \in D} \min_{|\xi|=1} |\mathcal{H}\xi| \geqslant K > 0$$

and $$(\xi, \mathcal{H}A\mathcal{H}'\xi) = (\mathcal{H}'\xi, A\mathcal{H}'\xi) \geqslant \lambda |\mathcal{H}'\xi|^2 \geqslant K\cdot\lambda|\xi|^2, \quad K > 0.$$

Here we have denoted by $\mathcal{H}\xi$ the matrix $\mathcal{H}$ operating on the vector ξ of R^n, (.,.) is the inner product in R^n, and A is the matrix a^{ij}. Thus (7.3.1) is uniformly elliptic in D'. Also since $\partial\Phi/\partial x_n \geqslant \frac{1}{2}$, $H^{-1}(Y)$ is Hölder continuous, in fact with exponent one. The h_j^l being Hölder continuous by 1°, we see that $\hat{a}^{kl}$ are too. Using 2° and 3° it is also easy to check that the growth properties of $\hat{F}$ and $\hat{b}^k$ are the right ones.

Thus $v(Y)$ satisfies the requirements of 7.2 and we can conclude that v has non-tangential limits almost everywhere in $|Y'| \leqslant \varrho/2$. Again since $\partial\Phi/\partial x_n \geqslant \frac{1}{2}$ the image of an essential part of every truncated cone with vertex in $|X - X_0| < \varrho/2$ is contained in some cone $V_h(Y')$. The theorem is proved with the observation that sets of measure zero in $\mathcal{A}$ correspond to null sets in $|Y'| \leqslant \varrho/2$ and inversely, due to 1° and the fact that $\partial\Phi/\partial x_n \geqslant 1$ on $\mathcal{A}$.

Remark. It is easy to see that the mapping H works with solutions of (1.1B) also.

The question might be asked, whether the hypotheses on the equation can be weakened while the theorems just proved still hold. It is not difficult to see that if we assume the same growth conditions as in the discussion after Theorem 5.1 then $u \in H^p$ with $p > 1$ implies the existence of non-tangential boundary values of u. It seems probable that $p = 1$ or $u \geqslant 0$ would suffice in this case also. However, Theorem 5.2 shows that in general no more is true. In particular $u = \cos\log x_n$ is a bounded solution of

$$\Delta u + \frac{1}{x_n} u_{x_n} + \frac{1}{x_n^2} u = 0$$

in R_+^n without boundary values.

On the other hand, if we consider the equation

$$Lu = \Delta u + \frac{k}{x_n} u_{x_n} = 0$$

with $k < 1$, it is well known that the boundary value problem

$$Lu = 0 \quad \text{in} \quad R_+^n,$$

$$u = \varphi(X') \quad X' \in R^{n-1},$$

where φ has suitable properties, has the solution

$$u(Y) = K\cdot y_n^{1-k} \int_{R^{n-1}} \frac{\varphi(X')\,dX'}{\left[\sum_{i=1}^{n-1}(y_i - x_i)^2 + y_n^2\right]^{(n-k)/2}},$$

see Weinstein [26]. Now it is easy to see that it is possible to represent a positive solution of $Lu=0$ in a similar manner with a positive measure $d\mu(X')$ instead of $\varphi(X')dX'$, and in a standard way it follows that u has non-tangential boundary values almost everywhere.

Whether the uniform ellipticity is necessary is not clear. If we keep the other conditions, and a^{ij} satisfy only

$$a^{ij}\xi_i\xi_j \geqslant \varepsilon(x_n)|\xi|^2,$$

where $\varepsilon(t)\searrow 0$, then it is necessary that

$$\int_0 \frac{\varepsilon(t)}{t}\,dt=\infty$$

since else $u=\sin(\log x_n)$ is a solution of

$$\frac{\partial^2 u}{\partial x_1^2}+\dots+\varepsilon(x_n)\frac{\partial^2 u}{\partial x_n}=f,$$

where
$$|f|=\left|-\frac{\varepsilon(x_n)}{x_n^2}(\sin\log x_n+\cos\log x_n)\right|\leqslant\frac{2\varepsilon(x_n)}{x_n^2}.$$

The author hopes to return to the question of the "right" conditions in this context.

