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NONDIFFERENTIABLE REVERSE CONVEX PROGRAMS AND FACETIAL CONVEXITY CUTS VIA A DISJUNCTIVE CHARACTERIZATION

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Disjunctive Programs can often be transcribed as reverse convex constrained problems with nondifferentiable constraints and unbounded feasible regions. We consider this general class of nonconvex programs, called Reverse Convex Programs (RCP), and show that under quite general conditions, the closure of the convex hull of the feasible region is polyhedral. This development is then pursued from a more constructive standpoint, in that, for certain special reverse convex sets, we specify a finite linear disjunction whose closed convex hull coincides with that of the special reverse convex set. When interpreted in the context of convexity/intersection cuts, this provides the capability of generating any (negative edge extension) facet cut. Although this characterization is more clarifying than computationally oriented, our development shows that if certain bounds are available, then convexity/intersection cuts can be strengthened relatively inexpensively.

Key words: Disjunctive programming, convexity/intersection cuts, facet inequalities, reverse convex programs.

1. Introduction

The paper considers mathematical programs in the presence of reverse convex constraints. By a reverse convex constraint, we mean a constraint $g(x)\leq 0$, where $g(\cdot)$ is a real valued concave function. The earliest work on problems with such constraints may be traced to Rosen (1966), Avriel and Williams (1970), Dembo (1972) and Avriel (1973). Most of this work was motivated by complementary geometric programs. Specialized reverse convex programs have been investigated by Bansal and Jacobsen (1975a, b) and by Hillestad (1975). The motivation for these papers were economic models with budget constraints in the presence of economies of scale. Perhaps the most signficant contributions for this class of problems appear in Hillestad and Jacobsen (1980a, b), wherein a characterization of optimal solutions and global algorithms for single and multiple reverse convex constrained programs are provided. Assuming differentiability of constraints and compactness of the feasible region, Hillestad and Jacobsen (1980a) show that the convex hull of the feasible region is polyhedral.

The class of problems we will consider may be written as follows:

$$\text{RCP} \qquad \operatorname*{Min}_{x} \{cx \colon x \in S\}$$

where

$$S = \{x \in R^n \colon g_i(x) \leq 0,\ i \in I = \{1, \ldots, p, p+1, \ldots, p+m\}\}, \tag{1.1}$$

and where $g_i : R^n \to R$ are concave functions for $i \in I$, and in particular, for $i = p+1, \ldots, p+m$, $g_i(\cdot)$ are affine functions. We will also assume that the linear constraints include nonnegativity restrictions on the x-variables. When convenient, we will also denote

$$P = \{x \colon g_i(x) \leq 0,\ i = p+1, \ldots, p+m\} \equiv \{x \colon Ax \leq b\}. \tag{1.2}$$

The above form of stating problem RCP with additional linear constraints is quite convenient for a discussion of cutting plane algorithms. Thus $Ax \leq b$ may often be interpreted as containing previously generated cuts.

This paper first exhibits some connections between disjunctive programs (DP), and reverse convex programs (RCP). This connection turns out to be fruitful for both DP and RCP. As for DP, deep cuts can be obtained as a result of viewing them as RCP. On the other hand, several results from DP, especially on convergence of cutting plane methods and derivation of valid inequalities, can be generalized for RCP.

Crucial to the exploitation of the above connection, is a study of nondifferentiable RCP. To appreciate this, note that DP's can be transcribed as RCP's, mainly through the use of nondifferentiable reverse convex constraints. Indeed, a wide class of DP's called Facial Disjunctive Programs can be represented as RCP's by using only one reverse convex constraint. We term problems with one reverse convex constraint ($p = 1$) as Barely RCP (BRCP). Section 2 of the paper motivates the study of nondifferentiable RCP's by illustrating some applications in DP. These connections also serve to illustrate that known instances of nonconvergence in DP can also be utilized for RCP, thus answering some convergence issues raised by Hillestad and Jacobsen (1980a)

The task of generalizing previously known results to the nondifferentiable case is undertaken in Section 3. Under the assumption that g_i are differentiable (actually pseudoconcave), and S is compact, Hillestad and Jacobsen (1980a) have established that the convex hull of S is polyhedral. We show that this result is easily extended to the case of interest, namely problems with nondifferentiable concave functions and an unbounded feasible region. We also generalize certain algebraic concepts introduced by Hillestad and Jacobsen (1980a). In particular, we introduce an appropriate concept of a quasi Basic Feasible Solution (QBFS) under nondifferentiability, which is algebraically characterized through any system of supports (of S)

whose vectors form a matrix of rank n. Some results related to the properties of QBFS are also proven.

The orientation of the paper in Sections 2 and 3 is to study nondifferentiable RCP, with the understanding that these results are also applicable to DP. In Section 4, our outlook is reversed. In this section we show how results from DP may impinge upon RCP. We provide a useful and very clarifying construction that can provide the closure of the feasible region of a particular elementary reverse convex set through a *finite* disjunction. Thus our study also provides new constructions for RCP, that are based on DP. Finally, Section 5 integrates both points of view by examining an application to a general linear complementarity (GLC) problem. A GLC problem, which is basically a DP, is reformulated as an RCP. Consequently, the developments of Section 4 are applicable. The main advantage of this representation is that it permits us to derive cuts that simultaneously consider multiple disjunctions. This is in contrast to the conventional algorithmic use of Balas' Disjunctive Cut Principle (Balas (1975)), wherein cut derivation utilizes only one violated disjunction at a time (Ramarao and Shetty (1984)). Hence, we are able to derive cuts that are stronger than any of the conventional cuts.

