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Polar sets and removable singularities of partial differential equations

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0. Introduction

The question of removable singularities for partial differential equations is essentially the following: If u is a solution of such an equation in a domain $V \subset R^n$ with a closet set S (of measure zero) removed; and if u is assumed to belong to a class limiting its size near S (for example $u \in L_p$), what conditions can be put on the size of S to insure that u (after being redefined in S) is a solution in all of V? For example, every bounded harmonic function in a punctured disk has a removable singularity at the "puncture". Here the class limiting the size of u may be taken to be the class of bounded functions; and S may be taken to be a single point.

L. Carleson [4] has shown that if u is harmonic in $V-S$ (V a bounded n-dimensional domain, S a compact set $\subset V$) and S has finite $n-2p'$ dimensional Hausdorff measure $(1/p+1/p'=1)$ then the singularities of u on S are removable provided $u \in L_p$. Serrin [10] has extended this result to second order linear elliptic equations with Hölder continuous coefficients, and has given a different sufficient condition for second order (linear or quasilinear) elliptic equations [11].

Our aim is to treat linear equations of arbitrary order. (Some results of this nature are contained in [3]). We begin by observing (in section 1) that the question of removable singularities of solutions in L_p is closely tied to the notion of "$m-p$ polar" sets in R^n (A compact set S is $m-p$ polar if every element in $H_{-m,p'}(R^n)$ with support in S vanishes), a notion apparently first introduced by Hörmander and Lions [6]. The relationships between the two concepts is expressed in theorems 1 and 2. These theorems are proved in sections 1 and 2 where they are applied to second order equations.

In section 3 generalizations of the $H_{m,p}$ spaces called "A spaces" are introduced, and, using these, sharper results are obtained for equations which are assumed to be of a more special form. Section 4 deals with geometric sufficient conditions for $m-p$ polarity of sets, which seems to be of interest independently of the question of removable singularities (see for example [5]); while in section 5 similar results are obtained for the A-spaces introduced earlier. Finally, as an illustration, the latter results are applied to give geometric conditions for removability of singularities of the heat equation.

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1. Basic notions

Throughout this paper, R^n will denote Euclidean n-space, S a compact subset of R^n, U an open subset of R^n containing S, and V a bounded domain in R^n with C^∞ boundary, containing S.

If B is a Banach space, its dual space will be denoted by B'.

By a *solution* to a partial differential equation we shall always mean a weak solution.

We use the standard definitions of the spaces $C_0^\infty(U) \equiv \mathcal{D}(U)$, $H_{m,p}(U) \equiv H_{m,L_p}(U)$, $\mathring{H}_{m,p}(U)$, as contained for example in [1].

Let B be a Banach space such that $C_0^\infty(U)$ is contained densely in B. We assume that the topology of $C_0^\infty(U)$ is stronger than that of B.

Definition. S is said to be polar with respect to B if the only element in B' having support in S is the zero element. A set polar with respect to $H_{m,p}(R^n)$ is called $m-p$ polar.

Let L be a linear partial differential operator which we write in the form

$$Lu = \sum_{|\alpha| \leqslant m} D^\alpha (a_\alpha(x) u),$$

where,

$$D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{j=1}^{n} \alpha_j$$

and where the a_α are bounded measurable functions defined on V.

Suppose the following holds: If u is a (weak) solution to $Lu = 0$ in $V - S$ and $u \in L_p(V)$ then $Lu = 0$ in V. Then we say that S is removable with respect to (L, V, L_p).

By a weak solution to $Lu = T$ in an open set Ω, where T is a distribution, we mean a distribution u such that $\langle L^*\phi, u\rangle = \langle \phi, T\rangle$ for all ϕ in $C_0^\infty(\Omega)$. As stated earlier by "solution" we shall always mean "weak solution".

We shall assume $1 < p < \infty$, except in the remainder of this section, (where p may equal 1) or as specified. Also throughout $1/p + 1/p' = 1$.

The following close relationship between the two concepts introduced is an almost immediate consequence of the definitions:

Theorem 1. *A sufficient condition for S to be removable with respect to $[L, V, L_p]$ is that S be $m-p'$ polar.*

We shall need a few lemmas. The following lemma is essentially stated by Grusin [5].