Theorem 7.4. *If u is a solution of* (1.1B) *and belongs to H^p, $p>1$, then $u(X',x_n)$ converges in $L^p(\Omega)$ when $x_n\to 0$ to its almost everywhere boundary function $\bar u(X')$ for every bounded subdomain Ω of R^{n-1}.*

Proof. We shall prove that for every $\varrho>0$, $u(X',x_n^{(k)})$ converges to $\bar u(X')$ in $L^p(|X'|\leqslant\varrho/2)$ for every sequence $\{x_n^{(k)}\}_{k=1}^{\infty}$ tending to zero. Since the limit function is unique, the theorem follows.

By Egorov's theorem it is sufficient to find an L^p function which majorizes u independently of x_n. This majorant function is constructed with the help of the maximal functions of Hardy and Littlewood.

We use the representation formula from the proof of Theorem 7.1. Since $u\in H^p$, the choice of the limit measure $d\mu$ can be made in such a way that $d\mu=\bar u dS$ where $\bar u\in L^p(|X'|\leqslant\varrho)$. Using some by now evident estimates we get the following inequality:

$$\begin{aligned}|u(Y)|\leqslant K+K\int_{|X'|\leqslant\varrho}\frac{\partial G}{\partial\nu}(X,Y)|\bar u|\,dX'+K\cdot y_n^{1-n}\iint_{|X-Y|\leqslant y_n/2}\{x_n^{\alpha}|u_i|+x_n^{\alpha-1}|u|\}\,dX\\ +K\iint_D G(X,Y)\{|F|+x_n^{\alpha}|u_{ij}|\}\,dX+K\iint_{|X-Y|\leqslant y_n/2}G(X,Y)\,x_n^{\alpha}|u_{ij}|\,dX\\ +K\iint_{D-\{|X-Y|\leqslant y_n/4\}}\{|G_{ij}|\,|X-Y'|^{\alpha}|u|+|G_j|x_n^{\alpha-1}|u|+G\cdot x_n^{\alpha-1}|u_i|\}\,dX,\end{aligned}$$

where Y' is the orthogonal projection of Y on R^{n-1} and G is the Green function of $a^{ij}(Y')\,\partial^2/\partial^2 x_i\partial x_j$.

Define $$\varphi(X') = \int_0^{\varrho} \{x_n^{\alpha-1}|u| + x_n^{\alpha}|u_i| + x_n^{1+\alpha}|u_{ij}|\}\,dx_n$$

for $|X'| \leqslant \varrho$, $\varphi = 0$ elsewhere. Hölder's inequality shows that $\varphi \in L^p(R^{n-1})$. The maximal function $\bar{u}^*$ and φ^* of $|\bar{u}|$ and φ respectively, defined by

$$\varphi^*(Y') = \sup_{\sigma>0} \frac{1}{\omega_{n-1}\sigma^{n-1}} \cdot \int_{|X'-Y'|\leqslant\sigma} \varphi(X')\,dX',$$

$$\bar{u}^*(Y') = \sup_{\sigma>0} \frac{1}{\omega_{n-1}\sigma^{n-1}} \cdot \int_{|X'-Y'|\leqslant\sigma} |\bar{u}(X')|\,dX';$$

both belong to $L^p(R^{n-1})$, cf. Zygmund [28], p. 32. We shall prove that $u(Y) \leqslant K[1+\bar{u}^*(Y')+\varphi^*(Y')]$, $|Y'| \leqslant \varrho/2$. In fact, using the inequalities of Lemma 3.5 and choosing N suitably large,