2. The unifying role

The earlierst connection between a subclass of disjunctive programs and problem RCP was established by Raghavachari (1969), who observed that a zero-one integer requirement could be replaced by the constraints

$$0 \leqslant x_j \leqslant 1, \qquad g(x) = \sum_j x_j(1 - x_j) \leqslant 0. \tag{2.1}$$

Consequently, valid inequalities that may be generated for BRCP via (2.1) may be used within the framework of implicit enumeration algorithms for zero-one linear integer programs. Moreover, the structural properties for BRCP therefore also apply for this class of problems, and we may also conclude that the class of problems BRCP is at least as difficult as a zero-one linear integer program. The following proposition identifies a large class of disjunctive programs that may be formulated as problem BRCP.

Proposition 2.1. *Let $S = P \cap X$, where P is a convex polyhedron and*

$$X = \bigcap_i \left\{ \bigcup_{h \in Q_i} \{x \colon \alpha^h x \leqslant \beta^h\} \right\},$$

where Q_i are finite index sets, and $i = 1, 2, \ldots, m$. Assume that $P \subseteq \{x \colon \alpha^h x \geqslant \beta^h\}$, for all $h \in Q_i$, and all i. (A DP with this property is called a Facial Disjunctive Program (FDP).) Define

$$g_i(x) = \operatorname*{Min}_{h \in Q_i} \{\alpha^h x - \beta^h\} \quad \textit{and} \quad g(x) = \sum_i g_i(x).$$

Then $S = P \cap \{x \colon g(x) \leqslant 0\}$. Hence every FDP is a BRCP. □

We have illustrated above that while problem BRCP appears to be somewhat unpretentious, it is quite a general nonconvex progam and is consequently at least as difficult as some of its special cases.

Zero-one linear integer programs and generalized linear complementarity problems are important facial disjunctive programs. Thus to model complementarity restictions $x_j y_j = 0$, $x_j \geqslant 0$, $y_j \geqslant 0$, $j \in Q$, one may utilize $\sum_{j \in Q} \text{Min}\{x_j, y_j\} \leqslant 0$. Disjunctive programs that do not possess the facial property required in Proposition 2.1 may of course be formulated by requiring each $g_i(x) \leqslant 0$, where g_i is as defined in the proposition. However for FDP, we recommend the use of Proposition 2.1 in deriving cuts. This is because such a formulation allows one to derive cuts using all disjunctions in the problem. This is in contrast to conventional disjunctive methods for FDP (see for example Ramaro and Shetty (1984)) where each cut is derived by using one violated disjunction. Hence cuts from the BRCP formulation may be expected to be deeper than those obtained by using conventional methods using one disjunction. We illustrate this point in Section 5, by an application to linear complementarity problems.

The above connections also imply certain negative results. For example, Hillestad and Jacobsen (1980a) propose the solution of RCP by a cutting plane algorithm utilizing convexity cuts. They pose the question whether such an algorithm converges for problem BRCP. The examples of Sen and Sherali (1985a), which are discussed in the context of DP, indicate that in the absence of additional hypotheses, the answer to this question is no. Sen and Sherali (1985a) also provide a convergence theorem for DP, which indicates the kind of sufficient conditions that are required to ensure convergence. We note finally that since such diverse nonconvex problems as integer programs, signomial geometric programs etc. can be studied under the umbrella of RCP, this class of problems plays an important unifying role.

3. Characterizations of feasible regions and optimal solutions under nondifferentiability

The properties of the feasible region developed here are applicable to problems in which $g_i(\cdot)$ are simply concave and hence the nondifferentiable functions describing feasible sets of certain important disjunctive and related nonconvex programs are admissible. We also permit unbounded feasible regions. Hence, the results of this section are a generalization of those of Hillestad and Jacobsen (1980a). To account for nondifferentiability our developments utilize some elementary constructions from convex analysis (Rockafellar (1970), Stoer and Witzgall (1970)).

We begin by giving the following lemma which characterizes the essential difference between convex and reverse convex constraints.

Lemma 3.1. *Let $g(x) \leqslant 0$ denote a reverse convex constraint with $g(\cdot)$ concave. Let x^1, x^2 satisfy $g(x^1) = g(x^2) = 0$. Then every affine combination $x(\lambda) = x^1 + (1-\lambda)x^2$, with either $\lambda \geqslant 1$ or $\lambda \leqslant 0$, also satisfies $g(x(\lambda)) \leqslant 0$.*

Proof. If $\lambda \geqslant 1$, then $x^1 = [x(\lambda) + (\lambda - 1)x^2]/\lambda$, so that

$$g(x^1) \geqslant [g(x(\lambda)) + (\lambda - 1)g(x^2)]/\lambda.$$

Now utilizing the fact that $g(x^1) = g(x^2) = 0$, we have $g(x(\lambda)) \leqslant 0$. The case for $\lambda \leqslant 0$ follows in an analogous manner. □

Remark 3.1. The disjunctive nature of reverse convex constraints is quite readily apparent from the condition that $\lambda \geqslant 1$ or $\lambda \leqslant 0$. It is this observation that leads us to consider extensions of disjunctive programming as a methodology for these problems.