Lemma 1. *If multiplication by an arbitrary but fixed $\phi \in C_0^\infty(U)$ is continuous in B, then S is polar with respect to B if and only if there exists a sequence $\phi_\nu \in C_0^\infty(U)$ such that each $\phi_\nu = 1$ in some neighborhood of S (possibly depending on ν) and $|\phi_\nu| \to 0$.*

Proof. Suppose there exists such a sequence ϕ_ν and $T \in B'$ has support in S. We wish to show $T = 0$. It suffices to show $\langle \psi, T\rangle = 0$ for all $\psi \in C_0^\infty(U)$ since $C_0^\infty(U)$ is dense in B. Now

$$|\psi\phi_\nu|_B \leqslant \text{constant}(\psi) \, |\phi_\nu|_B \to 0$$

and since $\psi - \psi \cdot \phi_\nu$ vanishes near S and T has support on S,

$$\langle \psi - \psi\phi_\nu, T\rangle = 0 \quad \text{all } \nu;$$

But $\psi - \psi\phi_\nu \to \psi$ in B, hence

$$\langle \psi, T\rangle = \lim_{\nu\to\infty} \langle \psi - \psi\phi_\nu, T\rangle = 0.$$

Conversely, suppose S is polar. Let B_s denote the set of elements vanishing near S. We claim that B_s is dense in B. If this were not the case, there would exist a non zero $T \in B'$ such that $\langle \overline{B_s}, T\rangle = 0$ (bar = closure) and T would have to vanish. Now let $\psi \in C_0^\infty(U)$ such that $\psi = 1$ near S. Then there exists a sequence $\psi_\nu \in C_0^\infty(U)$ and vanishing near S such that $\psi_\nu \to \psi$ in B. The sequence $\psi - \psi_\nu \equiv \phi_\nu$ ($=1$ near S) has compact support and $|\phi_\nu|_B \to 0$.

Lemma 2. *S is polar with respect to $H_{m,p}(R^n) \equiv \mathring{H}_{m,p}(R^n)$ if and only if it is polar with respect to $\mathring{H}_{m,p}(U)$.*

Proof. The "if" part is immediate from lemma 1. Suppose S is $m-p$ polar. Then if ϕ_ν is the sequence guaranteed by lemma 1, $\zeta\phi_\nu$ is the sequence needed to conclude that S is polar with respect to $\mathring{H}_{m,p}(U)$, where $\zeta \in C_0^\infty$ and equals 1 near S.

2. m—p polarity and removable singularities

We first prove theorem 1: Suppose $u \in L^p(V)$ then $Lu \in H_{-m,p}(V)$ and has support in S. Now the dual of $H_{-m,p}(V)$ is $\mathring{H}_{m,p'}(V)$ and since, by lemma 2, S is polar with respect to $\mathring{H}_{m,p'}(V)$, Lu must be the zero element in $H_{-m,p}(V)$. Thus u is a weak solution to $Lu = 0$.

Next we wish to give a partial converse to Theorem 1. Let L^* denote the formal adjoint of L. We say that the *weak unique continuation property* holds for $L^*u = 0$ in the open set W if every solution to $L^*u = 0$ having compact support in W vanishes identically in W.

Theorem 2. *Let W be a bounded domain with C^∞ boundary containing V, but having no common boundary points with it. Suppose that L^* is strongly elliptic and its coefficients are Hölder continuous in W, that $L^* = 0$ has the weak unique continuation property*[1] *in W. Then the condition of theorem 1 is also necessary.*

Proof. First, suppose that the Dirichlet problem for the equation $L^*u = 0$ in V has a unique solution. Then L^* can be extended (see, for example [1]) as an isomorphism $\mathfrak{L}_1$

$$\mathfrak{L}_1\colon \quad H_{m,p'}(V) \cap \mathring{H}_{(m/2),p'}(V) \to L_{p'}(V) \text{ (onto).}$$

The Banach space adjoint $\mathfrak{L}_2$ of $\mathfrak{L}_1$ then is an isomorphism

$$\mathfrak{L}_2\colon \quad L_p(V) \to (H_{m,p'}(V) \cap \mathring{H}_{(m/2),p'})' \text{ (onto).}$$

