Mathematical Programming 27 (1983) 307–319
North-Holland

OPTIMALITY CONDITIONS FOR NONDIFFERENTIABLE CONVEX SEMI-INFINITE PROGRAMMING

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Received 29 July 1981
Revised manuscript received 3 January 1983

This paper gives characterizations of optimal solutions to the nondifferentiable convex semi-infinite programming problem, which involve the notion of Lagrangian saddlepoint. With the aim of giving the necessary conditions for optimality, local and global constraint qualifications are established. These constraint qualifications are based on the property of Farkas–Minkowski, which plays an important role in relation to certain systems obtained by linearizing the feasible set. It is proved that Slater's qualification implies those qualifications.

Key words: Semi-Infinite Programming, Convex Functions, Lagrangian Saddlepoints, Constraint Qualifications, Optimality Conditions, Farkas–Minkowski Systems.

1. Introduction

If $\mathbb{R}^n$ is taken as domain, the general Semi-Infinite Programming problem (SIP) takes the form:

$$\begin{aligned} &\text{Inf} \quad \psi(x) \\ &\text{s.t.} \quad f_t(x) \leq 0, \ t \in T, \end{aligned} \tag{P}$$

where $x \in \mathbb{R}^n$, and T is an infinite set.

A possible approach to the SIP would be to associate (P) with some finite programming problem:

$$\begin{aligned} &\text{Inf} \quad \psi(x) \\ &\text{s.t.} \quad f_t(x) \leq 0, \ t \in \tilde{T} \subset T, \ \tilde{T} \text{ finite} \end{aligned} \tag{$\tilde{\mathrm{P}}$}$$

so that (P) and ($\tilde{\mathrm{P}}$) have the same optimal value.

With this aim, Pshenichnyi [20] replaces (P) by:

$$\begin{aligned} &\text{Inf} \quad \psi(x) \\ &\text{s.t.} \quad f(x) \leq 0 \end{aligned} \tag{$\hat{\mathrm{P}}$}$$

where $f(x) := \sup_{t\in T} f_t(x)$.

Since f, in general, is nondifferentiable although the functions f_t, $t \in T$, are convex and differentiable, the optimality conditions for ($\hat{\mathrm{P}}$) involve the subdifferential of $f(x)$, which has been characterized in [22].

In [3], establishment of the direct association $(\mathrm{P})\leftrightarrow(\tilde{\mathrm{P}})$ is achieved using a Helly-type theorem demonstrated by Klee [18], which is applied to certain families of open convex sets. It should be pointed out that the optimality conditions are given after (P) has been reduced to a finite $(\tilde{\mathrm{P}})$, for which the theory of finite optimality is used.

Second-order optimality conditions have been deduced in [12] and in [4], while in [19] higher-order conditions are established by extension of the m-order abstract variational theory, introduced in [13].

A different approach to the convex SIP is shown in Ben-Tal, Kerzner and Zlobec [2], where, under the assumption of differentiability of all the functions, necessary and sufficient optimality conditions are obtained which do not assume constraint qualifications, but nonetheless with the additional requirement that the constraints fulfil the so-called 'uniform mean value property'.

Borwein in [5], using Helly's theorem for families of compact convex sets and observing certain conditions of regularity, obtains a finitely constrained subprogramme $(\tilde{\mathrm{P}})$ with the same optimal value. His particular approach, based on the convexity of level sets, permits him to extend the validity of his results to problems in which the functions involved are, in general, quasi-convex and those which intervene in the constraints of subprogramme $(\tilde{\mathrm{P}})$ are strictly quasi-convex. We can obtain a Lagrangian condition only when all of the functions of $(\tilde{\mathrm{P}})$, including the objective function, are convex and $v(\mathrm{P})$, the optimal value of (P), is finite:

There exist n points t_i, $i=1, 2, \ldots, n$, in T, and nonnegative scalars λ_i, $i=1, 2, \ldots, n$, such that

$$v(\mathrm{P})=\inf\left\{\psi(x)+\sum_{i=1}^{n}\lambda_i f_{t_i}(x)\,\middle|\,x\in\mathbb{R}^n\right\} \tag{1}$$

(always assuming that the domain of all the functions is $\mathbb{R}^n$).

Jeroslow, in [15], establishes a necessary and sufficient condition for which (1) is verified for a convex SIP, with $v(\mathrm{P})$ finite and for any objective function, in which case we can say that the system of constraints $\{f_t(x)\leqslant 0,\ t\in T\}$ satisfies the uniform convex duality. This can be accomplished through a reductionist procedure by which the uniform convex duality is related to the uniform duality of certain linear representations. It is proved in [7] that the uniform duality is verified if, and only if, the system of constraints possesses the Farkas–Minkowski (F–M) property, which has been studied and characterized in that paper as well as in [11].