In our development, we will utilize the following

Definition 3.1 (Hillestad and Jacobsen (1980a)). Let $X \subseteq R^n$. $\bar{x} \in X$ is said to be a *quasi vertex of* X if $\bar{x}$ cannot be written as a convex combination of two other points in X. Further, $\bar{x} \in X$ is said to be a *quasi local vertex of* X if there is an open ball $B(\bar{x}, \varepsilon)$, $\varepsilon > 0$, such that $\bar{x}$ is a quasi vertex of $B(\bar{x}, \varepsilon) \cap X$.

While the definition of a quasi vertex given above coincides with the definition of an extreme point or a vertex in the case of a convex set, the two are distinguished in that a vertex is a point that cannot be written as a convex combination of any number of points in X (Hillestad and Jacobsen (1980a)).

The next result is a generalization of Hillestad and Jacobsen (1980a). Hillestad and Jacobsen (1980b) provide a similar result for BRCP, where g is quasi-concave. Also to introduce some notation, consider S given by (1.1) and for $\bar{x} \in S$, let $I(\bar{x}) = \{i \in I\colon g_i(\bar{x}) = 0\}$.

Theorem 3.1. *Let $S \subseteq R^n$ be given as in* (1.1). *Then* clconv(S) *is polyhedral.*

Proof. Arguments similar to that in Hillestad and Jacobsen (1980a) can be used to show that the number of quasi-vertices of S are bounded above by 2^{m+p}. Now to account for unboundedness, define $S(M) = S \cap \{x\colon \sum_{i=1}^n x_i \leqslant M\}$. Since S is imbedded in the nonnegative orthant, note that for all $M > 0$, $S(M)$ is compact, and the number of vertices of clconv$[S(M)] \leqslant 2^{p+m+1}$. Consequently, since the extreme points of clconv$[S(M)]$ remain bounded by a *fixed integer* as $M \to \infty$, clconv$[S]$ must have a finite number of extreme directions. Hence, clconv(S) is polyhedral, and the proof is complete. □

Unfortunately, a computationally convenient characterization of the vertices of S is not readily available. However, such a characterization is indeed available for a superset of such points, namely, for the set of quasi local vertices. Toward this end, for any $\bar{x} \in S$ and $i \in I$, let

$$\Omega_i(\bar{x}) = \{\xi \in R^n\colon \xi \text{ is a subgradient of } g_i(\cdot) \text{ at } \bar{x}\}.$$

Furthermore, for $\bar{x} \in S$, define a family of $(m+p) \times n$ matrices as follows:

$\mathcal{G}(\bar{x}) \triangleq \{G$: G is a matrix of size $(m+p) \times n$, i.e. there is one row for every constraint in (1.2); such that for each $i = 1, \ldots, m+p$, G_i, the ith row of G, is some $\xi \in \Omega_i(\bar{x})$ if $i \in I(\bar{x})$, and 0 otherwise.$\}$

Definition 3.2. Let $S \subseteq R^n$. $x \in S$ is said to be a Quasi Basic Feasible Solution (QBFS) if $G \in \mathcal{G}(\bar{x}) \Rightarrow \operatorname{rank}(G) = n$.

The equivalence between QBFS and quasi local vertices is stated below. The proof is fashioned along the basic lines of the proof in Hillestad and Jacobsen (1980a), although the consideration of nondifferentiability requires a somewhat different approach.

Theorem 3.2. *Let $S \subseteq R^n$. $\bar{x} \in S$ is a QBFS iff $\bar{x}$ is a quasi local vertex of S.*

Proof. Although our definition of a QBFS is different from that in Hillestad and Jacobsen (1980a), the proof that $\bar{x}$ is quasi local vertex implies that $\bar{x}$ is a QBFS follows essentially the same arguments. We therefore omit this part of the proof. Hence consider the converse and let $\bar{x}$ be a QBFS of S. If $\bar{x}$ is not a quasi local vertex of S, then there exists a positive real sequence $\{\varepsilon_k\} \to 0$ and associated sequences $\{x_k^1\}$ and $\{x_k^2\}$ in $S \cap B(\bar{x}, \varepsilon_k)$ and $\{\lambda_k\}$ in $(0, 1)$ such that

$$\bar{x} = \lambda_k x_k^1 + (1 - \lambda_k) x_k^2 \quad \text{for all } k.$$

Over some appropriate subsequence, if necessary, let

$$\bar{y} = \lim_{k \to \infty} \frac{x_k^1 - \bar{x}}{\|x_k^1 - \bar{x}\|} = -\lim_{k \to \infty} \frac{x_k^2 - \bar{x}}{\|x_k^2 - \bar{x}\|}.$$