In particular, the equation $\mathfrak{L}_2 u = T$ can be solved for u in $L_p(V)$ provided $T \in \mathring{H}_{-m,p}(V)$. For in that case T is a bounded linear functional on $\mathring{H}_{m,p'}(V)$, hence it is also (by restriction) a bounded linear functional on $H_{m,p'}(V) \cap \mathring{H}_{(m/2),p'}$, and thus $\mathfrak{L}_2 u = T$ can be solved with $u \in L_p(V)$. It is easily checked that this solution is a weak solution of $Lu = T$. Now suppose S is not polar. Then there exists a non zero T in $H_{-m,p}(V)$ with support in S. But this, in turn, implies that $T \in \mathring{H}_{-m,p}(V)$ and hence we can solve $Lu = T$ with u in $L_p(V)$. Thus we have a u in $L_p(V)$ which satisfies $Lu = 0$ in

[1] This last assumption is not essential.

$V-S$ but not in V. In the above argument, V could have been replaced by W equally well.

Next suppose that the Dirichlet problem for L^* in W has a non-trivial null space. This must be finite dimensional. Suppose it is generated by eigenfunctions $v_1, v_2, \dots v_k$, which are assumed linearly independent in W. By the weak unique continuation property it follows that they must also be linearly independent in $W-V$. Let $v_j' = v_j$ in $W-V$, 0 otherwise. We may assume that the v_j' are orthonormal with respect to L_2. Now the equation $\mathcal{L}_2 u = T - \sum \alpha_j v_j'$ can be solved for u in $L_p(W)$ provided $0 = (T - \sum \alpha_j \cdot v_j', v_i)$ for $i = 1, \dots k$, i.e., if $\alpha_i = (T, v_i)$. With this choice of the α_j solve for u. Then in V $Lu = T$ and we argue as before.

Remark 1. *Applications to second order linear equations.*
For second order elliptic equations of the forms

$$\text{a)}\quad \textstyle\sum (a_{ij} u)_{x_i x_j} + \sum (a_i u)_{x_i} + au = 0,$$

$$\text{b)}\quad (b_{ij} u_{x_i})_{x_j} + b_i u_{x_i} + bu = 0,$$

$$\text{c)}\quad \textstyle\sum c_{ij} u_{x_i x_j} + c_i u_{x_i} + cu = 0$$

let us assume that the a's are bounded measurable; the b_{ij} and b_i have bounded measurable derivatives (for example if they are Lipschitz continuous) and b is bounded; or that the c_{ij} have bounded measurable second derivatives, the c_i bounded measurable first derivatives and the c_i are bounded. In that case it follows from what has been said that sets are removable for a, b, c provided they are for the Laplacian. Now L. Carleson [4] has shown that a sufficient condition for S to be removable with $u \in L_p$ is that S has finite $n-2p'$ Hausdorff measure. Thus the same conclusion holds for the above equations. Let us note that for case b) our results do not imply Serrin's, nor do his imply ours. For the case c), however, Serrin's results are stronger than ours.

Remark 2. Suppose that in theorem 1 instead of being told that the function u is in $L_p(V)$ we are told that $u \in H_{k,p}(V)$. Let us assume for simplicity that the coefficients are sufficiently smooth. Then $Lu \in H_{k-m,p}$ with support in S. Thus S is removable provided it is polar with respect to $\mathring{H}_{m-k,p'}$. Here k may be positive or negative. The converse holds under the same additional assumptions as occur in theorem 2.

3. A-polarity

We would like to improve the results so far obtained for equations which are not elliptic. To that end we first consider the following general situation. Suppose A is the closure of $C_0^\infty(V)$ in a certain norm $|\ \ |_A$. Suppose furthermore that for $u \in C_0^\infty(V)$ the following a priori inequalities are valid:

$$C_1 |u|_A \leqslant |L^* u|_{p'} \leqslant C_2 |u|_A. \tag{3.1}$$

Then L^* has a bounded extension $\mathcal{L}_1$

$$\mathcal{L}_1\colon\quad A \to L_{p'}(V)$$

with closed range. Hence the Banach Space adjoint operator $\mathcal{L}_2$

$$\mathfrak{L}_2:\quad L_p(V)\to A'$$

is also bounded and has a closed range. Since $\mathfrak{L}_1$ is one to one, $\mathfrak{L}_2$ is onto. This implies that there exists a weak solution in $L_p(V)$ to $Lu=T$ where $T\in A'$.