Another interesting contribution to the field of Lagrangian duality in SIP can be found in [17], through the approximation of (P) using finite subprogrammes and by application of the recession theory. The technique utilized here coincides with that used by the same author in extending the Clark–Duffin theorem to the semi-infinite case [16].

On the other hand, in [23] the convex SIP is modelled as a problem of abstract convex programming, which yields conditions of optimality from a strengthened optimality test, extending the results in [24].

In this paper, optimality conditions are presented for convex SIP which involve the notion of the standard Lagrangian saddlepoint. Local and global constraint qualifications are also established, which show the transcendence of the F–M property for certain linearized systems. This fact is also brought to light in [7] and in [15], in relation to uniform duality.

The structure of this paper is as follows: Section 2 introduces the notation used, as well as certain preliminary results.

Section 3 includes a characterization of the Lagrangian saddle points for nondifferentiable convex SIP. Two constraint qualifications are established, one local, called Lagrangian regularity, and one an extension of Slater's qualification. It is demonstrated that Slater's qualification implies Lagrangian regularity of all the feasible points.

In Section 4 a global constraint qualification is introduced, which is implied by Slater's qualification. We give a linearization of the feasible set S through the subgradients of the functions f_t which vanish at some point of the boundary of S. Thus, this type of representation, due to its clear geometrical meaning, seems preferable to other possible types.

In relation to the importance of the F–M property in the linearization of a convex SIP, our main conclusion is that the Slater's qualification is too restrictive for SIP, as previously conjectured in [2]. In Section 4, and in Theorem 3.2 of [15], it is shown that Slater's qualification yields canonically closed linear systems, and the F–M property is more general. These ideas concern different possibilities as to the attempt to utilize the topological characteristics of (P) in its dependence with T. Along these lines we might also mention the uniform mean value property of [2].

2. Notation and preliminary results

Let $\{f_t, t \in T\}$ be a family of finite convex functions in $\mathbb{R}^n$, where the index set T is infinite. We consider the system of convex inequalities in $\mathbb{R}^n$: $\{f_t(x) \leqslant 0, t \in T\}$.

Let S be the set of solutions of the system. The system is consistent if $S \neq \emptyset$. Two systems are called equivalent if they have the same solutions.

We shall consider elements of the set $\mathbb{R}_+^{(T)}$ defined as

$$\mathbb{R}_+^{(T)} := \{\lambda : T \to \mathbb{R}_+ \mid \lambda_t = 0, \text{ for all } t \text{ except for a finite number}\}$$

Given a nonempty set $C \subset \mathbb{R}^p$, $\langle C \rangle$ denotes the convex hull of C; $K(C)$ the convex cone generated by C; $\operatorname{cl} C$ the closure of C; $\operatorname{int} C$ the interior of C; and C^* is the dual cone of C.

The null vector in $\mathbb{R}^n$ will be denoted by 0_n.

Let $\{a_t'x \leqslant \beta_t, t \in T\}$ be a linear infinite system in $\mathbb{R}^n$. The relation $a'x \leqslant \beta$ is a consequence relation of the given system if every solution to it satisfies this relation.

Now we recall some known results, which are used below.

Lemma 2.1 (Lemma 14.1 in [9]). *$a'x \leqslant 0$ is a consequence relation of the system $\{a_t'x \leqslant 0, t \in T\}$ if, and only if, $a \in \operatorname{cl} K\{a_t, t \in T\}$.*

Lemma 2.2 (Lemma 14.2 in [9]). *$a'x \leqslant \beta$ is a consequence relation of the consistent system $\{a_t'x \leqslant \beta_t, t \in T\}$ if, and only if, the relation $a'x + \beta\tau \leqslant 0$ is a consequence of the system*

$$\begin{aligned} a_t'x + \beta_t\tau &\leqslant 0, \quad t \in T, \\ \tau &\leqslant 0. \end{aligned}$$

Theorem 2.3. *$a'x \leqslant \beta$ is a consequence relation of the consistent system $\{a_t'x \leqslant \beta_t, t \in T\}$ if, and only if,*

$$\begin{bmatrix} a \\ \beta \end{bmatrix} \in \operatorname{cl} K\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T; \begin{bmatrix} 0_n \\ 1 \end{bmatrix}\right\}.$$

Proof. $a'x \leqslant \beta$ is a consequence relation of the given system if, and only if,

$$\begin{bmatrix} a_t' & \beta_t \\ 0_n' & 1 \end{bmatrix}\begin{bmatrix} x \\ \tau \end{bmatrix} \leqslant \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{implies} \quad (a' \;\; \beta)\begin{bmatrix} x \\ \tau \end{bmatrix} \leqslant 0.$$

The result follows by applying Lemma 2.1.