$$\begin{aligned}
&\left|\int_{|X'|\leqslant\varrho} \frac{\partial G}{\partial \nu}(X',Y)|\bar{u}|\,dX'\right| \\
&\qquad\leqslant K \cdot \int_{|X'-Y'|\leqslant y_n} (\cdot)\,dX' + K \cdot \sum_{\nu=0}^{N} \int_{2^{\nu}y_n \leqslant |X'-Y'| \leqslant 2^{\nu+1}y_n} \frac{\partial G}{\partial \nu}(X',Y)|\bar{u}|\,dX' \\
&\qquad\leqslant K \cdot y_n^{1-n} \int_{|X'-Y'|\leqslant y_n} |\bar{u}|\,dX' + K \sum_{\nu=0}^{N} \frac{y_n}{(2^{\nu}y_n)^n} \cdot \int_{|X'-Y'|\leqslant 2^{\nu+1}y_n} |\bar{u}|\,dX' \\
&\qquad\leqslant K \cdot \bar{u}^* + K \sum_{\nu=0}^{N} 2^{-\nu}|\bar{u}|^* = K \cdot \bar{u}^*.
\end{aligned}$$

Moreover,

$$\iint_D (\cdot)\,dX \leqslant \iint_{|X-Y|\leqslant y_n/2} + \iint_{\{|X-Y|\geqslant y_n/2\}\cap D}$$

$$\begin{aligned}
\left|\iint_{|X-Y|\leqslant y_n/2}\right| &\leqslant K + K \cdot \sum_{\nu=1}^{\infty} \iint_{2^{-\nu-1}y_n \leqslant |X-Y| \leqslant 2^{-\nu}y_n} G[x_n^{\alpha-2}|u| + x_n^{\alpha-1}|u_i| + x_n^{\alpha}|u_{ij}|]\,dX \\
&\leqslant K \cdot \sum_{\nu=1}^{\infty} \frac{y_n^{-1}}{(2^{-\nu}y_n)^{n-2}} \iint_{|X-Y|\leqslant 2^{-\nu}y_n} \{x_n^{\alpha-1}|u| + x_n^{\alpha}|u_i| + x_n^{1+\alpha}|u_{ij}|\}\,dX \\
&\leqslant K \cdot \sum_{\nu=1}^{\infty} 2^{-\nu}\varphi^*(Y') = K \cdot \varphi^*(Y') + K \\
\left|\iint_{|X-Y|\geqslant y_n/2}\right| &\leqslant K + K \cdot \sum_{\nu=1}^{N} \frac{y_n}{(2^{\nu}y_n)^n} \iint_{|X-Y|\leqslant 2^{\nu}y_n} \{x_n^{\alpha-1}|u| + x_n^{\alpha}|u_i| + x_n^{1+\alpha}|u_{ij}|\}\,dX \\
&\leqslant K \cdot \sum_{\nu=1}^{N} 2^{-\nu}\varphi^*(Y') = K \cdot \varphi^*(Y') + K.
\end{aligned}$$

The remaining integrals are estimated essentially in the same way. As an example

$$\iint_{D-\{|X-Y|\leqslant y_n/2\}} |G_j| x_n^{\alpha-1}|u|\,dX \leqslant K \cdot \sum_{\nu=1}^{N} \frac{y_n}{(2^{\nu}y_n)^n} \iint_{|X-Y|\leqslant 2^{\nu}y_n} x_n^{\alpha-1}|u|\,dX \leqslant K \cdot \varphi^*(Y').$$

The theorem is proved.

Theorem 7.5. *If u is a bounded solution of* (1.1B) *in a Liapunov region Ω, or $u \in H^p$ in R^n_+, with $p > 1 + (n-1)/2$, and $u \to 0$ almost everywhere on a set E on the boundary, then $u \to 0$ uniformly on every subset F of E whose distance to the complement of E is greater than zero.*

Proof. If we use the mapping of Theorem 7.3 we see that it is sufficient to assume that u is a solution in $\{X \mid x_n > 0, |X| \leqslant 2\varrho\}$, that u tends to zero almost everywhere in $|X'| \leqslant \varrho$ and to prove the uniform convergence in $|X'| \leqslant \varrho/2$.