From now on we are going to specialize the space A. Suppose

$$L^*\equiv \sum_{\alpha\in J} a_\alpha(x)D^\alpha, \tag{3.2}$$

where J is some finite collection of n-tuples of non-negative integers $\alpha=(\alpha_1, \dots, \alpha_n)$. We suppose that the order of the differential operator (3.2) is m. We may then choose for $|\ \ |_A$ the norm

$$\sum_J |D^\alpha u|L_{p'}\equiv |u|_A. \tag{3.3}$$

Theorem 3. *Let A be the closure of $C_0^\infty(V)$ with respect to the norm* (3.3). *Then S is removable with respect to $[L, V, L_p]$ if S is polar with respect to A. If in addition, the inequality*

$$|L^*u|_{p'}\geqslant C|u|_A \tag{3.4}$$

holds for all $u\in C_0^\infty(V)$ then the converse holds.

Proof. Using the notation already introduced, $\mathfrak{L}_1$ is a bounded map from $A\to L_p'(V)$. Hence $\mathfrak{L}_2$ is bounded from $L_p(V)\to A'$. Now suppose $u\in L_p(V)$ and $Lu=0$ in $V-S$; then $\mathfrak{L}_2u=T\in A'$ and has support in S. But if S is polar with respect to A this implies that T must vanish and hence $\mathfrak{L}_2u=0$ i.e. $Lu=0$ in V (weakly).

To prove the converse, we note that under the additional assumptions made, (3.1) holds and hence the map $\mathfrak{L}_2$: $L_p(V)\to A'$ is onto. Thus if S is not polar we can find a $T\in A'$ with support in S and solve the equation $\mathfrak{L}_2u=T$ with $u\in L_p(V)$.

Example. The heat equation. $\Delta u=u_t$.

Here we take

$$|u|_A\equiv\sum_i\left|\frac{\partial^2 u}{\partial x_i^2}\right|_{L_{p'}}+\left|\frac{\partial u}{\partial t}\right|_{L_{p'}}.$$

From what has been said we see that *a sufficient condition for S to be removable with respect to [heat operator, V, L_p] is that there exist a sequence ϕ_j: of functions in $C_0^\infty(R^n)$ which equal* 1 *in a neighborhood of S such that* $|\phi_j|_A\to 0$. A different criterion has been given Aronson [2] and Pini [9]. The *necessity* of our condition follows from the L_p estimate for the heat equation of Jones [7] and the second part of theorem 3.

4. Sufficient conditions for m—p polarity

In this section we wish to derive a geometric criterion for a compact set S to be $m-p$ polar. Let us here note that from now on we interchange the roles of p and p' (for the purpose of simplicity).

Given a compact set S consider a covering of S by open spheres of radius r and let $N(r)$ denote the smallest number of such spheres (or radius r) required for such a covering. We then define

$$M^\alpha\equiv M^\alpha(S)\equiv\lim_{r\to 0}\inf N(r)r^\alpha.$$

Theorem 4. *If $M^\alpha(S)<\infty$ then S is $m-p$ polar for $n-mp\geqslant\alpha$* $(1<p<\infty)$. (For $p=1$ we have to assume $M^\alpha(S)=0$.)

Remark. Let us note here that, for integral α, if S is a compact set contained in a smooth α dimensional manifold then certainly $M^\alpha(S)<\infty$. (For this case, if $p=2$, theorem 4 was proved in [6].) However, fractional α is not devoid of meaning. For example the usual one dimensional Cantorset has $M^\alpha<\infty$ where $\alpha-\log 2/\log 3$. By changing the lengths of the intervals used in the definition of the Cantor set one can arrive at arbitrary α. Let us also note that M^α is not quite the same as α dimensional Hausdorff measure, which is a somewhat more refined measure of dimension.

Lemma 3. *A sufficient condition for S to be $m-p$ polar is that there exist a sequence ϕ_j of functions in $C_0^\infty(R^n)$ which equal 1 near S, are uniformly bounded in R^n, such that the measure of their supports $\to 0$ and $|\phi_j|_{m,p}\leqslant const.$*

Proof. We know $\phi_j\to 0$ in $L_p(R^n)$. Now there exists a subsequence converging weakly in $H_{m,p}$ to a limit ϕ. By the Banach Saks theorem there exists a (further) subsequence whose arithmetic means $\psi_j\to\phi$ strongly in $H_{m,p}$. But then this convergence must also take place strongly in L_p, $\psi_j\to\phi$, which implies $\phi=0$. The sequence ψ_j satisfies all requirements of lemma 1. Note: the above lemma is not valid for $p=1$. We now proceed with the proof of theorem 4 (we treat only the case $1<p<\infty$).