Similar versions of the last result, which constitutes a generalization of the nonhomogeneous Farkas theorem, can be found in [10] and [11].

We recall that a consistent system $\{a_t'x \leqslant \beta_t, t \in T\}$ satisfies the Farkas–Minkowski property if every consequence relation of the system is a consequence of a finite subsystem.

The system $\{a_t'x \leqslant \beta_t, t \in T\}$ is canonically closed (CC) if the following conditions hold:

(i) there is an algebraic interior point, i.e. for some $x^0 \in \mathbb{R}^n$, $a_t'x^0 < \beta_t$ for all $t \in T$.

(ii) the set

$$\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T\right\}$$

is compact.

Theorem 2.4 (in [11]). *The consistent system $\{a_t'x \leqslant \beta_t, t \in T\}$ satisfies the F–M property if, and only if,*

$$K\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T; \begin{bmatrix} 0_n \\ 1 \end{bmatrix}\right\}$$

is closed.

The F–M property guarantees that the corresponding system yields uniform LP duality [7].

Corollary 2.4.1 (in [11]). *The system* $\{a_t'x \leq 0, t \in T\}$ *satisfies the* F–M *property if, and only if,* $K\{a_t, t \in T\}$ *is closed.*

Corollary 2.4.2. *If the consistent system* $\{a_t'x \leq \beta_t, t \in T\}$ *satisfies one of the following conditions, then it is a F–M system*:

(i) $K\left\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix}, t \in T\right\}$ *is closed.*

(ii) *The system is canonically closed.*

For a straightforward proof derived from Theorem 2.4, see [11]. The condition (ii) is due to Duffin and Karlovitz [8].

3. Optimality conditions related to Lagrangian saddlepoints

Consider the convex SIP:

$$\min_{x \in S} \quad \psi(x) \tag{P}$$

where $S := \{x \in \mathbb{R}^n \mid f_t(x) \leq 0 \text{ for all } t \in T\}$. Here ψ and f_t, $t \in T$, are convex functions in $\mathbb{R}^n$, not necessarily differentiable.

We will associate the standard Lagrangian function to problem (P):

$$\Psi(x, \lambda) = \psi(x) + \sum_{t \in T} \lambda_t f_t(x), \qquad x \in \mathbb{R}^n, \quad \lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$$

(see e.g. [6]).

It can be easily proved that if $(\bar{x}, \bar{\lambda})$ is a saddlepoint of Ψ, then $\bar{x}$ is an optimal solution of (P). Moreover the complementarity condition is valid, i.e. $f_t(\bar{x}) < 0$ implies $\bar{\lambda}_t = 0$. In order to get a saddlepoint from an optimal solution, we have to assume that the constraints satisfy some additional conditions.

We denote by $\partial h(x)$ the subdifferential of the convex function h in x.

Let $T(\bar{x}) := \{t \in T \mid f_t(\bar{x}) = 0\}$ be the set of active constraints.

The following proposition characterizes the existence of saddlepoints that involve a given $\bar{x}$, and it constitutes an infinite dimensional version of Theorem 28.3 of [21].

Lemma 3.1. *Given* $\bar{x} \in \mathbb{R}^n$, *there will exist a Lagrangian saddlepoint associated with* $\bar{x}$ *if, and only if,* $\bar{x} \in S$, *and* $(\bar{\lambda}_t)_{t \in T(\bar{x})} \in \mathbb{R}_+^{(T(\bar{x}))}$ *exists so that*

$$0_n \in \partial\psi(\bar{x}) + \sum_{t \in T(\bar{x})} \bar{\lambda}_t \, \partial f_t(\bar{x}).$$

Proof. Suppose that $(\bar{x}, \bar{\lambda})$, with $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ is a Lagrangian saddlepoint. Since $\bar{x}$ is an optimal solution of (P), $\bar{x} \in S$.