We use the representation formula of Theorem 7.1 with $X_0' = Y' =$ the orthogonal projection of Y on R^{n-1}. Theorems 7.1 and 7.4 show that $\bar{u}$ is identically zero. We get the inequality

$$|u(Y)| \leqslant K \cdot y_n + \iint_D G|f|\,dX + K \iint_D G\{x_n^{\alpha-2}|u| + x_n^{\alpha-1}|u_i| + x_n^{\alpha}|u_{ij}|\}\,dX$$
$$+ K \iint_{|X-Y| \leqslant y_n/2} x_n^{\alpha-n}|u|\,dX + K \cdot \iint_{D-\{|X-Y| \geqslant y_n/4\}} \{|G_{ij}|\,|X-Y'|^{\alpha}|u|$$
$$+ |G_j|\,x_n^{\alpha-1}|u|\}\,dX$$

for $|Y'| \leqslant \varrho/2$. In the remaining part of the proof K will denote constants independent of Y'. The first two terms on the right hand side tend to zero uniformly with y_n. In the third term we divide the area of integration into two parts, $|X-Y| \leqslant y_n/2$ and $|X-Y| \geqslant y_n/2$. With Hölder's inequality

$$\left| \iint_{|X-Y| \leqslant y_n/2} (\cdot)\,dX \right|$$
$$\leqslant K \cdot y_n^{\alpha-2+(1/p)-(\alpha/p)} \left[\iint_D x_n^{2p-1+\alpha}\{|u_{ij}|^p + x_n^{-p}|u_i|^p + x_n^{-2p}|u|^p\}\,dX \right]^{1/p}$$
$$\times \left[\iint_{|X-Y| \leqslant y_n/2} |X-Y|^{q(2-n)}\,dX \right]^{1/q} \leqslant K \cdot y_n^{\alpha+(1-n-\alpha)/p} = K \cdot y_n^{\gamma},$$

where $\gamma > 0$, from the assumption on p

$$\left| \iint_{|X-Y| \geqslant y_n/2} (\cdot)\,dX \right| \leqslant K \cdot y_n \cdot \left[\iint_D x_n^{2p-1+\alpha}\{|u_{ij}|^p + x_n^{-p}|u_i|^p + x_n^{-2p}|u|^p\}\,dX \right]^{1/p}$$
$$\times \left[\iint_{|X-Y| \geqslant y_n/2} \frac{x_n^{\alpha-1}}{|X-Y|^{nq}}\,dX \right]^{1/q}$$
$$\leqslant K \cdot y_n \left[\sum_{\nu=-1}^{N} (2^{\nu} y_n)^{-nq} \iint_{|X-Y| \leqslant 2^{\nu+1} y_n} x_n^{\alpha-1}\,dX \right]^{1/q} \leqslant K \cdot y_n^{\gamma_1},\ \gamma_1 > 0.$$

The remaining integrals are treated similarly after which the theorem is proved.

8. In [22] Stein proved that a necessary and sufficient condition for the harmonic function u defined in R^n_+ to have non-tangential boundary values at almost every boundary point is that the "generalized area integral"

$$\iint_{V_h(X')} x_n^{2-n}|u_i|^2 dX$$

is finite for almost every X' ($h>0$ may vary with X'). This theorem had been proved in the case $n=2$ by Marcinkiewicz and Zygmund [17], and Spencer [21], and the necessity part of it for $n>2$ by Calderón [5]. Widman [27] proved the same theorem for regions other than a half space. In [4] Calderón proved that a sufficient condition for u to have non-tangential boundary values is that u is bounded in $V_h(X')$ for almost every X'. This was later generalized by Carleson [7], who proved that it is sufficient to assume boundedness below in almost every $V_h(X')$. See also the work of Brelot and Doob in [3]. We shall prove the theorems of Stein and Calderón in the case when u is a solution of (1.1). The reader will notice that the manner of proof is somewhat different in some aspects. Thus Stein uses Calderón's theorem in his proof, while we will get Calderón's theorem as a corollary of that of Stein. The chief difference lies in the sufficiency part of Stein's theorem. The proof of the theorem corresponding to Carleson's generalization has so far escaped our efforts.

