Mathematical Programming 25 (1983) 183–198
North-Holland Publishing Company

INTEGRAL DECOMPOSITION IN POLYHEDRA

Colin McDIARMID

Wolfson College, Oxford, England

Received 17 September 1981
Revised manuscript received 13 April 1982

We say that a polyhedron P satisfies weak integral decomposition if whenever an integral vector is the sum of k vectors in P it is also the sum of k integral vectors in P. This property is related to rounding results for packing and covering problems. We study the property and two related properties, and give results concerning integral polymatroids, totally unimodular matrices and network flows, pairs of strongly-base-orderable matroids, and branchings in directed graphs.

Key words: Convex Polyhedra, Integral Decomposition, Packing and Covering, Polymatroids, Totally Unimodular Matrices, Network Flows, Strongly-base-orderable Matroids, Branchings

1. Introduction

We say that a (convex) polyhedron P in $\mathbb{R}^n$ satisfies *weak integral decomposition* (WID) if whenever an integral vector c is the sum of k vectors in P (that is, $c \in kP$), then c is also the sum of k integral vectors in P. This property is closely related to certain rounding properties for packing and covering problems—see Section 2. Four of the most interesting examples of polyhedra satisfying WID are the following.

(a) Any matroid polyhedron, or more generally any integral polymatroid [4, 15, 19, 27].

(b) Any polyhedron of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ where A is totally unimodular $(m \times n)$ matrix and b is an integral vector in $\mathbb{R}^m$. A special case is the polyhedron of feasible circulations in a network [1, 2, 24, 30].

(c) The intersection of the polyhedra of two strongly-base-orderable matroids [20]. Recall that such matroids include transversal matroids and gammoids.

(d) The convex hull of the (0, 1)-incidence vectors of the branchings in a directed graph [4].

We shall see in Section 7 that also any integral polyhedron of dimension at most two satisfies WID.

In this paper we consider also a property stronger than the weak integral decomposition property WID, namely the *strong integral decomposition* property (SID), and an intermediate property *middle integral decomposition* (MID). We shall see easily that

$$\text{SID} \Rightarrow \text{MID} \Rightarrow \text{WID}; \tag{1}$$

that the polyhedra in example (a) satisfy SID; that the polyhedra in example (b) satisfy MID but not necessarily SID; and that the polyhedra in examples (c) and (d) satisfy WID but not necessarily MID. Thus all these polyhedra satisfy WID, and indeed all satisfy 'equitable' versions of WID (see below).

Recall that a polyhedron P in $\mathbb{R}^n$ is *integral* if every face contains an integral vector. If P is pointed, that is, if P has an extreme point, this is equivalent to saying that every extreme point of P is integral. Now suppose that P is integral and let Q be the polyhedron obtained by projecting P onto say its first m co-ordinates, that is,

$$Q = \{\boldsymbol{x} \in \mathbb{R}^m : (\boldsymbol{x}, \boldsymbol{y}) \in P \text{ for some } \boldsymbol{y} \in \mathbb{R}^{n-m}\}.$$

Then the polyhedron Q is also integral, since the set of points of P projecting onto any given face of Q forms a face of P. We shall use this observation several times below.

Before we define the properties MID and SID let us consider briefly the property WID. Recall that a polyhedron P satisfies *weak integral decomposition* (WID) if given any positive integer k and any integral vector $\boldsymbol{c}$ in $(k+1)P$ there exists $k+1$ integral vectors in P which sum to $\boldsymbol{c}$. A simple induction shows (as noted in [4]; see also the earlier paper [28]) that this is equivalent to insisting that if $\boldsymbol{c}$ is in $(k+1)P$ as above, then there is an integral vector $\boldsymbol{x}$ in P such that $\boldsymbol{c}-\boldsymbol{x}$ is in kP, that is, insisting that there is an integral vector $\boldsymbol{x}$ in the polyhedron $Q = P \cap (\boldsymbol{c} - kP)$. Now a vector $\boldsymbol{c}$ is in $(k+1)P$ if and only if the polyhedron Q above is non-empty. Thus the polyhedron P satisfies WID if and only if, given any positive integer k and any integral vector $\boldsymbol{c}$ such that the polyhedron $Q = P \cap (\boldsymbol{c} - kP)$ is non-empty, there exists an integral vector in Q.

We say that the polyhedron P satisfies *middle integral decomposition* (MID) if given any positive integer k and any integral vector $\boldsymbol{c}$ the polyhedron $Q = P \cap (\boldsymbol{c} - kP)$ is integral. It is immediate from the above reformulation of WID that if P satisfies MID, then it must satisfy WID. The property MID was introduced in [4], where it was shown that integral polymatroids satisfy it.

Finally we say that the polyhedron P in $\mathbb{R}^n$ satisfies *strong integral decomposition* (SID) if given any positive integer k and any integral vector $\boldsymbol{c}$ the polyhedron

$$Q_k(\boldsymbol{c}) = \left\{(\boldsymbol{x}^1, \dots, \boldsymbol{x}^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \in P \ (i = 1, \dots, k), \sum_i \boldsymbol{x}^i = \boldsymbol{c}\right\}$$

is integral. Note that if P satisfies SID, then it must satisfy MID; for the polyhedron $P \cap (\boldsymbol{c} - kP)$ is the projection onto its first n co-ordinates of the polyhedron $Q_{k+1}(\boldsymbol{c})$ above. Thus if P satisfies SID it must satisfy WID. This also follows immediately from the definitions, since if an integral vector $\boldsymbol{c}$ is the sum of k vectors in P then the polyhedron $Q_k(\boldsymbol{c})$ above is non-empty and so if it is integral then it must contain an integral point.

We have now as promised introduced the three integral decomposition properties WID, MID and SID, and noted that the implications (1) hold.

We need a last batch of definitions. We define below an 'equitable' and a 'nearly uniform' version of the weak integral decomposition property WID.