Moreover, $\bar{x}$ is a global minimum of the convex function $\psi + \sum_{t \in T(\bar{x})} \bar{\lambda}_t f_t$ and so

$$0_n \in \partial\Big(\psi + \sum_{t \in T(\bar{x})} \bar{\lambda}_t f_t\Big)(\bar{x}) = \partial\psi(\bar{x}) + \sum_{t \in T(\bar{x})} \bar{\lambda}_t\, \partial f_t(\bar{x}).$$

Conversely, since $0_n \in \partial\psi(\bar{x}) + \sum_{t \in T(\bar{x})} \bar{\lambda}_t\, \partial f_t(\bar{x})$, and if we define $\bar{\lambda}_t = 0$, $t \notin T(\bar{x})$, we have $\psi(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x) \geqslant \psi(\bar{x})$, for all $x \in \mathbb{R}^n$.

As $\bar{x} \in S$, $\sum_{t \in T} \lambda_t f_t(\bar{x}) \leqslant 0$ for any $\lambda \in \mathbb{R}_+^{(T)}$, then $(\bar{x}, \bar{\lambda})$ is a saddlepoint of Ψ.

When $T(\bar{x}) \neq \emptyset$, we define

$$B(\bar{x}) := \{u_t \mid u_t \in \partial f_t(\bar{x}),\, t \in T(\bar{x})\} = \bigcup_{t \in T(\bar{x})} \partial f_t(\bar{x}).$$

It can be easily proved, from the convexity of the subdifferential set, that the condition

$$0_n \in \partial\psi(\bar{x}) + \sum_{t \in T(\bar{x})} \bar{\lambda}_t\, \partial f_t(\bar{x}), \quad (\bar{\lambda}_t)_{t \in T(\bar{x})} \in \mathbb{R}_+^{|T(\bar{x})|}$$

is equivalent to the existence of some $\bar{u} \in \partial\psi(\bar{x})$ such that $-\bar{u} \in K\{B(\bar{x})\}$.

Remember the notion of the cone of tangents to S at $\bar{x}$ (see [1]):

$$T(S, \bar{x}) := \{z \mid z = \lim_{k \to \infty} \lambda_k (x^k - \bar{x}), \text{ where } \lambda_k > 0,\, x^k \in S \text{ and } \lim_{k \to \infty} x^k = \bar{x}\}.$$

$T(S, \bar{x})$ is a closed convex cone, since S is a convex set, and verifies

$$T(S, \bar{x}) = \mathrm{cl}\{\lambda(x - \bar{x}) : x \in S,\, \lambda > 0\} = \mathrm{cl}\, K(S - \bar{x}).$$

Lemma 3.2. *If $\bar{x} \in S$, then $T(S, \bar{x}) \subset B(\bar{x})^*$.*

Proof. $f_t(x^k) \geqslant f_t(\bar{x}) + u_t'(x^k - \bar{x})$ holds for $u_t \in \partial f_t(\bar{x})$. If $t \in T(\bar{x})$ and $x^k \in S$, then $0 \geqslant f_t(x^k) \geqslant u_t'(x^k - \bar{x})$. For $\lambda_k > 0$, $u_t'\{\lambda_k(x^k - \bar{x})\} \leqslant 0$ and, taking limits, $u_t' z \leqslant 0$. Then $z \in B(\bar{x})^*$.

Remark. The previous inclusion can be strict, as the following example proves.

Example:

$$n = 1, f_t(x) := \begin{cases} 0 & \text{if } x \leqslant \bar{0}, \\ x^t & \text{if } x > 0, \end{cases} \qquad t \in T := [2, 3].$$

For $\bar{x} = 0$, $T(S, 0) =]-\infty, 0]$ and $B(0)^* =]-\infty, +\infty[$.

Definition 3.3. A point $\bar{x} \in S$ is said to be a Lagrangian regular point if

(i) $T(S, x) = \mathbb{R}^n$ when $T(\bar{x}) = \emptyset$,

(ii-a) $B(\bar{x})^* \subset T(S, \bar{x})$ and (ii-b) $K\{B(\bar{x})\}$ is closed, when $T(\bar{x}) \neq \emptyset$.

The condition (ii-a) is an extension, to the nondifferentiable semi-infinite case, of Guignard's qualification, the weakest in a large set of qualifications in nonlinear programming (see [1]).

The condition (ii-b) is necessary and sufficient in that the system, in $\bar{x}$, $\{u'x \leq 0, u \in B(\bar{x})\}$ verifies the F–M property. For T finite and differentiable functions, (ii-b) holds since $K\{B(\bar{x})\}$ is a polyhedral cone.

The following result constitutes a necessary optimality condition.