Consider any $i \in I(\bar{x})$. Since $g_i(\cdot)$ is concave, $\Omega_i(\bar{x})$ is a convex, compact set, and $g_i(\bar{x}) = 0$, we obtain that (over appropriate subsequences if necessary)

$$\inf_{\xi \in \Omega_i(\bar{x})} \frac{\xi \bar{y}}{\|\bar{y}\|} = \inf_{\xi \in \Omega_i(\bar{x})} \lim_{k \to \infty} \frac{(x_k^1 - \bar{x})\xi}{\|x_k^1 - \bar{x}\|} = \lim_{k \to \infty} \inf_{\xi \in \Omega_i(\bar{x})} \frac{(x_k^1 - \bar{x})\xi}{\|x_k^1 - \bar{x}\|}$$

$$= \lim_{k \to \infty} - \frac{g_i(x_k^1) - g_i(\bar{x}) - \|x_k^1 - \bar{x}\| \{\text{a function approaching zero as } x_k^1 \to \bar{x}\}}{\|x_k^1 - \bar{x}\|}$$

$$= \lim_{k \to \infty} \frac{g_i(x_k^1)}{\|x_k^1 - \bar{x}\|}.$$

Hence, since $g_i(x_k^1) \leqslant 0$ for all k, the directional derivative of $g_i(\cdot)$ at $\bar{x}$ in the direction $\bar{y}$, which is $\inf\{\xi\bar{y}/\|\bar{y}\|: \xi \in \Omega_i(\bar{x})\} = \xi_i^1\bar{y}/\|\bar{y}\|$, say, must be nonpositive. Similarly, the directional derivation of $g_i(\cdot)$ at $\bar{x}$ in the direction $-\bar{y}$ is nonpositive, i.e., there exists a $\xi_i^2 \in \Omega_i(\bar{x})$ such that $-\xi_i^2 \cdot \bar{y} \leqslant 0$. Consequently, since $\Omega_i(\bar{x})$ is convex, there exists a $G_i \in \Omega_i(\bar{x})$ such that $G_i\bar{y} = 0$. Now using such vectors G_i for all $i \in I(\bar{x})$, and using 0-vectors for $i \notin I(\bar{x})$ in order to construct a $G = \bar{G} \in \mathscr{G}(\bar{x})$, we obtain that rank $\bar{G} < n$ since there exists a $\bar{y} \neq 0$ such that $\bar{G}\bar{y} = 0$. This contradicts that $\bar{x}$ is a QBFS and the proof is complete. □

It is interesting to note that in the nondifferentiable case, an algebraic characterization of quasi local vertices may be obtained by using *any* system of supports $G \in \mathscr{G}(\bar{x})$. For problem BRCP, the quasi local vertices satisfy a very special property. We state this interesting property as a Corollary to the next theorem.

Theorem 3.3. *Let $S \subseteq R^n$, let the matrix A in (1.2) be $m \times n$, $m > n$, and let $|I| = p + m$. Then every quasi local vertex of S belongs to a d-dimensional face of $P = \{x: Ax \leqslant b\}$, where $d = \text{Min}\{p, n\}$.*

Proof. Let $\bar{x} \in S$ be a quasi local vertex. Hence from Theorem 3.2, $G \in \mathscr{G}(\bar{x}) \Rightarrow$ rank $(G) = n$. If $d = n$, there is nothing to prove. If $d = p < n$, then the rank of the last m rows of G cannot be less than $n - d$, for otherwise, rank $(G) < n$. Hence $\bar{x}$ must belong to a d-dimensional face of P. □

Corollary 3.1. *For problem BRCP, every quasi local vertex of S belongs to an edge of the polyhedral set $P = \{x: Ax \leqslant b\}$.*

Proof. Apply Theorem 3.3 with $p = 1$. □

Remark 3.2. Hillestad (1975) proved a special case of Corollary 3.1, and Hillestad and Jacobsen (1980b) generalized this to Corollary 3.1. The algorithms developed in these papers search the edges of the set P in an attempt to solve problem BRCP. We believe that a more fruitful approach to solving BRCP is by combining Corollary 3.1 with branch and bound (B&B) methodology. In particular, B&B methods based on a facial decomposition of P (Sen and Sherali (1985b)) should be quite effective. In the case of the general problem RCP however, B&B principles based on facial decomposition are somewhat more difficult to apply. The reason is as follows. For problem BRCP, when the face corresponding to a particular node (on the B&B tree) is an edge, quasi local vertices (at most two) on the edge are easily enumerated. Hence a simple fathoming procedure becomes available. However, when quasi local vertices belong to $d \geqslant 2$ dimensional faces (Theorem 3.3), these are no longer easily enumerated. Hence fathoming rules are not as straightforward. A more effective branching principle for the general RCP problem is based on cone splitting (Sen and Whiteson (1985)).