Given a family $x^1, \dots, x^k$ of integral vectors in $\mathbb{R}^n$ we say that it is *equitable* if $x^i - x^j \leqslant \mathbf{1}$ for all $i, j = 1, \dots, k$; and that it is *nearly uniform* if further $\mathbf{1} \cdot x^i - \mathbf{1} \cdot x^j \leqslant 1$ for all $i, j = 1, \dots, k$. Here and elsewhere $\mathbf{1}$ denotes an appropriate vector with each co-ordinate equal to 1, and so $\mathbf{1} \cdot x$ is the co-ordinate sum of the vector x. Note that if the family $x^1, \dots, x^k$ is equitable and if $ka \leqslant \sum_i x^i \leqslant kb$ where a and b are integral vectors, then $a \leqslant x^i \leqslant b$ for each $i = 1, \dots, k$.

Let us say that a polyhedron P in $\mathbb{R}^n$ satisfies *equitable weak integral decomposition* (equitable WID) if whenever an integral vector is the sum of k vectors in P it is the sum of an equitable family of k integral vectors in P. We define *nearly uniform weak integral decomposition* (nearly uniform WID) analogously. It is straightforward to check the following proposition.

Proposition 1.1. *A polyhedron P in $\mathbb{R}^n$ satisfies equitable WID if and only if $P \cap B$ satisfies WID for every 'integral box' B of the form $\{x \in \mathbb{R}^n\colon a \leqslant x \leqslant b\}$ where a and b are integral. And P satisfies nearly uniform WID if and only if $P \cap B$ satisfies WID for every 'generalised integral box' B of the form $\{x \in \mathbb{R}^n\colon a \leqslant x \leqslant b, l \leqslant \mathbf{1} \cdot x \leqslant m\}$ where a and b are integral, and l and m are integers.*

Let us say now that a polyhedron P in $\mathbb{R}^n$ satisfies equitable MID if the intersection of P with any integral box $\{x \in \mathbb{R}^n : a \leqslant x \leqslant b\}$ satisfies MID, and let us define nearly uniform MID analogously. Further, let us say that P satisfies equitable SID if the intersection of the polyhedron $Q_k(c)$ with any integral box in $\mathbb{R}^{kn}$ is always integral, and let us define nearly uniform SID analogously. If P satisfies equitable SID, then of course it satisfies equitable MID, and so on.

The plan of the paper is as follows. In Section 2 we sketch the connection between the weak integral decomposition property WID and certain rounding properties for packing and covering problems. In Section 3 we show that integral polymatroids satisfy nearly uniform SID. In Section 4 we investigate polyhedra obtained from totally unimodular matrices, and show that although they satisfy equitable MID they do not necessarily satisfy SID. We also comment briefly on network flows. Then in Section 5 we consider the intersection P of the polyhedra of the two strongly-base-orderable matroids, and find that P satisfies nearly uniform WID but not necessarily MID. Indeed the example in [4] of a polyhedron which satisfies WID but not MID may be represented as a polyhedron P here. In Section 6 we consider polyhedra arising from branchings and find that they satisfy nearly uniform WID but need not satisfy MID. Finally in Section 7 we investigate when a polyhedron satisfying WID is integral and conversely. We do not consider algorithms here, but the reader is referred to the recent paper by Orlin [22].

2. Weak integral decomposition and rounding

We sketch here the connection between the weak integral decomposition property WID and certain rounding properties for packing and covering problems. We follow the treatment of Baum and Trotter [4].

Let M be an $(m \times n)$ matrix of non-negative rationals. We say that the *integer round-down* property (IRD) holds for M if for any integral vector $\boldsymbol{c}$ in $\mathbb{R}_+^n$ (that is, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{c} \geq \boldsymbol{0}$) the optimal value of the integral packing problem

$$\max\{\mathbf{1} \cdot y \colon yM \leq \boldsymbol{c},\ y \geq \boldsymbol{0},\ y \text{ integral}\}$$

is obtained by rounding down to the nearest integer the optimal value of the linear programming relaxation (where we drop the constraint that y be integral). Similarly, we say that the *integer round-up* property (IRU) holds for M if for any integral vector $\boldsymbol{c}$ in $\mathbb{R}_+^n$ the optimal value of the integral covering problem

$$\min\{\mathbf{1} \cdot y \colon yM \geq \boldsymbol{c},\ y \geq \boldsymbol{0},\ y \text{ integral}\}$$

is obtained by rounding up the optimal value of its linear programming relaxation. Packing and covering problems such as those above arise naturally in combinatorial optimisation, and many instances in which IRU or IRD hold have been studied (see [4]).

We need two more definitions. A polyhedron P in $\mathbb{R}_+^n$ is called *upper comprehensive* if $\boldsymbol{x} \in P,\ y \geq \boldsymbol{x} \Rightarrow y \in P$; and P is called *lower comprehensive* if $\boldsymbol{x} \in P,\ \boldsymbol{0} \leq y \leq \boldsymbol{x} \Rightarrow y \in P$. Note that blocking polyhedra are upper comprehensive and anti-blocking polyhedra are lower comprehensive and bounded.

Theorem 2.1 [4]. (a) *Let P be an upper comprehensive integral polyhedron in $\mathbb{R}_+^n$ which is non-empty and not equal to $\mathbb{R}_+^n$. Let the rows of matrix M be the minimal integral vectors of P. Then IRD holds for M if and only if P satisfies the weak integral decomposition property WID.*

(b) *Let P be a lower comprehensive integral polyhedron in $\mathbb{R}_+^n$ which is bounded and has non-empty interior. Let the rows of matrix M be the maximal integral vectors of P. The IRU holds for M if and only if P satisfies the weak integral decomposition property WID.*

3. Integral polymatroids

An *integral polymatroid* is a particularly well-behaved sort of polytope. It may be defined [8] as a polyhedron $P = P(E, f)$ of the form

$$\left\{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{x} \geq \boldsymbol{0},\ \sum_{i \in S} x_i \leq f(S)(S \subseteq E)\right\}$$

where $E = \{1, 2, \ldots, n\}$, and f is a non-negative integer-valued function defined on the subsets of E which is non-decreasing and submodular; that is for all $R, S \subseteq E$

$$R \subseteq S \Rightarrow f(R) \leq f(S), \quad f(R)+f(S) \geq f(R \cup S)+f(R \cap S).$$

A well-known instance of an integral polymatroid is obtained by taking f as the rank function of a matroid on E, in which case $P(E, f)$ is the familiar 'matroid polyhedron', whose extreme points are the incidence vectors of the independent sets of the matroid. It is also the case that in general integral polymatroids are integral polytopes. For further definitions examples and results useful below see [8, 17, 19, 21, 27].