Lemma 3.4. *Let $\bar{x}$ be an optimum of the convex* SIP. *If $\bar{x}$ is a Lagrangian regular point, $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ will exist such that $(\bar{x}, \bar{\lambda})$ is a saddlepoint of Ψ.*

Proof. Since $\bar{x}$ is an optimal solution, and from Pshenichnyi's condition ([20, p. 56]), there is a subgradient $\bar{u} \in \partial\psi(\bar{x})$ such that

$$-\bar{u} \in \{K(S - \bar{x})\}^* = \{\mathrm{cl}\, K(S - \bar{x})\}^* = T(S, \bar{x})^*.$$

If $T(\bar{x}) = \emptyset$, $\bar{u} = 0_n \in \partial\psi(\bar{x})$.

If $T(\bar{x}) \neq \emptyset$, $-\bar{u} \in B(\bar{x})^{**} = K\{B(\bar{x})\}$.

In both cases Lemma 3.1 can be applied.

Before generalizing the Slater qualification for the convex SIP, we shall analyse the relationship between the algebraic and topologic interior of the level sets of a convex function.

Lemma 3.5. *Let $f(x)$ be a convex function in $\mathbb{R}^n$, $S_0 := \{x \in \mathbb{R}^n \mid f(x) < 0\}$ and $S := \{x \in \mathbb{R}^n \mid f(x) \leq 0\}$. Then $S_0 \neq \emptyset$ implies* $\mathrm{int}\, S = S_0$.

Proof. By continuity $S_0 \subset \mathrm{int}\, S$. Given $y \in \mathrm{int}\, S$ and $x^0 \in S_0$, if $y \neq x^0$ we take $z \in S$ such that y belongs to the interior of the segment between x^0 and z. As $f(x^0) < 0$ and $f(z) \leq 0$, we have $f(y) < 0$, by convexity.

Remark. The last result also holds if $f(x)$ is a strictly quasi-convex upper semicontinuous function.

Definition 3.6. Slater's qualification for convex SIP is:

(i) $T \subset \mathbb{R}^m$ is a compact set,

(ii) $f_t(x)$ is a continuous function of (t, x) in $T \times \mathbb{R}^n$, and

(iii) there is a point x^0 such that $f_t(x^0) < 0$, for all $t \in T$.

Applying theorem 10.7 of [21], condition (ii), for a convex SIP verifying (i), could be replaced by: $f(\cdot, x)$ is a continuous function in t for each $x \in \mathbb{R}^n$.

From [5, p. 304], the conjunction of (i) and (ii) allows us to reestablish (iii) in such a way that we claim that every $n+1$ inequalities of the form $f_t(x) < 0$ have common solutions.

We need a preliminary lemma, for Theorem 3.8, which constitutes an extension of Gordan's Alternative Theorem.

Lemma 3.7. *Let $a_t \in \mathbb{R}^n$ for all $t \in T$. If $\langle\{a_t, t \in T\}\rangle$ is a closed set, then the equivalence holds between the negation of proposition* (I) *and proposition* (II).

$$\{a_t'x < 0, t \in T\} \text{ is consistent,} \tag{I}$$

$$0_n \in \langle\{a_t, t \in T\}\rangle. \tag{II}$$

Proof. We can suppose that $a_t \neq 0_n$ for all $t \in T$. Otherwise, the result is trivial.

First, we show that (II) implies the negation of (I). Let us suppose that $\bar{x}$ is a solution of the system considered in (I). By hypothesis, $0_n \in \langle\{a_t, t \in T\}\rangle$, then $\lambda \in \mathbb{R}_+^{(T)}$, $\sum_{t\in T}\lambda_t = 1$ is such that $0_n = \sum_{t\in T}\lambda_t a_t$. In this case $0 = 0_n'\bar{x} = \sum_{t\in T}\lambda_t(a_t'\bar{x}) < 0$.

Let us prove that the negation of (I) implies (II). If $0_n \notin \langle\{a_t, t \in T\}\rangle$, we can apply the strict separation theorem: there is some vector $c \in \mathbb{R}^n$, $c \neq 0_n$, such that $c'a < 0$ for all $a \in \langle\{a_t, t \in T\}\rangle$. Then $c'a_t < 0$ for all $t \in T$, hence c is a solution of the system in (I).

The assumption that $\langle\{a_t, t \in T\}\rangle$ is a closed set cannot be eliminated, as we show in the following example.

Let $a_t := (\cos t, \sin t)'$, $T := [-\frac{1}{2}\pi, \frac{1}{2}\pi[$. It is clear that (I) and (II) fail simultaneously, $\langle\{a_t, t \in T\}\rangle$ not being closed.