4. Valid inequalities and facetial convexity cuts

In this section we depart from the work of Hillestad and Jacobsen (1980a) and address the derivation of valid inequalities for problem RCP and, more importantly, provide a finite disjunctive characterization which is capable of generating all associated facets cuts. As far as a rudimentary strategy for solving RCP is concerned, one may solve the following relaxation LP_ν at iteration ν,

$$\mathrm{LP}_\nu: \qquad \mathrm{Min}\{cx: x \in P^\nu = \{A^\nu x \leq b^\nu\}\}, \tag{4.1}$$

where $P^1 \equiv P = \{x: Ax \leq b\}$ and $P^{\nu+1} = P^\nu \cap \{\alpha^\nu x \geq \alpha_0^\nu\}$, $\nu \geq 1$. Here $\alpha^\nu x \geq \alpha_0^\nu$ is a valid cutting plane that is derived if the solution x^ν to LP_ν does not belong to S. Note that by treating x^ν as the origin, one may derive cutting planes that are valid inequalities for the set

$$T \triangleq \{x: g(x) \leq 0, x \geq 0\}, \tag{4.2}$$

where $g(\cdot)$ is some reverse convex constraint indexed by I, rewritten with respect to x^ν as the origin, such that $g(0) > 0$. Clearly one may attempt to derive stronger inequalities by deriving cuts that are valid for more complicated sets, for example, by utilizing all violated constraints. However, such inequalities will be considerably more difficult to derive. By using a single violated inequality, one can derive the usual convexity cut (see Glover (1973)) by using the following convex set C in the fundamental convexity cut lemma.

$$C = \{x: g(x) \geq 0\}. \tag{4.3}$$

Of course, as in Hillestad and Jacobsen (1980a), the convergence of such an algorithm would be a principal issue. However, as noted earlier, Sen and Sherali (1985a) have recently provided illustrations of nonconvergence, unless some additional properties are satisfied by the procedure.

Below, we define a finite disjunction, which stems from the use of the set C in (4.3) in the spirit of convexity cuts, and which is capable of yielding all related facet inequalities. The *existence* of such a disjunction follows from the fact that clconv(T) has been shown to be polyhedral via Theorem 3.1. However, in order to be able to derive the facet cuts, one needs a more constructive approach which actually specifies such a finite disjunction. Besides primarily being a theoretical contribution, the immediate advantages of this approach over the conventional use of convexity/intersection cuts is that not only do we have access to cuts with negative cut coefficients, but more importantly, the methodology allows a variety of deep cuts to be derived. The former attribute may be regarded as an extension of the polyhedral negative edge extension cut of Glover (1974), and the more recent contribution of Sherali and Shetty (1980b). For a discussion of why such capabilities are important, see Sen and Sherali (1986). A relatively inexpensive cut strengthening scheme for convexity cuts is also validated by our development.

Let us first note that since we assume that the current linear programming solution, which may be considered as the origin in the transformed nonbasic variable space, satisfies $g(0) > 0$, we may represent every support of C in this space by an inequality $\pi x \leq \pi_0$, such that $\pi_0 > 0$. Hence there is no loss of generality in assuming that C can be written as the intersection of (perhaps uncountably infinite) inequalities of the form $\pi x \leq 1$ in the nonbasic variable space. For convenience, we will assume that the index set J of nonbasic variables is simply $\{1, \ldots, n\}$, so that the nonbasic variables are being designated as $x_1, \ldots, x_n$, and all functions and sets are transformed into this nonbasic variable space. Now, for each $j \in J$, determine

$$\beta_j = \mathrm{Sup}\{x_j \colon (0, \ldots, x_j, \ldots, 0) \in C\} \tag{4.4}$$

in the usual manner as with convexity cuts, where the x_j appears in the jth position in $(0, \ldots, x_j, \ldots, 0)$ above. Define

$$J^+ = \{j \in J \colon \beta_j < \infty\}, \quad \bar{J}^+ = J - J^+, \tag{4.5}$$

and note that the usual convexity cut is

$$\sum_{j \in J^+} x_j / \beta_j \geq 1. \tag{4.6}$$

Hence, if $J^+ = \emptyset$, the problem is infeasible; therefore, assume $J^+ \neq \emptyset$. For each $j \in J^+$, let

$$\alpha^j x \leq 1 \tag{4.7}$$

be a supporting hyperplane for C at $(0, \ldots, \beta_j, \ldots, 0)$, derived via any subgradient of $g(\cdot)$ at $(0, \ldots, \beta_j, \ldots, 0)$. Next, for the pairs $j \in \bar{J}^+$ and $k \in J^+$ compute (using the nonbasic variable space representation of C)

$$\beta_{jk} = \mathrm{Inf}\{(-\gamma_j / \gamma_k) \colon \operatorname*{Sup}_{\substack{x \in C \\ x \geq 0}} (\gamma \cdot x) = 1,\ \gamma_k \geq 0,\ \gamma \in R^n\} \quad \text{for } j \in \bar{J}^+,\ k \in J^+. \tag{4.8}$$