The fact that integral polymatroids satisfy the weak integral decomposition property WID was noted in [19] (see also [27]) and in [15]. In [4] it is shown further that they satisfy the middle integral decomposition property MID. The following result (with each $P_i = P$) shows that an integral polymatroid P actually satisfies the strong integral decomposition property SID.

Theorem 3.1. *Let $P_1, \ldots, P_k$ be integral polymatroids in $\mathbb{R}^n$ and let $\boldsymbol{c}$ be an integral vector in $\mathbb{R}^n$. Then the polytope*

$$Q=\left\{(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \in P_i (i=1, \ldots, k), \sum_i \boldsymbol{x}^i=\boldsymbol{c}\right\}$$

is integral.

Proof. It is easy to see that the polytopes

$$Q_1=\{(x^1, \ldots, x^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \in P_i (i=1, \ldots, k)\}$$

and

$$Q_2=\left\{(x^1, \ldots, x^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \geqslant \mathbf{0} (i=1, \ldots, k), \sum_i \boldsymbol{x}^i \leqslant \boldsymbol{c}\right\}$$

are both integral polymatroids. Hence by Edmonds' intersection theorem [8] the polytope $Q_1 \cap Q_2$ is integral. But the polytope Q is a face of $Q_1 \cap Q_2$ and so Q is integral.

Corollary 3.2. *For $i=1, \ldots, k$ let P_i be an integral polymatroid in $\mathbb{R}^n$, let $\boldsymbol{a}^i$ and $\boldsymbol{b}^i$ be integral vectors in $\mathbb{R}^n$, and let l_i and m_i be positive integers. Let $\boldsymbol{c}$ be an integral vector in $\mathbb{R}^n$. Then the polytope*

$$Q=\left\{(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \in P_i, \boldsymbol{a}^i \leqslant \boldsymbol{x}^i \leqslant \boldsymbol{b}^i, l_i \leqslant \mathbf{1} \cdot \boldsymbol{x}^i \leqslant m_i (i=1, \ldots, k), \sum_i \boldsymbol{x}^i=\boldsymbol{c}\right\}$$

is integral.

Proof. Let $y = (y^1, \dots, y^k)$ be an extreme point of Q. For $i = 1, \dots, k$ define an integral vector $\boldsymbol{d}^i$ in $\mathbb{R}^n$ by setting $d^i_j = a^i_j$ if $y^i_j = a^i_j$ and $d^i_j = b^i_j$ otherwise, and define an integer t_i by setting $t_i = l_i$ if $\mathbf{1} \cdot y^i = l_i$ and $t_i = m_i$ otherwise. Then $\boldsymbol{y}$ is an extreme point of the polyhedron

$$Q' = \left\{(\boldsymbol{x}^1, \dots, \boldsymbol{x}^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \in P_i, \boldsymbol{x}^i \leq \boldsymbol{d}^i, \mathbf{1} \cdot \boldsymbol{x}^i \leq t_i \ (i = 1, \dots, k), \sum_i \boldsymbol{x}^i = \boldsymbol{c}\right\}.$$

For $i = 1, \dots, k$ let P'_i be obtained from P_i by restricting to $\boldsymbol{d}^i$ and truncating to t_i, that is $P'_i = \{\boldsymbol{x} \in P_i : \boldsymbol{x} \leq \boldsymbol{d}^i, \mathbf{1} \cdot \boldsymbol{x} \leq t_i\}$. Then P'_i is an integral polymatroid [21] and

$$Q' = \left\{(\boldsymbol{x}^1, \dots, \boldsymbol{x}^k) \in \mathbb{R}^{kn} : \boldsymbol{x}^i \in P'_i \ (i = 1, \dots, k), \sum_i \boldsymbol{x}^i = \boldsymbol{c}\right\}.$$

Hence by Theorem 3.1 the polytope Q' is integral. Hence the vector $\boldsymbol{y}$ is integral and so the polytope Q is integral, as required.

Corollary 3.3. *An integral polymatroid satisfies nearly uniform SID.*

4. Totally unimodular matrices and network flows

Recall that a matrix is *totally unimodular* if the determinant of every square submatrix equals 0 or ± 1. The following results appear essentially in [24].

Theorem 4.1. *Let the $(m \times n)$ matrix A be totally unimodular and let b be an integral vector in $\mathbb{R}^m$. Then the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} \leq \boldsymbol{b}\}$ satisfies the middle integral decomposition property MID.*

Proof. Let $\boldsymbol{c}$ be an integral vector in $\mathbb{R}^n$ and let k be a positive integer. We must show that the polyhedron $Q = P \cap (\boldsymbol{c} - kP)$ is integral. But a vector $\boldsymbol{x} \in \mathbb{R}^n$ is in Q if and only if

$$A\boldsymbol{x} \leq \boldsymbol{b} \quad \text{and} \quad A\boldsymbol{c} - A\boldsymbol{x} \leq k\boldsymbol{b},$$

that is, if and only if

$$A\boldsymbol{c} - k\boldsymbol{b} \leq A\boldsymbol{x} \leq \boldsymbol{b}.$$

It follows [16, 14] that Q is integral, as required.

Corollary 4.2. *Let the $(m \times n)$ matrix A be totally unimodular and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be integral vectors in $\mathbb{R}^m$ with $\boldsymbol{a} \leq \boldsymbol{b}$. (We may allow $\boldsymbol{a}$ and $\boldsymbol{b}$ to have some co-ordinates infinite.) Then the polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} \leq A\boldsymbol{x} \leq \boldsymbol{b}\}$ satisfies equitable MID.*

Proof. Apply Theorem 4.1 to the totally unimodular $((2m+2n)\times n)$ matrix formed by stacking A, $-A$, I and $-I$ where I is the $(n\times n)$ identity matrix.