Theorem 3.8. *Under the assumption that Slater's qualification for convex SIP is satisfied, all feasible points will be Lagrangian regular points.*

Proof. Let $\bar{x} \in S$, $S := \{x \in \mathbb{R}^n \mid f_t(x) \leq 0 \text{ for all } t \in T\}$. We define $f(x) := \max_{t\in T} f_t(x)$.

We shall assume first $T(\bar{x}) = \emptyset$. By Lemma 3.5 $\operatorname{int} S = \{x \mid f(x) < 0\}$. We have $f(\bar{x}) < 0$, then $\bar{x} \in \operatorname{int} S$ and $K(S - \bar{x}) = \mathbb{R}^n = T(S, \bar{x})$.

Now let us assume $T(\bar{x}) \neq \emptyset$. Under assumptions (i) and (ii) of Slater's qualification, we know that $B(\bar{x}) = \bigcup_{t\in T(\bar{x})} \partial f_t(\bar{x})$ is a compact set and $\langle B(\bar{x})\rangle = \partial f(\bar{x})$ (see [14, p. 201–204]).

$T(\bar{x}) \neq \emptyset$ does not preclude $0_n \in \langle B(\bar{x})\rangle$ (e.g. min. $\{x \mid x^2 \leq 0\}$). Under the assumption (iii) of Slater's qualification we have $f(x^0) < 0 = f(\bar{x})$, and $\bar{x}$ is not a global minimum of $f(x)$. Consequently $0_n \notin \partial f(\bar{x}) = \langle B(\bar{x})\rangle$, and this fact implies that $K\{B(\bar{x})\} = K\{\langle B(\bar{x})\rangle\}$ is a closed cone (see [21, p. 79]).

Take $B(\bar{x})^{\circledS} := \{z \mid u'z < 0, u \in B(\bar{x})\}$. Since $0_n \notin \langle B(\bar{x})\rangle$, $\langle B(\bar{x})\rangle$ being a closed set, and applying Lemma 3.7, this implies that $B(\bar{x})^{\circledS} \neq \emptyset$. We can apply Lemma 3.5 to the support function of $B(\bar{x})$: $\operatorname{int} B(\bar{x})^* = B(\bar{x})^{\circledS}$.

Take $z \in \operatorname{int} B(\bar{x})^*$, i.e. $u'z < 0$, for all $u \in B(\bar{x})$ and also $u'z < 0$ for all $u \in \langle B(\bar{x})\rangle = \partial f(\bar{x})$. Since $f'(\bar{x}; z) = \max_{u\in\partial f(\bar{x})} u'z$, where $f'(\bar{x}; z)$ is the directional derivative of function f, then $f'(\bar{x}; z) < 0$, and $\delta > 0$ exists such that $f(\bar{x} + \lambda z) < 0$, for all λ, $0 < \lambda < \delta$, i.e. $\bar{x} + \lambda z \in S$ with $\lambda \in]0, \delta[$ and $z \in T(S, \bar{x})$. We conclude that $B(\bar{x})^* = \mathrm{cl}\{\mathrm{int}\, B(\bar{x})^*\} \subset T(S, \bar{x})$.

As a consequence of the conjunction of Theorem 3.8 and Lemma 3.4 we have that Slater's qualification for convex SIP guarantees that the optimal points permit us to obtain Lagrangian saddlepoints.

4. Global constraint qualification; Farkas–Minkowski linearization

For the convex SIP we establish a global constraint qualification, alternative to the local condition of Lagrangian regular point required in Section 3 at every optimal solution $\bar{x}$ to get the necessary optimality condition. In fact, Slater's qualification involves the set of all constraints and the qualification proposed in this Section is implied by that.

Definition 4.1. We associate with the nondifferentiable convex SIP the following system of linear inequalities:

$$\{u'x \leqslant u'y - f_t(y),\ (t, y) \in T \times \mathbb{R}^n,\ u \in \partial f_t(y)\}. \tag{2}$$

Obviously, for convexity, the set of solutions of the system (2) is S, the feasible set of the SIP. Consequently the system above is a linear representation of the constraints of the convex SIP.

Definition 4.2. 'Farkas–Minkowski qualification'. The constraints of the convex SIP satisfy the Farkas–Minkowski qualification if the equivalent linear system (2) is a Farkas–Minkowski system.

In the following we verify the character of qualification of the established condition (see also [11]), and we analyse its relationship with that of Slater, without requiring differentiability.