In connection with (4.8), first of all observe that the problem is feasible since $\gamma \equiv \alpha^k$ is a candidate solution in (4.8). Second, note that since any feasible solution γ must satisfy $\gamma x \leq 1$ for all $x \in C$, $x \geq 0$, we must have $\gamma_j \leq 0$ or else some point on the ray $(0, \ldots, x_j, \ldots, 0)$, $x_j \geq 0$ will violate $\gamma x \leq 1$, contradicting that $j \in \bar{J}^+$. Consequently, β_{jk} is bounded from below by zero. Hence, if there exists a supporting hyperplane for $C \cap \{x \in R^n \colon x \geq 0\}$ with $\gamma_k > 0$ and $\gamma_j = 0$, then this is optimal in (4.8). Now define

$$Q = \{(j, k) \colon j \in \bar{J}^+,\ k \in J^+ \text{ and an optimum solution exists in (4.8)}\}$$

and let

$$\gamma^{jk} x \leq 1 \quad \text{for all } (j, k) \in Q \tag{4.10}$$

denote the supports of $C \cap \{x \in R^n \colon x \geq 0\}$ which are determined as optimal solutions in (4.8).

The following theorem is the key to deriving facet inequalities for the polyhedral set clconv(T).

Theorem 4.1. *Let T be defined by* (4.2) *and assume that* $x \equiv (x_j, j \in J = \{1, \ldots, n\})$ *represents the nonbasic variables. Consider the disjunctive set D given by*

$$D = \{x \in R^n\colon x \geq 0, \text{ and at least one of the following hold}:$$

$$\alpha^j x \geq 1, \ j \in J^+ \tag{4.11}$$

$$\gamma^{jk} x \geq 1, \ (j, k) \in Q\}, \tag{4.12}$$

where (4.11) *and* (4.12) *are respectively* (4.7) *and* (4.10) *with the inequalities reversed. Then* clconv(T) = clconv(D).

Proof. First observe that clconv(D) $\subseteq$ clconv(T). To see this, note that every x that satisfies any single constraint in (4.11) or (4.12) belongs to T. Hence $D \subseteq T$, so that clconv(D) $\subseteq$ clconv(T).

Next, let us show that clconv(T) $\subseteq$ clconv(D). Note by Corollary 3.1 and the form of T in (4.2), that all quasi local vertices of T must lie on the edges of the nonnegative orthant. Since for $j \in \bar{J}^+$, the entire edge is infeasible to T (the edge being contained in the interior of C), and for $j \in J^+$, since only the section of the edge $(0, \ldots, x_j, \ldots, 0)$ with $x_j \geq \beta_j$ is feasible to T from (4.4), the quasi local vertices of T are of the form $(0, \ldots, \beta_j, \ldots, 0)$, $j \in J^+$. Since $\alpha^j(0, \ldots, \beta_j, \ldots, 0)' = 1$ from (4.7), the quasi local vertices of T belong to D and so, {the vertices of clconv(T)} $\subseteq$ {quasi local vertices of T} $\subseteq D \subseteq$ clconv(D).

Therefore, in order to complete the proof, it is sufficient to show that the recession directions of clconv(T) are also recession directions of clconv(D). Toward this end, consider any supporting hyperplane of the type $\pi x \leq 1$ of $C \cap \{x \in R^n\colon x \geq 0\}$, and let $T_\pi = \{x\colon \pi x \geq 1, x \geq 0\}$. Let $J_\pi^+ = \{j\colon \pi_j > 0\} \subseteq J^+$ and let $\bar{J}_\pi^+ = \{j\colon \pi_j \leq 0\} \subseteq J$. Observe that clconv($T$) is described by the closure convex hull of the union of sets of the type T_π defined over all such possible supports of C. Hence, if $T_\pi = \emptyset$, we may disregard such a set, and to complete the proof, it is sufficient to show that if $T_\pi \neq \emptyset$, then its extreme directions are also recession directions of clconv(D). Observe in this case, denoting e_r as the unit vector $(0, \ldots, 1, \ldots, 0)$ with the 1 in position r, that the extreme directions of T_π are given by

$$d^j = e_j \quad \text{for } j \in J^+, \tag{4.13}$$

$$d^{jk} = (-\pi_j/\pi_k)e_k + e_j \quad \text{for } j \in \bar{J}_\pi^+, \ k \in J_\pi^+. \tag{4.14}$$

Now, for d^j, $j \in J_\pi^+ \subseteq J^+$, $d^j = e_j$ is also an extreme direction of $\{x \geq 0\colon \alpha^j x \geq 1\} \neq \emptyset$ of (4.11), and so, it is a recession direction of clconv(D).

Next, consider the pair $j \in \bar{J}_\pi^+$, $k \in J_\pi^+ \subseteq J^+$. If $j \in J^+$ also, then d^{jk} can be generated by e_k and e_j which are both recession directions of clconv(D) as above. On the other hand, suppose that $j \notin J^+$. Three cases arise in this situation. *Case* (i): $(j, k) \in Q$. In this case, consider the nonempty set $\{x \geq 0\colon \gamma^{jk} x \geq 1\}$ of (4.12). Since $\gamma_j^{jk} \leq 0$,