By Corollary 4.2 the polyhedron P there of course satisfies equitable WID. This result is essentially due to Baranyai [1, Lemma 3] (see also [30, Lemma 2.1] and generalises the theorem that a unimodular hypergraph has an equitable k-colouring for any positive integer k [29, 5, 30]. The result may also be stated in the following more palatable (?) form.

Corollary 4.3. *Let the $(m\times n)$ matrix A be totally unimodular and let c be an integral vector in $\mathbb{R}^n$. Then for any positive integer k there exists integral vectors $x^1,\dots,x^k$ in $\mathbb{R}^n$ which sum to c and are such that both the families $x^1,\dots,x^k$ and $Ax^1,\dots,Ax^k$ are equitable.*

The following example shows that polyhedra arising as here from totally unimodular matrices need not satisfy the strong integral decomposition property SID. This result is not surprising, for if such polyhedra had satisfied equitable SID rather than just equitable MID then we would have been able to use linear programming in a straightforward manner to solve certain timetabling problems with preassignments which are known [10] to be NP-complete.

Example 4.4. The 2-dimensional (3×3) assignment polytope P is the set of all non-negative vectors $x=(x_{ij}: i,j=1,2,3)$ such that $\sum_i x_{ij}=\sum_j x_{ij}=1$. Thus

$$P\{x\in\mathbb{R}^9: x\geq \mathbf{0},\ Ax=\mathbf{1}\}$$

for a certain totally unimodular (6×9) matrix A, and so P satisfies MID. If P satisfied SID, then the 3-dimensional $(3\times 3\times 3)$ assignment polytope Q would be integral, where Q is the set of non-negative vectors

$$x=(x_{ijk}: i,j,k=1,2,3)\ \text{ such that } \sum_i x_{ijk}=\sum_j x_{ijk}=\sum_k x_{ijk}=1.$$

However, Q is not integral (as is well known). We may specify a non-integral vertex $y=(y_{ijk})$ of Q by giving the three (3×3) matrices $M_k=(y_{ijk}: i,j=1,2,3)$ for $k=1,2,3$:

$$M_1=\begin{pmatrix}1&0&0\\0&\frac12&\frac12\\0&\frac12&\frac12\end{pmatrix};\qquad M_2=\begin{pmatrix}0&\frac12&\frac12\\\frac12&\frac12&0\\\frac12&0&\frac12\end{pmatrix};\qquad M_3=\begin{pmatrix}0&\frac12&\frac12\\\frac12&0&\frac12\\\frac12&\frac12&0\end{pmatrix}.$$

It is easy to check that $y\in Q$, and that if $z\in Q$ and $y_{ijk}=0 \Rightarrow z_{ijk}=0$, then $z=y$. Thus y is a vertex of Q, as required.

Now let us consider network flows. Let D be a directed graph with vertex-set V and edge-set E, and let A be its vertex-edge incidence matrix, which is of course totally unimodular. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be integral vectors in $\mathbb{R}^E$ with $\boldsymbol{a} \leq \boldsymbol{b}$. (We may allow $\boldsymbol{a}$ and $\boldsymbol{b}$ to have some co-ordinates infinite.) A *feasible circulation* in D is a vector f in $\mathbb{R}^E$ satisfying $\boldsymbol{a} \leq f \leq \boldsymbol{b}$ and $Af = \boldsymbol{0}$. By Corollary 4.2 the polyhedron P of feasible circulations in D satisfies equitable MID, and thus of course it also satisfies equitable WID. Some variant of this last result has appeared in each of [13, 19, 26, 31] and no doubt elsewhere.

It is straightforward to adapt Example 4.4 to show that a polyhedron P of feasible circulations need not satisfy the strong integral decomposition property SID. Alternatively this may be shown for the polyhedron of non-negative circulations in the complete graph on four vertices. Note also that the polyhedron P need not satisfy nearly uniform WID—consider for example the directed graph consisting of two disjoint directed cycles, one with two edges and one with four.

We have noted that polyhedra of feasible circulations satisfy MID but need not satisfy SID. However, often we are interested less in circulations than in network flows and in particular in the vectors linked by such flows. Let us use the same notation as above, and say that a *feasible flow* in D is simply a vector f in $\mathbb{R}^E$ such that $\boldsymbol{a} \leq f \leq \boldsymbol{b}$. We say that two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^V$ are *linked* if there is a feasible flow f with $Af = \boldsymbol{x} - \boldsymbol{y}$.

Theorem 4.5. *The polyhedron P in $\mathbb{R}^V \times \mathbb{R}^V$ of linked pairs of vectors satisfies equitable SID.*

Proof. Let k be a positive integer and let $(\boldsymbol{x}^0, \boldsymbol{y}^0)$ be a fixed integral vector in kP. For $i = 1, \dots, k$ let $(\boldsymbol{c}^i, \boldsymbol{d}^i)$ and $(\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{d}}^i)$ be integral vectors in $\mathbb{R}^V \times \mathbb{R}^V$ with $(\boldsymbol{c}^i, \boldsymbol{d}^i) \leq (\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{d}}^i)$. Let the polyhedron Q consist of all k-tuples of vectors $(\boldsymbol{x}^i, \boldsymbol{y}^i)$ in P with $(\boldsymbol{c}^i, \boldsymbol{d}^i) \leq (\boldsymbol{x}^i, \boldsymbol{y}^i) \leq (\hat{\boldsymbol{c}}^i, \hat{\boldsymbol{d}}^i)$ which sum to $(\boldsymbol{x}^0, \boldsymbol{y}^0)$. We must show that Q is integral.

Form a directed graph D' as follows. Start with k disjoint copies of D. For each vertex v of D, add a vertex v^+ together with an edge (v^+, v^i) from v^+ to each of the k copies v^i of v, and add a vertex v^- together with the k edges (v^i, v^-). Finally add a vertex s together with all the edges (s, v^+) and (v^-, s). Let E' be the edge set of D', and define integral vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$ in $\mathbb{R}^{E'}$ as follows. Let $a'_{e'} = a_e$ and $b'_{e'} = b_e$ if e' is a copy of the edge e in D; let $a'_{(v^+, v^i)} = c^i_v$ and $b'_{(v^+, v^i)} = \hat{c}^i_v$; let $a'_{(v^i, v^-)} = d^i_v$ and $b'_{(v^i, v^-)} = \hat{d}^i_v$; and let $a'_{(s, v^+)} = b'_{(s, v^+)} = x^0_v$ and $a'_{(v^-, s)} = b'_{(v^-, s)} = y^0_v$.