Theorem 4.3. *If $\bar{x}$ is an optimum point of a convex SIP which satisfies the Farkas–Minkowski qualification, then some $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ will exist such that $(\bar{x}, \bar{\lambda})$ is a Lagrangian saddlepoint.*

Proof. Since $\bar{x}$ is an optimal solution of the problem: Min. $\psi(x)$ on S, there is some $\bar{u} \in \partial\psi(\bar{x})$ such that $\bar{u}'(x - \bar{x}) \geqslant 0$, for all $x \in S$.

If $\bar{u} = 0_n$, we let $\bar{\lambda}_t = 0$, for all $t \in T$. If $\bar{u} \neq 0_n$, we have $-\bar{u}'x \leqslant -\bar{u}'\bar{x}$, which is a consequent relation of system (2) and, if (2) has the property of F–M, there will be parameters $\bar{\lambda}_i > 0$, $i = 1, 2, \ldots, q$; $u_i \in \partial f_{t_i}(y_i)$, $i = 1, 2, \ldots, q$; $\mu \geq 0$, such that

$$\psi(\bar{x}) - \psi(x) \leq -\bar{u}'x + \bar{u}'\bar{x} = \sum_{i=1}^{q} \bar{\lambda}_i [f_{t_i}(y_i) + u_i'(x - y_i)] - \mu \leq \sum_{i=1}^{q} \bar{\lambda}_i f_{t_i}(x)$$

for all $x \in \mathbb{R}^n$.

For $x = \bar{x}$, as $\bar{\lambda}_i > 0$, we shall have $t_i \in T(\bar{x})$. Consequently

$$\psi(\bar{x}) + \sum_{i=1}^{q} \bar{\lambda}_i f_{t_i}(\bar{x}) = \psi(\bar{x}) \leq \psi(x) + \sum_{i=1}^{q} \bar{\lambda}_i f_{t_i}(x)$$

for all $x \in \mathbb{R}^n$.

We let $\bar{\lambda}_t = \bar{\lambda}_i$ if $t = t_i$, and $\bar{\lambda}_t = 0$ if $t \neq t_i$, $i = 1, 2, \ldots, q$; then $(\bar{x}, \bar{\lambda})$ is a saddlepoint of Ψ.

In the proof of our main result, Theorem 4.5, which establishes the relationship between Slater and Farkas–Minkowski qualifications, we need a preliminary lemma, proved in [11], in which S^b denotes the set of boundary points of S.

Lemma 4.4. *Let $S \subset \mathbb{R}^n$ be a closed convex set and $\{c_t'x \leq \delta_t, t \in T\}$ a system such that*

(I) *every point of S is a solution to it;*

(II) *there is a $x^0 \in S$ such that $c_t'x^0 < \delta_t$, $t \in T$; and*

(III) *given any $y \in S^b$, there is some $t \in T$ such that $c_t'y = \delta_t$.*

Then $S = \{x \in \mathbb{R}^n \mid c_t'x \leq \delta_t, t \in T\}$.

We now state the main result in this section, in which the finite nature of the Slater's qualification is shown. From this result the significance of the F–M property arises.

Theorem 4.5. *If the convex SIP, S being bounded, satisfies the Slater qualification, then it will satisfy the Farkas–Minkowski qualification.*

Proof. We recall that $f(x) = \max_{t \in T} f_t(x)$ and $S^b = \{y \in S \mid f(y) = 0\}$.

The system

$$\{u'x \leq u'y,\ y \in S^b,\ u \in \partial f(y)\} \tag{3}$$

constitutes a linear representation of S. We shall see that the assumptions of Lemma 4.4 hold:

(I) every point of S is a solution of (3);

(II) x^0 is an algebraic interior point of (3); and

(III) given $y \in S^b$, there are relations of the system (3) which are satisfied with equality.

We shall prove that the system (3) is canonically closed, and, consequently, of Farkas–Minkowski. Take the set

$$A := \left\{ \begin{bmatrix} u \\ u'y \end{bmatrix} \Big/ y \in S^{\mathrm{b}}, u \in \partial f(y) \right\}.$$

We must prove that A is compact.

We know that $\partial f(S^{\mathrm{b}}) := \bigcup_{y \in S^{\mathrm{b}}} \partial f(y)$ is compact (see [21, p. 237]). We shall consider a sequence in A,

$$\begin{bmatrix} u_k \\ u'_k y_k \end{bmatrix}, \quad k = 1, 2, \ldots,$$

such that

$$\lim_{k \to \infty} \begin{bmatrix} u_k \\ u'_k y_k \end{bmatrix} = \begin{bmatrix} u \\ \alpha \end{bmatrix}.$$

Obviously $\lim_{k \to \infty} u_k = u$ and therefore $u \in \partial f(S^{\mathrm{b}})$.