We consider the grid G_r which divides R_n into cubes of side length r, with sides parallel to the coordiante axes and with one cube centered at the origin. Denoting by $N(r)$ the minimal number of cubes of this grid needed to cover S, we similarly define

$$\liminf_{r\to 0} \underline{N}(r)r^\alpha\equiv\underline{M}^\alpha\equiv\underline{M}(S).$$

It is easily seen that the following inequalities hold with positive constants C depending only on n.

$$C_1 M^\alpha(S)\leqslant\underline{M}^\alpha(S)\leqslant C_2 M^\alpha(S).$$

Let $\alpha(t)$ be a C^∞ function of t having the following properties

1. $\alpha(t)$ is symmetric about $t=0$.
2. $\alpha(t)=1$ for $|t|<\frac{1}{4}$.
3. $0\leqslant\alpha(t)\leqslant 1$ for $\frac{1}{4}\leqslant t\leqslant\frac{3}{4}$.
4. $\alpha(t)=0$ for $|t|\geqslant\frac{3}{4}$.
5. $\alpha(t)+\alpha(t-1)=1$ for $\frac{1}{4}\leqslant t\leqslant\frac{3}{4}$.

(This will then automatically hold for all t in

$$-\tfrac{1}{4}\leqslant t\leqslant\tfrac{5}{4}.)$$

Define $\beta(x)=\alpha(x_1)\cdot\alpha(x_2)\ \ldots\ \cdot\alpha(x_n)$. Now pick an $r>0$. Consider the collection C of all cubes in G_r covering S and add all cubes in G_r having at least one common boundary point with the cubes in C to obtain the (larger) collection C' of cubes. Similarly let C'' denote the (even larger) collection obtained by adding to C' all cubes with common boundary points. Let H denote the set of all centers of cubes in C'. Then define

$$\phi_r(x)\equiv\sum_{h\in H}\beta\left(\frac{x}{r}-h\right).$$

The function ϕ_r is C^∞, equals 1 in a neighborhood of S, never exceeds 1, and vanishes in the complement of the union of the cubes in C'', hence at points whose distance from S is greater than $3r\sqrt{n}$.

Next let us estimate $|\phi_r|^p_{m,p}$. From dimensional considerations it follows that for a fixed $h \in H$

$$\left|\beta\left(\frac{x}{r}-h\right)\right|^p_{m,p} \leqslant \text{const. } r^{n-mp}.$$

Now in each cube c in C'' all but at most a finite number (which depends only on n) of the terms in the sum defining ϕ_r vanish. Hence

$$|\phi_r|^p_{m,p,c} \leqslant \text{const. } r^{n-mp}.$$

Now, since $M^\alpha(S)$ is finite, we can find a sequence r_j such that

$$\underline{N}(r_j) \leqslant \text{const } r_j^{-\alpha},$$

hence
$$|\phi_{r_j}|^p_{m,p} \leqslant \text{const } r_j^{n-mp-\alpha}.$$

If, as has been assumed, $n-mp \geqslant \alpha$, it follows that the norms on the left hand side are bounded and hence the sequence ϕ_{r_j} satisfies all the requirements of the lemma.

Remark. The above theorem remains true if m is fractional, where the $|\ \ |_{m,p}$ norm is defined by complex interpolation, for example [8]. To see this, let m_0 and m_1 be two consecutive integers such that $m = m_0 + \theta(m_1 - m_0)$. Then we have as before

$$|\phi_{r_j}|_{m_0,p} \leqslant \text{const. } r_j^{((n/p)-m_0-(\alpha/p))},$$

the same inequality for m_1, hence

$$|\phi_{r_j}|_{m,p} \leqslant \text{const. } |\phi_{r_j}|^{1-\theta}_{m_0,p}\,|\phi_{r_j}|^{\theta}_{m_1,p} \leqslant \text{const. } r^{(1/p)(n-mp-\alpha)} \leqslant \text{const.}$$

and the proof proceeds as before.