Then the polyhedron Q is the projection onto the co-ordinates corresponding to the edges (v^+, v^i) and (v^i, v^-) of the polyhedron of circulations f' in D' with $\boldsymbol{a}' \leq f' \leq \boldsymbol{b}'$; and so Q is integral, as required.

The polyhedron P in the theorem above is an example of an 'integral polylinking system' [23], and the theorem actually holds for such polyhedra. This result may be deduced from the results of Section 3 or proved along similar lines.

5. Strongly-base-orderable matroids

We shall identify a matroid (see for example Welsh [27]) with its collection of independent sets. Given a family $\mathcal{A}$ of subsets of a set E the associated *transversal matroid* is the collection of partial transversals of $\mathcal{A}$. A *strongly-base-orderable matroid* on a set E may be defined as a non-empty collection $\mathcal{M}$ of subsets of E such that given any two sets A, $B \in \mathcal{M}$ there is a transversal matroid $\mathcal{I}$ such that A, $B \in \mathcal{I} \subseteq \mathcal{M}$. Thus a strongly-base-orderable matroid is a 'locally transversal' matroid, and any transversal matroid is of course strongly-base-orderable.

Suppose that we have a directed graph with vertex set V, and two subsets X and Y of V. The associated *gammoid* on X is the collection of subsets of X which are linked by vertex-disjoint paths to subsets of Y. Any transversal matroid is a gammoid, but not conversely. However any gammoid is still strongly-base-orderable [6, 27].

Now Corollary 4.3 together with a straightforward graph construction will show that the intersection of the polyhedra of two gammoids satisfies nearly uniform WID [25, 18]. However, one may use results from [7] together with a replication argument as in [20] to prove the following stronger theorem.

Theorem 5.1. *The intersection P of the polyhedra of two strongly-base-orderable matroids satisfies nearly uniform WID.*

In [20] this theorem is used to obtain various results on integral and real covering and packing with sets independent in two strongly-base-orderable matroids. These results extend the work of Weinberger [25] on transversal matroids. The example in [7] shows that if one of the matroids in Theorem 5.1 is not strongly-base-orderable then we may not have the weak integral decomposition property WID.

We have seen in Theorem 5.1 that the intersection P of the polyhedra of two strongly-base-orderable matroids satisfies WID. We note below that the polyhedron in [4] which satisfies WID but not MID may be represented as the intersection of the polyhedra of two transversal matroids. Thus our polyhedron P need not satisfy the middle integral decomposition property MID.

Example 5.2. Let the collection $\mathcal{C}$ of subsets of $\{1, 2, \dots, 6\}$ consist of the five sets $\{1, 3, 4\}$, $\{1, 2, 5\}$, $\{2, 3, 6\}$, $\{4, 5, 6\}$ and $\{1, 2, 3\}$ together with all their subsets, and let the polyhedron P in $\mathbb{R}^6$ be the convex hull of their incidence vectors. Following [4] we note that the vector $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ is in the polyhedron $Q = P \cap (\mathbf{1} - P)$ but the only integral vectors in Q are $(1, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1, 1)$. Thus Q is not integral and so P does not satisfy MID.

But now consider the transversal matroids M_1 and M_2 of the families $\{1, 6\}$, $\{2, 4\}$, $\{3, 5\}$ and $\{2, 3, 6\}$, $\{1, 3, 4\}$, $\{1, 2, 5\}$ respectively. The common independent sets of M_1 and M_2 form precisely the collection $\mathscr{C}$, and so by Edmond's intersection theorem the polyhedron P is the intersection of the polyhedra of M_1 and M_2.

6. Branchings

Let G be a directed graph with edge-set E. A *branching* in G is a set of edges which forms a forest in the underlying undirected graph (that is, contains no cycles) and is such that no two edges are directed towards the same vertex. A branching is said to be *rooted* at a vertex r if each vertex other than r has an edge directed towards it.

Now suppose that there is a branching in G rooted at r, and let the matrix A have as rows the (0, 1)-incidence vectors of the branchings in G rooted at r. Results of Edmonds [8, 9] show that if c is an integral vector in $\mathbb{R}_+^E$, then the (packing) linear program

$$\max\{\mathbf{1}\cdot y : yA \leqslant c, y \geqslant \mathbf{0}\}$$

always has an integral optimal vector y. (Indeed if the matrix C has as rows the incidence vectors of the minimal cuts in G rooted at r, then both the pairs (A, C) and (C, A) satisfy the strong min–max equality [11, 12].) But for the matrix B with as rows the incidence vectors of all the branchings in G the corresponding packing linear program as above does not necessarily have an integral optimal vector—consider for example the graph with three edges forming a directed cycle [11].

Now let the polyhedra $P_r(G)$ and $P(G)$ in $\mathbb{R}^E$ be the convex hulls of the rows of A and of B respectively. Since each rooted branching has the same size it follows easily from the above that $P_r(G)$ satisfies WID—note that if $c \in kP_r(G)$, then the maximum value in the packing problem above is k. (Further $P_r(G)$ must of course satisfy nearly uniform WID.) Baum and Trotter [4] have shown that the polyhedron $P(G)$ also satisfies WID. (They attribute their proof to R. Giles.) We show below that $P(G)$ actually satisfies nearly uniform WID, but that neither $P_r(G)$ nor $P(G)$ need satisfy MID.

Example 6.1. Let G be the directed graph in Fig. 1(a), with edges labelled $1, \ldots, 9$. Let the polyhedron $P = P(G)$ in $\mathbb{R}^9$ be the convex hull of the incidence vectors of the branchings in G. Let y be the vector $(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2})$ in $\mathbb{R}^9$—see Fig. 1(b). We shall show that y is an extreme point of $P \cap (\mathbf{1} - 2P)$.