$\gamma_k^{jk} > 0$ in (4.10), it follows that $D^{jk} = (-\gamma_j^{jk}/\gamma_k^{jk})e_k + e_j$ is a recession direction of this set and hence of clconv(D). However, from (4.8), $0 \leq (-\gamma_j^{jk}/\gamma_k^{jk}) \leq (-\pi_j/\pi_k)$, and since e_k is also a recession direction of clconv(D), once again we obtain that d^{jk} of (4.14) may be generated using D^{jk} and e_k and is therefore a recession direction of clconv(D). *Case (ii)*: $(j, k) \notin Q$, but there exists a $(j, r) \in Q$ such that $\beta_{jr} = 0$ in (4.8). In this case, $\gamma_j^{jr} = 0$ in the inequality $\gamma^{jr}x \geq 1$ in (4.12), and since $\{x \geq 0: \gamma^{jr}x \geq 1\} \neq \emptyset$, e_j is a recession direction of clconv(D). Since e_k is also a recession direction of clconv(D), we obtain as above that so is d^{jk}. *Case (iii)*: $(j, k) \notin Q$ and for any $(j, r) \in Q$, $\beta_{jr} > 0$ in (4.8). Let us show that this case cannot arise. Since $(j, k) \notin Q$, there must exist supports $\gamma x = 1$ of $C \cap \{x \in R^n: x \geq 0\}$ feasible in (4.8) with $\gamma_j = 0$ and furthermore, for all such supports, we must have $\gamma_k = 0$ as well. But by the enunciation of Case (iii), we must in addition have $\gamma_r \leq 0$ for all $r \in J^+$. For if there is $r \in J^+$, such that $\gamma_r > 0$, then we would have $(j, r) \in Q$ with $\beta_{jr} = 0$. Noting that $\gamma_j \leq 0$ for all $j \in \bar{J}^+$, it follows that $\gamma \leq 0$, which contradicts that $\gamma x = 1$ supports $C \cap \{x \in R^n: x \geq 0\}$, and completes the proof.

The constructive nature of Theorem 4.1 provides the facility to generate facets of clconv(T) by generating facets of clconv(D), and further any facet is accessible by using the methodology of Sen and Sherali (1986). Of course the computational attractiveness of using (4.8) is quite questionable in that it requires the solution of semiinfinite programs, in general. Note, however, that (4.8) can be written as a convex semiinfinite program by utilizing the inequality $\text{Sup}\{\gamma x: x \in C, x \geq 0\} \leq 1$ instead of the corresponding equality used in (4.8). When an optimum is obtained for the inequality constrained semiinfinite program, one must, however, rescale the inequality $\gamma x \leq \gamma_0$ in the manner $(\gamma/\gamma_0)x \leq 1$, where $0 < \gamma_0 = \text{Max}\{\gamma x: x \in C, x \geq 0\} \leq 1$. Upon rescaling, we may utilize the form in (4.12). At any rate, it is the semiinfiniteness that poses the actual computational difficulty, although for some special cases (e.g. when C is polyhedral), (4.8) may be computationally tractable. Alternatively, from the viewpoint of generating only a single valid deep cut, the foregoing development asserts that the following modification of (4.6) is valid.

$$\sum_{j \in J^+} \frac{x_j}{\beta_j} - \sum_{j \in \bar{J}^+} \frac{x_j}{\beta_j} \geq 1 \tag{4.15}$$

where for $j \in \bar{J}^+$,

$$\beta_j \geq \{\text{intercept on } j\text{th axis made by any supporting hyperplane } \pi x \leq 1 \text{ to } C \text{ for which the set } \{x \geq 0: \pi z \geq 1\} \neq \emptyset\}. \tag{4.16}$$

(Here, β_j could be infinitely large, where $1/\beta_j$ is to be taken as zero.) To explain the validity of (4.15), note from the application of the fundmental disjunctive cut principle (Balas (1975)) to the disjunction D defined in (4.11) and (4.12) that a valid cut obtained has coefficients $1/\beta_j$ for $j \in J^+$, by virtue of (4.4) and (4.7). Further, for $j \in \bar{J}^+$, the coefficient of x_j would be the (algebraically) largest, or with a negative sign, the numerically smallest of the coefficients appearing in (4.11) or (4.12). Hence it is valid to take β_j as the numerically largest of the inverse of these coefficients (where $1/0 \equiv \infty$). Since taking β_j larger than this is also valid, the validity of (4.15) is established. The advantage of (4.15) over (4.6) is that it is deeper since it permits negative cut coefficients, and its advantage over the computational use of Theorem 4.1 in general situations is that the bound (4.16) may be more accessible.

5. A numerical illustration using a linear complementarity problem

The fruitfulness of the approach of studying connections between DP and RCP is illustrated in this section, by an application to a general linear complementarity problem (Balas (1975) and Jeroslow (1978)). In general, complementarity conditions arise in numerous disjunctive programs (see Sherali and Shetty (1980a)). It is well known that certain other nonconvex programs like the bilinear program, Gallo and Ulkucu (1977), and the linear bilevel program, Bard and Falk (1982a), and the indefinite quadratic programming problem, Van de Panne (1974), can be transformed to problems with complementarity restrictions.