Moreover, $\lim_{k \to \infty} u'_k y_k = \alpha$. Since $\{y_k\} \subset S^{\mathrm{b}}$ there will be a subsequence $\{y_{k_i}\}$ such that $\lim_{i \to \infty} y_{k_i} = y \in S^{\mathrm{b}}$. Therefore, $\alpha = \lim_{i \to \infty} u'_{k_i} y_{k_i} = u'y$.

On the other hand, we know that the graph of ∂f on $\mathbb{R}^n \times \mathbb{R}^n$ is a closed set and hence $u \in \partial f(y)$ (see [21, p. 233]).

We conclude that A is closed. As a consequence of the Schwarz inequality this set is bounded. Take $\nu := \max_{u \in \partial f(S^{\mathrm{b}})} \|u\|$ and $\eta := \max_{y \in S^{\mathrm{b}}} \|y\|$. The norm of the vectors of A will be bounded above by $(\nu(\nu + \eta))^{1/2}$.

We conclude that system (3) has the F–M property, which is equivalent to saying that

$$K\left\{ A \cup \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \right\} = K\left\{ \left\langle A \cup \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \right\rangle \right\}$$

is a closed cone. Now, in our assumptions, $u \in \partial f(y)$ if and only if $u \in \langle \bigcup_{t \in T(y)} \partial f_t(y) \rangle$ where $T(y) = \{t \in T \mid f_t(y) = 0\}$ and therefore

$$\left\langle A \cup \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \right\rangle = \left\langle \tilde{A} \cup \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \right\rangle,$$

where

$$\tilde{A} := \left\{ \begin{bmatrix} u \\ u'y \end{bmatrix} \Big/ y \in S^{\mathrm{b}}, u \in \partial f_t(y), t \in T(y) \right\}.$$

Consequently, the equivalent system to (3),

$$\{u'x \leqslant u'y;\ y \in S^{\mathrm{b}}, u \in \partial f_t(y), t \in T(y)\}, \qquad (\tilde{3})$$

has the (F–M) property, and then so has system (2), since it is obtained by the addition of consequent relations to an F–M system.

Under the Slater qualification we can always associate a Lagrangian saddlepoint with the optimum $\bar{x}$:

(a) If S is bounded, we can apply Theorem 4.5.

(b) If S is not bounded, we define a new problem by the addition of the constraint $g(x) := x'x - 2x'\bar{x} + \bar{x}'\bar{x} - \rho^2 \leqslant 0$, with $\rho > \|\bar{x} - x^0\|$.

The new problem satisfies all the conditions of the Theorem 4.5. Then, there are some $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ and $\bar{\mu} \geqslant 0$ such that

$$\psi(\bar{x}) + \sum_{t \in T} \lambda_t f_t(\bar{x}) - \mu\rho^2 \leqslant \psi(\bar{x}) \leqslant \psi(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x) + \bar{\mu} g(x)$$

for all $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, all $\mu \geqslant 0$ and all $x \in \mathbb{R}^n$.

As $g(x) \leqslant 0$ is not active in $\bar{x}$ and if we take $\mu = 0$, we conclude that $(\bar{x}, \bar{\lambda})$ is a saddlepoint for the original SIP.

Theorem 4.3 continues to be valid for the case in which the F–M property is demanded only for any linearization of the set of solutions S, $\{a_p'x \leqslant \beta_p, p \in P\}$, which verifies the following property:

For every $p \in P$ there exists $t_p \in T$ such that $a_p'x - \beta_p \leqslant f_{t_p}(x)$, for all $x \in \mathbb{R}^n$. (4)

Any such linear system constitutes a particular case of the positive derivant introduced in [15]. Naturally, systems (2) and (3) verify property (4) in relation with the convex SIP.

According to this possible reformulation, the F–M qualification is quite weak, almost constituting a necessary and sufficient condition in which an optimal point will always yield a saddlepoint, whatever the objective function. Within this order of ideas, if the original system of constraints exhibits uniform convex duality (in which case it becomes evident that every optimum yields a saddlepoint, by applying characterization Theorem 3.1 of [15] and Theorem 2.2 of [7]), we can conclude that it is possible to associate a linear positive derivant, which constitutes an F–M system (apply Corollary 2.4.2(i)), to the original system of constraints.

Acknowledgement

We are very grateful to the referees for their comments and suggestions, and especially to the one who suggested the proof of Theorem 2.3, the shortest of all known proofs, which we have consequently included.

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