Now $\{1, 4, 5\}$ and $\{2, 8, 9\}$ are branchings in G, and y is half the sum of their incidence vectors; and thus $y \in P$. Also $\{1, 3, 6\}$, $\{2, 3, 5\}$, $\{4, 7, 8\}$ and $\{6, 7, 9\}$ are branchings in G, and $\mathbf{1} - y$ is half the sum of their incidence vectors; and thus $\mathbf{1} - y \in 2P$. Hence $y \in P \cap (\mathbf{1} - 2P)$.

The following nine inequalities are satisfied by each vector x in $P \cap (\mathbf{1} - 2P)$,

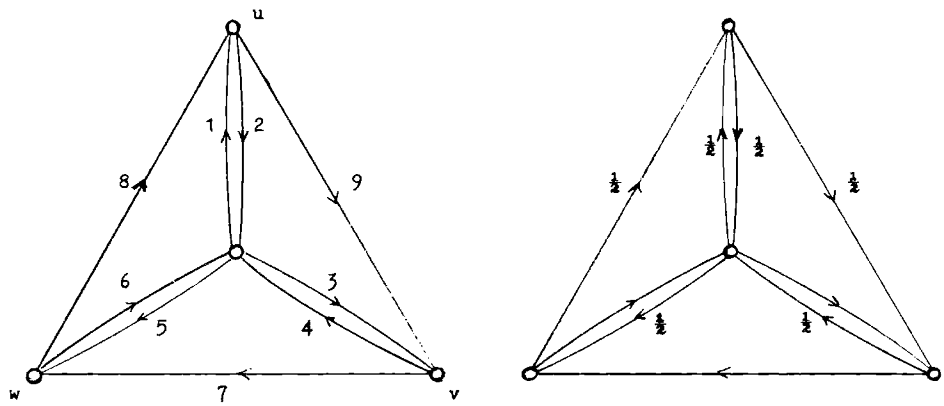

Fig. 1(a) The directed graph G; (b) the vector y.

and are satisfied at equality by the vector y.

$$x_3 \geq 0, \quad x_6 \geq 0, \quad x_7 \geq 0,$$
$$x_1 + x_2 \leq 1, \quad x_2 + x_4 \leq 1, \quad x_1 + x_8 \leq 1,$$
$$x_1 + x_2 + x_4 + x_9 \leq 2,$$
$$x_1 + x_2 + x_5 + x_8 \leq 2,$$
$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 1.$$

Further it is routine to check that the (9×9) matrix of coefficients is non-singular. Hence the non-integral vector y is an extreme point of $P \cap (\mathbf{1} - 2P)$. Thus the polyhedron $P(G)$ does not satisfy the middle integral decomposition property MID.

Let G' be obtained from G by adding a new vertex r together with edges labelled 10, 11 and 12 to the three outer vertices u, v and w of G. Extend y to a vector in $\mathbb{R}^{12}$ by setting $y_{10} = 0$ and $y_{11} = y_{12} = \frac{1}{2}$. Then we may check as above that y is an extreme point of the polyhedron $P' \cap (\mathbf{1} - 2P')$, where $P' = P_r(G')$. Thus the polyhedron $P_r(G')$ does not satisfy MID.

It remains now to show that the polyhedron $P(G)$ generated by the branchings in G satisfies nearly uniform WID, that is, if an integral vector c is the sum of k vectors in $P(G)$, then there exist k branchings $B_1, \ldots, B_k$ in G such that c is the sum of their incidence vectors, and $|B_i| - |B_j| \leq 1$ for $i, j = 1, \ldots, k$. This follows from the result above that $P(G)$ satisfies WID, together with the Lemma 6.2.

Lemma 6.2. *Let A and B be branchings in a directed graph G and suppose that $|A| < |B|$. Then there exist sets $X \subseteq A \setminus B$ and $Y \subseteq B \setminus A$ with $|Y| = |X| + 1$ such that both $(A \setminus X) \cup Y$ and $(B \setminus Y) \cup X$ are branchings.*

Proof. Clearly we may assume that the head of each edge b in B is incident with an edge in A, for otherwise we could simply transfer b to A. Also we may assume that A and B are disjoint, for otherwise we may consider the branchings $A \setminus B$ and $B \setminus A$ in the directed graph obtained by contracting the edges in $A \cap B$.

Consider a component K of the undirected graph corresponding to the branching A. Suppose that K has k edges and thus has $k+1$ vertices. Then the number of edges in B with their heads in K is at most $k+1$. Since $|B| > |A|$ there must be some component K with $|Y| = |X| + 1$, where X is the set of edges in A in K and Y is the set of edges in B with their heads in K.

We may complete the proof by showing that both the sets $A' = (A \setminus X) \cup Y$ and $B' = (B \setminus Y) \cup X$ are branchings in G. But it is immediate from their definitions that in each set no two edges can be directed towards the same vertex. Further any cycle C is the undirected graph corresponding to $A \cup B$ must contain an edge b in B followed by an edge in A incident with the head of b; and so C is not contained in A' or in B'. Hence A' and B' are both branchings in G, as required.

7. Integral polyhedra and weak integral decomposition

Consider a polyhedron P in $\mathbb{R}^n$. If P satisfies the weak integral decomposition property WID then we might expect P to be integral, and conversely. For example it is easy to see that if P satisfies WID and contains a rational point, then it must contain an integral point; and if P satisfies WID, then so does every face of P. Thus if P satisfies WID and has 'enough' rational points it must be integral. In this section we shall see that if a pointed polyhedron P satisfies WID, then its 'rationally spanned part' (defined below) must be integral. After that we show conversely that any integral polyhedron of dimension at most two satisfies WID.

Let C be a convex set in $\mathbb{R}^n$ and let Q be the set of rational points in C. Let us define the *rationally spanned part* rats(C) of C to be $C \cap \bar{Q}$ where $\bar{Q}$ denotes the closure of Q. For example if the polyhedron P in $\mathbb{R}^3$ is the convex hull of the vectors $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 1, \sqrt{2})$, then rats(P) is the convex hull of the first two vectors. The following result shows that the rationally spanned part of a polyhedron is another polyhedron.