Consider the following linear complementarity problem, taken from Bard and Falk (1982b). Find x_1, x_2, x_3, x_4 that satisfy

$$-x_1 + x_2 - x_3 \qquad = -1, \tag{5.1a}$$

$$2x_1 - x_2 \qquad -x_4 = 1, \tag{5.1b}$$

$$x_1, \ldots, x_r \geq 0 \tag{5.1c}$$

and

$$x_1 x_3 = 0, \quad x_2 x_4 = 0. \tag{5.2}$$

As noted by Bard and Falk (1982b), both Lemke's method, Lemke (1965), and the principal pivoting algorithm, Cottle and Dantzig (1968), fail to solve the above problem. Let us illustrate the development of cutting planes via the use of a reverse convex constraint. Let us use some linear form, say Min $x_1 + x_2$ to obtain the first extreme point of 5.51). The resulting extreme point is $x_1 = \frac{1}{2}$, $x_2 = 0$, $x_3 = \frac{1}{2}$, $x_4 = 0$ and the corresponding tableau rrepresentation is

$$x_1 = \tfrac{1}{2} + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4, \tag{5.3}$$

$$x_3 = \tfrac{1}{2} + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_4. \tag{5.4}$$

Since $x_1 x_3 = \frac{1}{4} \neq 0$, we derive a cut by using the reverse convex constaint

$$g(x) \triangleq \operatorname{Min}\{x_1, x_3\} + \operatorname{Min}\{x_2, x_4\} \leq 0$$

or

$$\operatorname{Min}\{\tfrac{1}{2} + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4, \tfrac{1}{2} + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_4\} + \operatorname{Min}\{x_2, x_4\} \leq 0.$$

Formulating the set C, we obtain $\beta_4 = 1$, and the corresponding inequality $\alpha^4 x \geq 1$ is given by (corresponding to the subgradient $(\frac{3}{2}, -\frac{1}{2})$ at the point $(x_2, x_4) \equiv (0, 1)$)

$$-3x_2 + x_4 \geq 1. \tag{5.5}$$

Next, to obtain the term β_{24} and the corresponding inequality $\gamma^{24}x \geq 1$, we solve the semiinfinite program

$$\operatorname*{Inf}_{\gamma}\{-\gamma_2/\gamma_4 \colon h(\gamma_2, \gamma_4) \leq 1, \gamma_4 \geq 0\},$$

where $h(\gamma_2, \gamma_4) = \operatorname{Sup}\{\gamma_2 x_2 + \gamma_4 x_4 \colon (x_2, x_4) \geq 0, (x_2, x_4) \in C\}$. Recall that the solution to the above problem will have to be scaled to conform to the form (4.12). We note here that the set C may be written as

$$\begin{aligned} C = \{x \colon & \tfrac{1}{2} + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_4 \geq 0, \\ & \tfrac{1}{2} + \tfrac{3}{2}x_2 - \tfrac{1}{2}x_4 \geq 0, \\ & \tfrac{1}{2} + \tfrac{1}{2}x + \tfrac{3}{2}x_4 \geq 0, \\ & \tfrac{1}{2} + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4 \geq 0\}. \end{aligned}$$

Of course, there are other equivalent representations. Clearly the polyhedral nature of C ensures that a cutting plane algorithm of the type given in Gustafson and Kortanek (1973) is finite. In any case, the solution to the above problem yields the inequality

$$-3x_2 + x_4 \geq 1. \tag{5.6}$$

Utilizing (5.5) and (5.6) to formulate the disjunction, and then applying the disjunctive cut principle as in Balas (1975), we obtain the cut

$$-3x_2 + x_4 \geq 1. \tag{5.7}$$

It is interesting to note that the above cut uniformly dominates the convexity cut $x_4 \geq 1$. Furthermore, if one simply utilizes the disjunctive cut principle on the disjunction: $x_1 \leq 0$ or $x_3 \leq 0$, then utilizing (5.3), (5.4) and the fact that $x_2, x_4 \geq 0$, we obtain the disjunctive cut $-x_2 + x_4 \geq 1$. Note once again that (5.7) dominates this cut. Note that on including (5.7) as a cut, the only point that remains feasible is the point $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 1$, the solution to the linear complementarity problem. (The other two alternative cuts above still admit infeasible points, although the minimization of $x_1 + x_2$ obtains the solution to (5.1), (5.2).)

Conclusion

Reverse Convex Programs (RCP) form an important class of nonconvex programs which subsume such diverse problems as signomial geometric programs and disjunctive programs. The latter class of problems, which itself subsumes facial disjunctive programs such as zero-one integer programs and linear complmentarity problems, gives rise to reverse convex constraints which are almost invariably nondifferentiable. Our development allows us to consider such disjunctive programs as special cases of RCP. Through appropriate generalizations of earlier results, we show that the closure of the convex hull of the feasible region of ECP is polyhedral and we provide an algebraic characterization for a set of solutions which contain the set of optimal solutions. Further, we show that certain special reverse convex sets can be represented by means of a finite linear disjunctive statement. This characterization provides the machinery for generating strong, facetial, convexity or intersection or disjunctive cutting planes. We have illustrated by an example that the cuts obtained through our development can dominate the conventional disjunctive and convexity cuts; however, the actual computation of these may be time consuming. Computational methods for problem RCP are addressed in Sen and Whiteson (1985).

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