Theorem 8.1. *Suppose u is a solution of* (1.1B), *in a Liapunov region Ω, with the property that for almost every $X_0\in\partial\Omega$ there is an $h>0$ such that u is bounded in $V_h(X_0)$. Then*

$$\iint_{V_k(X_0)} \delta^{2-n}(X)|u_i|^2 dX<\infty$$

for all $k>0$ and almost all $X_0\in\partial\Omega$.

Proof. Using the mapping of Theorem 7.3 we realize that it is sufficient to consider the case $\Omega=R^n_+$.

By Theorem 4.8 we see that the boundedness of u in $V_h(X')$ implies that

$$\iint_{V_{h'}(X')} x_n^{p-n+\gamma}|u_i|^p dX<\infty,$$

$$\iint_{V_{h'}(X')} x_n^{2p-n+\gamma}|u_{ij}|^p dX<\infty$$

for all $\gamma>0$, $h'>h$ and $p>1$. Moreover by Theorem 4.5 $x_n|u_i(X)|\leqslant K<\infty$ in each $V_{h'}(X')$. If we take an arbitrary $\varepsilon>0$, an arbitrary $\varrho>0$ and an arbitrary $k>0$ we can find a closed set $F\subset\{|X'|\leqslant\varrho\}$ such that $\operatorname{mes}(F)>\omega_{n-1}\cdot\varrho^{n-1}/(n-1)-\varepsilon$, and such that

$$\iint_{W_k(F)} x_n^{2p-1+\gamma}|u_{ij}|^p dX<\infty,$$

$|u|$ and $x_n|u_i|\leqslant K$ in $W_k(F)$. In order to be able to integrate partially in $W_k(F)$ we approximate the irregular part of the boundary of $W_k(F)$ with a sequence of regular surfaces Γ_ν: $x_n=\varphi_\nu(X')\in C^\infty$, the normal of Γ_ν always making an angle with the x_n-axis which is bounded above by $\pi/2-\varkappa$, where $\varkappa>0$ depends on k. For this construction, see Stein [22]. The Γ_ν's are constructed in such a way that $W_k^\nu\subset W_k^{\nu+1}$ and $\bigcup_\nu W_k^\nu=W_k(F)$ where W_k^ν denotes $W_k(F)\cap\{X\,|\,x_n>\varphi_\nu(X')\}$. If ε is small enough, and ν large enough, the boundary of W_k^ν consists of two parts, namely Γ_ν and a section of the hyperplane $x_n = 1$.

Now multiply the regularized equation by $x_n \cdot u$ and integrate partially in W_k^ν.

$$\iint_{W_k^\nu} x_n u[\bar{a}^{ij} u_{ij} - F + (a^{ij} - \bar{a}^{ij}) u_{ij}] \, dX = 0,$$

$$\int_{\partial W_k^\nu} x_n u \bar{a}^{ij} u_i \, dX_{(j)} - \iint_{W_k^\nu} x_n \bar{a}^{ij} u_i u_j \, dX - \frac{1}{2} \int_{\partial W_k^\nu} \bar{a}^{in} u^2 \, dX_{(i)} + \frac{1}{2} \iint_{W_k^\nu} \bar{a}_i^{in} u^2 \, dX$$

$$- \iint_{W_k^\nu} x_n \bar{a}_j^{ij} u u_i \, dX + \iint_{W_k^\nu} x_n u[(a^{ij} - \bar{a}^{ij}) u_{ij} - F] \, dX = 0.$$

By the ellipticity

$$\iint_{W_k^\nu} x_n u_i^2 \, dX \leqslant K + K \iint_{W_k^\nu} \bar{a}^{ij} u_i u_j x_n \, dX \leqslant K \cdot \int_{\partial W_k^\nu} x_n |u_i| |u| \, dS$$

$$+ K \int_{\partial W_k^\nu} u^2 \, dS + K \iint_{W_k^\nu} \{x_n^{\alpha-1} |u|^2 + x_n^{\alpha-1} |x_n \cdot u_i| \cdot |u| + x_n |f| \cdot |u| + |u| \cdot x_n^{1+\alpha} |u_{ij}|\} \, dX.$$

The right hand side of this inequality is bounded independently of ν, since the area of Γ_ν is bounded by a fix constant over $\cos(\frac{1}{2}\pi - \varkappa)$, and hence

$$\iint_{W_k(F)} x_n |u_i|^2 \, dX < \infty.$$

By Lemma 3.2 we conclude that

$$\iint_{V_h(X')} x_n^{2-n} |u_i|^2 \, dX < \infty$$

for almost every point of F and every $h > 0$. As the measure of F differs from that of $\{|X'| \leqslant \varrho\}$ by the arbitrary ε, the theorem is proved.