Proposition 7.1. *Let C be a convex set in $\mathbb{R}^n$ and let Q be the set of rational points in C. Then*

$$C \cap \bar{Q} = C \cap \operatorname{aff}(Q),$$

where aff(Q) *denotes the affine hull of Q.*

Proof. Since aff Q is closed and contains Q it is immediate that $C \cap \operatorname{aff} Q \supseteq C \cap \bar{Q}$. Conversely, suppose that the convex set $D = C \cap \operatorname{aff} Q$ is non-empty and let

$x \in D$. We must show that $x \in \bar{Q}$. Note first that since $Q \subseteq D \subseteq \text{aff}\, Q$ we have $\text{aff}\, D = \text{aff}\, Q$. Let $\epsilon > 0$. Then there exists a point y in the relative interior of D with $\|x - y\| < \frac{1}{2}\epsilon$; and there exists a $\delta > 0$, $\delta < \frac{1}{2}\epsilon$ such that

$$z \in \text{aff}\, D, \qquad \|y - z\| < \delta \;\Rightarrow\; z \in D.$$

But since $y \in \text{aff}\, Q$ certainly there is a rational point z in $\text{aff}\, Q = \text{aff}\, D$ with $\|y - z\| < \delta$. But now $z \in D$ and so $z \in Q$, and $\|x - z\| < \epsilon$. Hence $x \in \bar{Q}$, as required.

Now if P is a polyhedron and k is a positive integer clearly the polyhedra kP and $k(\text{rats}(P))$ contain precisely the same integral points. Hence we have the following.

Proposition 7.2. *A polyhedron P in $\mathbb{R}^n$ satisfies WID if and only if its rationally spanned part* rats(P) satisfies WID.

Let us say that a convex set C in $\mathbb{R}^n$ is *rationally spanned* if it equals its rationally spanned part rats(C); that is, if the set Q of rational points in C is dense in C, or equivalently if C is contained in the affine hull of Q. For example, any convex set with non-empty interior is rationally spanned. Also, the convex hull of any set of rational points is rationally spanned, and thus so is any polyhedron P specified by a finite number of rational constraints, since clearly we could assume that P is bounded. In particular any (bounded) integral polytope is rationally spanned. Also of course the rationally spanned part of any convex set is rationally spanned. By Proposition 7.2 we may from now on restrict our attention to rationally spanned polyhedra.

Consider then a rationally spanned polyhedron P which satisfies the weak integral decomposition property WID. Must P be integral? Proposition 7.3 shows that if P is pointed (that is if P has an extreme point), then indeed P must be integral; and Example 7.5 shows that if P is not pointed, then it need not be integral. Proposition 7.3 is related to the converse parts of [2, Theorem 2] and [30, Theorem 2.2].

Proposition 7.3. *Let the pointed polyhedron P be rationally spanned and satisfy the weak integral decomposition property WID. Then P is integral.*

Proof. Suppose that some extreme point x of P is not integral. Then there is a non-zero vector a such that

$$a \cdot x > b = \max\{a \cdot z \colon z \in P, z \text{ integral}\}.$$

Since P is rationally spanned there is a rational vector $y \in P$ such that $a \cdot y > b$. Choose a positive integer k such that ky is integral. Then since P satisfies WID there exist integral vectors $x^1, \ldots, x^k$ in P such that $ky = \sum_i x^i$. But now

$$kb < a \cdot ky = \sum_i a \cdot x^i \leq kb,$$

a contradiction.

From Propositions 7.3 and 1.1 we have immediately the following.

Proposition 7.4. *If a rationally spanned polyhedron satisfies the equitable weak integral decomposition property, then it must be integral.*

Example 7.5. We give an example of a rationally spanned polyhedron P which satisfies WID but is not integral. Of course by Propositions 7.3 and 7.4 the polyhedron has no extreme points and does not satisfy equitable WID. We let P be the half space in $\mathbb{R}^2$

$$P = \{(x, y) \in \mathbb{R}^2 \colon x + \sqrt{2}y \leq \sqrt{3}\}.$$

Clearly P is rationally spanned and is not integral. Let k be a positive integer and let a and b be integers with $a + \sqrt{2}b \leq k\sqrt{3}$. In order to show that P satisfies WID it suffices for us to show that the inequalities

$$a + \sqrt{2}b - (k-1)\sqrt{3} \leq x + \sqrt{2}y \leq \sqrt{3}$$

have a solution x, y in integers. But this is true since $a + \sqrt{2}b - (k-1)\sqrt{3} < \sqrt{3}$ and the numbers $x + \sqrt{2}y$ for x, y integers are dense in $\mathbb{R}$.

It remains for us to prove the (partial) converse implication that any integral polyhedron of dimension at most 2 must satisfy WID. Note first the following simple example.

Example 7.6. Let P be the convex hull of the vectors (0, 0, 0), (1, 1, 0), (0, 1, 1) and (1, 0, 1) in $\mathbb{R}^3$. Then (1, 1, 1) is in $2P$ but it is not the sum of two integral vectors in P. Thus the 3-dimensional integral polytope P does not satisfy WID.

Proposition 7.7. *Any integral polyhedron P of dimension at most 2 satisfies WID.*

Proof. It is sufficient to prove the result for a polyhedron of the form

$$Q_1 = \mathrm{conv}(\boldsymbol{a}, \boldsymbol{b}) + C$$

and one of the form

$$Q_2 = \mathrm{conv}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are integral vectors (not necessarily distinct) and C is cone (perhaps $C = \{\boldsymbol{0}\}$); for by Carathéodory's theorem, given a vector $\boldsymbol{x}$ in P there is a polyhedron Q of the form Q_1 or Q_2 such that $\boldsymbol{x} \in Q \subseteq P$.

Consider first the polyhedron Q_1. Let k be an integer ≥ 2 and let $\boldsymbol{x}$ be an integral vector in kP. Thus

$$x = \alpha a + \beta b + c$$

for some α, $\beta \geq 0$, $\alpha + \beta = k$ and some $c \in C$. But then either α or β is at least 1 and so either $x - a$ or $x - b$ is in $(k-1)P$, and we may complete the proof by induction.