5. Sufficient conditions for A-polarity

We would like to apply the methods in the preceding section to derive sufficient conditions for sets to be polar with respect to spaces of the type A discussed earlier. Here we take A to be the completion of $C_0^\infty(R^n)$ with respect to the norm defined by

$$|u|^p_A \equiv \sum_{\alpha \in K} |D^\alpha_u|^p_{Lp},$$

where K is some finite subset of all n-tuples of non-negative integers. We proceed as in the last section, except that instead of coverings by spheres or cubes we now consider coverings by rectangular solids having side lengths

$$r^{s_1} r^{s_2} \ldots r^{s_n},$$

where the s's are fixed non-negative numbers $\sum s_i = n$. We define $\underline{N}_s(r)$ as before and let

$$M^\alpha_s \equiv \liminf_{r\to\infty} \underline{N}_s(r) r^\alpha.$$

We proceed as before and estimate

$$\left| D^k \beta\left(\frac{x}{r} - h\right)\right|^p_{L_p} \leqslant \text{const } r^n \cdot r^{-(s\cdot k)p},$$

where $\quad \dfrac{x}{r} - h = \left(\dfrac{x_1}{r} - h_1, \dfrac{x_2}{r} - h_2 \ldots\right)$ and $(s \cdot k) \equiv s_1 k_1 + \ldots s_n k_n.$

Thus
$$\left|\beta\left(\frac{x}{r} - h\right)\right|^p_A \leqslant \text{const. } \max_{k \in K} r^{n-(s\cdot k)p},$$

$$\leqslant \text{const. } r^{n - m_{s,k}}$$

where we let
$$m_{s,k} \equiv \max_{k \in K} (s \cdot k).$$

Proceeding as before, it follows that

$$|\phi_r|_A \leqslant \text{const. } r^{n-p\cdot m_{s,k}} \underline{N}_s(r).$$

And we culminate with the

Theorem 5. *Suppose S is a compact set such that for some choice of $s = (s_1 \ldots s_n)$* (with $\sum s = n$)

$$M_s^{n-p\cdot m_{s,k}}(S) < \infty.$$

Then S is polar with respect to A.

Note. In proving theorem 5 lemma (3) used in the proof of theorem 4 must be modified. This causes no trouble.

Example. The heat equation.

The one dimensional heat equation leads us to choose

$$|u|^p_A = |u_t|^p_{L_p} + |u_{xx}|^p_{L_p}, \quad x = x_1, \quad t = x_2.$$

Now suppose S is a compact set situated on the x axis, and $M^\alpha(S) < \infty$. How small does α have to be to insure that S is A-polar? The problem is to choose s in such a way that theorem 5 will give optimal results. To that end, consider a covering of S by $N^\alpha(\bar{r})$ equal rectangles of length $\bar{r} = r^{s_1}$ and height r^{s_2} $(s_1 + s_2 = 2)$. Now suppose first that

$$N(\bar{r}) \leqslant \text{const. } \bar{r}^{-\alpha}. \tag{*}$$

Then
$$N_s(r) \leqslant \text{const. } r^{-\alpha s_1}.$$

It thus would suffice to have

$$\alpha s_1 \leqslant 2 - p \max [2s_1, 2 - s_1],$$

where the maximum on the right is not taken with respect to s_1. We wish to choose s to maximize the allowable α, i.e., we wish to maximize the expression

$$\frac{1}{s_1}(2 - p \max [2s_1, 2 - s_1])$$

over the interval $(0 \leqslant s_1 \leqslant 2)$. This maximum is attained at $s_1 = \frac{2}{3}$ $s_2 = \frac{4}{3}$ and the best α is $3-2p$. Thus if $1 < p < \frac{3}{2}$ and $M^{3-2p}(S) < \infty$ then S is polar with respect to A, provided S lies on the x axis. (If (*) does not hold for all $0 < r < 1$, we simply pick a sequence of $\bar{r}_i$ for which it holds and argue as before.)

For the corresponding situation in higher dimensions we conclude by similar calculations that if S lies in the hyperplane $t=0$ then S is A polar if $M^{n+1-2p}(S) < \infty$. This shows that if u satisfies the n dimensional ($n-1$ space dimensions) heat equation in $R^n - S$, S lies on $t=0$, and $u \in L_{p'}(R^n)$, then u is a solution in all R^n provided $M^{n+1-2p}(S) < \infty$. This is certainly true if, for example S lies in an $[n+1-2p]$ dimensional surface.

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