Theorem 8.2. *If u is a solution of* (1.1B) *in a Liapunov region Ω, and for almost every $X_0 \in \partial\Omega$ there is an $h > 0$ such that*

$$\iint_{V_h(X_0)} \delta^{2-n}(X) |u_i|^2 \, dX < \infty, \tag{8.2.1}$$

then u has a non-tangential limit at almost every boundary point.

Proof. As in the proof of Theorem 8.1, it is sufficient to consider the case $\Omega = R_+^n$. Taking arbitrarily $\varepsilon > 0$ and $\varrho > 0$, and using Lemma 3.3 and Theorem 4.8 we see that to every $k > 0$ we can find a closed set $F \subset \{|X'| \leqslant \varrho\}$ whose measure differs from that of $\{|X'| \leqslant \varrho\}$ by at most ε, such that

$$\iint_{W_k(F)} x_n^{2p-1+\alpha}\{x_n^{-2p}|u|^p + x_n^{-p}|u_i|^p + |u_{ij}|^p\}\,dX < \infty, \tag{8.2.2}$$

$$\iint_{W_k(F)} x_n|u_i|^2 dX < \infty \tag{8.2.3}$$

with $p > n/2$ and also with $p = 2$. Moreover, by Theorem 4.9 and Lemma 3.4 we may assume $x_n|u_i| = \sigma(1)$ when $x_n \to 0$ uniformly in $W_k(F)$.

We shall construct a representation formula similar to that of Theorem 7.1, but first we have to prove that $u \in H^2$, in a certain sense at least. To that effect, let $W_{k,t}(F) = \{X \mid (x_1, ..., x_n - t) \in W_k(F)\}$. Then approximate the lower part of $\partial W_{k,t}(F)$ by surfaces $\Gamma_{t,\nu}$ like in the previous proof. If $W_{k,t}^\nu$ is that part of $W_{k,t}$ which lies above $\Gamma_{t,\nu}$, $W_{k,t}^\nu$ is connected if ε is small enough, and $\partial W_{k,t}^\nu$ consists of $\Gamma_{t,n}$ and a section of the hyperplane $x_n = 1 + t$. In $W_{k,t}^\nu$ we integrate partially to get, with $[x_n]_j = \partial x_n/\partial x_j$

$$\iint_{W_{k,t}^\nu} \bar{a}^{ij} 2uu_i[x_n]_j dX = \int_{\partial W_{k,t}^\nu} \bar{a}^{ij}u^2[x_n]_j dX_{(i)} - \iint_{W_{k,t}^\nu} \bar{a}_i^{ij}u^2[x_n]_j dX.$$

On the other hand

$$\iint_{W_{k,t}^\nu} \bar{a}^{ij} 2uu_i[x_n]_j dX$$
$$= \int_{\partial W_{k,t}^\nu} \bar{a}^{ij} 2uu_i x_n dX_{(j)} - \iint_{W_{k,t}^\nu} \{\bar{a}_j^{ij} 2uu_i x_n + \bar{a}^{ij} 2u_{ij}ux_n + \bar{a}^{ij}2u_iu_jx_n\}\,dX.$$