Now consider the polyhedron Q_2. By the above we may assume that a, b and c are distinct and indeed are affinely independent. Also by translating if necessary we may assume that $c = 0$. Suppose that Q_2 does not satisfy WID and further that of all such polyhedra Q_2 contains the least number of integral vectors. Let the integer $k \geq 2$ be minimal such that some vector x in kQ_2 cannot be decomposed as required. Note that $x = \alpha a + \beta b$ for some α, $\beta \geq 0$ with $\alpha + \beta \leq k$. Clearly $\alpha < 1$ and $\beta < 1$ since otherwise we would contradict the minimality of k. Further $\alpha + \beta > 1$. Now consider the integral vector $y = a + b - x = (1-\alpha)a + (1-\beta)b$. Then $y \in Q_2$ and $y \neq 0, a, b$. Thus each of the three integral polyhedra $P_1 = \text{conv}(y, a, b)$, $P_2 = \text{conv}(y, a, 0)$ and $P_3 = \text{conv}(y, b, 0)$ contains fewer integral points than Q_2 and so must satisfy WID. But $Q_2 = P_1 \cup P_2 \cup P_3$ and so Q_2 satisfies WID, which is a contradiction.

References

[1] Z. Baranyai, "The edge-coloring of complete hypergraphs, I", *Journal of Combinatorial Theory* B26 (1979) 276–294.

[2] S. Baum and L.E. Trotter, "Integer rounding and polyhedral decomposition for totally unimodular systems", in: R. Henn, B. Korte and W. Oettli, eds., *Arbeitstagung über Operations Research und Optimierung* (Springer, Berlin, 1978) pp. 15–23.

[3] S. Baum and L.E. Trotter, "Finite checkability for integer rounding properties in combinatorial programming problems", *Mathematical Programming* (to appear).

[4] S. Baum and L.E. Trotter, "Integer rounding for polymatroid and branching optimisation problems", *SIAM Journal on Algebraic and Discrete Methods* (to appear).

[5] C. Berge, *Graphs and hypergraphs* (North-Holland, Amsterdam, 1973).

[6] R.A. Brualdi, "Induced matroids", *Proceedings of the American Mathematical Society* 29 (1971) 213–221.

[7] J. Davies and C. McDiarmid, "Disjoint common transversals and exchange structures", *Journal of the London Mathematical Society* 14(2) (1976) 55–62.

[8] J. Edmonds, "Submodular functions, matroids, and certain polyhedra", in: R. Guy, H. Hanani, N. Sauer and J. Schonheim, eds., *Combinatorial structures and their applications* (Gordon and Breach, New York, 1970) pp. 69–87.

[9] J. Edmonds, "Edge-disjoint branchings", in: R. Rustin, ed., *Combinatorial algorithms* (Algorithmics Press, New York, 1972) pp. 91–96.

[10] S. Even, A. Itai and A. Shamir, "On the complexity of timetable and multicommodity flow problems", *SIAM Journal on Computing* 5 (1976) 691–703.

[11] D.R. Fulkerson, "Blocking and anti-blocking pairs of polyhedra", *Mathematical Programming* 1 (1971) 168–194.

[12] D.R. Fulkerson, "Packing rooted directed cuts in a weighted directed graph", *Mathematical Programming* 6 (1974) 1–13.

[13] D.R. Fulkerson and D.B. Weinberger, "Blocking pairs of polyhedra arising from network flows", *Journal of Combinatorial Theory* 18 (1975) 265–283.

[14] R.S. Garfinkel and G.L. Nemhauser, *Integer programming* (Wiley, New York, 1972).
[15] F.R. Giles, "Submodular functions, graphs and integer polyhedra", Ph.D. thesis, University of Waterloo, Waterloo, Ontario, Canada (1975).
[16] A.J. Hoffman and J.G. Kruskal, "Integral boundary points of convex polyhedra", in: H.W. Kuhn and A.W. Tucker, eds., *Linear inequalities and related systems*, Annals of Mathematics Studies 38 (Princeton University Press, Princeton, NJ, 1956) pp. 223–246.
[17] E.L. Lawler, *Combinatorial optimization: Networks and matroids* (Holt, Rinehart and Winston, New York, 1976).
[18] M.D. McDaniel, "Network models for linear programming problems with integer rounding properties", M.S. thesis, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY (1981).
[19] C.J.H. McDiarmid, "Rado's theorem for polymatroids", *Mathematical Proceedings of the Cambridge Philosophical Society* 78 (1975) 263–281.
[20] C.J.H. McDiarmid, "On pairs of strongly-base-orderable matroids", Tech. Rep. No. 283, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY (1976).
[21] C.J.H. McDiarmid, "Blocking, anti-blocking and pairs of matroids and polymatroids", *Journal of Combinatorial Theory* B25 (1978) 313–325.
[22] J.B. Orlin, "A polynomial algorithm for integer programming covering problems satisfying the integer round-up property", Sloan School of Management Tech. Rept., MIT, Cambridge, MA (1980).
[23] A. Schrijver, "Matroids and linking systems", *Mathematical Centre Tracts* 88, Mathematisch Centrum, Amsterdam (1977).
[24] L.E. Trotter and D.B. Weinberger, "Symmetric blocking and anti-blocking relations for generalized circulations", *Mathematical Programming Study* 8 (1978) 141–158.
[25] D.B. Weinberger, "Transversal matroid intersections and related packings", *Mathematical Programming* 11 (1976) 164–176.
[26] D.B. Weinberger, "Network flows, minimum coverings, and the four-color conjecture", *Operations Research* 24 (1976) 272–290.
[27] D.J.A. Welsh, *Matroid theory* (Academic Press, London, 1976).
[28] D. de Werra, "Balanced schedules", *INFOR* 9 (1971) 230–237.
[29] D. de Werra, "Equitable colorations of graphs", *Revue Française d'Informatique et de Recherche Opérationnelle* R-3 (1971) 3–8.
[30] D. de Werra, "On some characterisations of totally unimodular matrices", *Mathematical Programming* 20 (1981) 14–21.
[31] D. Woodwall, "The induction of matroids by graphs", *Journal of the London Mathematical Society* 10(2) (1975) 27–35.