We can assume that t, k and ε beforehand were chosen so small that $\bar{a}^{ij}[x_n]_j dX_{(i)} \geqslant \lambda/4 dS$ on $\Gamma_{t,n}$ which implies, using (8.2.2), (8.2.3), and the fact that u is a solution,

$$\int_{\Gamma_{t,\nu}} u^2 dS \leqslant K + K\iint_{W_{k,t}^\nu} \{x_n^{\alpha-1}|u|^2 + x_n^{\alpha-1}|u| + x_n|u_i|^2\}\,dX + K\cdot\int_{\Gamma_{t,\nu}} |u|\,dS$$
$$+ \left[\iint_{W_{k,t}^\nu} x_n^{\alpha-1}|u|^2 dX\right]^{\frac12}\left[\iint_{W_{k,t}^\nu} x_n^{3+\alpha}|u_{ij}|^2 dX\right]^{\frac12} \leqslant K + K\int_{\Gamma_{t,\nu}} |u|\,dS.$$

Since the K's can be chosen so as not to depend on ν or t, we conclude that

$$\int_{\Gamma_{t,\nu}} u^2 dS \leqslant K < \infty.$$

In particular,

$$\int |u^t(X')|^2 dX' = \int |u(x_1, ..., x_{n-1}, \varphi(x_1, ..., x_{n-1}) + t)|^2 dX' \leqslant K < \infty,$$

where $x_n = \varphi(x_1, ..., x_{n-1})$ is the equation of that part of $\partial W_k(F)$ which does not lie in $x_n = 1$, and the integration is performed over the domain of this function. Thus we may select a subsequence $t_i \searrow 0$ such that u^{t_i} converges weakly to a function $\bar{u}(X')$ in L^2. We choose a point X_0' which is in the Lebesgue set of $\bar{u}$ and which is also a point of density of F. Let $V_h(X_0')$ be any appropriate cone. Y is a point of V_h and Y_t denotes the point $Y+(0, \ldots, t)$. Using the Green function G^t of the operator $a^{ij}(X_0')\partial^2/\partial x_i \partial x_j$ in $\{X \mid x_n > t\}$, and integrating partially in $W^\nu_{k,t} - B_{\sigma,t}$, where $B_{\sigma,t} = \{X \mid \ |X - Y_t| \leqslant \sigma\}$, we find

$$\omega_n u(Y_t)$$

$$= \int_{\partial W^\nu_{k,t}} \frac{\partial G^t}{\partial \nu}(X, Y_t)\, u(X)\, dS_X - \int_{\partial W^\nu_{k,t}} G^t \bar{a}^{ij} u_i\, dX_{(j)} - \int_{\partial W^\nu_{k,t}} G^t_j [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u\, dX_{(i)}$$

$$+ \int_{\partial B_{\sigma,t}} G^t [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u_i\, dX_{(j)} - \int_{\partial B_{\sigma,t}} G^t_j [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u\, dX_{(i)}$$

$$+ \iint_{W^\nu_{k,t}} G^t\{F - [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u_i\}\, dX_{(j)} + \iint_{B_{\sigma,t}} G^t [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u_{ij}\, dX$$

$$+ \iint_{W^\nu_{k,t} - B_{\sigma,t}} \{G^t_{ji} [\bar{a}^{ij}(X_0') - \bar{a}^{ij}(X)]\, u + G^t_j \bar{a}^{ij}_i u - G^t \bar{a}^{ij}_j u_i\}\, dX.$$

Now consider the projection onto R^{n-1} of the absolutely continuous measure $(\partial G^t/\partial \nu)(X, Y_t)\, dS$ on $\Gamma_{\nu,t}$. We observe that this measure is independent of t, and is bounded by

$$\frac{K \cdot y_n}{|\varphi_\nu(X') - Y|^n}\, dX'. \tag{8.2.4}$$

Thus we can select a subsequence ν_j converging weakly in L^2, say, to the measure $\psi(X')\, dX'$. We use the fact that

$$\omega_n \bar{u}(X_0') = \int_{\partial W^\nu_{k,t}} \frac{\partial G^t}{\partial \nu}(X, Y_t)\, \bar{u}(X_0')\, dS,$$

integrate with respect to σ and let $\nu_j \to \infty$. After some obvious estimates we get the following inequality:

$$\omega_n |u(Y_t) - \bar{u}(X_0')| \leqslant K \cdot y_n + \left| \int [u^t(X') - \bar{u}(X_0')]\, \psi(X')\, dX' \right|$$

$$+ K \int_{\Gamma_t} G^t(X, Y_t)\, |u_i|\, dS + K \int |\varphi_\nu(X') - X_0'|^\alpha\, |u^t(X')|\, \psi(X')\, dX'$$

$$+ K \iint_{|X - Y_t| \leqslant y_n/2} \{y_n^{1-n} |X - X_0'|^\alpha |u_i| + y_n^{-n} |X - X_0'|^\alpha |u|\}\, dX$$

$$+ K \iint_{W_{k,t}} G^t |F - [\bar{a}^{ij} - a^{ij}]\, u_{ij}|\, dX + K \iint_{W_{k,t} - \{|X - Y_t| \geqslant y_n/4\}} \{G^t x_n^{\alpha-1} |u_i|$$

$$+ |G^t_{ij}|\, |X - X_0'|^\alpha |u| + |G^t_j|\, |x_n^{\alpha-1}|\, |u|\}\, dX.$$

Since X_0' is a point of density of F we find that $x_n |u_i| < \varepsilon$ on Γ_t, if t is small enough and $|X - X_0'| < \delta$, say. Thus on this part of Γ_t

$$\int G^t |u_i|\, dS \leqslant \varepsilon \cdot \int \frac{y_n}{|X - Y_t|^n}\, dS \leqslant K \cdot \varepsilon.$$

On the remaining part of Γ_t

$$\int G^t |u_i| dS \leqslant y_n \cdot K(\delta).$$

Now put $t = t_i$, use the estimates just obtained, let $i \to \infty$ and use the weak convergence of u^t; it is easy to check that all the integrals involved are convergent. We get

$$\omega_n |u(Y) - \bar{u}(X_0')| \leqslant K(\delta) \cdot y_n + K \cdot \varepsilon + \left| \int [\bar{u}(X') - \bar{u}(X_0')] \psi(X') \, dX' \right| + \int |X' - X_0'| \, |\bar{u}(X')| \, |\psi(X')| dX' + \dots.$$

Assuming that X_0' does not belong to a certain subset of F having measure zero, as in the proof of Theorem 7.1 we find that the integrals represented by dots tend to zero as $Y \to X_0'$ inside the given cone. The following estimate presents no difficulties, if we use (8.2.4)

$$\left| \int [\bar{u}(X') - \bar{u}(X_0')] \psi(X') \, dX' \right| \leqslant K \cdot \sum_{\nu=0}^{N} \frac{y_n}{(2^\nu l)^n} \int_{|X' - X_0'| \leqslant 2^\nu l} |\bar{u}(X') - \bar{u}(X_0')| \, dX'$$
$$+ K(\delta) \cdot y_n \int_{|X'| \geqslant \delta} |\bar{u}(X') - \bar{u}(X_0')| dX' \leqslant K \cdot \varepsilon + K(\delta) \cdot y_n,$$

where $l = |X_0' - Y|$. Hence

$$\limsup_{\substack{Y \to X_0' \\ Y \in V_h(X_0')}} |u(Y) - \bar{u}(X_0')| \leqslant K \cdot \varepsilon \, .$$

ε being arbitrarily small we see that u has a finite non-tangential limit at almost all points of F. But the difference in measure between F and $|X'| \leqslant \varrho$ can also be made arbitrarily small, and the theorem is proved.

As immediate corollaries of Theorems 8.1 and 8.2 we get the following two theorems.

Theorem 8.3. *Suppose u is a solution of* (1.1 B) *in a Liapunov region Ω. A necessary and sufficient condition for u to have non-tangential boundary values almost everywhere is that to almost every $X_0 \in \partial\Omega$ there is an $h > 0$ such that*

$$\iint_{V_h(X_0)} \delta^{2-n}(X) |u_i|^2 dX < \infty.$$

Theorem 8.4. *Suppose u is a solution of* (1.1 B) *in a Liapunov region Ω. If to almost every $X_0 \in \partial\Omega$ there is an $h > 0$ such that u is bounded in $V_h(X_0)$, then u has non-tangential boundary values almost everywhere.*

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